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The reflector design problem

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Abstract

In this paper we review the mathematical advances achieved in recent years on a reflector design problem. This problem can be reduced to the solvability of a fully nonlinear partial differential equation of Monge-Ampere type, subject to a second boundary condition. In the far field case the existence and regularity of solutions was established in [W1]. In the near field case, the existence of weak solutions was obtained in [KO]. The regularity is a very complicated issue but we found precise conditions for it [KW]. In this paper we also prove the C^1 regularity of the reflector, assuming less regular energy distributions.

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1. Introduction

The reflector system we study in this paper consists of a light source at the origin O, a reflecting surface Γ and a bounded, smooth object Σ to be illuminated. Assume that Γ is a radial graph over a domain U in the unit sphere S^n, namely

$$\Gamma = \Gamma_\rho = \{X\rho(X) \mid X \in U\}. \tag{1.1}$$

Let $f \in L^1(U)$ be the density of light radiated from O, and let $g \in L^1(\Sigma)$ be a nonnegative function on Σ. We are concerned with the existence and regularity of the reflector Γ such that the light from O is reflected off to Σ and the distribution of the reflected light is equal to g.

A special and important case is the so called *far field case*, such as a reflector antenna, which can be regarded as the limit of the above

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problem with $\Sigma = \{dX : X \in V\}$, $d \to \infty$, where V is a domain in S^n. Accordingly we may regard g as a function in V, $g \in L^1(V)$. In contrast, the above general reflector problem is often referred to as the *near field case*.

Due to its applications in electro-magnetics and optics, the reflector design is a very practical problem, and has been extensively studied in engineering. This problem has also attracted attentions from mathematicians, see, e.g., problem 21 in [Y]. There are numerous works on numerical analysis and computation for the problem. The law of reflection, namely the angle of reflection is equal to that of incidence, is simple and elegant. However, mathematically the problem is extremely complicated. It involves a fully nonlinear partial differential equation of Monge-Ampere type, subject to a nonlinear second boundary condition.

The Monge-Ampere type equation in the near field case was first derived in [KW]. In the far field case, the equation was derived in [ON, W1], and was also obtained in [We] for a dual problem. The existence of weak solutions can be obtained by approximation by piece-wise paraboloidal surfaces in the far field case [CO, W1] or by piece-wise ellipsoidal surfaces [KO, KW], as for the standard Monge-Ampere equation. The main issue is the regularity. In the far field case the interior regularity was established in [W1] in dimension 2 and in [GW] in high dimensions, and the regularity near the boundary was recently obtained in [TW2]. But in the near field case it is a very complicated issue. We found that

- the regularity depends on the position of the reflector Γ;
- it also depends on the position and geometry of the object Σ;
- it also depends on the geometry of the boundary $\partial\Sigma$.

More precisely, we show that there is a region $\mathcal{D}$ in the cone

$$\mathcal{C}_U = \{tX : t > 0, X \in U\}, \tag{1.2}$$

which is independent of the distributions f and g (but we assume that f, g are positive and smooth), such that the part of the reflector Γ lying in $\mathcal{D}$ is smooth, and the part staying outside $\overline{\mathcal{D}}$ may not be C^1 smooth for some smooth, positive f and g. Moreover, the region $\mathcal{D}$ varies if one translates, rotates, or bends the surface Σ. Also $\mathcal{D}$ varies if one deforms smoothly the boundary $\partial\Sigma$. These phenomena show that regularity of the reflector problem in the near field case is extremely complicated, much more complicated than that in the far field case. However we found precise conditions for a point to be in the region $\mathcal{D}$ [KW]. See §5 below for details. We emphasize that all conditions in §5 are sharp.

This paper is arranged as follow. In Section 2 we introduce the equation, which was derived in [KW]. It is interesting to point out that when the receiving surface Σ is a plane passing through the origin, the equation is the standard Monge-Ampere equation

$$\det D^2 u = h \tag{1.3}$$

for some h depending on f, g, u and Du, where $u = \rho^{-1}$. In Section 3 we deal with the existence of weak solutions. In Section 4 we establish the a priori estimates. Section 5 is devoted to the regularity of weak solutions. We show the part of the reflector lying in the region $\mathcal{D}$ is smooth, provided f, g are smooth and positive, and show that the part staying outside $\overline{\mathcal{D}}$ may not be C^1 smooth for some smooth, positive f, g. All these results are included in [KW].

In Section 6 we include a new C^1 regularity result under weaker assumptions on f, g. We use a similar proof as Loeper [L] for the optimal transportation problem. We note that in the near field case, the C^1 regularity is also local, i.e., it holds only in the region $\mathcal{D}$. Therefore we need to show that a local supporting ellipsoid at some point in $\mathcal{D}$ must be a global one.

In Section 7 we briefly discuss the far field case of the reflector problem, which has been studied by many more people. As mentioned above, the existence and regularity of weak solutions were obtained in [W1]. Theoretically one may also consider the case when $U = V = S^n$ and the reflector Γ is a closed radial graph without boundary. In this case, a weak solution was obtained in [CO], and a smooth solution was obtained in [GW]. An important property in the far field case is that it is an optimal transportation problem (Theorem 4.1, [W2]), and so it becomes a linear programming problem. See [W3,GO] for details. In the far field case the C^1 regularity was proved in [CGH] and later in [TW3], and the $C^{1,\alpha}$ regularity was obtained in [L], and also later in [Liu], for measurable densities f and g.

In this paper we will consider the reflector problem in Euclidean space $\mathbb{R}^{n+1}$ for any $n \geq 2$, as the dimension $n > 2$ does not bring any substantial new difficulty to our treatment.

2. The equation

Let $\Gamma = \Gamma_\rho$ be a reflector. Suppose a ray $X = (x_1, \cdots, x_n, x_{n+1})$ is reflected off at a point $X\rho(X) \in \Gamma$ in direction $Y = (y_1, \cdots, y_n, y_{n+1})$ and reaches a point $Z = (z_1, \cdots, z_n, z_{n+1}) \in \Sigma$. Denote by γ the unit normal of Γ and by $T = T_\rho : X \to Z$ the reflection mapping. Then by the reflection law,

$$\begin{aligned} Y &= X - 2(X \cdot \gamma)\gamma, \\ Z &= X\rho + Yd, \end{aligned}$$

where $d = |Z - X\rho|$ is the distance from $X\rho$ to Z, and $X \cdot \gamma$ denotes the inner product in $\mathbb{R}^{n+1}$.

Let Ω be the projection of U on $\{x_{n+1} = 0\}$, so that

$$x = (x_1, \cdots, x_n) \in \Omega$$
$$\Updownarrow$$
$$X = (x, x_{n+1}) \in U,$$

where $x_{n+1} = \sqrt{1 - |x|^2}$. In the following we may also regard ρ as a function on Ω, and T as a mapping on Ω. By restricting to a subset we may assume that U is in the north hemisphere.

Case I: $\Sigma \subset \{x_{n+1} = 0\}$. That is when the receiving surface lies in a plane passing through the origin. In this case the Jacobian determinant of the mapping T is given by

$$\frac{dS_\Sigma}{dS_\Omega} = \det Dz,$$

where dS denotes the surface area element, $z = (z_1, \cdots, z_n)$. Let f and g be the energy distributions on U and Σ respectively. Assume there is no loss of energy in the process of reflection. Then by the energy conservation,

$$\int_U f = \int_\Sigma g. \tag{2.1}$$

Note that $dS_\Omega = \omega dS_U$, where $\omega(x) = \sqrt{1 - |x|^2}$. Hence we have the equation

$$\det Dz = \frac{f}{\omega g}.$$

After a long and tricky computation, we obtain

$$\det\big\{ - D^2\rho + 2\rho^{-1} D\rho \otimes D\rho \big\} = \frac{-a^{n+1}}{2^n \rho^{2n} b} \frac{f}{\omega g}, \tag{2.2}$$

where $D\rho = (\partial_1 \rho, \cdots, \partial_n \rho)$ is the gradient of ρ, $\hat{D}\rho = (D\rho, 0)$, $D^2\rho = (\partial_i \partial_j \rho)$ is the Hessian matrix of ρ,

$$a = |D\rho|^2 - (\rho + D\rho \cdot x)^2,$$
$$b = |D\rho|^2 + \rho^2 - (D\rho \cdot x)^2.$$

Let $u = \frac{1}{\rho}$. Then u satisfies the standard Monge-Ampère equation

$$\det D^2 u = h(x, u, Du). \tag{2.3}$$

Case II: The receiving surface Σ is given by

$$\Sigma = \{p \in \mathbb{R}^{n+1} : \ \psi(p) = 0\}. \tag{2.4}$$

We may assume that $\psi_{n+1} = \partial_{x_{n+1}}\psi < 0$, by replacing ψ by $-\psi$ if necessary. A special case is when Σ is a Euclidean graph of the form

$$\Sigma = \{X \in \mathbb{R}^{n+1} : \ x_{n+1} = \varphi(x), x \in \Omega^*\} \tag{2.5}$$

or a radial graph of the form

$$\Sigma = \{X\varphi(X) : \quad X \in V\}, \tag{2.6}$$

where Ω^* is a domain in $\mathbb{R}^n$ and V is a domain in the unit sphere S^n.

Computing the Jacobian of the mapping T we get

$$\frac{dS_\Sigma}{dS_\Omega} = -\frac{|\nabla\psi|}{\psi_{n+1}}\det Dz.$$

So we have the equation

$$\det Dz = -\frac{f\psi_{n+1}}{\omega g|\nabla\psi|}.$$

To compute Dz, let Z_0 be the intersection of the reflected ray with the plane $\{x_{n+1} = 0\}$ and denote

$$t = \frac{|Z - X\rho|}{|Z_0 - X\rho|}. \tag{2.7}$$

By a direct but very tricky computation, we obtain

$$\det\Big\{ -D^2\rho + \frac{2}{\rho}D\rho \otimes D\rho + \frac{a(1-t)}{2t\rho}\mathcal{N}\Big\} = h, \tag{2.8}$$

where

$$\begin{aligned} h &= -\frac{a^{n+1}}{2^n t^n \rho^{2n+1} b}\frac{f}{\omega^2 \beta g|\nabla\psi|}, \\ \mathcal{N} &= I + \frac{x \otimes x}{1 - |x|^2}. \end{aligned}$$

From (2.8), we also obtain the equation on the sphere

$$\det\Big\{ -D^2\rho + \frac{2}{\rho}D\rho \otimes D\rho - \frac{\cos\theta}{\sin\theta}|D\rho|I\Big\} = h \ \text{ in } \ U. \tag{2.9}$$

where I is the unit matrix, D is the covariant derivative in a local orthonormal coordinates, and θ is the angle between OX and OZ.

Equation (2.9) is a fully nonlinear partial differential equation of Monge-Ampere type. A special but natural boundary condition for the reflector problem is

$$T(U) = \Sigma. \tag{2.10}$$

This is a second boundary condition. For example, if $\Sigma \subset \{x_{n+1} = 0\}$ and Σ is the unit ball in $\mathbb{R}^n$, then (2.10) is equivalent to $|T(X)| = 1$ for any $X \in \partial U$.

3. Existence of weak solutions

3.1. Ellipsoid of revolution

In the study of the reflector problem, ellipsoid of revolution, namely ellipsoid obtained by rotating an ellipse along its major axis, plays a crucial role (in the far field case it is paraboloid of revolution). Such an ellipsoid E has two foci F_1, F_2. A ray from one focus will always be reflected to the other one.

Let $F_1 = 0$ be the origin of the coordinates. Then in the polar coordinate system, E can be represented as

$$E = \{Xe(X) : \ X \in S^n\}$$

with

$$\begin{aligned} e(X) &= \frac{a(1-\varepsilon^2)}{1-\varepsilon X\cdot\ell} \\ &= \frac{a^2-c^2}{a-cX\cdot\ell}, \end{aligned} \tag{3.1}$$

where a is the major axis, which equals half of the diameter of E, $c = \frac{1}{2}|F_2|$ is the distance from the center of E to its foci, $\varepsilon = \frac{c}{a}$ is the eccentricity, and $\ell = F_2/|F_2|$.

If the reflector Γ locally coincides with the ellipsoid E and if F_2 is a point on the receiving surface Σ, then the matrix

$$\mathcal{W} = \{-D^2\rho + \frac{2}{\rho}D\rho\otimes D\rho + \frac{a(1-t)}{2t\rho}\mathcal{N}\} \tag{3.2}$$

vanishes identically.

This property is also true for more general mappings, such as the reflection in the far field case and the mappings in optimal transportation. More precisely, if the image of a mapping is a fixed point, then the Jacobian matrix of the mapping vanishes identically.

3.2. Terminology related to convexity

We look for solutions to (2.9) (2.10) such that the matrix $\mathcal{W}$ is positive definite. Inspired by early works on Monge-Ampere type equations [P, W1, CO, KO], we introduce the following terminologies.

Supporting ellipsoid. An ellipsoid $E = \{Xe(X) : X \in S^n\}$ is a supporting ellipsoid of $\Gamma = \Gamma_\rho$ at $\bar{X}\rho(\bar{X})$ if one of its foci is at the origin and the other one on Σ, and E satisfies

$$\begin{aligned}&\rho(\bar{X}) = e(\bar{X}),\\ &\rho(X') \le e(X') \quad \forall\ X' \in U.\end{aligned} \tag{3.3}$$

R-convexity of function. We say ρ, or $\Gamma = \Gamma_\rho$, is *R-convex* (with respect to Σ) if for any point $\bar{X} \in U$, there is a supporting ellipsoid at $\bar{X}\rho(\bar{X})$.

R-polyhedron. We say Γ is an R-polyhedron (relative to Σ) if it is R-convex and is a piecewise ellipsoidal surface, namely

$$\Gamma = \bigcup\nolimits_{i=1}^{k} (E_i \cap \Gamma). \tag{3.4}$$

Moreover, for each ellipsoid E_i above, one of its foci is located at the origin and the other one on Σ.

If Γ is R-convex, it can be approximated by R-polyhedra. The approximation sequence can be obtained by choosing finitely many points $p_1, \cdots, p_m \in \Gamma$, and shrinking the supporting ellipsoids of Γ at these points slightly.

Reflection cone. Let γ_1 and γ_2 be two unit vectors ($\gamma_1 \neq \gamma_2$). Let $p \neq 0$ be a point in $\mathcal{C}_U$. The reflection cone $\mathcal{C}_{p,\gamma_1,\gamma_2}$ is the set of points $q \in \mathbb{R}^{n+1}$ which satisfy

$$\frac{p-q}{|p-q|} = 2\frac{c_1\gamma_1 + c_2\gamma_2}{|c_1\gamma_1 + c_2\gamma_2|} - \frac{p}{|p|} \tag{3.5}$$

for all possible constants c_1, c_2.

Remark 3.1. The geometric meaning of the reflection cone $\mathcal{C}_{p,\gamma_1,\gamma_2}$ is as follows. Let Γ_{ρ_1} and Γ_{ρ_2} be two surfaces passing through the point p, whose normals at p are γ_1 and γ_2, respectively. Let $\rho_t = t\rho_1 + (1-t)\rho_2$, where $-\infty < t < \infty$. Obviously Γ_{ρ_t} passes through the point p. Then a ray from the origin, reflected by Γ_{ρ_t} at p, will fall in the cone $\mathcal{C}_{p,\gamma_1,\gamma_2}$.

One can verify that $\mathcal{C}_{p,\gamma_1,\gamma_2}$ is a convex cone ([KW], Lemma 4.7). It becomes a plane if and only if the vectors γ_1, γ_2 and $\vec{Op}$ lies in a 2-plane.

R-convexity of boundary. We say $\partial\Sigma$ is R-convex with respect to a point $p \in \mathcal{C}_U$ if for any unit vectors γ_1, γ_2, the intersection $\mathcal{C}_{p,\gamma_1,\gamma_2} \cap \Sigma$ is connected. We say $\partial\Sigma$ is R-convex with respect to $\mathcal{C}_U$, or simply R-convex, if it is R-convex with respect to all points $p \in \mathcal{C}_U$.

3.3. Weak solutions

We can introduce two weak solutions. Recall that in the theory of convex bodies, one can introduce respectively the curvature measure and the area measure. For the standard Monge-Ampère equation, one can introduce a weak solution of Aleksandrov, which corresponds to the curvature measure, and a weak solution of Brenier, which corresponds to the area measure. Here we introduce a type A weak solution and a type B weak solution, corresponding respectively to the weak solutions of Aleksandrov and Brenier.

First we define two multiple valued maps, $T : U \to \Sigma$ and $\tau : \Sigma \to U$. For any $X \in U$,

$$T(X) = \{Z \in \Sigma : \ Z \text{ is a focus of a SE of } \Gamma \text{ at } X\rho(X)\},$$
$$\tau(Z) = \{X \in U : \ \exists \text{ a SE of } \Gamma \text{ at } X\rho(X) \text{ with } Z \text{ as its focus}\}.$$

where SE means supporting ellipsoid. Note that at any differentiable point of ρ, T is single valued and is exactly the reflection mapping. For any subset $\omega \subset U$ we denote $T(\omega) = \cup_{X\in\omega} T(X)$. Similarly we extend the definition of τ to subsets of Σ. If Γ is smooth and T is one-to-one, then τ is the inverse of T.

Let Γ be an R-polyhedron, as given in (3.4). Let $Z_1, \cdots, Z_k \in \Sigma$ be the foci of the ellipsoids $E_1, \cdots, E_k$. Then for any $Z = Z_k$, $\tau(Z) = \Gamma \cap E_k$ and $\tau(\Sigma')$ has measure zero, where $\Sigma' = \Sigma - \{Z_1, \cdots, Z_k\}$. By approximation one sees that if Γ is an R-convex surface, then for any Borel set $\omega \subset \Sigma$, $\tau(\omega)$ is also Borel. Therefore we may define

$$\mu_b(\omega) = \int_{\tau(\omega)} f \quad \forall\ \omega \subset \Sigma. \tag{3.6}$$

For any two Borel sets $\omega_1, \omega_2 \subset \Sigma$ with $\omega_1 \cap \omega_2 = \emptyset$, the set $\tau(\omega_1) \cap \tau(\omega_2)$ has measure zero, as ρ is twice differentiable a.e.. Hence μ_b is countably additive and so it is a measure. If

$$\mu_b(\omega) = \int_\omega g, \tag{3.7}$$

for any Borel set $\omega \subset \Sigma$, we say that Γ, or equivalently ρ, is a *weak solution of type B* to the reflector problem.

Similarly we can define a measure μ_a on U, that is

$$\mu_a(\omega) = \int_{T(\omega)} g \quad \forall\ \omega \in U. \tag{3.8}$$

We say that Γ, or equivalently ρ, is a *weak solution of type A* to the reflector problem if

$$\mu_a(\omega) = \int_\omega f \tag{3.9}$$

for any Borel set $\omega \subset \Omega$.

An advantage of type B weak solution is that one can easily prove the weak continuity, and so also the countable additivity, of the measure μ_b. One can also prove the weak continuity of μ_a by an idea in [TW1]. If f, g are positive, then two types of weak solutions are equivalent.

3.4. Uniform and gradient estimates

Assume that the reflector Γ is a radial graph given by (1.1) and Σ is contained in the cone $\mathcal{C}_V = \{tX : \ t > 0, X \in V\}$, where V is a domain on the sphere S^n. Suppose

$$\overline{U} \cap \overline{V} = \emptyset. \tag{3.10}$$

Then if Γ_ρ is a weak solution to the reflector problem, we have the Harnack type inequality

$$\sup_{X \in U} \rho(X) \leq \frac{2}{1-\beta} \inf_{X \in U} \rho(X), \tag{3.11}$$

where $\beta = \sup\{X \cdot Y : \ X \in U, Y \in V\}$.

This is because at any point $p \in \Gamma$, there is a supporting ellipsoid at p. Hence the above estimate follows from the expression (3.1). Similarly by the supporting ellipsoid, we also obtain the gradient estimate

$$\sup_{X \in U} |D\rho|(X) \leq C, \tag{3.12}$$

where C depends only on $\sup_U \rho$, β and

$$d_0 = \sup\{|Z| : \ Z \in \Sigma\}. \tag{3.13}$$

If assumption (3.10) is not satisfied, one may consider large reflector, namely solution ρ satisfying

$$\inf \rho(X) > d_0 \tag{3.14}$$

and establish similar estimates.

3.5. Existence of weak solutions

In the far field case, the existence of a type A weak solution was proved in [W1], and that of type B weak solution for closed reflector was proved in [CO], which was extended to the near field case in [KO]. In [KO] the authors discussed the existence of large reflectors only, namely they considered reflector Γ_ρ with $\inf \rho > 2d_0$. The following existence theorem is an extension of that in [KO].

Theorem 3.1. *Consider the reflector problem with distributions f and g satisfying the balance condition (2.1).*
(a) For any point $p \in \mathcal{C}_U$ with $|p| > 2d_0$, there is a weak solution $\rho = \rho_p$, such that the reflector Γ_ρ passes through the point p.
(b) Suppose that $\Sigma \subset \mathcal{C}_V$ and (3.10) holds. Then for any point $p \in \mathcal{C}_U$, there is a weak solution $\rho = \rho_p$ such that the reflector Γ_ρ passes through the point p.

To prove the theorem, choose a sequence of discrete measures g_k, which satisfies (2.1) and converges weakly to g. One can prove that for a fixed point $p \in \mathcal{C}_U$, there is a sequence of type B weak solutions ρ_k with distribution g_k, which passes through the point p. By the weak continuity of the measure μ_b and the uniform and gradient estimates, ρ_k converges to a weak solution ρ which passes through the point p.

3.6. The boundary condition

Suppose both f and g are positive. By the energy conservation (2.1), it is easy to show that a weak solution obtained above satisfies $T(\Omega) \supset \Sigma$ and the set $\{x \in \Omega : T(x) \notin \Sigma\}$ has measure zero. Moreover, if ρ is differentiable at $x_0 \in \Omega$, then $T_\rho(x_0) \subset \overline{\Sigma}$.

4. A priori estimates

4.1. A priori estimates

Consider the more general equation

$$\det\{D^2u - A(x,u,Du)\} = h(x,u,Du) \quad \text{in} \quad B_r(x_0), \tag{4.1}$$

where h is a positive, smooth function, $A = (A_{ij}(x,u,p))$ is a symmetric $n \times n$ matrix satisfying

$$A_{ij,p_kp_l}\xi_i\xi_j\eta_k\eta_l \geq c_0|\xi|^2|\eta|^2 \tag{4.2}$$

for any vectors $\xi, \eta \in \mathbb{R}^n$, $\xi \perp \eta$, where c_0 is a positive constant, and $A_{ij,p_kp_l} = \frac{\partial^2}{\partial p_k \partial p_l} A_{ij}$.

Lemma 4.1. *Let $u \in C^4$ be a solution to (4.1) such that the matrix $\mathcal{W} = D^2u - A(x,u,Du)$ is positive definite. Suppose $h \in C^{1,1}$, $h \geq h_0$ for some constant $h_0 > 0$. Then we have the estimate*

$$|D^2u|(x) \leq C \quad \forall\ x \in B_{r/2}(x_0), \tag{4.3}$$

for some C independent of u.

Proof. Denote $F[\mathcal{W}] = \log \det \mathcal{W}$, $\mathcal{W} = \{W_{ij}\}$, and $\bar{h} = \log h$. Then we have the equation

$$F[\mathcal{W}] = \bar{h}. \tag{4.4}$$

Differentiating the equation twice, and by the concavity of F, we get

$$F^{ij} W_{ij,kk} = -\frac{\partial^2 [\det \mathcal{W}]^{1/n}}{\partial W_{ij} \partial W_{rs}} W_{ij,k} W_{rs,k} + D_k^2 \bar{h} \geq D_k^2 \bar{h},$$

where $W_{ij,k} = \partial_{x_k} W_{ij}$, $F^{ij} = \frac{\partial}{\partial W_{ij}} F[\mathcal{W}]$. Let $z(x, \xi) = \rho^2 \xi_i \xi_j W_{ij}$. Suppose $\sup\{z(x,\xi) : \ x \in B_r(0), |\xi| = 1\}$ is attained at $\bar{x}$ and $\xi = (1, 0, \cdots, 0)$, where $\rho(x) = (1 - \frac{|x|^2}{r^2})^+$. By a rotation of axes we assume that $\{W_{ij}\}$ is diagonal at $\bar{x}$ such that $W_{11} \geq \cdots \geq W_{nn}$. Then at $\bar{x}$, F^{ij} is diagonal and

$$\begin{aligned} 0 &= (\log z)_i = \frac{2\rho_i}{\rho} + \frac{W_{11,i}}{W_{11}}, \\ 0 &\geq (\log z)_{ii} = (\frac{2\rho_{ii}}{\rho} - \frac{6\rho_i^2}{\rho^2}) + \frac{W_{11,ii}}{W_{11}}. \end{aligned}$$

We have

$$\begin{aligned} W_{11,ii} &= u_{11ii} - A_{11,p_i p_i} u_{ii}^2 - A_{11,p_k} u_{kii} + O(1 + u_{11}), \\ W_{ii,11} &= u_{11ii} - A_{ii,p_1 p_1} u_{11}^2 - A_{ii,p_k} u_{k11} + O(1 + u_{11}). \end{aligned}$$

By $(\log z)_i = 0$, we have $|u_{11k}| \leq C(1 + W_{11})/\rho$. Note that

$$\begin{aligned} F^{ii} u_{kii} &= F^{ii} W_{ii,k} + F^{ii} \partial_{x_k} A_{ii} = \partial_{x_k} F[\mathcal{W}] + O(1 + \mathcal{F}\mathcal{T}), \\ F^{ii} u_{ii}^2 &\leq F^{ii} W_{ii}^2 + O(1 + \mathcal{F}) \leq O(1 + \mathcal{F} + \mathcal{T}), \end{aligned}$$

where $\mathcal{F} = \sum F^{ii}$ and $\mathcal{T}$ is the trace of the matrix $\mathcal{W}$. Hence

$$\begin{aligned} F^{ii} W_{11,ii} &= F^{ii} W_{ii,11} + F^{ii} A_{ii,p_1 p_1} u_{11}^2 + O(\frac{1}{\rho}(1 + \mathcal{F}\mathcal{T})) \\ &\geq F^{ii} A_{ii,p_1 p_1} u_{11}^2 + O(\frac{1}{\rho}(1 + \mathcal{F}\mathcal{T})). \end{aligned}$$

By (4.2), $A_{ii,p_1 p_1} \geq c_0 > 0$. Hence

$$F^{ii} A_{ii,p_1 p_1} \geq c_0 \sum_{i>1} F^{ii} = \frac{1}{2} c_0 \mathcal{F} > 1$$

provided W_{11} is large enough. We obtain

$$\begin{aligned} 0 &\geq \sum_i F^{ii} (\log z)_{ii} \geq -\frac{C}{\rho^2} \mathcal{F} + F^{ii} \frac{W_{11,ii}}{W_{11}} \\ &\geq -\frac{C}{\rho^2} \mathcal{F} + c_0 W_{11} \mathcal{F} + O(\frac{1}{\rho}(1 + \mathcal{F})) \end{aligned}$$

We obtain $\rho^2 W_{11}(x) \le \rho^2 W_{11}(\bar{x}) \le C$ for any $x \in B_r(x_0)$. Hence (4.3) holds. □

We remark that similar proof of Lemma 4.1 can be found in [W1, GW, MTW]. Once the second derivative estimate is established, the least eigenvalue of the matrix $\mathcal{W}$ has a positive lower bound and so equation becomes uniformly elliptic. By Evans-Krylov's regularity for fully nonlinear uniformly elliptic equations, we obtain the higher order derivative estimate.

Theorem 4.1. *Under the assumption of Lemma 4.1,*

$$\|u\|_{C^{3,\alpha}(B_{r/2}(0))} \le C \tag{4.5}$$

4.2. Verification of (4.2)

To apply Theorem 4.1 to equation (2.8), let $u = \frac{1}{\rho}$. Then equation (2.8) becomes

$$\det\left\{D^2 u - \frac{\hat{a}(t-1)}{2ut}\mathcal{N}\right\} = h \tag{4.6}$$

for a function $h = h(x, u, Du)$, defined on $\Omega \times \mathbb{R}^1 \times \mathbb{R}^n$, where

$$\hat{a} = |Du|^2 - (u - Du \cdot x)^2.$$

Denote

$$\tau = \frac{(t-1)\hat{a}}{t}. \tag{4.7}$$

Then τ is a function of x, u and $p := Du$. Condition (4.2) is equivalent to

$$\{\tau_{p_i p_j}\} \ge c_0 I \tag{4.8}$$

By some very tricky computation, we have

$$\tau_{p_k p_l} = -\frac{\hat{a}}{t^2(\nabla\psi \cdot \xi)}(Z'_{p_k}\nabla^2\psi Z_{p_l}) + \frac{2}{t}\frac{\nabla\psi \cdot Z}{\nabla\psi \cdot \xi}(\delta_{kl} - x_k x_l), \tag{4.9}$$

where Z' is the transpose of Z, $Z_{p_k} = \partial_{p_k} Z$. Note that $Z = T(X)$ is a point of Σ, which is determined by X, Γ and Σ. Therefore we have

Theorem 4.2. *Suppose Σ is convex radial graph given by (2.6). Suppose that*

$$|(q-p) \cdot \nu| > 0 \tag{4.10}$$

for any point $p \in \mathcal{C}_U$, $q \in \Sigma$, where ν is the normal of Σ at q. Then the a priori estimate (4.3) and (4.5) hold for any R-convex solution to (4.6).

Indeed, when Σ is a convex radial graph given by (2.6), then the matrix $\{Z'_{p_k}\nabla^2\psi Z_{p_l}\}$ is nonnegative, and $\hat{a} < 0$, $\nabla\psi \cdot Z > 0$, $\nabla\psi \cdot \xi > 0$.

Hence (4.8) is satisfied. Assumption (4.10) implies that the right hand side function h is positive.

After more careful computation, we find that (4.8) is equivalent to

$$II + \frac{\sin\theta\cos\beta}{4\sin\alpha\cos^3\alpha}\frac{|q|^2}{|p|\,|p-q|^2}I > 0, \tag{4.11}$$

for any $q \in \Sigma$, where I and II are the first and the second fundamental forms of Σ along the direction $p - q$, θ is the angles between Op and Oq, α is the angle of reflection, and β is the angle between Oq and the normal of Σ at q. Hence we have

Theorem 4.3. *Suppose (4.10) and (4.11) holds. Then the a priori estimate (4.3) and (4.5) hold for any R-convex solution to (4.6).*

5. Regularity of solutions

5.1. The regularity domain $\mathcal{D}$

The domain $\mathcal{D}$ introduced in the introduction is defined as follow. A point $p \in \mathcal{D}$ if and only if the following conditions are satisfied:

(i) (4.10) holds for all $q \in \Sigma$;
(ii) (4.11) holds for all $q \in \Sigma$;
(iii) $\partial\Sigma$ is R-convex with respect to points in $\mathcal{C}_U$ near p.

Assumption (i) and (ii) are required in the a priori estimates (Theorem 4.3). Assumption (iii) is needed for the comparison principle below. It is easy to see that the set $\mathcal{D}$ is in general not a cone, and is in general a proper subset of $\mathcal{C}_U$. Therefore the near field case is not an optimal transportation problem.

5.2. A comparison principle

Let ρ be a weak solution in Theorem 3.1. By §3.6 we have $T(U) \supset \Sigma$ and $|\{x \in U :\ T(x) \notin \Sigma\}| = 0$. To prove the regularity of solutions, we show that in a sufficiently small ball, the weak solution coincides with a smooth solution. For this purpose we need a comparison principle. A crucial ingredient for the comparison principle is the inclusion

$$T(U) \subset \overline{\Sigma} \tag{5.1}$$

We show that the inclusion (5.1) holds, provided that $\partial\Sigma$ is R-convex. Indeed, if Γ is C^1, then (5.1) holds, since $T_\rho(x_0) \subset \overline{\Sigma}$ at any differentiable point x_0. If Γ is not C^1 at some point $p \in \Gamma$, let E_0, E_1 be two supporting ellipsoids at p, with normals γ_0, γ_1. Let $\mathcal{C}_{p,\gamma_0,\gamma_1}$ be the reflection cone (see §3.2). Since $\partial\Sigma$ is R-convex, $\mathcal{C}_{p,\gamma_0,\gamma_1}\cap\Sigma$ is connected, which means (5.1).

By (5.1) we have the following comparison principle.

Lemma 5.1. *Let ρ_1, ρ_2 be weak solutions of (2.8) in Ω with $f = f_1, f_2$, respectively, where Ω is a smooth domain in $B_1(0) \subset \mathbb{R}^n$. Suppose that $T_{\rho_1}(\Omega) \subset \overline{\Sigma}$. Suppose $f_1 < f_2$ in Ω and $\rho_1 \le \rho_2$ on $\partial\Omega$. Suppose Γ_1, the graph of ρ_1, lies in the region $\mathcal{D}$. Then we have $\rho_1 \le \rho_2$ in Ω.*

Indeed, if the lemma is not true, denote $\omega = \{x \in \Omega : \ \rho_1(x) > \rho_2(x)\}$. Then we have

$$T_{\rho_2}(\omega) \subset T_{\rho_1}(\omega). \tag{5.2}$$

Hence

$$\int_\omega f_2 > \int_\omega f_1 = \int_{T_{\rho_1}(\omega)} g \ge \int_{T_{\rho_2}(\omega)} g, \tag{5.3}$$

which is in contradiction with the definition of weak solutions.

Note that we have used the inclusion (5.1) for the equality in (5.3). The assumption that Γ_1 is contained in the region $\mathcal{D}$ is such that a local supporting ellipsoid is also a global one (see §6.2 below), which is used in (5.2). If ρ_1 is C^1 smooth, then there is a unique supporting ellipsoid at every point and a local supporting ellipsoid is automatically a global one. Hence Lemma 5.1 holds if ρ_1 is C^1 smooth.

5.3. Regularity of weak solutions

Theorem 5.1. *Let ρ be the weak solution in Theorem 3.1. Suppose that f, g are positive and smooth. Then if $p \in \Gamma_\rho$ is a point in $\mathcal{D}$, ρ is smooth near p.*

To prove the theorem, let B_r is a small ball such that the point p is contained in the graph of ρ in B_r. Consider equation (2.8) in B_r with the Dirichlet boundary condition $\rho = \varphi_\varepsilon$. We choose φ_ε properly, e.g., φ_ε is the mollification of ρ. Then one can establish the global a priori estimates to the Dirichlet problem as in [W1]. Hence by the continuity method, there is a smooth solution ρ_ε to the Dirichlet problem. By the interior a priori estimates (Theorem 4.3) and the comparison principle, ρ_ε converges to ρ in B_r. Namely the weak solution ρ is smooth in B_r.

5.4. Assumptions (i)-(iii) in §5.1 are sharp

To show condition (iii) is sharp, we use the idea from [W1] (page 362), see also §7.3 of [MTW]. The argument is roughly as follows.

- Choose a sequence of positive distributions $g_k \in C^\infty(\Sigma)$ which converges weakly to $g = \delta_{q_1} + \delta_{q_2}$, where q_1, q_2 are two points on Σ. Let ρ_k be the solution corresponding to the distributions f and g_k.

- If $\rho_k \in C^1$, then $T_{\rho_k}(\Omega) \subset \overline{\Sigma}$. Hence if $\rho_k \in C^1$ for all k, then $T_{\rho_0}(\Omega) \subset \overline{\Sigma}$. But since Γ_{ρ_0} reflects almost all rays to either q_1 or q_2, we have either $\rho_0 = e_1$ or $\rho_0 = e_2$, where e_i is an ellipsoid with one focus at the origin and the other at q_i, $i = 1, 2$.
- Let $x_0 \in \Omega$ such that $e_1(x_0) = e_2(x_0)$. Let P_t be the plane passing through the point $p = X_0 e_1(X_0)$ with normal $t\gamma_1 + (1-t)\gamma_2$, where $t \in (0,1)$ and $X_0 = (x_0, \sqrt{1-|x_0|^2})$. If $\rho_k \in C^1$, then for any $0 < t < 1$, the reflected ray by P_t at the point p will hit Σ. But if $\partial\Sigma$ is not R-convex with respect to p, we may choose Z_1, Z_2 such that the reflected ray by P_t at p will miss the object Σ. Hence ρ_k is not C^1 for large k.

One can show that (i) (ii) are sharp in a similar way as above [KW].

The case when the receiving surface $\Sigma \subset \{x_{n+1} = 0\}$ is at the borderline for the condition (ii). In this case the equation becomes the standard Monge-Ampere equation (2.3), subject to the boundary condition (2.10). Condition (2.10) is different from the boundary condition

$$Du(\Omega) = \Sigma, \tag{5.4}$$

studied in [C1, C2]. Therefore even if Σ is a convex domain in $\{x_{n+1} = 0\}$ (convexity in the usual sense), $\partial\Sigma$ may not be R-convex and the regularity of Caffarelli may not apply. In other words, the regularity of solutions to the boundary value problem (2.3) (2.10) requires a separate treatment.

6. C^1 regularity of reflector

In the far field case, the C^1 regularity of reflector for non-smooth distributions f, g was obtained in [CGH]. As the reflector problem in the far field case is an optimal transportation problem, the C^1 or $C^{1,\alpha}$ regularity for potential functions [L, Liu, TW3] also apply to the reflector problem in the far field case. But as indicated above, the reflector problem in the near field case is not an optimal transportation problem anymore.

To establish the C^1 regularity for nonsmooth distributions f, g in the near field case, we use a similar argument as in [L]. Our proof consists of the following three steps:
(i) verify a geometric property of (4.8);
(ii) show that a local supporting ellipsoid is a global one;
(iii) establish the continuity estimate.

6.1. A geometric interpretation of (4.8)

Let

$$E_i = \{X e_i(X) : \ X \in S^n\}, \quad i = 0, 1,$$

be two ellipsoids with one focus at the origin and the other one Z_i on the receiving surface Σ, where e_i is a function of the form (3.1). Denote $\mathcal{T} = \{X \in \mathbb{R}^{n+1} : e_0(X) = e_1(X)\}$ the intersection of E_0 and E_1. Let $p \in \mathcal{T}$ be a given point and let γ_0 and γ_1 be the normal of E_0 and E_1 at p. Let $\mathcal{C}_{p,\gamma_1,\gamma_2}$ be the reflection cone defined in §3.2. Denote $\ell = \Sigma \cap \mathcal{C}_{p,\gamma_1,\gamma_2}$. Then for any point $Z \in \ell$ between Z_0 and Z_1, there is a unique ellipsoid $E = E_{p,Z}$ with foci O and Z, passing through the point p. By the reflection property of ellipsoid, E is tangent to $\mathcal{T}$ at p.

Suppose the ellipsoid $E = \{X\, e(X) : X \in S^n\}$. Denote $w(x) = 1/e(x)$, $w_i(x) = 1/e_i(x)$, $i = 0, 1$, where as before, x is the projection of X on the plane $\{x_{n+1} = 0\}$. Since E is tangent to $\mathcal{T}$ at p, we have

$$Dw = \theta D w_1 + (1-\theta) D w_0 \tag{6.1}$$

for some $\theta \in (0,1)$. Conversely, for any $\theta \in (0,1)$, there is a point $Z \in \ell$ such that (6.1) holds.

Choose a proper coordinate system such that p is on the positive x_{n+1}-axis and $\mathcal{T}$ is tangential to the plane $\{x_n = 0\}$. Then

$$D(w_1 - w_0) = (0, \cdots, 0, \alpha)$$

for some $\alpha \neq 0$. Note that the matrix $\mathcal{W} \equiv 0$, see (3.2). We have

$$D^2 w = \tau(Dw)\mathcal{N}, \tag{6.2}$$

where τ is the function in (4.7). If (4.8) holds, we differentiate (6.2) in θ to get

$$\frac{d^2}{d\theta^2} D^2 w = \tau_{p_n p_n} |D(w_1 - w_0)|^2 \mathcal{N} > 0. \tag{6.3}$$

The above inequality implies that near $x = 0$,

$$w < \theta w_1 + (1-\theta) w_0 \tag{6.4}$$

on the plane $x_n = 0$. In particular we have

$$w(x) < \max(w_1(x), w_0(x)) \quad \text{for } x \text{ near } 0, \neq 0. \tag{6.5}$$

Remark 6.1. Inequality (6.5) corresponds to Loeper's geometric property of (A3) in optimal transportation. In optimal transportation, we have a Monge-Ampere type equation of the form (4.1) with $A(x, Du) = D_x^2 c(x, T(x, Du))$, where $c(x, y)$ is the cost function and T is the mapping determined by the potential u. By Remark 4.1 in [MTW], condition (A3) is equivalent to (4.2) above. Let $\varphi_1 = c(\cdot, y_1) + a_1$ and $\varphi_2 = c(\cdot, y_2) + a_2$. Suppose $\varphi_1 = \varphi_2$ at some point x_0. Let $\{y_t : t \in [1,2]\}$ be a c-segment (with respect to x_0) connecting y_1 and y_2 and let $\varphi_t = c(\cdot, y_t) + a_t$, where a_t is chosen such that $\varphi_t(x_0, y_t) = \varphi_1(x_0, y_1)$. Then since the matrix $\{D^2\varphi_t - A(x, D\varphi_t)\} \equiv 0$, differentiating the matrix in t we obtain

$$\frac{d^2}{dt^2} D^2\varphi_t = A_{p_k p_l} \partial_k(\varphi_2 - \varphi_1)\partial_l(\varphi_2 - \varphi_1). \tag{6.6}$$

From (6.6) one obtain Loeper's geometric interpretation of (A3).

6.2. Local supporting function is global

Let Γ_ρ be a weak solution to the reflector problem. Suppose that E_0, E_1 are two supporting ellipsoids at the point p. Inequality (6.5) implies that E is a local supporting ellipsoid of Γ_ρ at p. We claim that E is a global c-support at p as well.

Lemma 6.1. *Let w_0, w_1 and w be as above. Suppose (6.5) holds near 0. Then it holds for all $x \in \Omega$.*

Proof. By (3.1) we have the expressions

$$w = c_0 + \sum_{k=1}^{n} c_k x_k + c_* \sqrt{1 - |x|^2}, \tag{6.7}$$

$$w_i = c_0^i + \sum_{k=1}^{n} c_k^i x_k + c_*^i \sqrt{1 - |x|^2}, \quad i = 0, 1, \tag{6.8}$$

where c_k, c_* are constants. Since E is tangential to $\mathcal{T}$, we have $\partial_k w = \partial_k w_0 = \partial_k w_1$ for $k = 1, \cdots, n-1$ at $x = 0$, namely

$$c_k = c_k^0 = c_k^1, \quad k = 1, \cdots, n-1.$$

By (6.1)

$$c_n = \theta c_n^1 + (1-\theta) c_n^0.$$

Since all ellipsoids E_0, E_1, E pass through the point p,

$$c_0 + c_* = c_0^0 + c_*^0 = c_0^1 + c_*^1.$$

By (6.5) we also have

$$c_* > \theta c_*^1 + (1-\theta) c_*^0.$$

Therefore from the expressions (6.7) and (6.8) we have

$$w < w_\theta := \theta w_1 + (1-\theta) w_0$$

for all $x \in \Omega, x \neq 0$. □

Let E^s denote the solid ellipsoid enclosed by E. Then Lemma 6.1 implies that

$$E_0^s \cap E_1^s \subset E^s.$$

In optimal transportation, in order that a local c-support is a global one, one needs to assume the assumption (A3) in the whole domain Ω. For our reflector problem, the condition (4.8), which corresponds to (A3), is assumed in a subdomain $\mathcal{D}$ and we still have the property that a local support is a global one. This is not a surprise, due to ellipsoid's nice geometry. Obviously such property is also true for paraboloid.

6.3. Continuity estimate

We prove the following C^1 regularity. The proof is similar to that in [L].

Theorem 6.1. *Assume that $f \geq 0, \in L^p(U)$ with $p \geq \frac{n+1}{2}$, and $c_0 \leq g \leq c_1$ for some positive constants c_0, c_1. Then the part of the reflector Γ_ρ located in $\mathcal{D}$ is C^1.*

Proof. Let p_0 be an any given point in Γ_ρ located in $\mathcal{D}$. Choose a coordinate system such that p_0 is on the positive x_{n+1}-axis. Let $u = \frac{1}{\rho}$. To show that Γ_ρ is C^1 at p_0, it suffices to show that u is C^1 at the origin.

Suppose to the contrary that u is not C^1 at O. Then there is two supporting ellipsoids $E_i = \{Xe_i(X) : X \in S^n\}$, $i = 0, 1$, with focus at $Z_i \in \Sigma$ (the other one is at the origin). By previous discussions, all local supports are global. Hence to avoid the possible complexity of the geometry of Σ, we may assume that Z_0 and Z_1 are relatively close. As above let γ_0 and γ_1 be the normal of E_0 and E_1 at p_0, and $\mathcal{C}_{p_0,\gamma_1,\gamma_2}$ be the reflection cone defined in §3.2. Denote $\ell = \Sigma \cap \mathcal{C}_{p_0,\gamma_1,\gamma_2}$. Let $Z \in \ell$ be a middle point between Z_0 and Z_1 (by arc-length of ℓ). Then there is a unique ellipsoid $E = E_{p_0,Z}$ with foci O and Z, which is also a supporting ellipsoid of Γ_ρ at p_0, and is tangent to $\mathcal{T}$ at p_0, where $\mathcal{T}$ is the intersection of E_0 and E_1.

Suppose E is given by $E = \{Xe(X) : X \in S^n\}$, with

$$e(X) = \frac{a^2 - c^2}{a - cX \cdot \ell}$$

where a is the major axis of E, $c = \frac{1}{2}|Z|$, and $\ell = F_2/|F_2|$. We shrink the ellipsoid E by a small factor $\delta > 0$, to get a new ellipsoid E_δ given by

$$e_\delta(X) = \frac{(a-\delta)^2 - c^2}{(a-\delta) - cX \cdot \ell}.$$

Then the point p_0 is located outside of E. Let G be the component of $\{p \in \Gamma_\rho : p \notin E^s\}$ which contains p_0, where as above, E^s denotes the solid body enclosed by E. It is easy to verify that

$$|G| = C\delta^{\frac{n+1}{2}} \tag{6.9}$$

for some $C > 0$ depending on $Z_0 - Z_1$. Let G' be the radial projection of G on S^n, namely $G = \{X\rho(X) :\ X \in G'\}$. Then the energy from the light source at the origin is approximately

$$\begin{aligned}\int_{G'} f &\le |G'|^{1-\frac{1}{p}} \int_{G'} f^p \\ &= o(1)|G'|^{1-\frac{1}{p}} = o(1)\delta^{\frac{p-1}{p}\frac{n+1}{2}},\end{aligned} \tag{6.10}$$

where $o(1) \to 0$ as $|G'| \to 0$. On the other hand, let Σ' be the set of points $Z' \in \Sigma$ which is a focus of a supporting ellipsoid of Γ_ρ at some point in G. By (6.5) and similarly to [L], one can show that Σ' contains the σ-neighborhood of ℓ', where $\ell' \subset \ell$ is a curve whose two endpoints are respectively the middle points between Z_0, Z and Z_1, Z, and $\sigma \ge C\delta^{1/2}$, where C depends on $Z_0 - Z_1$. Hence we have

$$\int_{\Sigma'} g \ge C\delta^{\frac{n-1}{2}}. \tag{6.11}$$

Therefore by the energy conservation, $\int_{G'} f = \int_{\Sigma'} g$, we obtain

$$\delta^{\frac{n-1}{2}} \le o(1)\delta^{\frac{p-1}{p}\frac{n+1}{2}}$$

When $p \ge \frac{n+1}{2}$, we reach a contradiction. $\square$

Similarly as in [L], we have $\rho \in C^{1,\alpha}$ provided $c_0 \le g \le c_1$, $f \in L^p$ with $p > \frac{n+1}{2}$.

7. The far field case

As mentioned before, the far field case of the reflector problem can be regarded as the limit of the near field case with $\Sigma = \{dX :\ X \in V\}$, $d \to \infty$. It can also be stated as follows. Given two energy densities $f \in L^1(U), g \in L^1(V)$, where $U, V \subset S^n$, $\overline{U} \cap \overline{V} = \emptyset$. Find a reflector Γ_ρ which reflects the light from O with distribution f such that direction of the reflected light falls in V and the distribution of reflected light is equal to g.

7.1. The equation

The equation can be obtained from (2.8) by taking limit [KW]. It can also be derived by direct computation [ON, W1]. Let $X = X(t^1, t^2)$

be a smooth parametrization of S^2. Denote by $e = e_{ij}dt^idt^j$ the first fundamental form of S^2. Put $(e^{ij}) = (e_{ij})^{-1}$. Denote $\partial_i = \partial/\partial t^i$, $\nabla = e^{ij}\partial_i X\partial_j$. Then the unit normal of Γ at $X\rho(X)$ is given by

$$\gamma = \frac{\nabla\rho - \rho X}{\sqrt{\rho^2 + |\nabla\rho|^2}} = -\frac{\nabla u + uX}{\sqrt{u^2 + |\nabla u|^2}},$$

where $u = \frac{1}{\rho}$. Suppose a ray $X \in S^n$ is reflected by Γ to a direction $Y \in S^n$. Then the reflection mapping $T:\ X \to Y$ is given by

$$Y = X - 2\langle X, \gamma\rangle\gamma = -\frac{1}{\eta}[\nabla u + (u-\eta)X],$$

where $\eta = \frac{1}{2u}(|\nabla u|^2 + u^2)$. Direct computation show that

$$\partial_i Y = -\frac{1}{\eta}q_{ij}(\partial_j X - \frac{u_j}{u}\beta), \tag{7.1}$$

where $\beta = \frac{1}{\eta}(\nabla u + uX)$, $q_{ij} = \nabla_{ij}u + (u-\eta)e_{ij}$. The equation is obtained by computing the Jacobian determinant $|\frac{dY}{dX}|$. By (7.1) we obtain the equation [ON, W1, GW]

$$\frac{\det(\nabla_{ij}u + (u-\eta)e_{ij})}{\eta^n \det(e_{ij})} = \frac{f(X)}{g(T(X))}. \tag{7.2}$$

The boundary condition is

$$T(U) = V. \tag{7.3}$$

As noted in §3.1, the matrix $\{\nabla_{ij}u + (u-\eta)e_{ij})\}$ vanishes completely if the reflector is a paraboloid of revolution with focus at the origin.

7.2. Terminology in the far field case

The terminologies are similar to those in §3.2. But there are also some differences.

Supporting paraboloid. An paraboloid $F = \{X\hat{f}(X):\ X \in S^n\}$ is a supporting paraboloid of $\Gamma = \Gamma_\rho$ at $\bar{X}\rho(\bar{X})$ if the focus of F is at the origin and F satisfies

$$\begin{aligned} &\rho(\bar{X}) = \hat{f}(\bar{X}), \\ &\rho(X') \le \hat{f}(X')\ \ \forall\ X' \in U. \end{aligned} \tag{7.4}$$

For such a paraboloid, the function $\hat{f}$ has the form, in the polar coordinates,

$$\hat{f}(X) = \frac{C}{1 - \langle X, Y\rangle}, \tag{7.5}$$

where $Y \in S^n$ is the axis of F.

R-convexity of function. We say ρ, or Γ_ρ, is *R-convex* if for any point $\bar{X} \in U$, there is a supporting paraboloid at $\bar{X}\rho(\bar{X})$. An R-convex function is called admissible in [W1].

Reflection cone. Let γ_1 and γ_2 be two unit vectors ($\gamma_1 \neq \gamma_2$). Let $p \neq 0$ be a point in $\mathcal{C}_U$. The reflection cone $\mathcal{C}^*_{p,\gamma_1,\gamma_2}$ is a translation of $\mathcal{C}_{p,\gamma_1,\gamma_2}$ such that its vertex is at the origin, where $\mathcal{C}_{p,\gamma_1,\gamma_2}$ is defined in §3.2.

R-convexity of boundary. We say ∂V is R-convex if for any point $p \in \mathcal{C}_U$ and any unit vectors γ_1, γ_2, the intersection $\mathcal{C}^*_{p,\gamma_1,\gamma_2} \cap V$ is connected.

Remark 7.1. We make the translation of the reflection cone, because in the far field case, we are only concerned with the direction of the reflected rays, so we need to move the starting point of the reflected ray to the origin. The R-convexity of ∂V should replace the condition (C) in [W1].

Equivalently, ∂V is R-convex if the intersection $\mathcal{C} \cap V$ is connected for any round convex cone $\mathcal{C}$ with vertex at O containing a point in U.

7.3. Existence and regularity of solutions

Similarly to the treatment in §3 above for the near field case, one can introduce two different weak solutions, namely type A and type B weak solutions, for the far field case. A weak solution of type A was introduced in [W1], and a weak solution of type B was introduced in [CO].

In [W1] we proved the existence of a type A weak solution to the above problem, in a similar way as in §3 above. When ∂V is R-convex, the regularity was also established in [W1], in a similar way as in §4 and §5. That is, one establishes the a priori estimates as in §4.1, and proves the weak solution is smooth in a small ball by solving a Dirichlet problem, as in §5. In particular it was also shown in [W1] that the R-convexity of ∂V is necessary for the regularity.

The regularity in the far field case also follows from the arguments in §4 and §5 above. Recall that the far field case can be regarded as the limit of the near field case with $\Sigma = \{dX : X \in U\}$, $d \to \infty$. When d is sufficiently large (compared with with $\sup \rho$), conditions (4.2) and (4.8) are automatically satisfied, so the solution is smooth provided ∂V is R-convex.

The existence of a type B weak solutions was obtained in [CO] in the case when $U = V = S^n$. The a priori estimates (as in §4.1) were established in [GW], and the existence of a smooth solution was obtained by the continuity method.

Remark 7.2. When the densities f, g are not smooth, Caffarelli, Gutierrez, and Huang [CGH] proved the solution is C^1 smooth if $c_1 \le f, g \le c_2$ for some positive constants c_1, c_2. The C^1 regularity was also obtained in [TW3] by a different approach. On the other hand, Loeper proved the $C^{1,\alpha}$ regularity, assuming that $f \in L^p$ for p large enough. Recently, Jiakun Liu [Liu] also proved the $C^{1,\alpha}$ regularity by a different proof.

7.4. Duality of the reflector problem

In the far field case, there is a dual problem to the above problem. Let ρ be an R-convex function. Let

$$\rho^*(Y) = \inf\{\frac{1}{\rho(X)}\frac{1}{1-\langle X, Y\rangle} : \quad X \in \Omega\}. \tag{7.6}$$

This transform was first introduced in Lemma 1.1, [W1]. Then we have (see [GW])

- ρ^* is an R-convex function in V;
- T_ρ is the inverse of T_{ρ^*};
- if $u = \frac{1}{\rho}$ satisfies equation (7.2), then $u^* = \frac{1}{\rho^*}$ satisfies the same equation, namely

$$\frac{\det(\nabla_{ij}u^* + (u^* - \eta^*)e_{ij})}{\eta^{*n}\det(e_{ij})} = \frac{g(Y)}{f(T_{\rho^*}(Y))}, \tag{7.7}$$

where $\eta^* = \frac{1}{2u^*}(|\nabla u^*|^2 + u^{*2})$.

Denote

$$c(X, Y) = -\log(1 - \langle X, Y\rangle). \tag{7.8}$$

and denote $\varphi = \log u$, $\psi = \log \rho^*$. Taking logarithm in (7.6) we have the formula

$$\psi(Y) = \inf\{c(X, Y) - \varphi(X) : \quad X \in U\} \tag{7.9}$$

From the above formula one sees that the far field reflector problem is an optimal transportation problem. Therefore we have the following theorem, which was included in [W2].

Theorem 7.1. *Suppose $f \in L^1(U), g \in L^1(V)$ are two densities satisfying the energy conservation (2.1). Then there exists a Lipschitz continuous (φ_1, ψ_1) of the linear functional*

$$I(\varphi, \psi) = \int_\Omega f(X)\varphi(X) + \int_{\Omega^*} g(Y)\psi(Y)$$

in the convex set

$$K = \{(\varphi, \psi);\ \varphi \in C(\bar U), v \in C(\bar V),\ \text{and}$$
$$\varphi(X) + \psi(Y) \le c(X, Y) \quad \forall\ X \in U, Y \in V\},$$

such that $\rho_1 = e^{\varphi_1}$ *is a weak solution to the reflector problem. Moreover, the solution is unique, in the sense that if* ρ *is a smooth solution, then either* $\rho = C\rho_1$ *for some constant* $C > 0$.

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