Up-to boundary regularity for a singular perturbation problem of $p$-Laplacian type

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Received 20 April 2005; revised 12 October 2005
Available online 21 November 2005

Abstract

In this paper we are interested in establishing up-to boundary uniform estimates for the one phase singular perturbation problem involving a nonlinear singular/degenerate elliptic operator. Our main result states: if $\Omega \subset \mathbb{R}^n$ is a $C^{1,\alpha}$ domain, $f \in C^{1,\alpha}(\overline{\Omega})$ for some $0 < \alpha < 1$ and $u^\varepsilon$ verifies

$$\text{div} A(x, u^\varepsilon, \nabla u^\varepsilon) = \beta^\varepsilon(u^\varepsilon) \quad \text{in} \ \Omega, \ 0 \leq u^\varepsilon \leq 1 \quad \text{in} \ \Omega, \ u^\varepsilon = f \quad \text{on} \ \partial \Omega,$$

where $\varepsilon > 0$, $\beta_\varepsilon(t) = \frac{1}{\varepsilon} \beta\left(\frac{t}{\varepsilon}\right)$ and

$$0 \leq \beta(t) \leq B\chi_{\{0 < t < 1\}}, \quad \int_{\mathbb{R}} \beta_\varepsilon(t) \, dt = M > 0,$$

with some positive constants $B$ and $M$, then there exists a constant $C > 0$ independent of $\varepsilon$ such that

$$\|\nabla u^\varepsilon\|_{L^\infty(\overline{\Omega})} \leq C.$$
1. Introduction

In this paper we prove an up to boundary uniform gradient estimate for solutions to one phase singular perturbation problem involving nonlinear degenerate elliptic operator. This estimate then allows us to obtain existence for corresponding free boundary problem. In what follows we focus only on the case of model equation

\[ \Delta_p u = \text{div}(\lvert \nabla u \rvert^{p-2} \nabla u), \quad 1 < p < \infty, \]

the \( p \)-Laplace operator.

We recall here that \( u \in W^{1,p}(\Omega) \) is said to be \( p \)-harmonic in \( \Omega \) if

\[ \int_{\Omega} \lvert \nabla u \rvert^{p-2} \nabla u \nabla \varphi = 0 \]

for every \( \varphi \in C_0^\infty(\Omega) \). One of the most important properties of \( p \)-harmonic functions is Harnack inequality (see [12,13]), which will be used throughout of the paper. It is worth noting that our technique admits generalization to operators of the general type

\[ \mathcal{L}u = \text{div}(A(x,u,\nabla u)), \]

having \( p \)-Laplace type structure (see [7,12,13] and references therein).

The solution of the free boundary problem in question is expected to verify

\[ \begin{cases} \Delta_p u = 0 & \text{in } \Omega \cap \{u > 0\}, \\ |\nabla u| = c & \text{on } \Omega \cap \partial \{u > 0\}, \\ u = f & \text{on } \partial \Omega. \end{cases} \]

(2)

Here \( c \) is a positive constant. The problem above arises in combustion theory and has been intensively studied by several authors for linear elliptic and parabolic operators, that is when in our case \( p = 2 \).

Following [3], the solution to (2) is derived from an approximating family of functions, which are solutions to some Dirichlet problems. More precisely, let \( u^\varepsilon \) satisfy the following singular perturbation problem

\[ \begin{cases} \Delta_p u^\varepsilon = \beta_\varepsilon(u^\varepsilon) & \text{in } \Omega, \\ 0 \leq u^\varepsilon \leq 1 & \text{in } \Omega, \\ u^\varepsilon = f & \text{on } \partial \Omega, \end{cases} \]

(see Section 3 for details), then our main result says, that gradients of \( u^\varepsilon \) are uniformly bounded in \( \Omega \). This, in turn, implies that for a subsequence \( u^\varepsilon \rightarrow u \), uniformly in \( \overline{\Omega} \), and \( u \) solves the free boundary problem (2) in some weak sense.

The study was initiated by pioneering work of [1] for nonnegative solutions of (3) for linear uniformly elliptic operators under zero oblique derivative condition on the boundary. The interior Lipschitz regularity of \( u^\varepsilon \) has been explored by several authors (see [2,3] and references therein). It is also worth pointing out, that boundary regularity in some extent
appears also in [1], since zero oblique derivative condition considered there with half ball mappings allows reflecting solution to be defined in a whole ball and hence reducing the boundary case to the local one.

In contrast to linear operators, even the local analysis of the nonlinear problem is not so well developed. As to \( p \)-Laplace operator, the (3) has been studied by [4], where they extend the local results of [3] to this case.

The first complete treatment of up to the boundary regularity for two phase linear elliptic case was obtained by Gurevich [6] under assumption, that the gradient of boundary data vanishes whenever the function does it, i.e.

\[
\nabla f(x) = 0 \quad \text{whenever} \quad f(x) = 0.
\]

He has also shown that Lipschitz regularity breaks down if condition (4) is violated for arbitrary boundary data. One of the main techniques used there is a splitting argument for \( u^\varepsilon \) similarly to that of used in [2,8]. That is writing \( u^\varepsilon = u_1^\varepsilon + u_2^\varepsilon \), where \( u_1^\varepsilon \) is harmonic with the same boundary values as \( u^\varepsilon \), while \( u_2^\varepsilon \) vanishes on the boundary and it solves the semilinear equation.

In our set-up we consider arbitrary \( C^{1,\alpha} \) Dirichlet data. It is shown in [10,14–16], that solutions to Dirichlet problems for \( p \)-Laplace type equations have at most \( C^{1,\alpha} \) regularity, for some \( \alpha > 0 \). Therefore our assumptions on the boundary data are optimal.

Moreover in view of the nonlinearity of \( \Delta_p \), the splitting argument of [6] does not work in our case and we need to use nonlinear techniques. In particular we are using scaling argument and sharp gradient estimates of [11]. These methods are in some extent advanced applications of Krylov-type Harnack inequality [9]. Fortunately many of the geometrical arguments of [6] can successfully be used.

2. Preliminaries

In this section we start with introducing basic notations, used throughout the paper and two technical lemmas.

2.1. Notations

\( \Omega \) \quad Open connected set in \( n \)-dimensional Euclid space \( \mathbb{R}^n \), \( n \geq 2 \).
\( \Pi \) \quad Hyperplane \( \{x_n = 0\} \), \( x = (x_1, \ldots, x_n) \).
\( \hat{x} \) \quad Projection of \( x \) on \( \Pi \).
\( \nu \) \quad Inner normal to a boundary point of \( \Omega \).
\( \Gamma_x \) \quad Cone with vertex at point \( x \in \Pi \), such that \( |y - \hat{y}| \geq \frac{1}{2} |y - x| \).
\( B_r(x) \) \quad Ball with center at \( x \) and radius \( r \).
\( B_r \) \quad Ball with center at origin.
\( B_r^+(x) \) \quad \( B_r(x) \cap \{x_n > 0\} \).
\( B_r^+(x) \) \quad Ball with center at \( x \) and radius \( r \) in \( \Pi \).
\( \Omega_{\varepsilon}(u) \) \quad The set of the points \( x \in \Omega \), where \( u(x) \leq \varepsilon \).
2.2. Background results

We need two preliminary lemmas, both quite standard in uniform elliptic theory.

**Lemma 2.1.** Suppose that \( u \geq 0, \Delta_p u = 0 \) in \( B_1^+ \) and \( u(x) \geq \sigma > 0 \) for any \( x \in B_1^\prime \). Then there exists a constant \( c = c(n, p) \), such that

\[
u(x) \geq c\sigma, \quad x \in B_{3/4}^+.
\]

**Proof.** Suppose that \( w \) is the solution to Dirichlet problem

\[
\begin{aligned}
\Delta_p w &= 0 \quad \text{in } B_1^+, \\
w &= \sigma \quad \text{on } B_1^\prime, \\
w &= 0 \quad \text{on } \partial B_1 \cap \{x_n > 0\}.
\end{aligned}
\]

(5)

According to [11, Lemma 2], \( w \in C^{1,\alpha}(\overline{B}_{3/4}^+) \). Furthermore by virtue of maximum principle we infer that

\[
0 \leq w \leq \sigma
\]

in \( B_1^+ \). Now let us define \( \tilde{w} \) in whole \( B_1 \) in the following way

\[
\tilde{w}(x) = \begin{cases} 
  w(x) & \text{if } x \in B_1^+, \\
  2\sigma - w(x_1, \ldots, -x_n) & \text{if } x \in B_1 \cap \{x_n < 0\}.
\end{cases}
\]

It is easy to see, that \( \tilde{w} \) is \( p \)-harmonic in \( B_1 \) and according to (6)

\[
\sigma \leq \tilde{w} \leq 2\sigma
\]

in the lower half ball. Hence \( 0 \leq \tilde{w} \leq 2\sigma \) in \( B_1 \). Therefore from local Harnack inequality [12,13]

\[
\sup_{B_{3/4}} \tilde{w} \leq c \inf_{B_{3/4}} \tilde{w},
\]

and in particular

\[
u(x) \geq c^{-1}\sigma \quad \text{in } B_{3/4}^+.
\]

Coupling this inequality with comparison principle the result follows. \( \Box \)

**Lemma 2.2.** Assume that \( \Delta_p u = 0 \) in \( B_r(x) \) and \( u \geq 0 \) in \( B_r(x) \). Assume also that for some \( x_0 \in \partial B_r(x) \)

\[
u(x_0) = 0 \quad \text{and} \quad \frac{\partial u(x_0)}{\partial v} \leq k.
\]
where \( v \) is the inward normal direction at \( x_0 \) on \( \partial B_r \). Then there is a constant \( c = c(n, p) \) such that

\[
u(x) \leq ckr.
\]

**Proof.** This lemma is a slight modification of Theorem 2.2 in [1]. However for completeness we present the proof. Observe, that considering the scaled function

\[
v(y) = \frac{u(x + ry)}{r}, \quad y \in B_1
\]

we can reduce the general case to the unit ball \( B_1 \). Introduce the function \( w(y) = \gamma(e^{-\lambda|y|^2} - e^{-\lambda}) \), where the positive constants \( \gamma \) and \( \lambda \) will be determined below. Computing the derivatives of \( w \) we have

\[
\frac{\partial w}{\partial y_i} = -2\gamma\lambda y_i e^{-\lambda|y|^2},
\]

\[
\frac{\partial^2 w}{\partial y_i \partial y_j} = 4\gamma\lambda^2 y_i y_j e^{-\lambda|y|^2} - 2\gamma\lambda \delta_{i,j} e^{-\lambda|y|^2},
\]

\[
|\nabla w| = 2\gamma\lambda e^{-\lambda|y|^2} |y|.
\]

Invoking the explicit form of \( p \)-Laplace operator we get

\[
\Delta_p w = |\nabla w|^{p-2} \Delta w + (p - 2)|\nabla w|^{p-4} \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j w_{i,j} = \gamma\lambda e^{-\lambda|y|^2} (2\gamma\lambda e^{-\lambda|y|^2} |y|)^{p-2} [4\lambda(p-1)|y|^2 - 2(n + p - 2)].
\]

Therefore \( \Delta_p w \geq 0 \) in \( B_1 \setminus B_{1/2} \) if \( \lambda \geq \frac{2(n + p - 2)}{p - 1} \). Now by Harnack inequality

\[
v(0) \leq \sup_{B_{1/2}} v \leq c \inf_{B_{1/2}} v.
\]

Hence \( v(y) \geq \frac{1}{c} v(0) \) in \( B_{1/2} \). Choosing \( \gamma = \frac{v(0)}{c(e^{-\lambda/4} - e^{-\lambda})} \) we have \( w \leq v \) on \( \partial B_1 \cup \partial B_{1/2} \) and comparison principle gives, that \( w \leq v \) in \( B_1 \setminus B_{1/2} \). Then

\[
\frac{\partial w(x_0)}{\partial \nu} \leq \frac{\partial v(x_0)}{\partial \nu}.
\]

Explicitly this means, that \( \gamma \lambda e^{-\lambda} \leq k \), i.e.

\[
v(0) \leq \frac{kc(e^{-\frac{\lambda}{4}} - e^{-\lambda})}{\lambda e^{-\lambda}}.
\]

Returning to the function \( u \) the assertion follows. \( \Box \)
3. Uniformly Lipschitz estimate

3.1. Problem set-up

Let \( \Omega \) be a smooth bounded domain in \( \mathbb{R}^n \). Assume that \( f \in C^{1,\alpha}(\overline{\Omega}) \). Let us introduce the family of approximate identities \( \{\beta_{\varepsilon}\} \) defined as

\[
\beta_{\varepsilon}(t) = \frac{1}{\varepsilon^\beta} \left( \frac{t}{\varepsilon} \right),
\]

where \( \beta \) is a nonnegative bounded continuous function on \( \mathbb{R}^n \), \( \beta \leq B \), \( \text{supp} \beta \subset [0, 1] \), and

\[
\int_{\mathbb{R}} \beta = M > 0.
\]

Apparantely \( \text{supp} \beta_{\varepsilon} \subset [0, \varepsilon] \) and

\[
\int_{\mathbb{R}} \beta_{\varepsilon} = M > 0.
\]

3.2. Main result

Theorem 3.1. Suppose \( \Omega \subset \mathbb{R}^n \) is a bounded \( C^{1,\alpha} \) domain and \( f \in C^{1,\alpha}(\overline{\Omega}) \), \( \| f \|_{C^{1,\alpha}(\overline{\Omega})} \leq R, 0 < \alpha < 1 \). Let \( u_{\varepsilon} \) be a solution to the following singular perturbation problem

\[
\begin{aligned}
\Delta_p u &= \beta_{\varepsilon}(u_{\varepsilon}) \quad \text{in } \Omega, \\
0 &\leq u_{\varepsilon} \leq 1 \quad \text{in } \Omega, \\
u_{\varepsilon} &= f \quad \text{on } \partial \Omega.
\end{aligned}
\]

Then there is a constant \( C = C(n, p, B, R) \), independent of \( \varepsilon \), such that

\[
\| \nabla u_{\varepsilon} \|_{L^\infty(\overline{\Omega})} \leq C.
\]

Remark 3.2. It is proved in [4, Theorem 2.1], that for any compact \( K \subset \Omega \) there is a positive constant \( C_K \) depending only on \( n, p, M \) and distance between \( \Omega \) and \( K \) such that

\[
\| \nabla u_{\varepsilon} \|_{L^\infty(K)} \leq C_K.
\]

Hence our theorem generalizes this result up to \( \partial \Omega \) for smooth enough \( \partial \Omega \) and data.

Proof of Theorem 3.1. Without loss of generality (see [11]) we can restrict ourselves to the case of upper half unit ball. Indeed since \( \Omega \) is a smooth domain then we can using a smooth map reduce the general case to that one on \( B_1^+ \) and the boundary data will be given on \( B_1 \cap \{x_n = 0\} \).
We consider $B_1^+$ as a union of two sets $\Omega_\epsilon(u^\epsilon)$ and $B_1^+ \setminus \Omega_\epsilon(u^\epsilon)$. In what follows we write $\Omega_\epsilon$ instead of $\Omega_\epsilon(u^\epsilon)$ for short. □

### 3.3. Lipschitz regularity in $\Omega_\epsilon = \{u^\epsilon \leq \epsilon\}$

**Proposition 1.** For $x \in B_{1/2}^+ \cap \Omega_\epsilon$ there is a constant $C_1 = C_1(n, p, B, R)$ such that

$$|\nabla u^\epsilon(x)| \leq C_1.$$

**Proof.** Assume that $x \in B_{1/2}^+ \cap \Omega_\epsilon$ and $\text{dist}(x, \Pi) \geq \epsilon$, where $\Pi = \{x_n = 0\}$. Then consider scaled functions

$$v^\epsilon(y) = \frac{u^\epsilon(x + \epsilon y)}{\epsilon}, \quad y \in B_1$$

and apply [4, Lemma 2.2]. This gives

$$v^\epsilon(y) \leq c, \quad y \in B_{1/2}.$$

By local gradient estimates (see, e.g., [5]), we have then

$$|\nabla v^\epsilon(0)| = |\nabla u^\epsilon(x)| \leq c(n, p, B).$$

In order to prove the assertion for the case $\text{dist}(x, \Pi) < \epsilon$ we need the following lemma.

**Lemma 3.3.** Assume that $x \in \Omega_\epsilon \cap B_{1/2}^+$ and $\text{dist}(x, \Pi) < \epsilon$. Then there exists a constant $c(n, p, B)$, such that

$$u^\epsilon(\hat{x}) \leq c\epsilon.$$

(8)

Here $\hat{x}$ is the projection of $x$ on the hyperplane $\Pi$.

**Proof.** Indeed suppose that our assertion fails, then for some $\epsilon$ we have

$$u^\epsilon(\hat{x}) \geq N\epsilon,$$

for some large $N$. Denote $d_0 = \text{dist}(\hat{x}, \Omega_\epsilon)$ and suppose for some $x_0 \in \Omega_\epsilon \cap \partial B_{d_0}(\hat{x})$ we have

$$d_0 = |x_0 - \hat{x}|.$$

Let now $\Gamma_{\hat{x}}$ be the cone with vertex at $\hat{x} \in \Pi$ such that $|y - \hat{y}| \geq \frac{1}{2}|y - \hat{x}|$ for every $y \in \Gamma_{\hat{x}}$. 
Case 1. \( x_0 \in \Gamma' \). Let us consider ball \( B_{d_0/2}(x_0) \). Since \( x_0 \in \Gamma' \), then clearly \( B_{d_0/2}(x_0) \subset B_1^+ \). For the auxiliary function \( v^\varepsilon(y) = \frac{u^\varepsilon(x_0 + (d_0/2)y)}{\varepsilon} \), \( y \in B_1 \), we have

\[
\Delta_p v^\varepsilon(y) = \left( \frac{d_0}{2} \right)^p \beta(v^\varepsilon) = M
\]
since \( d_0 < \varepsilon \). Also \( v^\varepsilon(0) \leq 1 \) and hence we can apply [4, Lemma 2.2] to conclude \( v^\varepsilon(y) \leq c \) for \( y \in B_{1/2} \), i.e.

\[
u^\varepsilon(x) \leq c \varepsilon, \quad x \in B_{d_0/4}(x_0).
\]

On the other hand, for \( z \in B_{d_0}'(\hat{x}) \) we have

\[
f(z) \geq f(\hat{x}) - R|z - \hat{x}| \geq N\varepsilon - Rd_0 \geq (N - R)\varepsilon
\]
since \( d_0 < \varepsilon \). Hence the scaled function \( w^\varepsilon(y) = \frac{u^\varepsilon(\hat{x}_0 + \varepsilon y)}{\varepsilon} \) is \( p \)-harmonic in \( B_1^+ \), and \( w^\varepsilon \geq N - R \) on \( B_1' \) therefore according to Lemma 2.1 \( w^\varepsilon \geq c(N - R) \) in \( B_{3/4}^+ \), i.e.

\[
u^\varepsilon(x) \geq c(N - R) \quad \text{in} \quad B_{3/4d_0}^+(\hat{x}).
\]

Hence for the point \( \xi \) belonging to \( \partial B_{3d_0/4}(\hat{x}) \cap \partial B_{d_0/4}(x_0) \) we have \( \varepsilon c(N - R) \leq u^\varepsilon(\xi) \leq c \varepsilon, \) which is a contradiction if \( N \) is too large.

Case 2. \( x_0 \notin \Gamma' \). Let \( x_1 \in \Omega_\varepsilon \) be such that \( d_1 = \text{dist}(\hat{x}_0, \Omega_\varepsilon) \) is realized. Observe that

\[
|x_1 - \hat{x}| \leq |x_1 - \hat{x}_0| + |\hat{x}_0 - \hat{x}| \leq d_1 + d_0 \leq \frac{d_0}{2} + d_0
\]
since \( d_1 \leq \frac{d_0}{2} \). If \( x_1 \in \Gamma' \) we are done. Otherwise let \( x_2 \) be such that \( d_2 = \text{dist}(\hat{x}_1, \Omega_\varepsilon) \) is realized. Observe that

\[
|x_2 - \hat{x}| \leq |\hat{x}_1 - x_2| + |\hat{x}_1 - \hat{x}| \leq d_2 + |x_1 - \hat{x}| \leq \frac{d_0}{4} + \frac{d_0}{2} + d_0
\]
since \( d_2 \leq \frac{d_1}{2} \leq \frac{d_0}{4} \). There are two possibilities: either after finite steps we come to Case 1 or we have a sequence of points \( x_i \in \partial \Omega_\varepsilon, \ i = 0, 1, 2, \ldots, x_{i+1} \notin \Gamma' \) and

\[
d_{i+1} \leq \frac{d_i}{2} \leq \frac{d_0}{2^{i+1}},
\]

\[
|x_i - \hat{x}| \leq d_0 + d_0 \sum_{k=1}^{i} \frac{1}{2^k} \leq 2d_0.
\]
Therefore at least for a subsequence, again denoted \( x_i \), we have \( x_i \to \xi \in B'_{2d_0} \), \( f(\xi) = \varepsilon \).

But

\[
f(\xi) \geq f(\hat{x}) - R|\hat{x} - \xi| \geq \varepsilon(N - 2R),
\]

which contradicts to \( f(\xi) = \varepsilon \), if \( N \) is too large. Hence the result follows in this case too.

\Box

To continue the proof of Proposition 1, let \( w \) solve the following Dirichlet problem

\[
\begin{align*}
\Delta_p w &= 0 \quad \text{in } B_1^+, \\
 w &= u^\varepsilon \quad \text{on } \partial B_1^+.
\end{align*}
\]

Then \( w \in C^{1,\alpha}(\overline{B}_{3/4}^+) \) and by comparison principle we have \( u^\varepsilon \leq w \). Note that by [11, Lemma 2],

\[
|\nabla w| \leq C(\text{osc}_{B_1^+} w + \|f\|_{C^{1,\alpha}}) \leq C \quad \text{in } B_{3/4}^+
\]

which in conjunction with (8) yields

\[
u^\varepsilon(y) \leq w(y) \leq w(\hat{x}) + C|y - \hat{x}| \leq C\varepsilon, \quad \text{if } y \in B_{2\varepsilon}^+(\hat{x}).
\]

Then again applying the gradient estimates of [11], the result follows.

\Box

3.4. Lipschitz regularity in \( B_{1/8}^+ \setminus \Omega_{\varepsilon} \)

Lemma 3.4. For \( x \in B_{1/4}^+ \setminus \Omega_{\varepsilon} \) there exists a constant \( A = A(n, p) > 0 \) such that

\[
f(x) - \varepsilon \leq A \text{dist}(x, \Omega_{\varepsilon}). \quad (10)
\]

Proof. Assume that \( x_0 \in B_{1/4}^+ \setminus \Omega_{\varepsilon} \) and inequality (10) is violated. Then for some \( \varepsilon \) we have

\[
f(x_0) - \varepsilon \geq N \text{dist}(x_0, \Omega_{\varepsilon}),
\]

where \( N \) is large. Let \( d_\varepsilon = \text{dist}(x_0, \Omega_{\varepsilon}) \) and \( x_\varepsilon \in \partial \Omega_{\varepsilon} \) be such that the distance is realized, i.e., \( d_\varepsilon = |x_0 - x_\varepsilon| \).

Case 1. \( x_\varepsilon \in \Gamma_{x_0} \). Let us define \( v^\varepsilon \) to be the scaled function

\[
v^\varepsilon(y) = \frac{u^\varepsilon(x_0 + d_\varepsilon y) - \varepsilon}{d_\varepsilon} \quad \text{in } B_1^+.
\]

Obviously \( v^\varepsilon(y) \geq 0, \Delta_p v^\varepsilon = 0 \) in \( B_1^+ \). Observe, that for \( x \in B_{d_\varepsilon}^+(x_0) \) we have

\[
f(x) \geq f(x_0) - R|x - x_0| \geq \varepsilon + Nd_\varepsilon - Rd_\varepsilon \geq \varepsilon + \frac{N}{2}d_\varepsilon
\]
(if $N$ is large), that is $v^\varepsilon(y) \geq cN$ for $y \in B'_1$. Hence we can apply Lemma 2.1 to conclude that $v^\varepsilon \geq cN$ in $B'_{3/4}$ or in terms of $u^\varepsilon$

$$u^\varepsilon(x) \geq \varepsilon + cNd_\varepsilon, \quad x \in B'_{3d_\varepsilon/4}(x_0). \quad (11)$$

Now let $B''$ be the ball with radius $d_\varepsilon/4$ and centered at $\bar{x}_\varepsilon = x_\varepsilon + \frac{x_0 - x_\varepsilon}{d_\varepsilon}$. Observe that $u^\varepsilon - \varepsilon$ satisfies to the conditions of Lemma 2.2 since $x_\varepsilon \in \partial B''$. This gives

$$u^\varepsilon \leq \varepsilon + Cd_\varepsilon. \quad (12)$$

Now at the point $\bar{x}_\varepsilon$ belonging to $\partial B'_{3d_\varepsilon/4}(x_0)$ we have (according to (11) and (12))

$$\varepsilon + Ncd_\varepsilon \leq u^\varepsilon(\bar{x}_\varepsilon) \leq cd_\varepsilon,$$

which is a contradiction if $N$ is chosen large enough.

**Case 2.** $x_\varepsilon \notin \Gamma_{x_0}$. In this case the proof goes in the same way as in Case 2 of Lemma 3.3.

**Proposition 2.** If $x \in B_{1/8}^+ \setminus \Omega_\varepsilon$ then there is a constant $C_2 = C_2(n, p, R)$ such that

$$|\nabla u^\varepsilon(x)| \leq C_2.$$

**Proof.** Let $x \in B_{1/8}^+ \setminus \Omega_\varepsilon$ and $d_\varepsilon = \text{dist}(x, \Omega_\varepsilon)$ and $d = \text{dist}(x, \Pi)$.

**Case 1.** $d_\varepsilon \leq d$. Without loss of generality we may assume, that $d_\varepsilon \leq \frac{1}{8}$. Indeed if $d_\varepsilon > \frac{1}{8}$, then gradient bound follows from local estimates of [4]. Therefore let us assume, that $d_\varepsilon \leq \frac{1}{8}$ and let $x_\varepsilon \in \partial \Omega_\varepsilon$ be such that $d_\varepsilon = |x - x_\varepsilon|$. Observe

$$|x_\varepsilon| \leq |x| + d_\varepsilon \leq \frac{1}{4},$$

and hence by Proposition 1

$$|\nabla u^\varepsilon(x_\varepsilon)| \leq C_1.$$

Consider

$$v^\varepsilon(y) = \frac{u^\varepsilon(x + d_\varepsilon y) - \varepsilon}{d_\varepsilon}, \quad y \in B_1.$$

Clearly for these functions we have

$$\begin{align*}
\Delta_p v^\varepsilon(y) &= 0, \quad y \in B_1, \\
v^\varepsilon(y_\varepsilon) &= 0, \quad y_\varepsilon \in \partial B_1, \\
|\nabla v^\varepsilon(y_\varepsilon)| &\leq C_1, \\
v^\varepsilon(y) &\geq 0, \quad y \in B_1,
\end{align*} \quad (13)$$

$$\begin{align*}
\Delta_p v^\varepsilon(y) &= 0, \quad y \in B_1, \\
v^\varepsilon(y_\varepsilon) &= 0, \quad y_\varepsilon \in \partial B_1, \\
|\nabla v^\varepsilon(y_\varepsilon)| &\leq C_1, \\
v^\varepsilon(y) &\geq 0, \quad y \in B_1,
\end{align*} \quad (13)$$
where \( y_\varepsilon = \frac{x - x}{d_\varepsilon} \in \partial B_1 \). According to Lemma 2.2

\[
v^\varepsilon(0) \leq c
\]
for any \( \varepsilon \). Moreover by Harnack’s inequality \( v^\varepsilon(y) \leq c \), when \( y \in B_{1/2} \). Hence applying local gradient estimates of [5], we get \( |\nabla v^\varepsilon(0)| \leq c \) for some \( c = c(n, p) \), which completes the proof of Case 1.

**Case 2.** \( d < d_\varepsilon \leq 4d \). Without loss of generality we may assume, that \( d_\varepsilon \leq \frac{1}{8} \), otherwise, as in the Case 1 the gradient bound follows from local estimates of [4]. Let \( w \) be the solution to Dirichlet problem

\[
\begin{align*}
\Delta_p w(x) &= 0, \quad x \in B_1^+, \\
w(x) &= u^\varepsilon(x), \quad x \in \partial B_1^+.
\end{align*}
\tag{14}
\]

Since \( 0 \leq u^\varepsilon \leq 1 \) then by [11, Lemma 2] \( w \in C^{0,1}(\overline{B}_{3/4}) \) and

\[
|\nabla w(x)| \leq C(\text{osc} w + \|f\|_{C^1,\alpha}) \leq C(2 + R).
\]

Using comparison principle, we can estimate \( u^\varepsilon \) by \( w \) from above as follows

\[
u^\varepsilon(x) \leq w(x) \leq w(\hat{x}) + C(2 + R)|x - \hat{x}| \leq f(\hat{x}) + C(2 + R)d.
\tag{15}
\]

Observe that \( |\hat{x}| \leq |x| + d \leq \frac{1}{4} \) and hence we can apply Lemma 3.4:

\[
f(\hat{x}) \leq \varepsilon + A \text{dist}(\hat{x}, \Omega_\varepsilon) \leq \varepsilon + A(d_\varepsilon + d) \leq \varepsilon + 5Ad.
\tag{16}
\]

Coupling (15) and (16) we get \( u^\varepsilon(x) \leq \varepsilon + (5A + C(2 + R))d \).

Now let us consider the following auxiliary function

\[
v^\varepsilon(y) = \frac{u^\varepsilon(x + dy) - \varepsilon}{d}, \quad y \in B_1.
\]

Clearly \( \Delta_p v^\varepsilon(y) = 0 \) in \( B_1 \) and by Harnack inequality

\[
v^\varepsilon(y) \leq C v^\varepsilon(0) \leq C(5A + C(2 + R)), \quad y \in B_{1/2}.
\]

Finally, applying local gradient estimates in \( B_{1/2} \) the proof of Case 2 follows.

**Case 3.** \( 4d < d_\varepsilon \). First let us assume, that \( d_\varepsilon \leq \frac{1}{8} \). Note that if \( y \in B_{d_\varepsilon/2}(\hat{x}) \) then

\[
|y| \leq |y - x| + |x| \leq 2\frac{d_\varepsilon}{2} + |x| \leq \frac{1}{4}.
\]

Now using Lemma 3.4 and majorizer \( w \) (see (14)) we can estimate \( u^\varepsilon \) in \( B_{d_\varepsilon/2}(\hat{x}) \) as follows
Since distance function is Lipschitz continuous with Lipschitz constant 1 then we have

\[ \text{dist}(\hat{y}, \Omega_\varepsilon) \leq d_\varepsilon + |\hat{y} - x| \leq 2d_\varepsilon. \]

Therefore

\[ u_\varepsilon(y) \leq \varepsilon + \left(2A + \frac{C(2 + R)}{2}\right)d_\varepsilon = \varepsilon + cd_\varepsilon. \]

Now we can apply [11, Lemma 2] for function \( v_\varepsilon = u_\varepsilon - \varepsilon \) in \( B_{d_\varepsilon/2}(\hat{x}) \), which yields

\[ |\nabla v_\varepsilon(x)| = |\nabla u_\varepsilon(x)| \leq C(c + R). \]

Now assume that \( d_\varepsilon \geq \frac{1}{8} \). Then observe that \( B_{1/16}(\hat{x}) \subset B_1 \setminus \Omega_\varepsilon \), since

\[ |x| \leq |\hat{x} - x| + |\hat{x}| \leq \frac{3}{16}. \]

In this case we have

\[ \begin{cases}
\Delta_p u_\varepsilon = 0 & \text{in } B_{1/16}^+(\hat{x}), \\
0 \leq u_\varepsilon \leq 1 & \text{in } B_{1/16}^+(\hat{x})
\end{cases} \tag{17} \]

and the estimate follows from [11, Lemma 2]. \[\square\]

4. Existence

Having in our disposal the uniformly Lipschitz regularity for \( \{u_\varepsilon\} \), employing Ascoli–Arzela lemma it is easy to infer, that there exists a solution \( u \) for problem (2).

**Theorem 4.1.** Assume that \( \{u_\varepsilon\} \) is a family of solution to (3). Then for every sequence \( \varepsilon_j \to 0 \) there exists a subsequence \( \varepsilon'_j \to 0 \) and Lipschitz continuous function \( u \) in \( \overline{\Omega} \) such that

(i) \( u^{\varepsilon'_j} \to u \) uniformly in \( \overline{\Omega} \),
(ii) \( \Delta_p u = 0 \) in \( \Omega \cap \{u > 0\} \).

Proof of this theorem is similar to Lemma 3.1 in [4].
Definition 3. A unit vector \( \eta \in \mathbb{R}^n \) is said to be the inward normal in measure theoretic sense to the free boundary \( \partial \{ u > 0 \} \) at a point \( x_0 \in \partial \{ u > 0 \} \) if

\[
\lim_{r \to 0} \frac{1}{r^n} \int_{B_r(x_0)} |\chi_{\{u>0\}} - \chi_{\{x-(x-x_0) \cdot \eta > 0\}}| \, dx = 0. \tag{18}
\]

Definition 4. Let \( v \) be a continuous function in \( B_1^+ \). We say that \( v \) is not degenerate at point \( x_0 \in B_1^+ \cap \{ v = 0 \} \) if there exists \( c, r_0 > 0 \) such that

\[
\frac{1}{r^n} \int_{B_r(x_0)} v \, dx \geq cr, \quad \text{for any } r \in (0, r_0). \tag{19}
\]

Theorem 4.2. Assume that \( u^{\varepsilon}_j \) is a solution to (3) in \( \Omega \) such that \( u^{\varepsilon}_j \to u \) uniformly in \( \Omega \) and \( \varepsilon_j \to 0 \). Let \( x_0 \in \Omega \cap \partial \{ u > 0 \} \) be such that \( \partial \{ u > 0 \} \) has an inward unit normal \( \eta \) in the measure theoretic sense at \( x_0 \) and suppose that \( u \) is nondegenerate at \( x_0 \). Under this assumptions, we have

\[
u(x) = \left( \frac{M_p}{p-1} \right)^{\frac{1}{p}} [(x-x_0) \cdot \eta]^+ + o(x-x_0). \tag{20}
\]

Here \( M = \int_{\mathbb{R}^2} \beta \).

The proof of last theorem follows immediately from [4, Theorem 4.3] if we consider a ball \( B_r(x_0) \) such that \( B_r(x_0) \subset \Omega \).

References