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Calculus of Variations

On derivation of Euler–Lagrange equations for incompressible energy-minimizers

Nirmalendu Chaudhuri · Aram L. Karakhanyan

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Abstract We prove that any distribution q satisfying the grad-div system $\nabla q = \operatorname{div} \mathbf{f}$ for some tensor $\mathbf{f} = (f^i_j)$, $f^i_j \in h^r(U)$ $(1 \leq r < \infty)$ -the *local Hardy space*; q is in h^r and q is locally represented by the sum of singular integrals of f^i_j with Calderón-Zygmund kernel. As a consequence, we prove the existence and the local representation of the hydrostatic pressure p (modulo constant) associated with incompressible elastic energy-minimizing deformation $\mathbf{u}$ satisfying $|\nabla \mathbf{u}|^2$, $|\mathrm{cof}\ \nabla \mathbf{u}|^2 \in h^1$. We also derive the system of Euler–Lagrange equations for volume preserving local minimizers $\mathbf{u}$ that are in the space $K^{1,3}_{\mathrm{loc}}$ [defined in (1.2)]—partially resolving a long standing problem. In two dimensions we prove partial $C^{1,\alpha}$ regularity of weak solutions provided their gradient is in L^3 and p is Hölder continuous.

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1 Introduction

Let $\Omega \subset \mathbb{R}^n$, $n = 2, 3$ be a bounded Lipschitz material body. For Neo-Hookean or Mooney-Rivlin materials [1,17,19] such as vulcanized rubber, in the equilibrium state one is interested in minimizing the elastic energy

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N. Chaudhuri (✉)
School of Mathematics and Applied Statistics, University of Wollongong,
Wollongong, NSW 2522, Australia
e-mail: chaudhur@uow.edu.au

A. L. Karakhanyan
Department of Mathematics, University of Texas at Austin, Austin, TX 78712, USA
e-mail: aram@math.utexas.edu

$$E[\mathbf{w}] := \int_{\Omega} L(\nabla \mathbf{w}(x))dx \tag{1.1}$$

for incompressible $W^{1,2}$-deformations $\mathbf{w} : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ subject to its own boundary condition, and corresponding to a given smooth bulk energy $L : \mathbb{M}^{n\times n} \to \mathbb{R}$. Let us define the subspace $K^{1,r}$ for $1 \le r < \infty$, by

$$K^{1,r}(\Omega, \mathbb{R}^n) := \left\{\mathbf{w} \in W^{1,r}(\Omega, \mathbb{R}^n) : \mathrm{cof}\, \nabla\mathbf{w} \in L^r(\Omega, \mathbb{M}^{n\times n})\right\}, \tag{1.2}$$

where $W^{1,r}$ denotes the usual *Sobolev spaces* (see for example, [14, Chapter 7]) and cof P is the *cofactor* matrix of P. Using the identity $P^t \,\mathrm{cof}\, P = Id_n \det P$, it follows that $\det \nabla\mathbf{w} \in L^1$ for any $\mathbf{w} \in K^{1,2}$. Since $|P| = |\mathrm{cof}\, P|$ for any $P \in \mathbb{M}^{2\times 2}$, the function spaces $K^{1,r}$ and $W^{1,r}$ are equal in $\mathbb{R}^2$. Let us denote the admissible set of deformations

$$\mathcal{A} := \left\{\mathbf{w} \in K^{1,2}(\Omega, \mathbb{R}^n) : \det \nabla\mathbf{w} = 1 \quad \text{a.e. in } \Omega\right\}. \tag{1.3}$$

We call $\mathbf{u} \in \mathcal{A}$ to be a *local minimizer* of $E[\cdot]$ if and only if

$$E[\mathbf{u}] \le E[\mathbf{w}] \quad \text{for all } \mathbf{w} \in \mathcal{A} \quad \text{and} \quad \mathrm{supp}\,(\mathbf{w} - \mathbf{u}) \subset \Omega. \tag{1.4}$$

Under the hypothesis that the energy density L is smooth, *polyconvex* (convex function of minors) [1] and satisfies the growth condition

$$C_1(|X|^2 + |\mathrm{cof}\, X|^2) - C_2 \le L(X) \le C_3(1 + |X|^2 + |\mathrm{cof}\, X|^2), \tag{1.5}$$

for all $X \in \mathbb{M}^{n\times n}$, for some $C_1 > 0$, $C_2 \ge 0$, $C_3 > 0$, where $X : Y := \mathrm{trace}(X^tY) = \sum_{ij} x^i_j y^i_j$ is the scalar product on $\mathbb{M}^{n\times n}$ and $|X|^2 := X : X$; using direct methods in the calculus of variations together with weak continuity of the determinant, Ball [1] proved the existence of local minimizers $\mathbf{u} \in \mathcal{A}$ of the energy $E[\cdot]$. An example of polyconvex L satisfying the growth condition (1.5) is the stored-energy for incompressible isotropic Mooney-Rivlin materials in $\mathbb{R}^3$, given by

$$L(X) = \frac{\mu_1}{2}(I_1(X) - 3) + \frac{\mu_2}{2}(I_2(X) - 3), \tag{1.6}$$

where $I_1(X) := \mathrm{trace}(C) = |X|^2$, $I_2(X) := \frac{1}{2}[(\mathrm{trace}(C))^2 - \mathrm{trace}(C^2)] = |\mathrm{cof}\, X|^2$ are the first two principle invariants of the right Cauchy-Green strain tensor $C := X^tX$ and μ_1, μ_2 are positive material constants.

Though the existence of the local minimizers of $E[\cdot]$ in $\mathcal{A}$ is known for over 30 years, the existence of integrable hydrostatic pressure, the derivation of system of Euler–Lagrange equations and determining partial regularity for such minimizers remains a challenging open problem. In this article we prove the following results:

- **(I)** The h^r $(1 \le r < \infty)$—integrability and local representation of any distribution q satisfying the grad-div system $\nabla q = \mathrm{div}\, \mathbf{f}$, where $\mathbf{f} := (f^i_j)$, $f^i_j \in h^r$—the *local r-Hardy space* (Theorem 2.2).
- **(II)** Existence of a hydrostatic pressure (Lagrange multiplier) p satisfying an equation of the form $\nabla p = \mathrm{div}\, \sigma$ where $\sigma := (DL(\nabla\mathbf{u}))^t\nabla\mathbf{u}$ is the Cauchy-Green strain tensor associated with the volume preserving minimizer $\mathbf{u}$ of $E[\cdot]$. L^r estimates on σ yields L^r estimates on p if $r > 1$. The borderline case: a h^1-Hardy estimate on σ leads to a h^1 estimate for p (Theorem 3.1).
- **(III)** Validity of Euler–Lagrange equations if the minimizer $\mathbf{u}$ is in $K^{1,3}_{\mathrm{loc}}$. The pair $(\mathbf{u}, p)$ satisfies the system

$$\operatorname{div}(DL(\nabla \mathbf{u}(x)) - p(x)\operatorname{cof}\nabla \mathbf{u}(x)) = \mathbf{0} \quad \text{in} \quad \Omega, \tag{1.7}$$

in the sense of distribution, where the divergence is taken in each rows (Theorem 4.1).

(IV) Partial $C^{1,\alpha}$ regularity in two dimensions for weak solutions of (1.7) provided their gradient is in L^3 and p is Hölder continuous with exponent $0 < \alpha < 1$ (Theorem 5.1).

L^2-version of the result in (**I**) is classical (see [23, Remark 1.4, p. 11]) and plays an important role in incompressible fluids [23]. The result in (**I**) is a crucial ingredient in proving (**II**). The h^1-version of (**I**) is quite delicate and to the best of our knowledge it is new, and may be of independent interest. For the case $r > 1$, it follows that $\nabla q \in W^{-1,r}$; adapting the classical functional-analytic approach demonstrated for $r = 2$ (see [17,23]), or arguing directly by duality and solving the Bogovskii [2] problem of the type

$$\operatorname{div}\mathbf{w} = f \quad \text{in} \quad V \subset\subset U, \quad \mathbf{w} = 0 \quad \text{in} \quad \partial V,$$

(see for example, [7, p. 472–474]) one can prove that $q \in L^r_{\text{loc}}(U)$. However, both of these approaches fail to provide information for the critical case $r = 1$ and do not give a representation of q. Whereas, our unified singular integral approach is self-contained and provide local h^r-estimates of q, as well as a representation of q. Main ideas in our proof is to represent the localized-mollified distribution of q in terms of the Newtonian potential in $\mathbb{R}^n$ and finding its uniform h^r estimates, by using Calderón–Zygmund estimate [4,11]. Finally we show that the local representation of q consists the sum of Calderón-Zygmund type singular integrals of the tensor $\mathbf{f}$ (see Eq. (2.27) in Sect. 4).

In two dimensions, under the stronger hypothesis that the local minimizers of $E[\cdot]$ are classical ($C^{1,\alpha}$-diffeomorphism), namely in the Sobolev space $W^{2,r}$ for some $r > 2$, LeTallec and Oden [17] established the system of equations in (1.7). For $n = 2$, Bauman, Owen and Phillips [3] proved that if a minimizer is in $W^{2,r}$ for some $r > 2$, then it is smooth. For such $W^{2,r}$, $r > 2$ minimizers, the authors in [3] argued directly on the level of the Euler–Lagrange equations exploring the existence of integrable hydrostatic pressure. Evans and Gariepy [9] proved that any *non-degenerate*, Lipschitz area-preserving local minimizers of $E[\cdot]$ are in $C^{1,\alpha}(\Omega_0)$, for some $0 < \alpha < 1$ and a dense open subset $\Omega_0 \subset \Omega$. We believe that the Euler–Lagrange equations (1.7) that we derived for $K^{1,3}$-minimizers may be useful in understanding the partial regularity of such minimizers, as evidenced by the result in (**IV**).

In order to prove the existence of an integrable pressure p associated with an incompressible local energy-minimizer $\mathbf{u}$, we require the additional mild assumptions that $|\nabla \mathbf{u}|^2 \log(2 + |\nabla u|^2)$ and $|\operatorname{cof}\nabla \mathbf{u}|^2 \log(2 + |\operatorname{cof}\nabla \mathbf{u}|^2)$ are locally integrable. For $n = 2$, to derive the system of equilibrium equations (1.7) for $(\mathbf{u}, p)$ in Ω, we need $\mathbf{u}$ to be in $W^{1,3}$; whereas the previous results in this direction were for $W^{2,r}$-minimizers, $r > 2$.

We organize the paper as follows. In Sect. 2, we prove (**I**); in Sect. 3, we prove (**II**); in Sect. 4, we prove (**III**), and finally in Sect. 5, we prove (**IV**). Throughout this article C is a generic absolute constant depending on n, U, Ω, $\mathbf{u}(\Omega)$, $V \subset\subset \mathbf{u}(\Omega)$, r, and L. Its value can vary from line to line, but each line is valid with C being a pure positive number.

2 Local integrability of solutions of $\nabla q = \operatorname{div}\mathbf{f}$

We recall some of the basic definitions and terminologies of Hardy spaces. Let $1 \le r < \infty$. A distribution f belongs to $H^r(\mathbb{R}^n)$ if and only if $f \in L^r(\mathbb{R}^n)$ and $R_j(f) \in L^r(\mathbb{R}^n)$ (see for example, [21, Proposition 3, p. 123]) for $j = 1, \ldots, n$, where R_j is the Riesz transform of f given by

$$R_j(f)(x) := \lim_{\varepsilon \to 0} c_n \int_{|y| \geq \varepsilon} \frac{y_j}{|y|^{n+1}} f(x-y)\,dy\,, \quad c_n := \frac{\Gamma(\frac{n+1}{2})}{\pi^{(n+1)/2}},$$

so that $\widehat{R_j(f)}(\xi) = i\frac{\xi_j}{|\xi|}\widehat{f}$. In short, we will write $H^r(\mathbb{R}^n)$ as simply H^r. For $f \in H^r$, the norm is defined as

$$\|f\|_{H^r} := \|f\|_{L^r} + \sum_{j=1}^{n} \|R_j(f)\|_{L^r}.$$

A standard result [20, p. 237] states that a positive function f, the Riesz transform $R_j f \in L^1_{\text{loc}}$ if and only if $f \log(2+f) \in L^1_{\text{loc}}$, if and only if the maximal function

$$(Mf)(x) := \sup_{\rho > 0} \frac{1}{\text{meas } B_\rho(x)} \int_{B_\rho(x)} |f(y)|\,dy$$

is locally integrable. For $1 < r < \infty$, a classical result asserts that $f \in H^r$ if and only if $f \in L^r$, see [20, p. 220]. The celebrated Fefferman duality theorem (see [10], [11, Theorem 2], [21, Theorem 1, p. 142]) asserts that the dual of H^1 is the BMO—the functions of bounded mean oscillations. The following theorem is due to Calderón-Zygmond [4], Fefferman and Stein [11, Corollary 1, p. 149–151] and Stein [20, Theorem 3, p. 39].

Theorem 2.1 (Calderón-Zygmond, Fefferman-Stein) *Let* $1 \leq r < \infty$ *and* $f \in H^r$. *Let* G *be a* C^1 *function on* $\mathbb{R}^n \setminus \{0\}$ *homogeneous of degree* 0 *with mean value* 0 *over the unit sphere* $\mathbb{S}^{n-1}$, *that is*

$$\int_{\mathbb{S}^{n-1}} G(x)\,d\sigma(x) = 0. \tag{2.1}$$

Then the function defined as

$$T_0 f(x) := \lim_{\delta \to 0} \int_{|y| \geq \delta} \frac{G(y)}{|y|^n} f(x-y)\,dy \tag{2.2}$$

exists a.e. and furthermore,

$$\|T_0 f\|_{H^r} \leq C_{n,r} \|f\|_{H^r}. \tag{2.3}$$

In particular, R_j's are bounded linear operator on H^r for any $1 \leq r < \infty$. Let us recall the definition of *local Hardy spaces* introduced by Goldberg [13]. A distribution f on $\mathbb{R}^n$ is said to be in the local r-Hardy space, written as $f \in h^r$, if and only if the maximal function

$$\mathcal{M}_{\text{loc}} f(x) := \sup_{0 < \varepsilon < 1} |(\rho_\varepsilon * f)(x)|$$

is in L^r, where $\rho_\varepsilon := \varepsilon^{-n}\rho(x/\varepsilon)$ is a standard approximation of the identity. The h^r norm of f is defined to be the L^r norm of the maximal function $\mathcal{M}_{\text{loc}} f$. It follows that if $f \in h^r$ then $\eta f \in h^r$ for any smooth cut-off function η and $H^r \subset h^r$. For bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, we adopt the definition of Hardy spaces $h^r(\Omega)$ introduced by Miyachi [18]. A distribution f on Ω is said to be in $h^r(\Omega)$ if f is the restriction to Ω of a distribution F in $h^r(\mathbb{R}^n)$, i.e.,

$$\begin{aligned} h^r(\Omega) &:= \left\{ f \in \mathcal{D}'(\Omega) : \exists\, F \in h^r(\mathbb{R}^n), \text{ such that } F\big|_\Omega = f \right\} \\ &= h^r(\mathbb{R}^n) \big/ \{F \in h^r(\mathbb{R}^n) \, : \, F = 0 \text{ on } \Omega\}. \end{aligned}$$

The norm on this space is the quotient norm: the infimum of h^r norms of all possible extensions of f in $\mathbb{R}^n$. For $1 < r < \infty$ the spaces $h^r(\Omega)$ is equivalent to $L^r(\Omega)$. For smooth bounded domains Ω, the Theorem 2.1 is valid for $f \in h^1(\Omega)$, see [5,18].

Theorem 2.2 *Let $U \subset \mathbb{R}^n$, $n \geq 2$ be a bounded Lipschitz domain (open and connected) and $1 \leq r < \infty$. Let $\boldsymbol{f} = (f^i_j)$ be such that $f^i_j \in h^r(U)$ for $1 \leq i, j \leq n$. Then any distribution q (modulo a constant) on $C^\infty_0(U)$ satisfying the linear system of equations*

$$\nabla q = \operatorname{div} \boldsymbol{f} \ \text{ in } \mathcal{D}'(U, \mathbb{R}^n) \iff \langle \nabla q, \boldsymbol{v} \rangle = -\int_U \boldsymbol{f}(x) : \nabla \boldsymbol{v}(x)\, dx \tag{2.4}$$

for $\boldsymbol{v} \in C^\infty_0(U, \mathbb{R}^n)$, is in $h^r(V)$ for any $V \subset\subset U$. Furthermore, q is locally represented by sum of singular integrals of f^i_j (see Eq. (2.27)), and for any $V \subset\subset U$, there exists $C > 0$ depending only on U, V and r, such that

$$\|q\|_{h^r(V)/\mathbb{R}} \leq C \|\boldsymbol{f}\|_{h^r(U, \mathbb{M}^{n\times n})/\mathcal{V}},$$

where $h^r(V)/\mathbb{R} := \{q \in h^r(V) : \int_V q = 0\}$ and $\mathcal{V} := \{\boldsymbol{g} \in h^r(U, \mathbb{M}^{n\times n}) : \operatorname{div} \boldsymbol{g} = \boldsymbol{0}\}$.

Proof of Theorem 2.2 Let $U \subset \mathbb{R}^n$, $n \geq 2$ be a bounded Lipschitz domain. Let $\mathbf{f} := (f^i_j) \in \mathbb{M}^{n\times n}$ and $f^i_j \in h^r(U)$ for $1 \leq r < \infty$ and $1 \leq i, j \leq n$. Let $q \in \mathcal{D}'(U)$ be such that

$$\nabla q = \operatorname{div} \mathbf{f} \quad \text{in} \quad \mathcal{D}'(U, \mathbb{R}^n). \tag{2.5}$$

Our idea is to mollify the equations in (2.5) and use Calderón-Zygmund estimate to obtain uniform bound for the mollified q. Let $V \subset\subset U$ be a sub-domain and $0 < \varepsilon < \operatorname{dist}(V, \partial U)$. Let ρ_ε be the usual mollification kernel, and define the convolution $q_\varepsilon : V \to \mathbb{R}$ by

$$q_\varepsilon(x) = (q * \rho_\varepsilon)(x) := \langle q, (\rho_\varepsilon)_x \rangle \quad \text{for} \quad x \in V, \quad \text{where} \quad (\rho_\varepsilon)_x(y) := \rho_\varepsilon(y - x), \quad y \in U.$$

Then by the standard properties of the mollification [6, Proposition 1, p. 492], q_ε is smooth and for any $1 \leq i \leq n$

$$\frac{\partial}{\partial x_i}(q * \rho_\varepsilon) = \frac{\partial q}{\partial x_i} * \rho_\varepsilon = q * \frac{\partial \rho_\varepsilon}{\partial x_i}.$$

Thus, mollifying the system of equations in (2.5) yields

$$\nabla q_\varepsilon = \operatorname{div} \mathbf{f}_\varepsilon \quad \text{in } V, \tag{2.6}$$

where the divergence is taken in each rows of $\mathbf{f}_\varepsilon := \left((f^i_j)_\varepsilon\right)$, here $(f^i_j)_\varepsilon := f^i_j * \rho_\varepsilon$ are the mollification of f^i_j. Since $f^i_j \in h^r(U)$, we conclude that

$$(f^i_j)_\varepsilon \to f^i_j \quad \text{strongly in} \quad h^r(V) \quad \text{as } \varepsilon \to 0, \tag{2.7}$$

for all $1 \leq i, j \leq n$. Applying the divergence operator to the both sides of the Eq. (2.6), we obtain

$$\Delta q_\varepsilon = \operatorname{div}(\operatorname{div} \mathbf{f}_\varepsilon) \quad \text{in } V. \tag{2.8}$$

Since there is no control on the boundary values of q_ε, we need to localize the Eq. (2.8). Let $W \subset\subset V \subset\subset U$. Let $\eta \in C^\infty_0(\mathbb{R}^n)$, $0 \leq \eta \leq 1$ be a cut-off function such that $\eta \equiv 1$ in W and $\eta \equiv 0$ outside V. Let $\bar{q}_\varepsilon := \eta q_\varepsilon$ be the localization of q_ε. Then $\bar{q}_\varepsilon$ is the solution of Poisson equation

$$\Delta \bar{q}_\varepsilon = \bar{f}_\varepsilon \quad \text{in } \mathbb{R}^n, \tag{2.9}$$

where

$$\begin{aligned}\bar{f}_\varepsilon &:= \eta\Delta q_\varepsilon + 2\langle\nabla q_\varepsilon, \nabla\eta\rangle + q_\varepsilon\Delta\eta \\ &= \eta\,\mathrm{div}(\mathrm{div}\,\mathbf{f}_\varepsilon) + 2\langle\mathrm{div}\,\mathbf{f}_\varepsilon, \nabla\eta\rangle + q_\varepsilon\Delta\eta.\end{aligned} \tag{2.10}$$

Therefore $\bar{q}_\varepsilon$ is represented by the Newtonian potential in $\mathbb{R}^n$. In other words,

$$\bar{q}_\varepsilon(x) = -\int\limits_{\mathbb{R}^n} \Phi(x-y)\bar{f}_\varepsilon(y)\,dy\,, \tag{2.11}$$

where Φ is the fundamental solution of the Laplace equation in $\mathbb{R}^n$ given by

$$\Phi(x) := \begin{cases} -\frac{1}{2\pi}\log|x| & \text{if } n=2 \\ \frac{1}{n(n-2)\alpha(n)}\,\frac{1}{|x|^{n-2}} & \text{if } n\geq 3, \end{cases} \tag{2.12}$$

for $x \in \mathbb{R}^n\backslash\{0\}$, and $\alpha(n) := \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}$ is the volume of the unit ball in $\mathbb{R}^n$. Using (2.10) in (2.11), we obtain

$$\begin{aligned}\bar{q}_\varepsilon(x) &= -\int\limits_{\mathbb{R}^n}\eta(y)\Phi(x-y)\,\mathrm{div}(\mathrm{div}\,\mathbf{f}_\varepsilon(y))\,dy - 2\int\limits_{\mathbb{R}^n}(\langle\mathrm{div}\,\mathbf{f}_\varepsilon, \nabla\eta\rangle + q_\varepsilon\Delta\eta)\,\Phi(x-y)\,dy \\ &:= -I^1_\varepsilon(x) - 2I^2_\varepsilon(x) - I^3_\varepsilon(x),\end{aligned} \tag{2.13}$$

where

$$I^1_\varepsilon(x) := \int\limits_{\mathbb{R}^n}\eta(y)\,\Phi(x-y)\;\mathrm{div}(\mathrm{div}\,\mathbf{f}_\varepsilon(y))\,dy\,, \tag{2.14}$$

$$I^2_\varepsilon(x) := \int\limits_{\mathbb{R}^n}\langle\mathrm{div}\,\mathbf{f}_\varepsilon(y), \nabla\eta(y)\rangle\;\Phi(x-y)\,dy\,, \tag{2.15}$$

$$I^3_\varepsilon(x) := \int\limits_{\mathbb{R}^n} q_\varepsilon(y)\,\Phi(x-y)\,\Delta\eta(y)\,dy. \tag{2.16}$$

By direct computations, observe that, for $1 \leq i, j \leq n$

$$(\Phi)_{y_i} = \eta_{y_i}\Phi(y) - \frac{1}{\omega_n}\frac{\eta\,y_i}{|y|^n}, \tag{2.17}$$

$$(\eta\,\Phi)_{y_iy_j} = \eta_{y_iy_j}\,\Phi(y) - \frac{1}{\omega_n}\frac{y_i\eta_{y_j} + y_j\eta_{y_i}}{|y|^n} - \frac{1}{\omega_n}\left(\delta_{ij} - n\frac{y_iy_j}{|y|^2}\right)\frac{\eta}{|y|^n}, \tag{2.18}$$

where δ_{ij} is the Krönecker delta and $\omega_n := n\alpha_n$ is the surface area of the unit sphere $\mathbb{S}^{n-1}$. We now establish an uniform local h^r-estimates ($1 \leq r < \infty$) for q_ε through the following steps:

Step 1: Limit of I^3_ε. Let us fix $x \in W \subset\subset V \subset\subset U$. Since $\Delta\eta = 0$ on W, the integrand in $I^3_\varepsilon(x)$ is smooth. Since q_ε is determined up to a constant, by adding a constant, if necessary, we can assume $\int_{\mathbb{R}^n} q_\varepsilon(y)\,dy = 0$, so that

$$I^3_\varepsilon(x) = \int\limits_{\mathbb{R}^n} q_\varepsilon(y)\left(\Phi(x-y)\Delta\eta(y)\,dy - \int\limits_{\mathbb{R}^n}\Phi(x-z)\Delta\eta(z)\,dz\right)dy.$$

Thus we can add $-\int_{\mathbb{R}^n} \Phi(x-z)\Delta\eta(z)\,dz$ to the function $y \mapsto \Delta\eta(y)\Phi(x-y)$, if nessecary, to ensure that it has vanishing integral. For each fixed $x \in W$, let $\mathbf{v}_x : V \to \mathbb{R}^n$ be the solution of the Bogovskii problem

$$\begin{cases} \operatorname{div} \mathbf{v}_x(y) = \Delta\eta(y)\Phi(x-y) & \text{for } y \in V \\ \mathbf{v}_x = 0 & \text{on } \partial V. \end{cases} \tag{2.19}$$

Then using (2.19), integrating by parts and the convergence of $\mathbf{f}_\varepsilon$, we obtain

$$\begin{aligned} I_\varepsilon^3(x) &= \int_{\mathbb{R}^n} q_\varepsilon(y)\Delta\eta(y)\Phi(x-y)\,dy \\ &= \int_{\mathbb{R}^n} q_\varepsilon(y) \operatorname{div} \mathbf{v}_x(y)\,dy \\ &= -\int_{\mathbb{R}^n} \langle \nabla q_\varepsilon(y), \mathbf{v}_x(y)\rangle\,dx \\ &= -\int_{\mathbb{R}^n} \langle \operatorname{div} \mathbf{f}_\varepsilon(y), \mathbf{v}_x(y)\rangle\,dy \\ &= \int_{\mathbb{R}^n} \mathbf{f}_\varepsilon(y) : \nabla_y \mathbf{v}_x(y)\rangle\,dy \\ &\to \int_{\mathbb{R}^n} \mathbf{f}(y) : \nabla_y \mathbf{v}_x(y)\,dy \qquad \text{as } \varepsilon \to 0 \\ &:= I_0^3(x) \quad \text{for} \quad x \in W \subset\subset V. \end{aligned} \tag{2.20}$$

Thus, the strong convergence of $\mathbf{f}_\varepsilon \to \mathbf{f}$ in $h^r(V, \mathbb{M}^{n\times n})$ yields strong convergence of $I_\varepsilon^3 \to I_0^3$ in $h^r(W)$ as $\varepsilon \to 0$.

Step 2: Limit of I_ε^2. Let us fix $x \in W \subset\subset V \subset\subset U$. Integrating by parts, invoking (2.17) and letting $\varepsilon \to 0$ we have

$$\begin{aligned} I_\varepsilon^2(x) &= \int_{\mathbb{R}^n} \langle \operatorname{div} \mathbf{f}_\varepsilon(y), \Phi(x-y)\,\nabla\eta(y)\rangle\;dy \\ &= -\int_{\mathbb{R}^n} \mathbf{f}_\varepsilon(y) : \nabla_y\left(\Phi(x-y)\,\nabla\eta(y)\right)\,dy \\ &= -\int_{\mathbb{R}^n} \mathbf{f}_\varepsilon : \left(\Phi(x-y)\,\nabla^2\eta - \frac{(y-x)\otimes\nabla\eta}{\omega_n\,|y-x|^n}\right)dy \\ &\to -\int_{\mathbb{R}^n} \mathbf{f} : \left(\Phi(x-y)\,\nabla^2\eta - \frac{(y-x)\otimes\nabla\eta}{\omega_n\,|y-x|^n}\right)dy \\ &:= I_0^2(x) \quad \text{for } x \in W, \end{aligned} \tag{2.21}$$

where $a \otimes b := (a_i b_j)_{1\le i,j\le n}$ for $a, b \in \mathbb{R}^n$. Using the strong convergence of $\mathbf{f}_\varepsilon$ in $h^r(V)$, it follows that $I_\varepsilon^2 \to I_0^2$ in $h^r(W)$ as $\varepsilon \to 0$.

Step 3: Limit of I^1_ε. Integrating by parts twice and invoking (2.18) we have

$$
\begin{aligned}
I^1_\varepsilon(x) &= \int_{\mathbb{R}^n} \eta(y)\Phi(x-y)\ \operatorname{div}(\operatorname{div}\mathbf{f}_\varepsilon(y))\,dy \\
&= \int_{\mathbb{R}^n} \mathbf{f}_\varepsilon(y) : \nabla^2_y\left(\eta(y)\Phi(x-y)\right)\,dy \\
&= \int_{\mathbb{R}^n} \mathbf{f}_\varepsilon(y) : \left(\Phi(x-y)\,\nabla^2\eta(y) - \frac{1}{\omega_n}\frac{\nabla\eta\otimes(y-x)+(y-x)\otimes\nabla\eta}{|x-y|^n}\right)dy \\
&\quad -\frac{1}{\omega_n}\int_{\mathbb{R}^n} \mathbf{f}_\varepsilon(y) : \left(Id_n - n\frac{(y-x)\otimes(y-x)}{|x-y|^2}\right)\frac{\eta}{|x-y|^n}\,dy \\
&:= I^{11}_\varepsilon(x) + I^{12}_\varepsilon(x), \quad \text{for } x\in W,
\end{aligned}
$$

where Id_n is the $n\times n$ identity matrix. Using the convergence of $\mathbf{f}_\varepsilon$, observe that as $\varepsilon\to 0$,

$$
\begin{aligned}
I^{11}_\varepsilon(x) &:= \int_{\mathbb{R}^n} \mathbf{f}_\varepsilon : \left(\Phi(x-y)\,\nabla^2\eta - \frac{\nabla\eta\otimes(y-x)+(y-x)\otimes\nabla\eta}{\omega_n\,|x-y|^n}\right)dy \\
&\to \int_{\mathbb{R}^n} \mathbf{f} : \left(\Phi(x-y)\,\nabla^2\eta - \frac{\nabla\eta\otimes(y-x)+(y-x)\otimes\nabla\eta}{\omega_n\,|x-y|^n}\right)dy \\
&:= I^{11}_0(x) \quad x\in W.
\end{aligned}
\tag{2.22}
$$

In order to estimate I^{12}_ε, define the kernels $\Omega_{ij} : \mathbb{R}^n\backslash\{0\}\to\mathbb{R}$ by

$$
\Omega_{ij}(y) := \delta_{ij} - n\,\frac{y_i y_j}{|y|^2}, \quad y\in\mathbb{R}^n\backslash\{0\}, \quad i,j=1,\dots,n. \tag{2.23}
$$

Since $n\alpha_n = \omega_n$, integrating by parts, observe that for any $i, j = 1,\dots,n$,

$$
\begin{aligned}
\int_{\mathbb{S}^{n-1}} \Omega_{ij}(y)\,d\sigma(y) &= \int_{\mathbb{S}^{n-1}} (\delta_{ij} - n y_i y_j)\,d\sigma(y) \\
&= \omega_n\delta_{ij} - n\int_{\mathbb{S}^{n-1}} y_i y_j\,d\sigma(y) \\
&= \omega_n\delta_{ij} - n\int_{B_1} \frac{\partial}{\partial y_j} y_i\,dy \\
&= \omega_n\delta_{ij} - n\delta_{ij}\alpha_n \\
&= 0.
\end{aligned}
$$

Hence each Ω_{ij} satisfies all the conditions of Calderón-Zygmund Kernel [20]. Therefore,

$$
I^{12}_\varepsilon(x) := -\frac{1}{\omega_n}\int_{\mathbb{R}^n} \eta\mathbf{f}_\varepsilon : \left(Id_n - n\frac{(y-x)\otimes(y-x)}{|x-y|^2}\right)\frac{dy}{|x-y|^n}\Bigg) \tag{2.24}
$$

is the sum of Calderón-Zygmund singular integrals with the homogeneous kernel Ω_{ij}. Since $\mathbf{f} \in h^r(U, \mathbb{M}^{n\times n})$ $1 \le r < \infty$, by Theorem 2.1 we conclude that $I^{12} \in h^r(W)$. Furthermore, the following sum of singular integrals

$$I_0^{12}(x) := -\frac{1}{\omega_n} \int_{\mathbb{R}^n} \eta \mathbf{f} : \left(Id_n - n \frac{(y-x) \otimes (y-x)}{|x-y|^2} \right) \frac{dy}{|x-y|^n} \tag{2.25}$$

exists for almost every $x \in W \subset\subset V$ and is in $h^r(W)$. From (2.24) and (2.25) we compute

$$I_\varepsilon^{12}(x) - I_0^{12}(x) = -\frac{1}{\omega_n} \sum_{i,j=1}^{n} \int_{\mathbb{R}^n} \left(\eta (f_j^i)_\varepsilon(y) - \eta f_j^i(y) \right) \frac{\Omega_{ij}(x-y)}{|x-y|^n}\, dy.$$

Hence by Theorem 2.1, there exists $C := C(V, W, r) > 0$ such that

$$\|I_\varepsilon^{12} - I_0^{12}\|_{h^r(W)} \le C \sum_{j=1}^{n} \|(f_j^i)_\varepsilon - f_j^i\|_{h^r(V)} \to 0 \quad \text{as } \varepsilon \to 0. \tag{2.26}$$

Step 4: Explicit representation of q. To complete the proof, let us define the potential $q : W \to \mathbb{R}$ by

$$q(x) := -\left(I_0^{11}(x) + I_0^{12}(x) + 2I_0^2(x) + I_0^3(x) \right).$$

Then from (2.20)–(2.22) and (2.26), we conclude that $q_\varepsilon \to q$ strongly in $h^r_{\text{loc}}(U)$ for any $1 \le r < \infty$ and q is represented as

$$\begin{aligned} q(x) = &\int_U \mathbf{f} : \left(\Phi(x-y) \nabla^2 \eta - \nabla_y \mathbf{v}_x \right) dy + \frac{1}{\omega_n} \int_U \mathbf{f} : (\nabla\eta \otimes (y-x) - (y-x) \otimes \nabla\eta) \frac{dy}{|x-y|^n} \\ &+ \frac{1}{\omega_n} \int_U \eta \mathbf{f} : \left(Id_n - n \frac{(y-x) \otimes (y-x)}{|x-y|^2} \right) \frac{dy}{|x-y|^n} \end{aligned} \tag{2.27}$$

for any $x \in W$. Since q is the strong limit of the family q_ε in W, it is independent of the choice of the cut-off function η. This completes the proof of Theorem 2.2. □

3 First variation of energy and the existence of hydrostatic pressure

Let $\Omega \subset \mathbb{R}^n$, $n = 2, 3$ be a smooth, simply connected and bounded domain and let $L : \mathbb{M}^{n\times n} \to \mathbb{R}$ be a smooth function. We are now in a position to establish the existence of integrable hydrostatic pressure associated with volume preserving local minimizers of the energy $E[\cdot]$ defined in (1.1). By direct computations, observe that the incompressible isotropic Mooney-Rivlin bulk-energy given by

$$L(P) = \frac{\mu_1}{2}(|P|^2 - 3) + \frac{\mu_2}{2}(|\text{cof } P|^2 - 3), \tag{3.1}$$

satisfies the following.

$$DL = \mu_1 P + \mu_2 \begin{pmatrix} \text{cof}(SQ)_1^1 : (SP)_1^1 & -\text{cof}(SQ)_2^1 : (SQ)_2^1 & \text{cof}(SQ)_3^1 : (SP)_3^1 \\ -\text{cof}(SQ)_1^2 : (SP)_1^2 & \text{cof}(SQ)_2^2 : (SP)_2^2 & -\text{cof}(SQ)_3^2 : (SP)_3^2 \\ \text{cof}(SQ)_1^3 : (SP)_1^3 & -\text{cof}(SQ)_2^3 : (SP)_2^3 & \text{cof}(SQ)_3^3 : (SP)_3^3 \end{pmatrix},$$

where $Q := \operatorname{cof} P$, and $(SX)^i_j$ is the 2×2 submatrix obtained by deleting the ith row and the jth column of the matrix $X \in \mathbb{M}^{3\times 3}$. Furthermore, the Cauchy-Green strain tensor is given by

$$(DL(P))^t P = \mu_1 P^t P + \mu_2 \begin{pmatrix} |Q_2|^2 + |Q_3|^2 & -\langle Q_1, Q_2\rangle & -\langle Q_1, Q_3\rangle \\ -\langle Q_1, Q_2\rangle & |Q_1|^2 + |Q_3|^2 & -\langle Q_2, Q_3\rangle \\ -\langle Q_1, Q_2\rangle & -\langle Q_2, Q_3\rangle & |Q_1|^2 + |Q_2|^2 \end{pmatrix}$$

for all $P \in \mathbb{M}^{3\times 3}$, where $Q_i := (\operatorname{cof} P)_i := ((\operatorname{cof} P)^i_1, (\operatorname{cof} P)^i_2, (\operatorname{cof} P)^i_3)$ is the ith row of $\operatorname{cof} P$, $i = 1, 2, 3$. Motivated by the above calculations, assume that L satisfies the following growth condition:

$$\max\left(|L(P)|, \left|(DL(P))^t P\right|\right) \le C\left(1 + |P|^2 + |\operatorname{cof} P|^2\right), \tag{3.2}$$

for some $C > 0$, for any $P \in \mathbb{M}^{n\times n}$.

Now we prove the existence of an integrable hydrostatic pressure q on the deformed domain $\mathbf{u}(\Omega)$ and establish an explicit representation of q in terms of Calderón-Zygmund singular integrals of the Cauchy-Green strain $\tilde{\sigma} := (DL(\nabla\mathbf{u}))^t \nabla\mathbf{u}) \circ \mathbf{u}^{-1}$ in $\mathbf{u}(\Omega)$. Our proof consists of deriving the first variation of the energy $E[\cdot]$, obtaining the equation $\nabla q = \operatorname{div}\tilde{\sigma}$ and finally to use Theorem 2.2 in establishing h^r estimates for q.

Theorem 3.1 *Let $L : \mathbb{M}^{n\times n} \to \mathbb{R}$, $n = 2, 3$ be smooth and satisfies the growth condition (3.2). Assume that $\boldsymbol{u} \in \mathcal{A}$ be a continuous and injective local minimizer of $E[\cdot]$ such that $|\nabla\boldsymbol{u}|^2, |\operatorname{cof}\nabla\boldsymbol{u}|^2 \in h^r_{\mathrm{loc}}(\Omega)$ for some $1 \le r < \infty$. Then there exists a scalar function $q \in h^r_{\mathrm{loc}}(\boldsymbol{u}(\Omega))$ satisfying the equation of the form $\nabla q = \operatorname{div}\tilde{\sigma}$ in $\mathcal{D}'(\boldsymbol{u}(\Omega), \mathbb{R}^n)$, such that*

$$\|q\|_{h^r(V)/\mathbb{R}} \le C\left(\left\||\nabla\boldsymbol{u}|^2\right\|_{h^r(\boldsymbol{u}^{-1}(U))} + \left\||\operatorname{cof}\nabla\boldsymbol{u}|^2\right\|_{h^r(\boldsymbol{u}^{-1}(U))}\right), \quad V \subset\subset U \subset\subset \boldsymbol{u}(\Omega),$$

for some $C > 0$ depending on r, V, U, n and $\boldsymbol{u}(\Omega)$, and the pair $(\boldsymbol{u}, q)$ satisfies the integral identity

$$\int_\Omega DL(\nabla\boldsymbol{u}(x)) : \nabla(\boldsymbol{v}\circ\boldsymbol{u})\, dx = \int_{\boldsymbol{u}(\Omega)} q(y)\, \operatorname{div}\boldsymbol{v}(y)\, dy \tag{3.3}$$

for all $\boldsymbol{v} \in C^\infty_0(\boldsymbol{u}(\Omega), \mathbb{R}^n)$.

Corollary 3.2 *Let $W \subset\subset V \subset\subset \boldsymbol{u}(\Omega)$ and let $\eta \in C^\infty_0(V)$ be a cut-off function such that $\eta \equiv 1$ on W. Then q is represented as*

$$q(x) = \int_V \tilde{\sigma} : \left(\Phi(x-y)\nabla^2\eta - \nabla_y \boldsymbol{v}_x\right) dy + \frac{1}{\omega_n}\int_V \tilde{\sigma} : (\nabla\eta \otimes (y-x) - (y-x)\otimes\nabla\eta)$$
$$\times \frac{dy}{|x-y|^n} + \frac{1}{\omega_n}\int_V \eta\tilde{\sigma} : \left(Id_n - n\frac{(y-x)\otimes(y-x)}{|x-y|^2}\right)\frac{dy}{|x-y|^n}, \tag{3.4}$$

for any $x \in W$, where Φ is the Newtonian potential in $\mathbb{R}^n$ defined in (2.12) and $\boldsymbol{v}_x$ as defined in (2.19).

Remark 3.3 In connection to the study of regularity of finite energy deformations, Šverák [22] proved that for any $W^{1,n}$-deformation $\mathbf{w}$ with $\det\nabla\mathbf{w}(x) > 0$, a.e., there exists a continuous function ω on $\mathbb{R}$ with $\omega(0) = 0$ such that

$$|\mathbf{w}(x) - \mathbf{w}(y)| \le \omega(|x-y|), \quad \text{for any } x, y \in \Omega \subset\subset \mathbb{R}^n.$$

For $n = 2$, Iwaniec and Šverák [16] proved that any non-constant $W^{1,2}$-deformation $\mathbf{w}$ with integrable *distortion* $K(\cdot, \mathbf{w}) := |\nabla \mathbf{w}(\cdot)|^2 / \det \nabla \mathbf{w}(\cdot)$, the Stoilow factorization holds, and therefore the map $\mathbf{w}$ can be written as a composition of a homeomorphism with a holomorphic function. Hence such maps $\mathbf{w}$ are open and discrete (may have isolated branch-points). Thus in particular, area-preserving $W^{1,r}$ $(r > 2)$-deformations in the plane are continuous and injective. It is now well-known (see [15,24]) that any non-constant $W^{1,n}$-deformation $\mathbf{w}$ for which the distortion function $K(\cdot, \mathbf{w}) := |\nabla \mathbf{w}(\cdot)|^n / \det \nabla \mathbf{w}(\cdot) \in L^r$ for some $r > n - 1$, the Stoilow factorization holds. However, for $n \geq 3$, deformations in $K^{1,2}$ may be totally discontinuous, see for example [22, p. 119].

In order to prove Theorem 3.1, we establish the following first variation of the energy integral $E[\cdot]$.

Lemma 3.4 (First Variation) *Let $\boldsymbol{u} \in \mathcal{A}$ be a local minimizer of $E[\cdot]$. We further assume that $\boldsymbol{u}$ is a continuous and an injective map. Then $\boldsymbol{u}$ satisfies the following integral identity*

$$\int_{\Omega} DL(\nabla \boldsymbol{u}(x)) : \nabla(\boldsymbol{v} \circ \boldsymbol{u})(x)\, dx = 0, \tag{3.5}$$

for all smooth, compactly supported and divergence free vector fields $\boldsymbol{v}$ on $\boldsymbol{u}(\Omega)$.

Proof By the invariance of domain theorem, $\mathbf{u}(\Omega)$ is open and $\mathbf{u} : \Omega \to \mathbf{u}(\Omega)$ is a homeomorphism. Let $\mathbf{v} \in C_0^\infty(\mathbf{u}(\Omega), \mathbb{R}^n)$ be a vector field with $\operatorname{div} \mathbf{v} = 0$. For each $y \in \mathbf{u}(\Omega)$ consider the unique smooth flow $\phi(y, \cdot) : \mathbb{R} \to \mathbf{u}(\Omega)$ given by

$$\frac{d\phi}{dt}(y, t) = \mathbf{v}(\phi(y, t)) \quad \text{in } \mathbb{R}, \quad \phi(y, 0) = y. \tag{3.6}$$

Using the relations $\frac{\partial}{\partial p^i_j} \det P = (\operatorname{cof} P)^i_j$ and $P\,(\operatorname{cof} P)^t = Id_n \det P$, by a direct calculations we observe that

$$\frac{d}{dt}\big(\det \nabla_y \phi(y, t)\big) = \det \nabla_y \phi(y, t) \operatorname{div} \mathbf{v} = 0. \tag{3.7}$$

Since $\det \nabla_y \phi(y, 0) = 1$, from (3.7) it follows that $\det \nabla_y \phi(y, t) = 1$ for all $t \in \mathbb{R}$ and $y \in \mathbf{u}(\Omega)$. Consider the map $\mathbf{w} : \Omega \times \mathbb{R} \to \mathbf{u}(\Omega)$ defined by

$$\mathbf{w}(x, t) := \phi(\cdot, t) \circ \mathbf{u}\,(x) = \phi(\mathbf{u}(x), t) \quad \text{for any } t \in \mathbb{R}, \quad x \in \Omega.$$

Let $V := \operatorname{supp} \mathbf{v} \subset \mathbf{u}(\Omega)$, then $\mathbf{v}(\mathbf{u}(x)) = 0$ for $\mathbf{u}(x) \notin V$. This in conjunction with the uniqueness of ϕ implies that $\phi(\mathbf{u}(x), t) = \mathbf{u}(x)$ for all points x such that $\mathbf{u}(x) \notin V$. Since Ω is bounded, $\mathbf{u}$ is continuous and V is compact, $\Omega' = \mathbf{u}^{-1}(V)$ is a compact subset of Ω. Hence $\operatorname{supp}(\mathbf{w}(x, t) - \mathbf{u}(x)) \subset \Omega'$. Furthermore, $\det \nabla_x \mathbf{w}(x, t) = \det \nabla_y \phi(y, t) \det \nabla \mathbf{u}(x) = 1$. Therefore, $\mathbf{w}(\cdot, t) \in \mathcal{A}$ and $\operatorname{supp}(\mathbf{u} - \mathbf{w}(\cdot, t)) \subset \Omega$ for all $t \in \mathbb{R}$. Since $\mathbf{u}$ is a local minimizer of $E[\cdot]$,

$$E[\mathbf{u}] \leq E[\mathbf{w}(\cdot, t)] \quad \text{for all } t \in \mathbb{R}.$$

Thus, for all smooth, compactly supported and divergence free vector fields $\mathbf{v}$ on $\mathbf{u}(\Omega)$, we have

$$
\begin{aligned}
0 &= \frac{d}{dt} \int_\Omega L(\nabla \mathbf{w}(x,t))\, dx \Big|_{t=0} \\
&= \sum_{i,j=1}^n \int_\Omega L^i_j(\nabla \mathbf{w}(x,t)) \frac{d}{dt}\left(\frac{\partial w^i}{\partial x_j}(x,t)\right) dx \Big|_{t=0} \\
&= \sum_{i,j=1}^n \int_\Omega L^i_j(\nabla \mathbf{w}(x,t)) \frac{\partial}{\partial x_j}\left(\frac{d\phi^i}{dt}(\mathbf{u}(x),t)\right) dx \Big|_{t=0} \\
&= \sum_{i,j=1}^n \int_\Omega L^i_j(\nabla \mathbf{w}(x,t)) \frac{\partial}{\partial x_j}\left(v^i(\phi(u(x),t)\right) dx \Big|_{t=0} \\
&= \sum_{i,j=1}^n \int_\Omega L^i_j(\nabla \mathbf{u}(x)) \frac{\partial}{\partial x_j}\left(v^i(\mathbf{u}(x))\right) dx \\
&= \int_\Omega DL(\nabla \mathbf{u}(x)) : \nabla(\mathbf{v}\circ \mathbf{u})(x)\, dx,
\end{aligned}
$$

where $L^i_j(P) := \frac{\partial L}{\partial p^i_j}(P)$. This proves the lemma. □

Proof of Theorem 3.1 Let $1 \le r < \infty$ and $U \subset\subset \mathbf{u}(\Omega)$. Let $\mathbf{u} \in \mathcal{A}$ be a local minimizer of $E[\cdot]$ such that $|\nabla \mathbf{u}|^2 \in h^r(U)$ and $|\text{cof } \nabla \mathbf{u}|^2 \in h^r(U)$ for some $1 \le r < \infty$. Assume further that $\mathbf{u} : \Omega \to \mathbf{u}(\Omega)$ is continuous and bijective map.

Now let us define $\mathbf{g} = (g^1, \ldots, g^n) : C^1_0(\mathbf{u}(\Omega), \mathbb{R}^n) \to \mathbb{R}$ by

$$\langle \mathbf{g}, \mathbf{v}\rangle := \int_\Omega DL(\nabla \mathbf{u}(x)) : \nabla(\mathbf{v}\circ \mathbf{u})(x)\, dx, \tag{3.8}$$

for all $\mathbf{v} = (v^1, \ldots, v^n) \in C^1_0(\mathbf{u}(\Omega), \mathbb{R}^n)$. In view of the volume constraint and growth condition (3.2), it follows that

$$|\langle \mathbf{g}, \mathbf{v}\rangle| \le C\left(1 + \|\nabla u\|^2_{L^2(\Omega)} + \|\text{cof } \nabla u\|^2_{L^2(\Omega)}\right) \|\nabla \mathbf{v}\|_{L^\infty(\mathbf{u}(\Omega))}, \tag{3.9}$$

for any $\mathbf{v} \in C^1_0(\mathbf{u}(\Omega), \mathbb{R}^n)$. Hence $\mathbf{g}$ is a continuous linear functional on $C^1_0(\mathbf{u}(\Omega), \mathbb{R}^n)$. Using the the first variation (3.5), we conclude that

$$\langle \mathbf{g}, \mathbf{v}\rangle = 0 \quad \forall \mathbf{v} \in C^1_0(\mathbf{u}(\Omega), \mathbb{R}^n) \text{ such that } \operatorname{div} \mathbf{v} = 0. \tag{3.10}$$

Hence there exists $q \in \mathcal{D}'(\mathbf{u}(\Omega))$ (see [23, Proposition 1.1, p. 10]), such that

$$\mathbf{g} = -\nabla q \quad \text{in } \mathcal{D}'(\mathbf{u}(\Omega), \mathbb{R}^n) \tag{3.11}$$

modulo translation of a constant. In order to obtain h^r estimates of q; for $1 \le i, j \le n$, we define $\sigma^i_j : \Omega \to \mathbb{R}$ by

$$\sigma^i_j(x) := \sum_{k=1}^n L^i_k(\nabla \mathbf{u}(x)) \frac{\partial u^j}{\partial x_k}(x) \quad \text{for } x \in \Omega, \tag{3.12}$$

so that, the Cauchy-Green strain tensor on Ω is given by

$$\sigma := \left(\sigma^i_j\right) = (DL(\nabla \mathbf{u}))^t \, \nabla \mathbf{u}. \tag{3.13}$$

Define the ijth component of the Cauchy-Green Strain tensor $\tilde{\sigma}^i_j$ on the deformed domain $\mathbf{u}(\Omega)$ by

$$\tilde{\sigma}^i_j := \sigma^i_j \circ \mathbf{u}^{-1} \quad \text{on } \mathbf{u}(\Omega), \quad i, j = 1, \ldots, n. \tag{3.14}$$

The growth condition $|\sigma^i_j| \le C(1 + |\nabla \mathbf{u}|^2 + |\mathrm{cof}\, \nabla \mathbf{u}|^2)$ and $|\nabla \mathbf{u}|^2, |\mathrm{cof}\, \nabla \mathbf{u}|^2 \in L \log L$ yields $\tilde{\sigma}^i_j \in h^1(U)$. If $\mathbf{u} \in K^{1,2r}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ for some $1 < r < \infty$, from the definition of σ^i_j, $\tilde{\sigma}^i_j$ and the condition (3.2) on L, it follows that

$$\begin{aligned} \int_U |(\tilde{\sigma}^i_j|^r &= \int_{\mathbf{u}^{-1}(U)} |\sigma^i_j|^r \\ &\le C\left(1 + \|\nabla \mathbf{u}\|^{2r}_{L^{2r}(\mathbf{u}^{-1}(U))} + \|\mathrm{cof}\, \nabla \mathbf{u}\|^{2r}_{L^{2r}(\mathbf{u}^{-1}(U))}\right), \end{aligned} \tag{3.15}$$

for any $U \subset\subset \mathbf{u}(\Omega)$. In conclusion, if $|\nabla \mathbf{u}|^2 \in h^r$ and $|\mathrm{cof}\, \nabla \mathbf{u}|^2 \in h^r_{\mathrm{loc}}$ for some $1 \le r < \infty$, we have

$$\sigma := \left(\sigma^i_j\right) \in h^r_{\mathrm{loc}}(\Omega, \mathbb{M}^{n\times n}) \quad \text{and} \quad \tilde{\sigma} := \left(\tilde{\sigma}^i_j\right) \in h^r_{\mathrm{loc}}(\mathbf{u}(\Omega), \mathbb{M}^{n\times n}).$$

Observe that, the definition of $\mathbf{g}$ in (3.8), σ^i_j in (3.12), $\tilde{\sigma}^i_j$ in (3.14) and the change of variables (see [22, Corollary 1]) yields,

$$\begin{aligned} \langle \mathbf{g}, \mathbf{v} \rangle &= \sum_{i,k=1}^n \int_\Omega L^i_k(\nabla \mathbf{u}(x)) \, \frac{\partial}{\partial x_k}(v^i \circ \mathbf{u})(x) \, dx \\ &= \sum_{i,j,k=1}^n \int_\Omega L^i_k(\nabla \mathbf{u}(x)) \, \frac{\partial v^i}{\partial y_j}(\mathbf{u}(x)) \frac{\partial u^j}{\partial x_k}(x) \, dx \\ &= \sum_{i,j=1}^n \int_\Omega \sigma^i_j(x) \, \frac{\partial v^i}{\partial y_j}(\mathbf{u}(x)) \, dx \\ &= \int_\Omega \sigma(x) : \nabla_{\mathbf{u}} \mathbf{v}(\mathbf{u}(x)) dx \\ &= \int_{\mathbf{u}(\Omega)} \tilde{\sigma}(y) : \nabla \mathbf{v}(y) \, dy \\ &= -\langle \mathrm{div}\, \tilde{\sigma}, \mathbf{v} \rangle \end{aligned} \tag{3.16}$$

for any $v \in C^1_0(\mathbf{u}(\Omega), \mathbb{R}^n)$. Hence

$$\mathbf{g} = -\,\mathrm{div}\, \tilde{\sigma} \quad \text{in } \mathcal{D}'(\mathbf{u}(\Omega), \mathbb{R}^n), \tag{3.17}$$

where the divergence is taken in each rows. Therefore, combining (3.11) and (3.17), we get

$$\nabla q = \mathrm{div}\, \tilde{\sigma} \quad \text{in } \mathcal{D}'(\mathbf{u}(\Omega), \mathbb{R}^n). \tag{3.18}$$

Applying Theorem 2.2 to (3.18), we conclude that q satisfies the local representation (3.4) and the estimate

$$\begin{aligned}\|q\|_{h^r(V)/\mathbb{R}} &\le C\|\tilde{\sigma}\|_{h^r(U,\mathbb{M}^{n\times n})/\mathcal{V}} \\ &\le C\left(\||\nabla \mathbf{u}|^2\|_{h^r(\mathbf{u}^{-1}(U))} + \||\mathrm{cof}\ \nabla \mathbf{u}|^2\|_{h^r(\mathbf{u}^{-1}(U))}\right), \end{aligned} \tag{3.19}$$

for any $V \subset\subset U \subset\subset \mathbf{u}(\Omega)$, for some $C > 0$ depending on r, V, U n and $\mathbf{u}(\Omega)$. Since $q \in L^1_{\mathrm{loc}}(\mathbf{u}(\Omega))$, from (3.11) it follows that

$$\langle \mathbf{g}, \mathbf{v}\rangle = -\langle \nabla q, \mathbf{v}\rangle = \langle q, \mathrm{div}\, \mathbf{v}\rangle = \int\limits_{\mathbf{u}(\Omega)} q(y)\ \mathrm{div}\, \mathbf{v}(y)\, dy$$

for any $\mathbf{v} \in C^1_0(\mathbf{u}(\Omega), \mathbb{R}^n)$. Hence

$$\int\limits_{\Omega} DL(\nabla \mathbf{u}(x)) : \nabla(\mathbf{v} \circ \mathbf{u})(x) dx = \int\limits_{\mathbf{u}(\Omega)} q(y)\, \mathrm{div}\, \mathbf{v}(y)\, dy, \tag{3.20}$$

for any $\mathbf{v} \in C^1_0(\mathbf{u}(\Omega), \mathbb{R}^n)$. This completes the Theorem. □

4 Derivation of Euler–Lagrange equations

Theorem 4.1 *Let $\Omega \subset \mathbb{R}^n$, $n = 2, 3$ be a smooth, simply connected and bounded domain. Let $\boldsymbol{u} \in \mathcal{A} \cap K^{1,s}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ for some $s \ge 3$ be a continuous and injective local minimizer of $E[\cdot]$. Then the hydrostatic pressure $p := q \circ \boldsymbol{u} \in L^{s/2}_{\mathrm{loc}}(\Omega)$, and the pair $(\boldsymbol{u}, p)$ satisfies*

$$\int\limits_{\Omega} DL(\nabla \boldsymbol{u}(x)) \ :\ \nabla\phi(x)\, dx = \int\limits_{\Omega} p(x)\, \mathrm{cof}\ \nabla \boldsymbol{u}(x)\ :\ \nabla\phi(x)\, dx, \tag{4.1}$$

for all $\phi \in C^1_0(\Omega, \mathbb{R}^n)$, where $q \in L^{s/2}_{\mathrm{loc}}(\boldsymbol{u}(\Omega))$ as in Theorem 3.1. In other words, the pair $(\boldsymbol{u}, p)$ satisfies the system of Euler–Lagrange equations

$$\mathrm{div}\ (DL(\nabla \boldsymbol{u}(x)) - p(x)\, \mathrm{cof}\ \nabla \boldsymbol{u}(x)) = \mathbf{0} \quad \text{in} \quad \mathcal{D}'(\Omega, \mathbb{R}^n).$$

Proof We recall that $K^{1,s}(\Omega, \mathbb{R}^n) := \{\mathbf{w} \in W^{1,s}(\Omega, \mathbb{R}^n) : \mathrm{cof}\ \nabla \mathbf{w} \in L^s(\Omega, \mathbb{M}^{n\times n}\}$ and $\mathcal{A} := \{\mathbf{w} \in K^{1,2}(\Omega, \mathbb{R}^n) : \det \nabla \mathbf{w} = \mathbf{1}$ a.e.$\}$. Let $\mathbf{u}$ be as in the statement of the theorem. By Theorem 3.1, there exists $q \in L^{s/2}_{\mathrm{loc}}(\mathbf{u}(\Omega))$ such that the pair $(\mathbf{u}, q)$ satisfies the identity (3.20). Let $\mathbf{u}^{-1} : \mathbf{u}(\Omega) \to \Omega$ be the inverse of $\mathbf{u}$. Then using the volume-constraint we obtain

$$\nabla_y \mathbf{u}^{-1}(y) = (\nabla_x \mathbf{u}(x))^{-1} = (\mathrm{cof}\ \nabla \mathbf{u}(x))^t\,, \quad y = \mathbf{u}(x),$$

and hence by the change of variables

$$\int\limits_{\mathbf{u}(\Omega)} |\nabla \mathbf{u}^{-1}(y)|^2 dy = \int\limits_{\Omega} |\mathrm{cof}\ \nabla \mathbf{u}(x)|^2 dx < \infty.$$

Using the relation cof $(XY) = \mathrm{cof}\, X\, \mathrm{cof}\, Y$, for $X, Y \in \mathbb{M}^{n\times n}$, observe that

$$Id_n = \mathrm{cof}\ \left(\nabla_y \mathbf{u}^{-1}\, \nabla \mathbf{u}\right) = \mathrm{cof}\ \nabla_y \mathbf{u}^{-1}\ \mathrm{cof}\ \nabla \mathbf{u} = \mathrm{cof}\ \nabla_y \mathbf{u}^{-1}\ (\nabla \mathbf{u})^{-t},$$

and hence

$$\mathrm{cof}\ \nabla \mathbf{u}^{-1} = (\nabla \mathbf{u})^t.$$

Since $\mathbf{u} \in K^{1,s}_{\text{loc}}(\Omega, \mathbb{R}^n)$, it follows that $\mathbf{u}^{-1} \in K^{1,s}_{\text{loc}}(\mathbf{u}(\Omega), \Omega)$ for $s \geq 3$. Let $V \subset\subset \mathbf{u}(\Omega)$ and $\phi \in C^1_0(\mathbf{u}^{-1}(V), \mathbb{R}^n)$. Then the composition $\phi \circ \mathbf{u}^{-1} \in W^{1,s}_0(V, \mathbb{R}^n)$. Hence there exists $\mathbf{v}_\varepsilon \in C^1_0(V, \mathbb{R}^n)$ such that $\mathbf{v}_\varepsilon \to \psi := \phi \circ \mathbf{u}^{-1}$ strongly in $W^{1,s}(V, \mathbb{R}^n)$ as $\varepsilon \to 0$. Let $U := \mathbf{u}^{-1}(V)$. Then Hölder inequality yields

$$\begin{aligned}\int_U DL(\nabla\mathbf{u}) : (\nabla(\mathbf{v}_\varepsilon \circ \mathbf{u}) - \nabla(\psi \circ \mathbf{u}))\, dx &= \int_U (\nabla\mathbf{u})^t DL(\nabla\mathbf{u}) : (\nabla_z \mathbf{v}_\varepsilon(\mathbf{u}) - \nabla_z \psi(\mathbf{u}))\, dx \\ &\leq C\|\nabla\mathbf{u}\|_{L^{2s'}(U)} \|\nabla(\mathbf{v}_\varepsilon - \psi)\|_{L^s(V)},\end{aligned}$$

where $s' := s/(s-1)$. Notice that $s \geq 3$ yields $2s' \leq s$ and hence $\nabla\mathbf{u} \in L^s_{\text{loc}}(\Omega) \subseteq L^{2s'}_{\text{loc}}(\Omega)$. Therefore, from (3.8) we obtain

$$\begin{aligned}\langle \mathbf{g}, \mathbf{v}_\varepsilon \rangle &= \int_{\mathbf{u}^{-1}(V)} DL(\nabla\mathbf{u}(x)) : \nabla(\mathbf{v}_\varepsilon \circ \mathbf{u})(x)\, dx \\ &\to \int_{\mathbf{u}^{-1}(V)} DL(\nabla\mathbf{u}(x)) : \nabla(\phi \circ \mathbf{u}^{-1} \circ \mathbf{u})(x)\, dx \quad \text{as} \quad \varepsilon \to 0 \\ &= \int_{\mathbf{u}^{-1}(V)} DL(\nabla\mathbf{u}(x)) : \nabla\phi(x)\, dx. \end{aligned} \tag{4.2}$$

Since we have $\nabla\mathbf{u}$, $\operatorname{cof} \nabla\mathbf{u} \in L^s_{\text{loc}}$, $q \in L^{s/2}_{\text{loc}}$ and $L^{s/2}_{\text{loc}} \subseteq L^{s/(s-1)}_{\text{loc}}$ for $s \geq 3$, making the change of variables in (3.20), and letting $\varepsilon \to 0$ we obtain

$$\begin{aligned}\langle \mathbf{g}, \mathbf{v}_\varepsilon \rangle &= \int_V q(y) \text{ trace } (\nabla\mathbf{v}_\varepsilon(y))\, dy. \\ &= \int_{\mathbf{u}^{-1}(V)} q(\mathbf{u}(x)) \text{ trace } (\nabla_{\mathbf{u}}\mathbf{v}_\varepsilon(\mathbf{u}(x)))\, dy \\ &= \int_{\mathbf{u}^{-1}(V)} q(\mathbf{u}(x)) \text{ trace } \left(\nabla(\mathbf{v}_\varepsilon \circ \mathbf{u})(x)\ (\operatorname{cof} \nabla\mathbf{u}(x))^t\right) dx \\ &= \int_{\mathbf{u}^{-1}(V)} q(\mathbf{u}(x)) \operatorname{cof} \nabla\mathbf{u}(x) : \nabla(\mathbf{v}_\varepsilon \circ \mathbf{u})(x)\, dx, \\ &\to \int_{\mathbf{u}^{-1}(V)} q(\mathbf{u}(x)) \operatorname{cof} \nabla\mathbf{u}(x) : \nabla(\phi \circ \mathbf{u}^{-1} \circ \mathbf{u})(x)\, dx \\ &= \int_{\mathbf{u}^{-1}(V)} q(\mathbf{u}(x)) \operatorname{cof} \nabla\mathbf{u}(x) : \nabla\phi(x)\, dx. \end{aligned} \tag{4.3}$$

Hence from (4.2) and (4.3) we obtain

$$\int_{\mathbf{u}^{-1}(V)} DL(\nabla\mathbf{u}(x)) : \nabla\phi(x)\, dx = \int_{\mathbf{u}^{-1}(V)} q(\mathbf{u}(x)) \operatorname{cof} \nabla\mathbf{u}(x) : \nabla\phi(x)\, dx,$$

for any $\phi \in C^1_0(\mathbf{u}^{-1}(V), \mathbb{R}^n)$. Finally, choose a sequence of smooth, simply connected sub-domains $V_k \subset\subset V_{k+1} \subset\subset \mathbf{u}(\Omega)$ such that $\mathbf{u}(\Omega) = \cup_{k=1}^\infty V_k$. Utilizing the foregoing arguments, there exists $q_k \in L^{s/2}(V_k)$, $k \geq 1$ such that

$$\int\limits_{\mathbf{u}^{-1}(V_k)} DL(\nabla \mathbf{u}) : \nabla\phi \, dx = \int\limits_{\mathbf{u}^{-1}(V_k)} q_k(\mathbf{u}) \text{ cof } \nabla \mathbf{u} : \nabla \phi \, dx \,, \tag{4.4}$$

for $\phi \in C_0^1(\mathbf{u}^{-1}(V_k), \mathbb{R}^n)$. Since $\mathbf{u}$ is locally volume-preserving homeomorphism, $\Omega = \cup_{k=1}^{\infty} \mathbf{u}^{-1}(V_k)$ is an open covering of Ω and $\mathbf{u}^{-1}(V_k) \subset\subset \mathbf{u}^{-1}(V_{k+1})$. Using the identity div cof $\nabla\mathbf{u}(x) = \mathbf{0}$ and the invertibility of $\nabla \mathbf{u}(x)$, from (4.4) it follows that q_k is unique up to a translation of a constant. Thus adding constant terms as necessary to each q_k, we deduce from (4.4) that for each fixed $k \geq 1$

$$q_i(z) = q_k(z) \quad \text{for } z \in V_i, \quad 1 \leq i \leq k.$$

We finally define $q : \mathbf{u}(\Omega) \to \mathbb{R}$ as $q(z) := q_k(z)$ for $z \in V_k$, so that $q \in L^{s/2}_{\text{loc}}(\mathbf{u}(\Omega))$. This proves that for any $\phi \in C_0^1(\Omega, \mathbb{R}^n)$, the pair $(\mathbf{u}, q)$ satisfies

$$\int\limits_{\Omega} DL(\nabla \mathbf{u}(x)) : \nabla \phi(x) \, dx = \int\limits_{\Omega} q(\mathbf{u}(x)) \text{ cof } \nabla \mathbf{u}(x) : \nabla\phi(x) \, dx.$$

Now let us define the hydrostatic pressure p on Ω by

$$p(x) := q(\mathbf{u}(x)) \quad \text{for } x \in \Omega.$$

Then for any $k \geq 1$,

$$\int\limits_{\mathbf{u}^{-1}(V_k)} |p(x)|^{s/2} = \int\limits_{\mathbf{u}^{-1}(V_k)} |q(\mathbf{u}(x))|^{s/2} dx = \int\limits_{V_k} |q(z)|^{s/2} dz < \infty.$$

Hence $p \in L^{s/2}_{\text{loc}}(\Omega)$ and the pair $(\mathbf{u}, p)$ satisfies

$$\int\limits_{\Omega} DL(\nabla \mathbf{u}(x)) : \nabla \phi(x) \, dx = \int\limits_{\Omega} p(x) \text{ cof } \nabla \mathbf{u}(x) : \nabla \phi(x) \, dx, \tag{4.5}$$

for any $\phi \in C_0^1(\Omega, \mathbb{R}^n)$. In other words, $(\mathbf{u}, p)$ satisfies the system of Euler–Lagrange equations

$$\text{div} \, (DL(\nabla \mathbf{u}(x)) - p(x) \, \text{cof } \nabla \mathbf{u}(x)) = \mathbf{0} \quad \text{in } \Omega,$$

in the sense of (4.5). This completes the proof. □

5 Partial regularity of area-preserving minimizers

In two dimensions, as a consequence of the Euler–Lagrange equations (1.7), together with the standard elliptic estimates [12], we establish the following theorem.

Theorem 5.1 *Let $\Omega \subset \mathbb{R}^2$ be a smooth, bounded simply connected domain and let $L : \mathbb{M}^{2\times 2} \to \mathbb{R}$ be smooth, uniformly convex, such that DL has linear growth and D^2L is bounded. Let $\boldsymbol{u} \in W^{1,3}(\Omega, \mathbb{R}^2)$ be an area-preserving minimizer of the energy $E[\cdot]$. Furthermore, assume that the associated hydrostatic pressure q on the deformed domain $\boldsymbol{u}(\Omega)$ is $C^{0,\alpha}$ for some $0 < \alpha < 1$. Then $\nabla \boldsymbol{u}$ is Hölder continuous on a dense open set $\Omega_0 \subset \Omega$.*

Proof Since $\mathbf{u} \in W^{1,3}(\Omega, \mathbb{R}^2)$ and $\mathbf{u}$ is area-preserving, $\mathbf{u}(\Omega)$ is open and $\mathbf{u}$ is a homeomorphism from Ω to $\mathbf{u}(\Omega)$. By Theorem 4.1, there exists $q \in L^{3/2}_{\text{loc}}(\mathbf{u}(\Omega))$ and the pair $(\mathbf{u}, q \circ \mathbf{u})$ satisfies the system

$$\sum_{j=1}^{2} \frac{\partial}{\partial x_j} \left(\frac{\partial L}{\partial p^i_j}(\nabla \mathbf{u}) - p(x)\, (\text{cof } \nabla \mathbf{u})^i_j \right) = 0, \quad \text{in } \Omega, \quad i = 1, 2, \tag{5.1}$$

where $p := q \circ \mathbf{u}$. Assume that $q \in C^{0,\alpha}(\mathbf{u}(\Omega))$. Since $\mathbf{u} \in W^{1,3}$, Sobolev imbedding theorem yields $\mathbf{u} \in C^{0,1/3}$, and hence p is Hölder continuous with the exponent $\alpha/3$. Let $F : \Omega \times \mathbb{M}^{2\times 2} \to \mathbb{R}$ be the free-energy defined as

$$F(x, P) := L(P) - p(x) \det P \quad x \in \Omega, \quad P \in \mathbb{M}^{2\times 2},$$

so that we can rewrite the nonlinear system (5.1) as

$$\sum_{j=1}^{2} \frac{\partial}{\partial x_j} \left(A^i_j(x, \nabla \mathbf{u}) \right) = 0, \quad \text{in} \quad \Omega, \quad i = 1, 2, \tag{5.2}$$

where

$$A^i_j(x, P) := \frac{\partial F}{\partial p^i_j}(x, P) = \frac{\partial L}{\partial p^i_j}(P) - p(x)(\text{cof } P)^i_j.$$

Let $U \subset\subset \Omega$. Since $|\text{cof } P| = |P|$ for any $P \in \mathbb{M}^{2\times 2}$, $|DL(P)| \le C(1 + |P|)$ and $D^2L(P)$ is bounded,

$$|A^i_j(x, P)| \le C(1 + |P|), \quad \left| \frac{\partial A^i_j}{\partial p^k_l}(x, P) \right| \le C, \tag{5.3}$$

for any $x \in U$, $P \in \mathbb{M}^{2\times 2}$. By Hölder continuity of p, it follows that

$$\begin{aligned} \frac{|A^i_j(x, P) - A^i_j(y, P)|}{1 + |P|} &= |p(x) - p(y)| \frac{\left|(\text{cof } P)^i_j\right|}{1 + |P|} \\ &\le C|x - y|^{\alpha/3}, \end{aligned} \tag{5.4}$$

for any $x \in U$, $P \in \mathbb{M}^{2\times 2}$. By direct calculations and the ellipticity of L it follows that

$$\begin{aligned} \sum_{i,j,k,l=1}^{2} \frac{\partial A^i_j}{\partial p^k_l}(x, P)\xi_{ij}\xi_{kl} &= \sum_{i,j,k,l=1}^{2} \frac{\partial^2 F}{\partial p^i_j p^k_l}(x, P)\xi_{ij}\xi_{kl} \\ &= \sum_{i,j,k,l=1}^{2} \frac{\partial^2 L}{\partial p^i_j p^k_l}(P)\xi_{ij}\xi_{kl} - 2p(x) \det \xi \\ &\ge \lambda_0 |\xi|^2 - 2p(x) \det \xi \\ &:= I(x, \xi), \quad \text{for } P = (p^i_j),\ \xi = (\xi_{ij}) \in \mathbb{M}^{2\times 2}, \end{aligned} \tag{5.5}$$

where $\lambda_0 > 0$ is the ellipticity constant of L. Completing squares, observe that

$$\begin{aligned}\frac{I(x,\xi)}{\lambda_0} &= |\xi|^2 - 2\frac{p(x)}{\lambda_0}\det\xi \\ &= \xi_{11}^2+\xi_{12}^2+\xi_{21}^2+\xi_{22}^2 - 2\frac{p}{\lambda_0}(\xi_{11}\xi_{22}-\xi_{12}\xi_{21}) \\ &= \left(\xi_{11}-\frac{p}{\lambda_0}\xi_{22}\right)^2+\left(\xi_{12}+\frac{p}{\lambda_0}\xi_{21}\right)^2+\left(1-\frac{p^2}{\lambda_0^2}\right)(\xi_{22}^2+\xi_{21}^2). \qquad (5.6)\end{aligned}$$

Similarly, we obtain

$$\frac{I(x,\xi)}{\lambda_0} = \left(\xi_{22}-\frac{p}{\lambda_0}\xi_{11}\right)^2+\left(\xi_{21}+\frac{p}{\lambda_0}\xi_{12}\right)^2+\left(1-\frac{p^2}{\lambda_0^2}\right)(\xi_{11}^2+\xi_{12}^2). \qquad (5.7)$$

Adding the identities (5.6) and (5.7), we obtain

$$\begin{aligned}2\frac{I}{\lambda_0} &= \left(\xi_{11}-\frac{p}{\lambda_0}\xi_{22}\right)^2+\left(\xi_{12}+\frac{p}{\lambda_0}\xi_{21}\right)^2+\left(\xi_{22}-\frac{p}{\lambda_0}\xi_{11}\right)^2 \\ &\quad+\left(\xi_{21}+\frac{p}{\lambda_0}\xi_{12}\right)^2+\left(1-\frac{p^2}{\lambda_0^2}\right)|\xi|^2 \\ &\geq \left(1-\frac{p^2}{\lambda_0^2}\right)|\xi|^2. \qquad (5.8)\end{aligned}$$

Thus from (5.5) and (5.8), it follows that the map $P \mapsto A(\cdot, P)$ is *strongly elliptic* if there exists $\mu_0 > 0$ such that

$$\sum_{i,j,k,l=1}^{2} \frac{\partial L^i_j}{\partial p^k_l}(x,P)\xi_{ij}\xi_{kl} \geq \frac{\lambda_0}{2}\left(1-\frac{p^2}{\lambda_0^2}\right)|\xi|^2 \geq \mu_0|\xi|^2, \quad \text{for } x\in\Omega,\ P,\xi\in\mathbb{M}^{2\times 2},$$

which is equivalent to assume that

$$p^2 \leq \lambda_0^2 - 2\lambda_0\mu_0 \implies (p-\mu_0)^2 \leq (\lambda_0-\mu_0)^2. \qquad (5.9)$$

Since p is defined up to addition of arbitrary constant, the inequality (5.9) is satisfied in subdomain $U \subset\subset \Omega$ if and only if

$$\operatorname{osc}_U p < \lambda_0. \qquad (5.10)$$

Since p is Hölder continuous, the estimate (5.10) holds for any subdomain $U \subset \Omega$ with sufficiently small diameter. Hence $A(x, P)$ is strongly elliptic in P for each $x \in U \subset\subset \Omega$, having sufficiently small diameter. This proves that $A^i_j(x, P)$ satisfies all the conditions of Giaquinta-Modica in [12] on $U \subset\subset \Omega$, with diameter of U being small. Hence by [12, Theorem 1], we conclude that $\nabla\mathbf{u}$ is Hölder continuous on a dense open subset U_0 of U. By Vitali's covering theorem [8, Corollary 2, p. 28] we conclude the proof. □

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