On the regularity of weak solutions to refractor problem

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July 24, 2008

Abstract

We introduce a simple method, allowing to simplify the regularity issues for weak solutions to refractor problem. It avoids the use of covariant derivatives and it is straightforward. Main idea is to use a suitable parametrization of unit sphere used in [KW] in connection to reflector problem.

1 Introduction and main result

It is well-known that ellipse and hyperbola have simple refraction properties, namely if rays of light diverge from one focus, then after refraction they pass parallel to the major axis (see figure 1). If the ellipse (resp. hyperbola) represents the boundary separating two medias, with refractive constants n_1, n_2 then according to refraction law

$$n_1 \sin \alpha = n_2 \sin \beta,$$

where α and β are the angles between normal and respectively the ray before and after refraction. Introduce the refractive index, $k = n_1/n_2$, then one can verify that $k = 1/\varepsilon$, where ε is the eccentricity of ellipse (resp. hyperbola) [M]. These properties are limiting cases of solutions to more general problems of determining the surface required to refract rays of light diverging from one point and after refraction covering a given set of directions on the unit sphere. More precisely let us assume we are given two sets Ω, Ω^* on unit sphere centered at origin, and nonnegative integrable functions f, g defined respectively on Ω and Ω^*. For every $X \in \Omega$ we issue a ray from origin passing through X, which after refraction from the unknown surface Γ is another ray given by $Y = Y(X) \in \Omega^*$. It is clear that mapping Y is determined by Γ. Now the problem is the following: given two pairs (Ω, f) and (Ω^*, g) satisfying to mass balance condition

$$\int_\Omega f = \int_{\Omega^*} g, \tag{1.1}$$

find a surface Γ, such that for corresponding mapping $Y(X)$ we have

$$Y(\Omega) = \Omega^*.$$

Suppose that $\Gamma = \{Z, Z = X\rho(X)\}$, then mathematically this problem is amount to solve a Monge-Ampère type equation

$$\det(D^2_{ij}\rho - \sigma_{ij}(x, \rho, D\rho)) = h(x, \rho, D\rho), \tag{1.2}$$

subject to boundary condition

$$Y(\Omega) = \Omega^*. \tag{1.3}$$

Here Ω is a subset of upper half sphere. The solution to (1.2), should be sought in the class of functions such that the matrix $D^2_{ij}\rho - \sigma_{ij}(x, \rho, D\rho) \geq 0$. If ρ is smooth and ρ_1 is the radial, smooth function defining Γ_1 such that Γ_1 touches Γ from above, moreover

$$D^2_{ij}\rho_1 - \sigma_{ij}(x, \rho_1, D\rho_1) = 0,$$

then it is easy to see that it implies $D^2_{ij}\rho - \sigma_{ij}(x, \rho, D\rho) \geq 0$. It turns out that the suitable support functions with above properties are ellipsoids and hyperboloids of revolution (see Section 2.1).

Recently C.Gutierrez and Q.Huang proved that the problem above is an optimal transfer problem [U] with cost function

$$c(X, Y) = \log \frac{1}{1 - \frac{1}{k}(X \cdot Y)}.$$

A similar cost function appears in reflector problem introduced by X-J. Wang [W1], [W2]. The regularity of the solutions to optimal transfer problems are discussed in [MTW] and

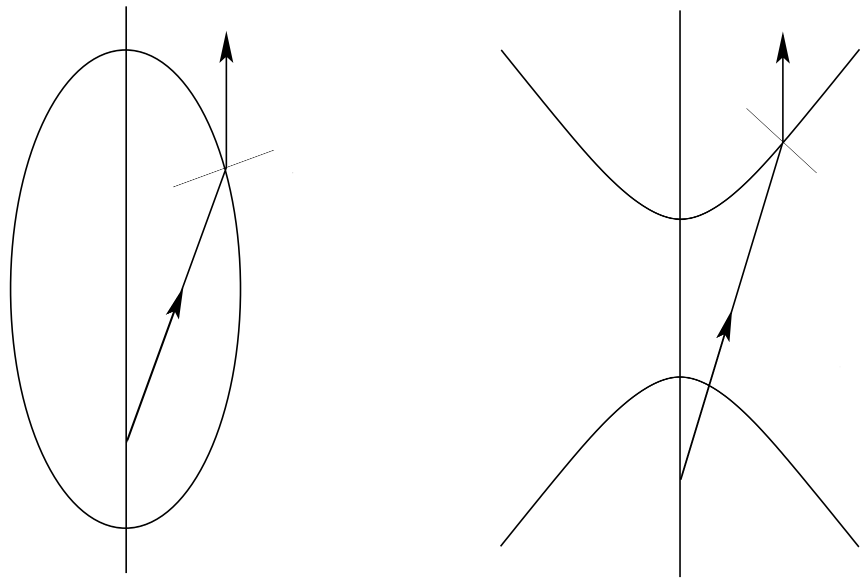

Figure 1: Refraction from ellipse and hyperbola.

[TW1]. The most important thing is the so-called A3 condition, imposed on matrix σ_{ij} [MTW]. As soon as one has it the rest of the regularity, both local and global will follow from the classical framework established in [MTW], [TW1] and [TW2]. In [GH] authors have verified the A3 condition, however without using Euclidian coordinates.

In this note we give a simple way of verifying the A3 condition, for $k > 1$ without invoking to covariant derivatives. It is also explicit, strict and straightforward (2.16). Main idea is to find a simple formula for mapping $Y(X)$ using a parametrization of upper unit half sphere, used in [KW]. Then the rest will follow along the arguments of [KW]. This method is very general and one can apply it to *far-field* problem. Indeed if one considers a map $z = \rho x + ty$, where t is the stretch function, then $\det Dz$ will give the equation for far-field problem. However we don't discuss this problem in present note. It is worth noting that, if support functions are hyperbolas, i.e. $k < 1$ the A3 condition is in general not fulfilled (see (2.16)). Our main result is contained in the following

Theorem 1 *If ρ is the radial function defining Γ, and $u = 1/\rho$, then u is a weak solution* *to*

$$det\left\{D^2u-(Id+\frac{x\otimes x}{1-|x|^2})\frac{1}{B}\right\}=h, \tag{1.4}$$
$$h=\frac{f(x)}{g(y)}\frac{|Y_{n+1}V|}{(k|\delta|)^n}\frac{1}{(1-|x|^2)|det\mu|},$$

where $b=u^2+|Du|^2-(Du\cdot x)^2, V=\sqrt{u^2-\delta b}$ *and* μ *is given by (2.8). Furthermore let's assume that* Ω_0^* *is* c^* *convex with respect to* Ω_0*, where* Ω_0 *and* Ω_0^* *are the orthogonal projections of respectively* Ω *and* Ω^* *onto hyperplane* $x_{n+1}=0$ *and*

$$c(x,y)=\log\left\{1-\varepsilon(x\cdot y+\sqrt{1-|x|^2}\sqrt{1-|y|^2})\right\}.$$

If $\varepsilon<1$*, then* B^{-1} *is concave in gradient, and hence the weak solution* u *is locally smooth, provided densities* $f\in C^2(\Omega), g\in C^2(\Omega^*)$ *and* $0<\lambda\le f,g\le\Lambda<\infty$*.*

For definition of c^* convexity we refer to [MTW].

1.1 Problem Set-up

Let us consider the case of two homogeneous medias, with refractive constants n_1 and n_2. Ω and Ω^* are two domains on the unit sphere $\mathcal{S}^n=\{x=(x_1,\dots,x_{n+1}),x_1^2+\dots+x_{n+1}^2=1\}$. For $X\in\mathcal{S}^n, x=(x_1,\dots,x_n,0)$. We also suppose that Ω is a subset of upper unit sphere $\mathcal{S}^n\cap\{x_{n+1}>0\}$. In what follows we consider ρ as a function of $x\in\Omega_0$, with Ω_0 as orthogonal projection of Ω on to hyperplane $x_{n+1}=0$. By $D\rho$ we denote the gradient of function ρ with respect to x variable $D\rho=(D_{x_1}\rho,\dots,D_{x_n}\rho)$. First let us derive a formula for unit vectors X and Y, using angles α and β. Since X,Y and outward unit vector γ lie in the same plane, we have

$$Y=C_1X+C_2\gamma \tag{1.5}$$

for two unknowns, C_1 and C_2 depending on X. If one takes the scalar product of Y with γ and then with e_{n+1}, then

$$\begin{cases}\cos\beta=C_1\cos\alpha+C_2\\ \cos(\alpha-\beta)=C_1+C_2\cos\alpha.\end{cases} \tag{1.6}$$

Multiplying the first equation by $\cos\alpha$ and subtracting from the second one we infer

$$C_1=\frac{\sin\beta}{\sin\alpha},\qquad\qquad C_2=\cos\beta-C_1\cos\alpha. \tag{1.7}$$

Introduce $k = n_1/n_2$, hence we find that $C_1 = k$ and $C_2 = \cos\beta - k\cos\alpha$, that is

$$Y = kX + (\cos\beta - k\cos\alpha)\gamma. \tag{1.8}$$

We can further manipulate (1.8). Note that

$$n_2^2 - n_2^2\cos^2\beta = n_2^2\sin^2\beta = n_1^2\sin^2\alpha = n_1^2 - n_1^2\cos^2\alpha. \tag{1.9}$$

Dividing the both sides by n_2^2 we obtain

$$k^2\cos^2\alpha = (k^2-1) + \cos^2\beta.$$

Returning to (1.8) we get

$$\begin{aligned} Y &= kX + (\sqrt{k^2\cos^2\alpha - (k^2-1)} - k\cos\alpha)\gamma = \\ &= k\left(X + [\sqrt{(X\cdot\gamma)^2 - \delta} - X\cdot\gamma]\gamma\right), \end{aligned} \tag{1.10}$$

where $\delta = (k^2-1)/k^2$. From [KW] we have

$$\gamma = -\frac{D\rho - X(\rho + D\rho\cdot x)}{\sqrt{\rho^2 + |D\rho|^2 - (D\rho\cdot x)^2}} \tag{1.11}$$

where $X = (x, \sqrt{1-|x|^2}), D\rho = (\rho_{x_1}, \ldots, \rho_{x_n})$. It is convenient to work with a new function $u = \rho^{-1}$. By direct computation we have that

$$\gamma = \frac{Du + X(u - Du\cdot x)}{\sqrt{u^2 + |Du|^2 - (Du\cdot x)^2}}. \tag{1.12}$$

Introduce $b = u^2 + |Du|^2 - (Du\cdot x)^2$, then

$$\begin{aligned} Y &= k\left(X + [\sqrt{(X\cdot\gamma)^2 - \delta} - X\cdot\gamma]\gamma\right) \\ &= k\left(X + [\sqrt{\frac{u^2}{b} - \delta} - \frac{u}{\sqrt{b}}]\gamma\right) \\ &= k\left(X + b^{-1}[\sqrt{u^2 - \delta b} - u][Du + X(u - Du\cdot x)]\right), \end{aligned} \tag{1.13}$$

where we used

$$X\cdot\gamma = \frac{u}{\sqrt{u^2 + |Du|^2 - (Du\cdot x)^2}} > 0. \tag{1.14}$$

It is worth to point out that $\cos^2\beta = k^2\cos^2\alpha - (k^2-1) \geq 0$ implies that $u^2 - \delta b \geq 0$. In its turn this gives a gradient estimate

$$|Du| \leq \frac{\varepsilon}{\sqrt{1-\varepsilon^2}} \frac{u}{X_{n+1}}$$

provided $\varepsilon < 1$ and hence $\delta = 1 - \varepsilon^2 > 0$. Before starting our computations let us note, that if $\mu = Id + C\xi \otimes \eta$ for some constant C and for any two vectors $\xi, \eta \in \mathbf{R}^n$, then one has

$$\mu^{-1} = Id - \frac{C\xi \otimes \eta}{1 + C(\xi \cdot \eta)}. \tag{1.15}$$

Recall that Y is a unit vector, hence $D_k Y_{n+1} = -y \cdot D_k y / Y_{n+1}$, where $y = (Y_1, Y_2, \ldots, Y_n, 0)$, so we conclude

$$\begin{aligned} \frac{dS_{\Omega^*}}{dS_\Omega} &= \begin{vmatrix} Y_{1,1}, & \cdots, & Y_{1,n}, & Y_1 \\ \vdots & \ddots & \ddots & \vdots \\ Y_{n,1}, & \cdots, & Y_{n,n}, & Y_n \\ Y_{n+1,1}, & \cdots, & Y_{n+1,n}, & Y_{n+1} \end{vmatrix} \\ &= \frac{1}{Y_{n+1}} \det Dy. \end{aligned} \tag{1.16}$$

In fact one needs to take the absolute value of the right hand side to obtain the Jacobian J.

2 Proof of Theorem 1

The aim of this section is to prove the following

Proposition 1 *If Y is given as above and*

$$y = k\left[x - \frac{\delta}{\sqrt{u^2 - \delta b} + u}(Du + x(u - Du \cdot x))\right], \tag{2.1}$$

then

$$\frac{Dy}{k} = \frac{\delta}{V}\mu[Id - x \otimes x]\left\{(Id + \frac{x \otimes x}{1 - |x|^2})\frac{1}{B} - D^2 u\right\}, \tag{2.2}$$

where $b = u^2 + |Du|^2 - (Du \cdot x)^2$.

Proof. Introduce $V = \sqrt{u^2 - \delta b} + u, z = Du + x(u - Du \cdot x)$. Using these notations one can rewrite

$$y = k[x - \frac{\delta}{V} z].$$

By a direct computation we have

$$\frac{Dy}{k} = \delta_{ij} - \frac{\delta}{V}(z^i_j - \frac{z^i V_j}{V}). \tag{2.3}$$

Note that

$$\begin{aligned} z^i_j &= u_{ij} - x_i x_m u_{m,j} + \delta(u - Du \cdot x), &(2.4)\\ V_j &= p u_j - q(u_m - (u_m - (Du \cdot x)x_m))u_{mj}, \\ p &= \frac{V - \delta(u - Du \cdot x)}{V - u}, q = \frac{\delta}{V - u}. \end{aligned}$$

Then

$$\begin{aligned} \frac{Dy}{k} &= \delta_{ij} - \frac{\delta}{V}\Big[(Id - x \otimes x)D^2 u + Id(u - Du \cdot x) - \frac{p}{V} z \otimes Du &(2.5)\\ &-\frac{q}{V} z \otimes (Du - (Du \cdot x)x)D^2 u\Big] &(2.6)\\ &= [1 - \frac{\delta}{V}(u - Du \cdot x)]\Big[Id + Az \otimes Du \\ &-B\left\{(Id - x \otimes x) + \frac{q}{V} z \otimes (Du - (Du \cdot x)x)\right\} D^2 u\Big], &(2.7) \end{aligned}$$

where we set

$$A = \frac{\frac{\delta p}{V^2}}{1 - \frac{\delta}{V}(u - Du \cdot x)}, B = \frac{\frac{\delta}{V}}{1 - \frac{\delta}{V}(u - Du \cdot x)}.$$

Lemma 1 *Let $\mu = Id + Az \otimes Du$, then*

$$\mu^{-1}\left\{(Id - x \otimes x) + \frac{q}{V} z \otimes (Du - (Du \cdot x)x)\right\} = Id - x \otimes x. \tag{2.8}$$

Proof. First by (1.15)

$$\mu^{-1} = Id - \frac{Az \otimes Du}{1 + A(z \cdot Du)}.$$

Let $\mathcal{N} = \left\{(Id - x \otimes x) + \frac{q}{V} z \otimes (Du - (Du \cdot x)x)\right\}$, then by a direct computation we have

$$\begin{aligned} \mu^{-1}\mathcal{N} &= (Id - x \otimes x) + \frac{q}{V} z \otimes (Du - (Du \cdot x)x) - \frac{Az \otimes Du}{1 + A(z \cdot Du)} &(2.9)\\ &+\frac{A}{1 + A(z \cdot Du)}[(Du \cdot x)z \otimes x - \frac{q}{V}(Du \cdot z)z \otimes (Du - (Du \cdot x)x)]. \end{aligned}$$

Let us sum up all $\otimes$ products with z, the resulting vector is

$$\frac{q}{V}(Du-(Du\cdot x)x)+\frac{A}{1+A(z\cdot Du)}[-Du+(Du\cdot x)x-\frac{q}{V}(Du\cdot z)(Du-(Du\cdot x)x)]$$
$$=[\frac{q}{V}-\frac{A}{1+A(z\cdot Du)}(1+\frac{q}{V}Du\cdot z)](Du-(Du\cdot x)x).$$

On the other hand

$$\frac{q}{V}-\frac{A}{1+A(z\cdot Du)}(1+\frac{q}{V}Du\cdot z)=\frac{1}{1+A(z\cdot Du)}[\frac{q}{V}-A]. \tag{2.10}$$

Using definitions of q, p and A we obtain that

$$\begin{aligned}\frac{q}{V}-A&=\frac{\delta}{V(V-u)}-\frac{\delta p}{V(V-\delta(u-Du\cdot x))} \qquad (2.11)\\ &=\frac{\delta}{V}\left(\frac{1}{V-u}-\frac{\frac{V-\delta(u-Du\cdot x)}{V-u}}{V-\delta(u-Du\cdot x)}\right)=0\end{aligned}$$

The lemma is proved. □

Summarizing we finally obtain

$$\begin{aligned}\frac{Dy}{k}&=[1-\frac{\delta}{V}(u-Du\cdot x)]B\mu[Id-x\otimes x]\left\{(Id+\frac{x\otimes x}{1-|x|^2})\frac{1}{B}-D^2u\right\} \qquad (2.12)\\ &=\frac{\delta}{V}\mu[Id-x\otimes x]\left\{(Id+\frac{x\otimes x}{1-|x|^2})\frac{1}{B}-D^2u\right\}.\end{aligned}$$

Now returning to Jacobian, we have the formula

$$\begin{aligned}J&=\left|\frac{\det Dy}{Y_{n+1}}\right| \qquad (2.13)\\ &=\frac{(k|\delta|)^n}{|Y_{n+1}V|}(1-|x|^2)\det\mu\left|\det\left\{(Id+\frac{x\otimes x}{1-|x|^2})\frac{1}{B}-D^2u\right\}\right|\\ &=\frac{f(x)}{g(y)}\end{aligned}$$

thus the equation is

$$\begin{aligned}\det\left\{D^2u-(Id+\frac{x\otimes x}{1-|x|^2})\frac{1}{B}\right\}&=h, \qquad (2.14)\\ h=\frac{f(x)}{g(y)}\frac{|Y_{n+1}V|}{(k|\delta|)^n}&\frac{1}{(1-|x|^2)|\det\mu|}.\end{aligned}$$

The reason why the Hessian of u in above equation comes first is because at each point where $\rho=1/u$ can be touched from above by an ellipsoid, the matrix $\mathcal{W}=D^2u-(Id+\frac{x\otimes x}{1-|x|^2})\frac{1}{B}$ is nonnegative.

2.1 Ellipsoid and hyperboloid of revolution

In this section we show that $\mathcal{W} \equiv 0$ for $u = \frac{1}{C}(1 - \varepsilon(\ell \cdot X))$, that is when $\rho = 1/u$ is the radial graph of ellipsoid or hyperboloid of revolution. To fix ideas we assume that $\ell = e_{n+1}$. Thus $u = \frac{1}{C}(1 - \varepsilon X_{n+1})$. It is enough to show that $B = CX_{n+1}/\varepsilon$. By direct computation

$$b = \frac{1}{C^2}(1 - 2\varepsilon X_{n+1} + \varepsilon^2) \tag{2.15}$$
$$u^2 - \delta b = \frac{\varepsilon^2}{C^2}(X_{n+1} - \varepsilon)^2.$$

Therefore $V = (1 - \varepsilon^2)/C$, which implies that

$$B = \frac{\delta}{V - \delta(u - Du \cdot x)} = \frac{CX_{n+1}}{\varepsilon}.$$

2.2 Verification of A3 condition

The equation (1.4) is generalized Monge-Ampère equation. To obtain smoothness of the solution, one needs to show, that B^{-1} is strictly concave in gradient. This is a necessary condition, called A3 and first introduced in [MTW], in order to obtain C^2 a priori estimates. It turns out that if $\delta > 0$, i.e. when support functions are ellipsoids of revolution, then B^{-1} is strictly concave in gradient. Recall that $B^{-1} = \delta^{-1}(\sqrt{u^2 - \delta b} + u)$, hence it is enough to show that $\sqrt{u^2 - \delta b}$ is concave in gradient. Let ξ be the dummy variable for Du, then we have

$$\frac{\partial}{\partial \xi_k}\sqrt{u^2 - \delta b} = -\frac{\delta}{\sqrt{u^2 - \delta b}} b_{p_k} \tag{2.16}$$
$$\frac{\partial^2}{\partial \xi_k \partial \xi_l}\sqrt{u^2 - \delta b} = -\frac{\delta}{\sqrt{u^2 - \delta b}}[b_{\xi_k \xi_l} + \delta \frac{b_{\xi_k} b_{\xi_l}}{u^2 - \delta b}].$$

On the other hand $b = u^2 + |\xi|^2 - (\xi \cdot x)^2$, which is strictly convex function of ξ, provided $|x| < 1$. Hence

$$\frac{\partial^2 B^{-1}}{\partial \xi_k \partial \xi_l} < 0.$$

From here the proof of Theorem 1 follows from [MTW] and [TW2]. □

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