Available online at www.sciencedirect.com

SCIENCE DIRECT®

J. Differential Equations 226 (2006) 558–571

Journal of Differential Equations

www.elsevier.com/locate/jde

Up-to boundary regularity for a singular perturbation problem of p-Laplacian type

Aram L. Karakhanyan

Center for Mathematics and Its Applications, The Australian National University, Canberra, ACT 0200, Australia

Received 20 April 2005; revised 12 October 2005

Available online 21 November 2005

Abstract

In this paper we are interested in establishing up-to boundary uniform estimates for the one phase singular perturbation problem involving a nonlinear singular/degenerate elliptic operator. Our main result states: if $\Omega \subset \mathbf{R}^n$ is a $C^{1,\alpha}$ domain, $f \in C^{1,\alpha}(\overline{\Omega})$ for some $0 < \alpha < 1$ and u^ε verifies

$$\operatorname{div} \mathbf{A}\big(x, u^\varepsilon, \nabla u^\varepsilon\big) = \beta_\varepsilon\big(u^\varepsilon\big) \quad \text{in } \Omega,\ 0 \leqslant u^\varepsilon \leqslant 1 \text{ in } \Omega,\ u^\varepsilon = f \text{ on } \partial\Omega,$$

where $\varepsilon > 0$, $\beta_\varepsilon(t) = \frac{1}{\varepsilon}\beta\big(\frac{t}{\varepsilon}\big)$ and

$$0 \leqslant \beta(t) \leqslant B\chi_{\{0<t<1\}}, \qquad \int\limits_{\mathbf{R}} \beta_\varepsilon(t)\,dt = M > 0,$$

with some positive constants B and M, then there exists a constant $C > 0$ independent of ε such that $\|\nabla u^\varepsilon\|_{L^\infty(\overline{\Omega})} \leqslant C$.

Keywords: Singular perturbation problem; Free boundary problem; p-Laplace operator; Global gradient bounds

E-mail address: aram.karakhanyan@maths.anu.edu.au.

0022-0396/$ – see front matter
doi:10.1016/j.jde.2005.10.014

1. Introduction

In this paper we prove an up to boundary uniform gradient estimate for solutions to one phase singular perturbation problem involving nonlinear degenerate elliptic operator. This estimate then allows us to obtain existence for corresponding free boundary problem. In what follows we focus only on the case of model equation

$$\Delta_p u = \operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big), \quad 1 < p < \infty, \tag{1}$$

the p-Laplace operator.

We recall here that $u \in W^{1,p}(\Omega)$ is said to be p-harmonic in Ω if

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\varphi = 0$$

for every $\varphi \in C_0^\infty(\Omega)$. One of the most important properties of p-harmonic functions is Harnack inequality (see [12,13]), which will be used throughout of the paper. It is worth noting that our technique admits generalization to operators of the general type

$$\mathcal{L}u = \operatorname{div}\big(\mathbf{A}(x, u, \nabla u)\big),$$

having p-Laplace type structure (see [7,12,13] and references therein).

The solution of the free boundary problem in question is expected to verify

$$\begin{cases} \Delta_p u = 0 & \text{in } \Omega \cap \{u > 0\}, \\ |\nabla u| = c & \text{on } \Omega \cap \partial\{u > 0\}, \\ u = f & \text{on } \partial\Omega. \end{cases} \tag{2}$$

Here c is a positive constant. The problem above arises in combustion theory and has been intensively studied by several authors for linear elliptic and parabolic operators, that is when in our case $p = 2$.

Following [3], the solution to (2) is derived from an approximating family of functions, which are solutions to some Dirichlet problems. More precisely, let u^ε satisfy the following singular perturbation problem

$$\begin{cases} \Delta_p u^\varepsilon = \beta_\varepsilon(u^\varepsilon) & \text{in } \Omega, \\ 0 \leqslant u^\varepsilon \leqslant 1 & \text{in } \Omega, \\ u^\varepsilon = f & \text{on } \partial\Omega, \end{cases} \tag{3}$$

(see Section 3 for details), then our main result says, that gradients of u^ε are uniformly bounded in Ω. This, in turn, implies that for a subsequence $u^\varepsilon \to u$, uniformly in $\overline{\Omega}$, and u solves the free boundary problem (2) in some weak sense.

The study was initiated by pioneering work of [1] for nonnegative solutions of (3) for linear uniformly elliptic operators under zero oblique derivative condition on the boundary. The interior Lipschitz regularity of u^ε has been explored by several authors (see [2,3] and references therein). It is also worth pointing out, that boundary regularity in some extent appears also in [1], since zero oblique derivative condition considered there with half ball mappings allows reflecting solution to be defined in a whole ball and hence reducing the boundary case to the local one.

In contrast to linear operators, even the local analysis of the nonlinear problem is not so well developed. As to p-Laplace operator, the (3) has been studied by [4], where they extend the local results of [3] to this case.

The first complete treatment of up to the boundary regularity for two phase linear elliptic case was obtained by Gurevich [6] under assumption, that the gradient of boundary data vanishes whenever the function does it, i.e.

$$\nabla f(x) = 0 \quad \text{whenever } f(x) = 0. \tag{4}$$

He has also shown that Lipschitz regularity breaks down if condition (4) is violated for arbitrary boundary data. One of the main techniques used there is a splitting argument for u^ε similarly to that of used in [2,8]. That is writing $u^\varepsilon = u_1^\varepsilon + u_2^\varepsilon$, where u_1^ε is harmonic with the same boundary values as u^ε, while u_2^ε vanishes on the boundary and it solves the semilinear equation.

In our set-up we consider arbitrary $C^{1,\alpha}$ Dirichlet data. It is shown in [10,14–16], that solutions to Dirichlet problems for p-Laplace type equations have at most $C^{1,\alpha}$ regularity, for some $\alpha > 0$. Therefore our assumptions on the boundary data are optimal.

Moreover in view of the nonlinearity of Δ_p, the splitting argument of [6] does not work in our case and we need to use nonlinear techniques. In particular we are using scaling argument and sharp gradient estimates of [11]. These methods are in some extent advanced applications of Krylov-type Harnack inequality [9]. Fortunately many of the geometrical arguments of [6] can successfully be used.

2. Preliminaries

In this section we start with introducing basic notations, used throughout the paper and two technical lemmas.

2.1. Notations

Ω	Open connected set in n-dimensional Euclid space $\mathbf{R}^n$, $n \geqslant 2$.
Π	Hyperplane $\{x_n = 0\}$, $x = (x_1, \dots, x_n)$.
$\hat{x}$	Projection of x on Π.
ν	Inner normal to a boundary point of Ω.
Γ_x	Cone with vertex at point $x \in \Pi$, such that $\lvert y - \hat{y}\rvert \geqslant \frac{1}{2}\lvert y - x\rvert$.
$B_r(x)$	Ball with center at x and radius r.
B_r	Ball with center at origin.
$B_r^+(x)$	$B_r(x) \cap \{x_n > 0\}$.
$B_r'(x)$	Ball with center at x and radius r in Π.
$\Omega_\varepsilon(u)$	The set of the points $x \in \Omega$, where $u(x) \leqslant \varepsilon$.

2.2. Background results

We need two preliminary lemmas, both quite standard in uniform elliptic theory.

Lemma 2.1. *Suppose that $u \geqslant 0$, $\Delta_p u = 0$ in B_1^+ and $u(x) \geqslant \sigma > 0$ for any $x \in B_1'$. Then there exists a constant $c = c(n, p)$, such that*

$$u(x) \geqslant c\sigma, \quad x \in B_{3/4}^+.$$

Proof. Suppose that w is the solution to Dirichlet problem

$$\begin{cases} \Delta_p w = 0 & \text{in } B_1^+, \\ w = \sigma & \text{on } B', \\ w = 0 & \text{on } \partial B_1 \cap \{x_n > 0\}. \end{cases} \tag{5}$$

According to [11, Lemma 2], $w \in C^{1,\alpha}(\overline{B}_{3/4}^+)$. Furthermore by virtue of maximum principle we infer that

$$0 \leqslant w \leqslant \sigma \tag{6}$$

in B_1^+. Now let us define $\tilde{w}$ in whole B_1 in the following way

$$\tilde{w}(x) = \begin{cases} w(x) & \text{if } x \in B_1^+, \\ 2\sigma - w(x_1, \dots, -x_n) & \text{if } x \in B_1 \cap \{x_n < 0\}. \end{cases}$$

It is easy to see, that $\tilde{w}$ is p-harmonic in B_1 and according to (6)

$$\sigma \leqslant \tilde{w} \leqslant 2\sigma$$

in the lower half ball. Hence $0 \leqslant \tilde{w} \leqslant 2\sigma$ in B_1. Therefore from local Harnack inequality [12,13]

$$\sup_{B_{3/4}} \tilde{w} \leqslant c \inf_{B_{3/4}} \tilde{w},$$

and in particular

$$w(x) \geqslant c^{-1}\sigma \quad \text{in } B_{3/4}^+.$$

Coupling this inequality with comparison principle the result follows. □

Lemma 2.2. *Assume that $\Delta_p u = 0$ in $B_r(x)$ and $u \geqslant 0$ in $B_r(x)$. Assume also that for some $x_0 \in \partial B_r(x)$*

$$u(x_0) = 0 \quad \textit{and} \quad \frac{\partial u(x_0)}{\partial \nu} \leqslant k,$$

where ν is the inward normal direction at x_0 on ∂B_r. Then there is a constant $c = c(n, p)$ such that

$$u(x) \leqslant ckr.$$

Proof. This lemma is a slight modification of Theorem 2.2 in [1]. However for completeness we present the proof. Observe, that considering the scaled function

$$v(y) = \frac{u(x + ry)}{r}, \quad y \in B_1$$

we can reduce the general case to the unit ball B_1. Introduce the function $w(y) = \gamma(e^{-\lambda|y|^2} - e^{-\lambda})$, where the positive constants γ and λ will be determined below. Computing the derivatives of w we have

$$\begin{aligned}
\frac{\partial w}{\partial y_i} &= -2\gamma\lambda y_i e^{-\lambda|y|^2}, \\
\frac{\partial^2 w}{\partial y_i \partial y_j} &= 4\gamma\lambda^2 y_i y_j e^{-\lambda|y|^2} - 2\gamma\lambda\delta_{i,j} e^{-\lambda|y|^2}, \\
|\nabla w| &= 2\gamma\lambda e^{-\lambda|y|^2}|y|.
\end{aligned}$$

Invoking the explicit form of p-Laplace operator we get

$$\begin{aligned}
\Delta_p w &= |\nabla w|^{p-2}\Delta w + (p-2)|\nabla w|^{p-4}\sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j w_{i,j} \\
&= \gamma\lambda e^{-\lambda|y|^2}\left(2\gamma\lambda e^{-\lambda|y|^2}|y|\right)^{p-2}\left[4\lambda(p-1)|y|^2 - 2(n+p-2)\right].
\end{aligned}$$

Therefore $\Delta_p w \geqslant 0$ in $B_1 \setminus B_{1/2}$ if $\lambda \geqslant \frac{2(n+p-2)}{p-1}$. Now by Harnack inequality

$$v(0) \leqslant \sup_{B_{1/2}} v \leqslant c \inf_{B_{1/2}} v.$$

Hence $v(y) \geqslant \frac{1}{c}v(0)$ in $B_{1/2}$. Choosing $\gamma = \frac{v(0)}{c(e^{-\lambda/4} - e^{-\lambda})}$ we have $w \leqslant v$ on $\partial B_1 \cup \partial B_{1/2}$ and comparison principle gives, that $w \leqslant v$ in $B_1 \setminus B_{1/2}$. Then

$$\frac{\partial w(x_0)}{\partial \nu} \leqslant \frac{\partial v(x_0)}{\partial \nu}.$$

Explicitly this means, that $\gamma\lambda e^{-\lambda} \leqslant k$, i.e.

$$v(0) \leqslant \frac{kc(e^{-\frac{\lambda}{4}} - e^{\lambda})}{\lambda e^{-\lambda}}.$$

Returning to the function u the assertion follows. □

3. Uniformly Lipschitz estimate

3.1. Problem set-up

Let Ω be a smooth bounded domain in $\mathbf{R}^n$. Assume that $f \in C^{1,\alpha}(\overline{\Omega})$. Let us introduce the family of approximate identities $\{\beta_\varepsilon\}$ defined as

$$\beta_\varepsilon(t) = \frac{1}{\varepsilon}\beta\left(\frac{t}{\varepsilon}\right),$$

where β is a nonnegative bounded continuous function on $\mathbf{R}^n$, $\beta \leqslant B$, $\operatorname{supp}\beta \subset [0, 1]$, and

$$\int_{\mathbf{R}} \beta = M > 0.$$

Apparently $\operatorname{supp}\beta_\varepsilon \subset [0, \varepsilon]$ and

$$\int_{\mathbf{R}} \beta_\varepsilon = M > 0.$$

3.2. Main result

Theorem 3.1. *Suppose $\Omega \subset \mathbf{R}^n$ is a bounded $C^{1,\alpha}$ domain and $f \in C^{1,\alpha}(\overline{\Omega})$, $\|f\|_{C^{1,\alpha}(\overline{\Omega})} \leqslant R$, $0 < \alpha < 1$. Let u^ε be a solution to the following singular perturbation problem*

$$\begin{cases} \Delta_p u = \beta_\varepsilon(u^\varepsilon) & \text{in } \Omega, \\ 0 \leqslant u^\varepsilon \leqslant 1 & \text{in } \Omega, \\ u^\varepsilon = f & \text{on } \partial\Omega. \end{cases} \tag{7}$$

Then there is a constant $C = C(n, p, B, R)$, independent of ε, such that

$$\left\|\nabla u^\varepsilon\right\|_{L^\infty(\overline{\Omega})} \leqslant C.$$

Remark 3.2. It is proved in [4, Theorem 2.1], that for any compact $K \Subset \Omega$ there is a positive constant C_K depending only on n, p, M and distance between Ω and K such that

$$\left\|\nabla u^\varepsilon\right\|_{L^\infty(K)} \leqslant C_K.$$

Hence our theorem generalizes this result up to $\partial\Omega$ for smooth enough $\partial\Omega$ and data.

Proof of Theorem 3.1. Without loss of generality (see [11]) we can restrict ourselves to the case of upper half unit ball. Indeed since Ω is a smooth domain then we can using a smooth map reduce the general case to that one on B_1^+ and the boundary data will be given on $B_1 \cap \{x_n = 0\}$.

We consider B_1^+ as a union of two sets $\Omega_\varepsilon(u^\varepsilon)$ and $B_1^+ \setminus \Omega_\varepsilon(u^\varepsilon)$. In what follows we write Ω_ε instead of $\Omega_\varepsilon(u^\varepsilon)$ for short. $\square$

3.3. Lipschitz regularity in $\Omega_\varepsilon = \{u^\varepsilon \leqslant \varepsilon\}$

Proposition 1. *For $x \in B_{1/2}^+ \cap \Omega_\varepsilon$ there is a constant $C_1 = C_1(n, p, B, R)$ such that*

$$\left|\nabla u^\varepsilon(x)\right| \leqslant C_1.$$

Proof. Assume that $x \in B_{1/2}^+ \cap \Omega_\varepsilon$ and $\operatorname{dist}(x, \Pi) \geqslant \varepsilon$, where $\Pi = \{x_n = 0\}$. Then consider scaled functions

$$v^\varepsilon(y) = \frac{u^\varepsilon(x + \varepsilon y)}{\varepsilon}, \quad y \in B_1$$

and apply [4, Lemma 2.2]. This gives

$$v^\varepsilon(y) \leqslant c, \quad y \in B_{1/2}.$$

By local gradient estimates (see, e.g., [5]), we have then

$$\left|\nabla v^\varepsilon(0)\right| = \left|\nabla u^\varepsilon(x)\right| \leqslant c(n, p, B).$$

In order to prove the assertion for the case $\operatorname{dist}(x, \Pi) < \varepsilon$ we need the following lemma.

Lemma 3.3. *Assume that $x \in \Omega_\varepsilon \cap B_{1/2}^+$ and $\operatorname{dist}(x, \Pi) < \varepsilon$. Then there exists a constant $c(n, p, B)$, such that*

$$u^\varepsilon(\hat{x}) \leqslant c\varepsilon. \tag{8}$$

Here $\hat{x}$ is the projection of x on the hyperplane Π.

Proof. Indeed suppose that our assertion fails, then for some ε we have

$$u^\varepsilon(\hat{x}) \geqslant N\varepsilon,$$

for some large N. Denote $d_0 = \operatorname{dist}(\hat{x}, \Omega_\varepsilon)$ and suppose for some $x_0 \in \Omega_\varepsilon \cap \partial B_{d_0}(\hat{x})$ we have

$$d_0 = |x_0 - \hat{x}|.$$

Let now $\Gamma_{\hat{x}}$ be the cone with vertex at $\hat{x} \in \Pi$ such that $|y - \hat{y}| \geqslant \frac{1}{2}|y - \hat{x}|$ for every $y \in \Gamma_{\hat{x}}$.

Case 1. $x_0 \in \Gamma_{\hat{x}}$. Let us consider ball $B_{d_0/2}(x_0)$. Since $x_0 \in \Gamma_{\hat{x}}$, then clearly $B_{d_0/2}(x_0) \subset B_1^+$. For the auxiliary function $v^\varepsilon(y) = \frac{u^\varepsilon(x_0+(d_0/2)y)}{\varepsilon}$, $y \in B_1$, we have

$$\Delta_p v^\varepsilon(y) = \left(\frac{d_0}{2}\right)^p \frac{\beta(v^\varepsilon)}{\varepsilon^p} \leqslant \frac{M}{2^p}$$

since $d_0 < \varepsilon$. Also $v^\varepsilon(0) \leqslant 1$ and hence we can apply [4, Lemma 2.2] to conclude $v^\varepsilon(y) \leqslant c$ for $y \in B_{1/2}$, i.e.

$$u^\varepsilon(x) \leqslant c\varepsilon, \quad x \in B_{d_0/4}(x_0).$$

On the other hand, for $z \in B'_{d_0}(\hat{x})$ we have

$$f(z) \geqslant f(\hat{x}) - R|z - \hat{x}| \geqslant N\varepsilon - Rd_0 \geqslant (N - R)\varepsilon$$

since $d_0 < \varepsilon$. Hence the scaled function $w^\varepsilon(y) = \frac{u^\varepsilon(\hat{x}+d_0 y)}{\varepsilon}$ is p-harmonic in B_1^+, and $w^\varepsilon \geqslant N - R$ on B'_1 therefore according to Lemma 2.1 $w^\varepsilon \geqslant c(N - R)$ in $B^+_{3/4}$, i.e.

$$u^\varepsilon(x) \geqslant \varepsilon c(N - R) \quad \text{in } B^+_{3/4d_0}(\hat{x}).$$

Hence for the point ξ belonging to $\partial B_{3d_0/4}(\hat{x}) \cap \partial B_{d_0/4}(x_0)$ we have $\varepsilon c(N - R) \leqslant u^\varepsilon(\xi) \leqslant c\varepsilon$, which is a contradiction if N is too large.

Case 2. $x_0 \notin \Gamma_{\hat{x}}$. Let $x_1 \in \Omega_\varepsilon$ be such that $d_1 = \operatorname{dist}(\hat{x}_0, \Omega_\varepsilon)$ is realized. Observe that

$$|x_1 - \hat{x}| \leqslant |x_1 - \hat{x}_0| + |\hat{x}_0 - \hat{x}| \leqslant d_1 + d_0 \leqslant \frac{d_0}{2} + d_0$$

since $d_1 \leqslant \frac{d_0}{2}$. If $x_1 \in \Gamma_{\hat{x}}$ we are done. Otherwise let x_2 be such that $d_2 = \operatorname{dist}(\hat{x}_1, \Omega_\varepsilon)$ is realized. Observe that

$$|x_2 - \hat{x}| \leqslant |\hat{x}_1 - x_2| + |\hat{x}_1 - \hat{x}| \leqslant d_2 + |x_1 - \hat{x}| \leqslant \frac{d_0}{4} + \frac{d_0}{2} + d_0$$

since $d_2 \leqslant \frac{d_1}{2} \leqslant \frac{d_0}{4}$. There are two possibilities: either after finite steps we come to Case 1 or we have a sequence of points $x_i \in \partial\Omega_\varepsilon$, $i = 0, 1, 2, \ldots$, $x_{i+1} \notin \Gamma_{\hat{x}_i}$ and

$$d_{i+1} \leqslant \frac{d_i}{2} \leqslant \frac{d_0}{2^{i+1}},$$

$$|x_i - \hat{x}| \leqslant d_0 + d_0 \sum_{k=1}^{i} \frac{1}{2^k} \leqslant 2d_0.$$

Therefore at least for a subsequence, again denoted x_i we have $x_i \to \xi \in B'_{2d_0}$, $f(\xi) = \varepsilon$. But

$$f(\xi) \geqslant f(\hat{x}) - R|\hat{x} - \xi| \geqslant \varepsilon(N - 2R),$$

which contradicts to $f(\xi) = \varepsilon$, if N is too large. Hence the result follows in this case too. $\square$

To continue the proof of Proposition 1, let w solve the following Dirichlet problem

$$\begin{cases} \Delta_p w = 0 & \text{in } B_1^+, \\ w = u^\varepsilon & \text{on } \partial B_1^+. \end{cases} \tag{9}$$

Then $w \in C^{1,\alpha}(\overline{B}_{3/4}^+)$ and by comparison principle we have $u^\varepsilon \leqslant w$. Note that by [11, Lemma 2],

$$|\nabla w| \leqslant C\big(\operatorname{osc}_{B_1^+} w + \|f\|_{C^{1,\alpha}}\big) \leqslant C \quad \text{in } B_{3/4}^+$$

which in conjunction with (8) yields

$$u^\varepsilon(y) \leqslant w(y) \leqslant w(\hat{x}) + C|y - \hat{x}| \leqslant C\varepsilon, \quad \text{if } y \in B_{2\varepsilon}^+(\hat{x}).$$

Then again applying the gradient estimates of [11], the result follows. $\square$

3.4. Lipschitz regularity in $B_{1/8}^+ \setminus \Omega_\varepsilon$

Lemma 3.4. *For $x \in B'_{1/4} \setminus \Omega_\varepsilon$ there exists a constant $A = A(n, p) > 0$ such that*

$$f(x) - \varepsilon \leqslant A \operatorname{dist}(x, \Omega_\varepsilon). \tag{10}$$

Proof. Assume that $x_0 \in B'_{1/4} \setminus \Omega_\varepsilon$ and inequality (10) is violated. Then for some ε we have

$$f(x_0) - \varepsilon \geqslant N \operatorname{dist}(x_0, \Omega_\varepsilon),$$

where N is large. Let $d_\varepsilon = \operatorname{dist}(x_0, \Omega_\varepsilon)$ and $x_\varepsilon \in \partial\Omega_\varepsilon$ be such that the distance is realized, i.e., $d_\varepsilon = |x_0 - x_\varepsilon|$.

Case 1. $x_\varepsilon \in \Gamma_{x_0}$. Let us define v^ε to be the scaled function

$$v^\varepsilon(y) = \frac{u^\varepsilon(x_0 + d_\varepsilon y) - \varepsilon}{d_\varepsilon} \quad \text{in } B_1^+.$$

Obviously $v^\varepsilon(y) \geqslant 0$, $\Delta_p v^\varepsilon = 0$ in B_1^+. Observe, that for $x \in B'_{d_\varepsilon}(x_0)$ we have

$$f(x) \geqslant f(x_0) - R|x - x_0| \geqslant \varepsilon + N d_\varepsilon - R d_\varepsilon \geqslant \varepsilon + \frac{N}{2} d_\varepsilon$$

(if N is large), that is $v^\varepsilon(y) \geqslant cN$ for $y \in B'_1$. Hence we can apply Lemma 2.1 to conclude that $v^\varepsilon \geqslant cN$ in $B^+_{3/4}$ or in terms of u^ε

$$u^\varepsilon(x) \geqslant \varepsilon + cNd_\varepsilon, \quad x \in B^+_{3d_\varepsilon/4}(x_0). \tag{11}$$

Now let B'' be the ball with radius $\frac{d_\varepsilon}{4}$ and centered at $\bar{x}_\varepsilon = x_\varepsilon + \frac{x_0 - x_\varepsilon}{4}$. Observe that $u^\varepsilon - \varepsilon$ satisfies to the conditions of Lemma 2.2 since $x_\varepsilon \in \partial B''$. This gives

$$u^\varepsilon \leqslant \varepsilon + Cd_\varepsilon. \tag{12}$$

Now at the point $\bar{x}_\varepsilon$ belonging to $\partial B_{3d_\varepsilon/4}(x_0)$ we have (according to (11) and (12))

$$\varepsilon + Ncd_\varepsilon \leqslant u^\varepsilon(\bar{x}_\varepsilon) \leqslant cd_\varepsilon,$$

which is a contradiction if N is chosen large enough.

Case 2. $x_\varepsilon \notin \Gamma_{x_0}$. In this case the proof goes in the same way as in Case 2 of Lemma 3.3. □

Proposition 2. *If $x \in B^+_{1/8} \setminus \Omega_\varepsilon$ then there is a constant $C_2 = C_2(n, p, R)$ such that*

$$\left|\nabla u^\varepsilon(x)\right| \leqslant C_2.$$

Proof. Let $x \in B^+_{1/8} \setminus \Omega_\varepsilon$ and $d_\varepsilon = \operatorname{dist}(x, \Omega_\varepsilon)$ and $d = \operatorname{dist}(x, \Pi)$.

Case 1. $d_\varepsilon \leqslant d$. Without loss of generality we may assume, that $d_\varepsilon \leqslant \frac{1}{8}$. Indeed if $d_\varepsilon > \frac{1}{8}$, then gradient bound follows from local estimates of [4]. Therefore let us assume, that $d_\varepsilon \leqslant \frac{1}{8}$ and let $x_\varepsilon \in \partial\Omega_\varepsilon$ be such that $d_\varepsilon = |x - x_\varepsilon|$. Observe

$$|x_\varepsilon| \leqslant |x| + d_\varepsilon \leqslant \frac{1}{4}$$

and hence by Proposition 1

$$\left|\nabla u^\varepsilon(x_\varepsilon)\right| \leqslant C_1.$$

Consider

$$v^\varepsilon(y) = \frac{u^\varepsilon(x + d_\varepsilon y) - \varepsilon}{d_\varepsilon}, \quad y \in B_1.$$

Clearly for these functions we have

$$\begin{cases} \Delta_p v^\varepsilon(y) = 0, & y \in B_1, \\ v^\varepsilon(y_\varepsilon) = 0, & y_\varepsilon \in \partial B_1, \\ |\nabla v^\varepsilon(y_\varepsilon)| \leqslant C_1, & \\ v^\varepsilon(y) \geqslant 0, & y \in B_1, \end{cases} \tag{13}$$

where $y_\varepsilon = \frac{x_\varepsilon - x}{d_\varepsilon} \in \partial B_1$. According to Lemma 2.2

$$v^\varepsilon(0) \leqslant c$$

for any ε. Moreover by Harnack's inequality $v^\varepsilon(y) \leqslant c$, when $y \in B_{1/2}$. Hence applying local gradient estimates of [5], we get $|\nabla v^\varepsilon(0)| \leqslant c$ for some $c = c(n, p)$, which completes the proof of Case 1.

Case 2. $d < d_\varepsilon \leqslant 4d$. Without loss of generality we may assume, that $d_\varepsilon \leqslant \frac{1}{8}$, otherwise, as in the Case 1 the gradient bound follows from local estimates of [4]. Let w be the solution to Dirichlet problem

$$\begin{cases} \Delta_p w(x) = 0, & x \in B_1^+, \\ w(x) = u^\varepsilon(x), & x \in \partial B_1^+. \end{cases} \tag{14}$$

Since $0 \leqslant u^\varepsilon \leqslant 1$ then by [11, Lemma 2] $w \in C^{0,1}(\overline{B}^+_{3/4})$ and

$$\left|\nabla w(x)\right| \leqslant C\left(\operatorname{osc} w + \|f\|_{C^{1,\alpha}}\right) \leqslant C(2+R).$$

Using comparison principle, we can estimate u^ε by w from above as follows

$$u^\varepsilon(x) \leqslant w(x) \leqslant w(\hat{x}) + C(2+R)|x - \hat{x}| \leqslant f(\hat{x}) + C(2+R)d. \tag{15}$$

Observe that $|\hat{x}| \leqslant |x| + d \leqslant \frac{1}{4}$ and hence we can apply Lemma 3.4:

$$f(\hat{x}) \leqslant \varepsilon + A\operatorname{dist}(\hat{x}, \Omega_\varepsilon) \leqslant \varepsilon + A(d_\varepsilon + d) \leqslant \varepsilon + 5Ad. \tag{16}$$

Coupling (15) and (16) we get $u^\varepsilon(x) \leqslant \varepsilon + (5A + C(2+R))d$.

Now let us consider the following auxiliary function

$$v^\varepsilon(y) = \frac{u^\varepsilon(x + dy) - \varepsilon}{d}, \quad y \in B_1.$$

Clearly $\Delta_p v^\varepsilon(y) = 0$ in B_1 and by Harnack inequality

$$v^\varepsilon(y) \leqslant C v^\varepsilon(0) \leqslant C\big(5A + C(2+R)\big), \quad y \in B_{1/2}.$$

Finally, applying local gradient estimates in $B_{1/2}$ the proof of Case 2 follows.

Case 3. $4d < d_\varepsilon$. First let us assume, that $d_\varepsilon \leqslant \frac{1}{8}$. Note that if $y \in B^+_{d_\varepsilon/2}(\hat{x})$ then

$$|y| \leqslant |y - x| + |x| \leqslant 2\frac{d_\varepsilon}{2} + |x| \leqslant \frac{1}{4}.$$

Now using Lemma 3.4 and majorizer w (see (14)) we can estimate u^ε in $B^+_{d_\varepsilon/2}(\hat{x})$ as follows

$$u(y) \leqslant u(\hat{y}) + C(2+R)\frac{d_\varepsilon}{2} \leqslant \varepsilon + A\,\mathrm{dist}(y, \Omega_\varepsilon) + C(2+R)\frac{d_\varepsilon}{2}.$$

Since distance function is Lipschitz continuous with Lipschitz constant 1 then we have

$$\mathrm{dist}(\hat{y}, \Omega_\varepsilon) \leqslant d_\varepsilon + |\hat{y} - x| \leqslant 2d_\varepsilon.$$

Therefore

$$u^\varepsilon(y) \leqslant \varepsilon + \left(2A + \frac{C(2+R)}{2}\right)d_\varepsilon = \varepsilon + cd_\varepsilon.$$

Now we can apply [11, Lemma 2] for function $v^\varepsilon = u^\varepsilon - \varepsilon$ in $B^+_{d_\varepsilon/2}(\hat{x})$, which yields

$$\left|\nabla v^\varepsilon(x)\right| = \left|\nabla u^\varepsilon(x)\right| \leqslant C(c+R).$$

Now assume that $d_\varepsilon \geqslant \frac{1}{8}$. Then observe that $B^+_{1/16}(\hat{x}) \subset B_1 \setminus \Omega_\varepsilon$, since

$$|x| \leqslant |\hat{x} - x| + |\hat{x}| \leqslant \frac{3}{16}.$$

In this case we have

$$\begin{cases} \Delta_p u^\varepsilon = 0 & \text{in } B^+_{1/16}(\hat{x}), \\ 0 \leqslant u^\varepsilon \leqslant 1 & \text{in } B^+_{1/16}(\hat{x}) \end{cases} \tag{17}$$

and the estimate follows from [11, Lemma 2]. □

4. Existence

Having in our disposal the uniformly Lipschitz regularity for $\{u^\varepsilon\}$, employing Ascoli–Arzela lemma it is easy to infer, that there exists a solution u for problem (2).

Theorem 4.1. *Assume that $\{u^\varepsilon\}$ is a family of solution to* (3). *Then for every sequence $\varepsilon_j \to 0$ there exists a subsequence $\varepsilon'_j \to 0$ and Lipschitz continuous function u in $\overline{\Omega}$ such that*

(i) $u^{\varepsilon'_j} \to u$ *uniformly in* $\overline{\Omega}$,
(ii) $\Delta_p u = 0$ *in* $\Omega \cap \{u > 0\}$.

Proof of this theorem is similar to Lemma 3.1 in [4].

Definition 3. A unit vector $\eta \in \mathbf{R}^n$ is said to be the inward normal in measure theoretic sense to the free boundary $\partial\{u>0\}$ at a point $x_0 \in \partial\{u>0\}$ if

$$\lim_{r\to 0}\frac{1}{r^n}\int\limits_{B_r(x_0)}\left|\chi_{\{u>0\}}-\chi_{\{x\,|\,(x-x_0)\cdot\eta>0)\}}\right|dx=0. \tag{18}$$

Definition 4. Let v be a continuous function in B_1^+. We say that v is not degenerate at point $x_0 \in B_1^+ \cap \{v=0\}$ if there exists $c, r_0 >$ such that

$$\frac{1}{r^n}\int\limits_{B_r(x_0)} v\,dx \geqslant cr, \quad \text{for any } r\in(0,r_0). \tag{19}$$

Theorem 4.2. *Assume that u_j^ε is a solution to* (3) *in Ω such that $u_j^\varepsilon \to u$ uniformly in $\overline{\Omega}$ and $\varepsilon_j \to 0$. Let $x_0 \in \Omega \cap \partial\{u>0\}$ be such that $\partial\{u>0\}$ has an inward unit normal η in the measure theoretic sense at x_0 and suppose that u is nondegenerate at x_0. Under this assumptions, we have*

$$u(x)=\left(\frac{Mp}{p-1}\right)^{\frac{1}{p}}\left[(x-x_0)\cdot\eta\right]^+ + o(x-x_0). \tag{20}$$

Here $M=\int_{\mathbf{R}}\beta$.

The proof of last theorem follows immediately from [4, Theorem 4.3] if we consider a ball $B_r(x_0)$ such that $B_r(x_0)\subset\Omega$.

References

[1] H. Berestycki, L.A. Caffarelli, L. Nirenberg, Uniform estimates for regularization of free boundary problems, in: C. Sadoski (Ed.), Analysis and Partial Differential Equations, in: Lecture Notes in Pure and Appl. Math., vol. 122, Dekker, New York, 1988, pp. 567–619.
[2] L.A. Caffarelli, C.E. Kenig, Gradient estimates for variable coefficient parabolic equations and singular perturbation problems, Amer. J. Math. 120 (1998) 391–439.
[3] L.A. Caffarelli, C. Lederman, N. Wolanski, Uniform estimates and limits for a two phase parabolic singular perturbation problem, Indiana Univ. Math. J. 46 (1997) 453–489.
[4] D. Danielli, A. Petrosyan, H. Shahgholian, A singular perturbation problem for the p-Laplace operator, Indiana Univ. Math. J. 52 (2003) 457–476.
[5] E. DiBenedetto, $C^{1+\alpha}$ local regularity of weak solutions of degenerate elliptic equations, Nonlinear Anal. 7 (8) (1983) 827–850.
[6] A. Gurevich, Boundary regularity for free boundary problems, Comm. Pure Appl. Math. 52 (3) (1999) 363–403.
[7] J. Heinonen, T. Kilpeläinen, O. Martio, Nonlinear Potential Theory of Degenerate Elliptic Equations, Oxford Science, New York, 1993.
[8] C.E. Kenig, Harmonic analysis techniques for second order elliptic boundary value problems, in: CBMS Reg. Conf. Ser. Math., vol. 83, Amer. Math. Soc., Providence, RI, 1994.
[9] N.V. Krylov, Boundedly inhomogeneous elliptic and parabolic equations in a domain, Izv. Akad. Nauk SSSR 47 (1983) 75–108 (in Russian).

[10] J.L. Lewis, Regularity of the derivatives of solutions to certain degenerate elliptic equations, Indiana Univ. Math. J. 32 (1983) 849–858.
[11] G.M. Lieberman, Boundary regularity for solutions of degenerate elliptic equations, Nonlinear Anal. 12 (11) (1988) 1203–1219.
[12] J. Serrin, Local behavior of solutions of quasi-linear equations, Acta Math. 111 (1964) 247–302.
[13] J. Serrin, The Dirichlet problem for quasilinear elliptic equations with many independent variables, Philos. Trans. Roy. Soc. London Ser. A 264 (1969) 413–496.
[14] P. Tolksdorf, Regularity for a more general class of quasilinear elliptic equations, J. Differential Equations 51 (1984) 126–150.
[15] K. Uhlenbeck, Regularity for a class of nonlinear elliptic systems, Acta Math. 138 (1977) 219–240.
[16] N.N. Uraltceva, Degenerate quasilinear elliptic systems, Zap. Nauchn. Sem. S.-Peterburg Otdel. Mat. Inst. Steklov. (POMI) 7 (1968) 184–222 (in Russian).