On the Lipschitz Regularity of Solutions of a Minimum Problem with Free Boundary

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Abstract

In this article under assumption of "small" density for negativity set, we prove local Lipschitz regularity for the one phase minimization problem with free boundary for the functional

$$\mathcal{E}_p(v,\Omega)=\int_\Omega |\nabla v|^p+\lambda_1^p\chi_{\{u\leq 0\}}+\lambda_2^p\chi_{\{u>0\}},\quad 1<p<\infty,$$

where λ_1,λ_2 are positive constants so that $\Lambda=\lambda_1^p-\lambda_2^p<0$, χ_D is the characteristic function of set D, $\Omega\subset\mathbf{R}^n$ is (smooth) domain and minimum is taken over a suitable subspace of $W^{1,p}(\Omega)$.

1 Introduction

Let $\mathcal{K}_g=\{v\in W^{1,p}(\Omega):v-g\in W_0^{1,p}(\Omega)\}$ for prescribed smooth function g $\Omega\subset\mathbf{R}^n$ and consider the energy minimization problem,

$$\mathcal{E}_p(u,\Omega)=\inf_{v\in\mathcal{K}_g}\mathcal{E}_p(v,\Omega),1<p<\infty \tag{1}$$

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with

$$\mathcal{E}_p(u,\Omega) = \int_\Omega |\nabla u|^p + \lambda_1^p \chi_{\{u\le 0\}} + \lambda_2^p \chi_{\{u>0\}}.$$

Here $\Omega \subset \mathbf{R}^n$ is bounded and smooth domain, λ_1, λ_2 are positive constants so that $\Lambda = \lambda_1^p - \lambda_2^p < 0$, χ_M is the characteristic function of the set $M \in \mathbf{R}^n$, i.e.

$$\chi_M = \begin{cases} 1 & \text{if } x \in M \\ 0 & \text{if } x \notin M. \end{cases}$$

The minimizer u is expected to verify to the following overdetermined problem

$$\Delta_p u = 0 \quad \text{in } u \neq 0, \qquad |\nabla u^+|^p - |\nabla u^-|^p = c \quad \text{on } \partial\{u>0\}, \qquad u = g \quad \text{on } \partial\Omega, \tag{2}$$

where the u^+, u^- are respectively the positive and negative parts of u, c is a positive constant and tha boundary data g is not necessarily nonnegative. This problem, usually termed Bernoulli-type problem, models for example cavitational flow of one or two perfect fluids, or equilibrium configuration for heat or electrostatic energy optimization. Weak solutions of problem 2 can be obtained by minimizing $\mathcal{E}_p$, (see theorem 2) and our objective here is to analyze the regularity of those solutions u.

Since u has a jump along the *free boundary* $\Gamma = \partial\{u > 0\}$, the best expected regularity for u is Lipschitz continuity. In the classical case $p = 2$, corresponding to usual Laplacian, this is proved in [ACF], and in [DP] for any $1 < p < \infty$ and $u^- \equiv 0$. The main complexity, in attacking the Lipschitz regularity for general case, is the lack of monotonicity formulas, firstly introduced in [ACF], and subsequently developed in [CJK], [CKS]. However we can still prove that $u \in C^{0,1}_{loc}$ if the negativity set $\Omega^-(u) = \{u < 0\}$ is reasonably small. The $C^{0,1}$ estimate plays vital role in establishing $C^{1,\alpha}$ regularity of free boundary near flat points. However here we solely focus upon proving local $C^{0,1}$ estimate for solutions. The present study has been inspired by a recent work [KKS] and by [LS], where similar result is proven for another overdetermined problem:

$$F(D^2u) = \chi_{\{D\}} \quad \text{in } B_1, \quad u = |\nabla u| = 0 \quad \text{in } B_1 \setminus D, \tag{3}$$

for a certain class of uniformly elliptic operators F. We observe here that unlike to (3) we don't have a pde, to which solutions u of (1) would verify in Ω.

2 Preliminaries

The following notations are used throughout the paper: $\Omega \subset \mathbf{R}^n$ is a smooth and bounded domain, g is a smooth function defined on some neighborhood of $\partial\Omega$, $W^{1,p}(\Omega), W^{1,p}_0(\Omega)$ are the usual Sobolev spaces, $B_R(y) = \{x \in \mathbf{R}^n : |x-y| < R\}$, $B_R = B_R(0)$, $u^{\pm}$ are respectively the positive and the negative parts of u, $\chi_{\{D\}}$ the characteristic function of D, $\Gamma = \partial\{u > 0\}$ free boundary. Let λ_1 and λ_2 be two positive constants so that $\Lambda = \lambda_1^p - \lambda_2^p < 0$ where $1 < p < \infty$. Consider functional

$$\mathcal{E}_p(u, \Omega) = \int_\Omega |\nabla u|^p + \lambda_1^p \chi_{\{u \le 0\}} + \lambda_2^p \chi_{\{u > 0\}}.$$

In what follows we denote by $\lambda(u)$ the following function:

$$\lambda(u) = \begin{cases} \lambda_1^p & \text{if } u \le 0 \\ \lambda_2^p & \text{if } u > 0. \end{cases}$$

As in the classical paper [ACF] we define $\lambda(0) = \lambda_1^p$ if $\Lambda < 0$ and $\lambda(0) = \lambda_2^p$ if $\Lambda > 0$. For brevity we focus on the case $\Lambda < 0$. Existence of solutions to (1) easily follows from the lower semicontinuity of $\mathcal{E}_p$ as in [ACF].

Theorem 1 *Let u be a (local) minimizer of $\mathcal{E}_p$. Then u is bounded.*

Proof: First let us observe that

$$\int_\Omega |\nabla u|^p + \lambda_1^p \chi_{\{u \le 0\}} + \lambda_2^p \chi_{\{u > 0\}} = \int_\Omega |\nabla u|^p + \Lambda \chi_{\{u \le 0\}} + \lambda_2^p \mathrm{meas} D. \qquad (4)$$

For given $D \subset \Omega$ let us consider the functional $I_0(u, D) = \int_D |\nabla u|^p + \Lambda \chi_{\{u \le 0\}}$. If u is a minimizer of $\mathcal{E}_p(u, D)$ then it is also a minimizer of $I_0(u, D)$ and vice versa since the difference between I_0 and $\mathcal{E}_p$ is a constant for given domain D.

Now take $u_\varepsilon = u + \varepsilon \min(M - u, 0)$, where $M = \sup g > 0$ and ε is a small positive number. Then taking $D = \Omega$ and testing u against u_ε we get

$$\int_\Omega |\nabla u|^p + \Lambda \chi_{\{u \le 0\}} \le \int_\Omega |\nabla u_\varepsilon|^p + \Lambda \chi_{\{u_\varepsilon \le 0\}}$$

Note that u and u_ε are different on the set $\{u > M\}$, therefore last inequality becomes

$$\int_{\Omega\cap\{u>M\}} |\nabla u|^p \leq \int_{\Omega\cap\{u>M\}} |\nabla u|^p(1-\varepsilon)^p + \Lambda\chi_{\{u_\varepsilon\leq 0\}}$$

which is a contradiction since $\Lambda < 0$ and hence $u \leq M$. Now take $u_\varepsilon = u - \min(u-m, 0)$ where $m = infg < 0$ and ε is a positive number. Again since u is a minimizer we have

$$\int_\Omega |\nabla u|^p + \Lambda\chi_{\{u\leq 0\}} \leq \int_\Omega |\nabla u_\varepsilon|^p + \Lambda\chi_{\{u_\varepsilon\leq 0\}}.$$

On the set $\{u < m\}$, where u and u_ε are different we have that

$$\int_{\Omega\cap\{u<m\}} |\nabla u|^p + \Lambda\chi_{\{u<m\}} \leq \int_{\Omega\cap\{u<m\}} |\nabla u_\varepsilon|^p(1-\varepsilon)^p + \Lambda\chi_{\{u\leq -\frac{\varepsilon m}{1-\varepsilon}\}}.$$

Note that $-\frac{\varepsilon m}{1-\varepsilon} > 0$ and therefore we get that

$$\begin{aligned}\int_{\Omega\cap\{u<m\}} |\nabla u|^p &\leq \int_{\Omega\cap\{u<m\}} |\nabla u_\varepsilon|^p(1-\varepsilon)^p + \Lambda\left[\chi_{\{u\leq -\frac{\varepsilon m}{1-\varepsilon}\}} - \chi_{\{u<m\}}\right] \\ &= \int_{\Omega\cap\{u<m\}} |\nabla u_\varepsilon|^p(1-\varepsilon)^p.\end{aligned}$$

This implies that $m \leq u$. □

Theorem 2 $u \in C^\alpha_{loc}(\Omega)$.

Proof: Let $B_R(y) \subset \Omega$ and w be the solution to the folloing Dirichlet problem

$$\Delta_p w = 0 \quad \text{in} \quad B_R(y), \quad w = u \quad \text{on} \quad \partial B_R(y). \tag{5}$$

Then we have that

$$\int_{B_R(y)} |\nabla u|^p + \lambda(u) \leq \int_{B_R(y)} |\nabla w|^p + \lambda(w)$$

where $\lambda(u) = \lambda_1^p\chi_{\{u\leq 0\}} + \lambda_2^p\chi_{\{u>0\}}$. Note that we also have

$$\int_{B_R(y)} |\nabla u|^p \geq \int_{B_R(y)} |\nabla w|^p.$$

Since $\lambda(u)$ is bounded it implies that

$$\int_{B_R(y)} [|\nabla u(x)|^p - |\nabla v(x)|^p]\, dx \leq CR^n \tag{6}$$

Furthermore one has from [DP]

$$\int_{B_R(y)} [|\nabla u|^p - |\nabla v|^p] \geq \begin{cases} c\left(\int_{B_R(y)} |\nabla(u-v)|^p\right)^{2/p} \left(\int_{B_R(y)} |\nabla u|^p\right)^{1-2/p}, & 1 < p \leq 2, \\ c\int_{B_R(y)} |\nabla(u-v)|^p, & 2 \leq p < \infty. \end{cases} \tag{7}$$

which together with (6) implies that

$$\int_{B_R(y)} |\nabla(u-w)|^p \leq \begin{cases} C\lambda_+^{p^2/2} R^{np/2} \left(\int_{B_R(y)} |\nabla u|^p\right)^{1-p/2}, & 1 < p \leq 2 \\ C\lambda_+^p R^n, & 2 \leq p < \infty. \end{cases} \tag{8}$$

Recall that from the gradient estimates for harmonic functions we have that

$$\sup_{B_{R/2}(y)} |\nabla w| \leq C \frac{\sup_\Omega |u|}{R}$$

Now for small R and $p > 2$ we have

$$\begin{aligned} \int_{B_{R/2}(y)} |\nabla u|^2 &\leq C\int_{B_{R/2}(y)} |\nabla(u-w)|^p + C\int_{B_{R/2}(y)} |\nabla w|^p \\ &\leq C\int_{B_{R/2}(y)} |\nabla(u-w)|^p + CR^{n-p}. \end{aligned} \tag{9}$$

Then combining (8) and (9) as in [DP] the result follows. □

Corollary 1 *u is p-subharmonic*

Proof: We first note that if v verifies to

$$\Delta_p v = 0 \ \text{ in } \ B_R(y), \quad v = u \ \text{ on } \ \partial B_R(y).$$

where $B_R(y) \subset \Omega$, then testing u against $\min(u, v)$ we have we find that

$$\int_{B_R(y)} [|\nabla u(x)|^p - |\nabla \min(u(x), v(x))|^p]\, dx \le \Lambda \int_{B_R \cap \{u>0\ge v\}} 1dx.$$

Since u is Hölder continuous, the set $\{u > v\}$ is open and we can apply (7) to infer that

$$\int_{B_R(y)} [|\nabla u|^p - |\nabla \min(u, v)|^p] > 0.$$

However $\Lambda < 0$, which yields $\max(u-v, 0) = 0$ in B_R, that is $u \le v$ in B_R. Hence u is p-subharmonic in Ω. $\square$

Before proceeding further we summarize some basic properties of solutions to (1).

Theorem 3 *Let u be the solution to (1). Then*

- $\Delta_p u = 0$ *in* $[\{u > 0\} \cup \{u < 0\}] \cap \Omega$,
- $\Delta_p u \ge 0$ *in* Ω,
- $\lim_{\varepsilon\downarrow 0} \int_{\partial\{u<-\varepsilon\}} ((p-1)|\nabla u|^p - \lambda_1^p)\, \nu\cdot\eta + \lim_{\delta\downarrow 0} \int_{\partial\{u>\delta\}} ((p-1)|\nabla u|^p - \lambda_2^p)\, \nu\cdot \eta = 0$ *for any* $\eta \in C_0^1(\Omega, \mathbf{R}^n)$ *provided meas*$\{u = 0\} = 0$.

The proof follows precisely as in [ACF].

3 Main result

In this section we assume that $\lambda_1 = 0$, since introducing $\lambda_0^p = \lambda_2^p - \lambda_1^p = -\Lambda > 0$ we can consider a new functional

$$\int_\Omega |\nabla u|^p + \lambda_0^p \chi_{\{u>0\}} = \mathcal{E}_p(u, \Omega) - \lambda_1^p \text{meas}\Omega$$

Therefore we identify $\mathcal{E}_p(u, \Omega)$ with $\int_\Omega |\nabla u|^p + \lambda^p \chi_{\{u>0\}}$ for some positive constant λ. Next we define the main class of functions that we are going to work with.

Definition 1 *Let z be a fixed point and $0 < r < 1$. u is said to be of class $\mathcal{Q}_r(z, M)$ if* *(i)* u *is a local minimizer of* $\mathcal{E}_p$ *in* $B_r(z)$,

(ii) $\sup_{B_r(z)} |u| \leq M$,

(iii) $z \in \partial\{u > 0\}$.

Let

$$\Theta(x_0, r) = \frac{\text{meas}\,(\{u < 0\} \cap B_r)}{\text{meas} B_r}, x_0 \in \partial\{u > 0\}$$

Theorem 4 *Let* $u \in \mathcal{Q}_1(x_0, M)$. *There exists a positive universal constant* $C > 0$ *such that*

$$|u(x)| \leq \frac{2M}{C}|x|$$

provided $\Theta(x_0, r) \leq C$ *for all* $0 < r < 1$.

Proof: Without loss of generality we may assume $x_0 = 0$. It is enough to prove that

$$\sup_{B_{2^{-(k+1)}}} |u(x)| \leq \max\left\{\frac{M}{C2^k}, \frac{S(k)}{2}, \ldots, \frac{S(k-m)}{2^{m+1}}, \ldots, \frac{S(0)}{2^{k+1}}\right\} \tag{10}$$

where $S(k) = \sup_{B_{2^{-k}}} |u|$. Assume a contradiction. Then there are integers $k_j, j = 1, 2, \ldots$ so that

$$\sup_{B_{2^{-(k_j+1)}}} |u_j(x)| > \max\left\{\frac{jM}{2^{k_j}}, \frac{S_j(k_j)}{2}, \ldots, \frac{S_j(k_j-m)}{2^{m+1}}, \ldots, \frac{S_j(0)}{2^{k_j+1}}\right\} \tag{11}$$

and

$$\Theta(0, 2^{-k_j}) \leq \frac{1}{j} \to 0. \tag{12}$$

Here

$$S_j(k_j - m) = \sup_{B_{2^{-(k_j-m)}}} |u_j|, m = 0, 1, 2, \ldots, k_j.$$

$u_j \in \mathcal{Q}_1(z, M)$. Observe that $|u_j| \leq M$ implies $k_j \to \infty$.

Consider auxiliary function v_j defined as

$$v_j(x) = \frac{u_j(x2^{-k_j})}{S_j(k_j+1)}$$

We start by proving $W^{1,p}$ estimates for v_j. Set $\sigma_j = 2^{-k_j} S_j^{-1}(k_j+1)$. Note that by (11) $\sigma_j \le j^{-1} \to 0$. For fixed $R_0 > 0$ we have

$$\int_{B_{R_0}} |\nabla v_j(x)|^p dx = \sigma_j^p \int_{B_{R_0}} |\nabla u_j(x2^{-k_j})|^p dx \qquad (13)$$
$$= \sigma_j^p 2^{nk_j} \int_{B_{R_0 2^{-k_j}}} |\nabla u_j(y)|^p dy$$

Let $\rho > 0$ and φ is the standard cut-off function of B_ρ. Then if $\eta = \varphi^p u_j^+$ is a admissible test function and (ii) yields

$$\int_{B_\rho} |\nabla u_j^+|^{p-2} \nabla u_j^+ \nabla \eta \le 0.$$

Rearranging the terms and after using Hölder inequality we get

$$\int_{B_\rho} \varphi^p |\nabla u_j^+|^p \le p \int_{B_\rho} |\nabla u_j^+|^{p-1} \varphi^{p-1} |\nabla \varphi| u_j^+ dx \le \qquad (14)$$
$$= p \left(\int_{B_\rho} |\nabla \varphi|^p (u_j^+)^p dx \right)^{\frac{1}{p}} \left(\int_{B_\rho} |\nabla u_j^+|^p \varphi^p dx \right)^{1-\frac{1}{p}}$$

So we get Caccioppoli's inequality

$$\int_{B_{\rho/2}} |\nabla u_j^+|^p \le \frac{c}{\rho^p} \int_{B_\rho} (u_j^+)^p \le c\rho^{n-p} \left(\sup_{B_\rho} |u_j| \right)^p. \qquad (15)$$

Let us take $\frac{\rho}{2} = \frac{R_0}{2^{k_j}}$ in the last inequality,

$$\int_{B_{R_0 2^{-k_j}}} |\nabla u_j^+|^p \le c \left(\frac{2R_0}{2^{k_j}} \right)^{n-p} \left(\sup_{B_{\frac{2R_0}{2^{k_j}}}} |u_j| \right)^p.$$

Choose $R_0 = 2^{l-1}$ for l, fixed integer $l < k_j$ we have then

$$\int_{2^{l-1}} |\nabla v_j^+|^p \le c \left[\frac{2^{-k_j}}{S_j(k_j+1)} \right]^p 2^{nk_j} 2^{(l-k_j)(n-p)} \left(\sup_{B_{2^{l-k_j}}} |u_j| \right)^p \le \qquad (16)$$
$$\le c \left[\frac{2^{-k_j}}{S_j(k_j+1)} \right]^p 2^{nk_j} 2^{(l-k_j)(n-p)} \left(2^{l+1} S_j(k_j+1) \right)^p =$$
$$= 2^{ln+p},$$

where the second inequality follows from (11). Therefore $\|\nabla v_j\|_{L^p}$ is locally bounded implying local uniform $W^{1,p}$ estimates for v_j for j large.

If $p > n$ then the Sobolev imbedding theorem implies uniform local C^α estimate for v_j, for j large. Suppose $1 < p \leq n$. Consider the scaled energy functional

$$\mathcal{E}_j(v, D) = \int_D |\nabla v|^p + \sigma_j^p \lambda^p \chi_{\{v>0\}} \tag{17}$$

First let us observe that a simple calculation gives

$$\mathcal{E}_j(v_j, B_{R_0}) = \sigma_j^p 2^{nk_j} \mathcal{E}_p(u_j, B_{R_0 2^{-k_j}}). \tag{18}$$

Therefore v_j is a solution to

$$\mathcal{E}_j(v_j) = \inf_{v \in \mathcal{K}_j} \mathcal{E}_j(v)$$

$$\mathcal{K}_j = \{v \in W^{1,p}(B_{2^{k_j}}), v - v_j \in W_0^{1,p}(B_{2^{k_j}})\}.$$

Applying Theorem 1 to v_j we have uniform C^α_{loc} estimate. Using uniform $W^{1,p}$ and C^α_{loc} estimates we have at least for a subsequence that

$$v_j \to v_\infty, \text{in } W^{1,p}(B_2) \cap C^\alpha(B_2) \tag{19}$$

Now we claim that v_∞ is a local minimizer of $D_p(v) = \int |\nabla v|^p$. Indeed, for any $\varphi \in C_0^\infty(B_1)$ we have

$$\int_{B_1} |\nabla v_j|^p + \sigma_j^p \lambda^p \chi_{\{v_j>0\}} \leq \int_{B_1} |\nabla(v_j + \varphi)|^p + \sigma_j^p \lambda^p \chi_{\{v_j+\varphi>0\}}$$

By (19) we have

$$\int_{B_1} |\nabla v_j|^p \to \int_{B_1} |\nabla v_\infty|^p$$

$$\int_{B_1} |\nabla(v_j + \varphi)|^p \to \int_{B_1} |\nabla(v_\infty + \varphi)|^p.$$

Since also $\sigma_j \leq \frac{1}{j}$, we get

$$\sigma_j^p \int_{B_1} \lambda^p \chi_{\{v_j>0\}} \to 0,$$

$$\sigma_j^p \int_{B_1} \lambda^p \chi_{\{v_j+\varphi>0\}} \to 0.$$

Hence we conclude that

$$\int_{B_1} |\nabla v_\infty|^p \leq \int_{B_1} |\nabla (v_\infty + \varphi)|^p.$$

In view of C^α regularity this yields that v_∞ is a local minimizer for $D_p(v)$ in B_1.

From definition of v_j and (12) we conclude:

- $0 \leq v_\infty \leq 2, \text{in } \ B_1$
- $\Delta_p v_\infty = 0, \text{in } \ B_1$
- $v_\infty(0) = 0$
- $\sup_{B_{\frac{1}{2}}} |v_\infty| = 1$

which contradicts to the strong maximum principle. □

Corollary 2 *Assume that* $\Theta_r(z, r) \leq C$ *for all* $z \in B_{1/2} \cap \Gamma$*, then* $u \in \mathcal{Q}_1(0, M)$ *is Lipschitz in* $B_{1/4}$*.*

Proof: Let $u(x) > 0$ and $d(x) = \text{dist}(x, \partial\{u > 0\})$. Let $z \in \partial\{u > 0\}$ so that $d(x) = |x - z|$. Then $u(x) \leq 2MC^{-1}d(x)$. By Harnack's inequality $u \leq 2cMC^{-1}d(x)$ in $B_{d(x)/2}$. Consider $v(y) = \frac{u(x+d(x)y)}{d(x)}$. Then

$$\Delta_p v = 0, \quad \text{in } \ B_1, \quad 0 \leq v(y) \leq 2cMC^{-1} \quad \text{in } \ B_{1/2}.$$

Then from local gradient estimate $|\nabla v(0)| \leq C(n, p, M, C)$. □

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