LIPSCHITZ CONTINUITY OF FREE BOUNDARY IN THE CONTINUOUS CASTING PROBLEM WITH DIVERGENCE FORM ELLIPTIC EQUATION

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Abstract. In this paper we are concerned with the regularity of weak solutions u to the one phase continuous casting problem

$$\operatorname{div}(A(x)\nabla u(X)) = \operatorname{div}[\beta(u)\mathbf{v}(X)], \quad X \in \mathcal{C}_L$$

in the cylindrical domain $\mathcal{C}_L = \Omega \times (0, L)$ where $X = (x, z), x \in \Omega \subset \mathbb{R}^{N-1}, z \in (0, L), L > 0$ with given elliptic matrix $A : \Omega \to \mathbb{R}^{N^2}, A_{ij}(x) \in C^{1,\alpha_0}(\Omega), \alpha_0 > 0$, prescribed convection $\mathbf{v}$, and the enthalpy function $\beta(u)$. We first establish the optimal regularity of weak solutions $u \geq 0$ for one phase problem. Furthermore, we show that the free boundary $\partial\{u > 0\}$ is locally Lipschitz continuous graph provided that $\mathbf{v} = \mathbf{e}_N$, the direction of x_N coordinate axis and $\partial_z u \geq 0$. The latter monotonicity assumption in z variable can be easily obtained for a suitable boundary condition.

1. Introduction

In this article we study the optimal regularity of weak solutions to the stationary Stefan problem, with prescribed convection, and the smoothness of free boundary. There are a number of phase transition problems in applied sciences that are encompassed by this mathematical model, among which is the thawing or freezing of the water where the liquid part is in motion, for more details we refer to [4], [1] Chapter 10.7, [11].

In general setting the convection term $\mathbf{v}$ is to be determined from a Navier-Stokes system [4], however in this paper we assume that $\mathbf{v}$ is given. Furthermore, in the study of regularity of free boundary we will consider constant convection vector $\mathbf{v}$ and take $f = 0$, [11]. The phase transition problems with prescribed convection is called the continuous casting problem, and appears for instance in metal production [11] page 32.

Here we focus on a model anisotropic stationary problem with uniformly elliptic matrix $A_{ij}(x)$ with $C^{1,\alpha}, \alpha > 0$ regular entries which are independent of "height" variable z.

2. Problem set up

We now turn to the mathematical formulation of the problem. Let $\Omega \subset \mathbb{R}^{N-1}$ be a bounded Lipschitz domain. Let $L > 0$ and set $\mathcal{C}_L = \Omega \times (0, L)$. The points in $\mathcal{C}_L$ will be denoted by $X = (x, z)$, where $x \in \Omega$ and $z \in (0, L)$. The partial derivatives of a function $u : \mathcal{C}_L \to \mathbb{R}$ are denoted by $\partial_{x_i} u, \partial_z u, i = 1, \ldots, N-1$. Sometimes we will write $\partial_i u$ or u_i instead of $\partial_{x_i} u$ for short.

Keywords: Continuous casting problem, Stefan problem, free boundary, convection.

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In this paper we study the following boundary value problem

$$
\left\{
\begin{array}{lll}
\operatorname{div}(A(x)\nabla u) = \partial_z \beta(u) + f & \text{in} & \mathcal{C}_L, \\
\left\{\begin{array}{lll} u(x,0) = 0 & \text{on} & \Omega\times\{0\}, \\ u(x,L) = m > 0 & \text{on} & \Omega\times\{L\}, \end{array}\right. & & \\
u(X) = g(X) & \text{on} & \partial\Omega\times(0,L),
\end{array}
\right.
\tag{2.1}
$$

where $m > 0$ is a positive constant, $g \in H^1(\mathcal{C}_L)$ such that g's trace vanishes on $\Omega\times\{0\}$ and equals $m > 0$ on $\Omega\times\{L\}$, β is the enthalpy defined by (2.3), $a > 0$ is a constant

The equation

$$
\operatorname{div}(A\nabla u) = \operatorname{div}[\beta \mathbf{v}] + f, \tag{2.2}
$$

emerges in the steady state heat transfer problems in anisotropic media in the presence of convection. Here A is the anisotropic thermal conductivity, ρ is the density, $\mathbf{v}$ the prescribed convection, f accounts for sources or sinks, β the enthalpy defined as

$$
\beta(s) = \left\{
\begin{array}{lll}
as & \text{if} & s < 0, \\
\in [0,1] & \text{if} & s = 0, \\
as + 1 & \text{if} & s > 0.
\end{array}
\right.
\tag{2.3}
$$

For more background on this problem see [10]. It follows from (2.2) that u satisfies

We will be also interested in the local behaviour of weak solutions of

$$
\operatorname{div}(A(x)\nabla u) = \partial_z \beta(u) + f \qquad \text{in} \qquad \mathcal{C}_L. \tag{2.4}
$$

with convection $\mathbf{v}$ being the constant vector $\mathbf{e}_N = (0,0,\dots,0,1)$.

Throughout this paper we make the following hypotheses on the matrix A:

$$
\begin{array}{ll}
\mathbf{A1} & A : \Omega \to \mathbb{R}^{N^2}, \\
\mathbf{A2} & \lambda|\zeta|^2 \le A_i j\zeta^i\zeta^j \le \Lambda|\zeta|^2, \lambda, \Lambda > 0, \\
\mathbf{A3} & A \in C^{1,\alpha_0}(\overline{\Omega}), \alpha_0 > 0.
\end{array}
\tag{2.5}
$$

In other worlds A is independent of z variable, uniformly elliptic with C^{1,α_0} continuous entries.

Proposition 1. *Let $g \in H^1(\overline{\mathcal{C}_L})$ such that g's trace vanishes on $\Omega\times\{0\}$ and has constant value $m > 0$ on $\Omega\times\{L\}$. Then there exists a weak solution $u \in H^1(\mathcal{C}_L)$ of (2.1). Moreover, if $g \in C^{0,1}(\overline{\mathcal{C}_L})$ and the resulted solution is $\alpha-$Hölder continuous in $\overline{\mathcal{C}_L}$ for any $\alpha \in (0,1)$.*

Proof. The proof, which we briefly sketch here, is standard and is based on penalisation method [3], [5]: for any $\varepsilon > 0$ we consider the boundary value problem

$$
\left\{
\begin{array}{ll}
\operatorname{div}(A(x)\nabla u^\varepsilon(x)) = \partial_z\left(au^\varepsilon + \frac{\ell}{2}\left(1 + \tanh\frac{u^\varepsilon}{\varepsilon}\right)\right) & \text{in } \mathcal{C}_L, \\
u^\varepsilon = g & \text{on } \partial\mathcal{C}_L.
\end{array}
\right.
$$

From (2.5) it follows that there is unique $u^\varepsilon \in H^1(\Omega)\cap C^{2,\alpha}(\mathcal{C}_L)$ for some $\alpha\le\alpha_0$. Furthermore, if one multiplies this equation by $u^\varepsilon - g$ then after standard manipulations we can get

$$\begin{aligned}
\lambda\int_{\mathcal{C}_L}|\nabla u^\varepsilon|^2 &\le \int_{\mathcal{C}_L}\nabla u^\varepsilon A\nabla u^\varepsilon=\int_{\mathcal{C}_L}\nabla gA\nabla u^\varepsilon+\left(au^\varepsilon+\frac{\ell}{2}\left(1+\tanh\frac{u^\varepsilon}{\varepsilon}\right)\right)(u^\varepsilon-g)_z\\
&\le \delta\int_{\mathcal{C}_L}|\nabla u^\varepsilon|^2+\frac{1}{\delta}\int_{\mathcal{C}_L}|\nabla g|^2+\frac{a}{2}\int_{\mathcal{C}_L}\partial_z(u^\varepsilon)^2+\frac{\ell}{2}(1+\tanh\frac{u^\varepsilon}{\varepsilon})\partial_z u^\varepsilon\\
&\quad -\int_{\mathcal{C}_L}au^\varepsilon g_z-\int_{\mathcal{C}_L}\frac{\ell}{2}(1+\tanh\frac{u^\varepsilon}{\varepsilon})g_z\\
&\le 2\delta\int_{\mathcal{C}_L}|\nabla u^\varepsilon|^2+\frac{1}{\delta}\int_{\mathcal{C}_L}(|\nabla g|^2+\ell^2)+\frac{a}{2}|\Omega|m^2\\
&\quad -a\int_\Omega m^2+a\int_{\mathcal{C}_L}u^\varepsilon_z g+\ell\int_{C_L}g_z.
\end{aligned}$$

Hence, choosing $\delta>0$ small enough and after rearranging the terms we get

$$\int_{\mathcal{C}_L}|\nabla u^\varepsilon|^2\le C\left(a|\Omega|m^2+\int_{\mathcal{C}_L}(g^2+|\nabla g|^2+\ell^2)\right)$$

with some tame constant C independent of ε. From here and Poincaré's inequality [6] we get the uniform estimate $\|u^\varepsilon\|_{H^1(\mathcal{C}_L)}\le C$. After passing to the limit one can readily verify that the limit function u solves the equation $\mathcal{L}_A u = au_z+\eta_z$ in the weak sense and η takes values only in the interval $[0,\ell]$. The Hölder continuity follows from the standard DeGiorgi type estimates. $\square$

Proposition 2. *Let $u\in H^1(\mathcal{C}_L)$ be a weak solution of $\mathcal{L}_A u=\partial_z\eta$ in $\mathcal{C}_L$ and $\eta\in\beta(u)$ and $u^\star\in H^1(\mathcal{C}_L)$ is a weak supersolution $\mathcal{L}_A u^\star\le\partial_z(\eta^\star)$ with $\eta^\star\in\beta(u^\star)$. Suppose that for some $\rho>0$ we have*

$$|u|+|u^\star|\ge\rho\quad\text{in }\Omega\times(L-\rho,L).\tag{2.6}$$

If $u^\star\ge u$ on $\partial\mathcal{C}_L$ then $u^\star\ge u$ in $\mathcal{C}_L$.

For reader's convenience I will give the proof of Proposition 2, which is similar that of [3] with slight amendments due to the anisotropy of A in the last section of the paper. Note that (2.6) is necessary for Proposition 2 to hold, see [3].

Corollary 1. *Retain the conditions of Proposition 1 and assume further that there is $c_0>0$ such that*

$$\liminf_{z\to z_0}\frac{g(x,z)-g(x,z_0)}{z-z_0}\ge c_0,\quad\forall x\in\partial\Omega, z, z_0\in[0,L].\tag{2.7}$$

Then u is monotone in z direction and $\partial\{u>0\}$ is C^α graph over $\overline{\Omega}$.

The proof of Corollary 1 follows from Proposition 2 and (2.7) and can be found in [3]. It is worth noting that the method in [3] gives the same degree of regularity for both the solution in $\overline{\mathcal{C}_L}$ and the free boundary on $\overline{\Omega}$. Unfortunately, the best global regularity for u one can expect, under condition of Proposition 2 is log-Lipschitz. On the other hand the best local regularity of u that is Lipschitz continuity, see Theorem 1. However, in local outset the strong monotonicity of u in $z-$variable does not follow immediately and some delicate analysis is required in order to obtain the strong monotonicity of u in the subdomains of $\mathcal{C}_L$.

Now we formulate our main results.

Theorem 1. *Let u be a non-negative bounded weak solution to (2.2). Then u is locally Lipschitz continuous in $\mathcal{C}_L$, provided that $\boldsymbol{v} \in \boldsymbol{L}^\infty(\mathcal{C}_L, \mathbb{R}^N)$ and $f \in C(\overline{\mathcal{C}_L})$.*

The local regularity for two phase problem is discussed in [8], and [9]. As for the regularity of free boundary, our main result here states that if u is a Lipschitz continuous solution of (2.2) and $\partial_z u \geq 0$, then the free boundary is a locally Lipschitz continuous graph in $z-$direction.

Theorem 2. *Let u be a nonnegative weak solution to (2.1) in $\mathcal{C}_L$ such that u is nondecreasing in $z-$direction. Then for any subdomain $D \subset \mathcal{C}_L$, $\Gamma(u) = \partial\{u > 0\} \cap D$ is locally a Lipschitz graph in $\boldsymbol{e}_N-$direction.*

Before entering into the details of the proof we would like to highlight the main ideas in the proof of Theorem 2. First we establish the non-degeneracy of u. Then it will be seen that $\partial_z u \geq 0$ implies strong monotonicity $\partial_z u \geq c_0 > 0$, for some $c_0 = c_0(D)$, locally for any subdomain $D \subset\subset \mathcal{C}_L$. Combining this with the Lipschitz continuity of u the proof will follow.

The paper is organised as follows: In Section 3 we prove the local optimal regularity of the weak solutions of (2.2). In Section 4 we introduce Baiocchi's transformation w of u which allows us to retrieve the non-degeneracy of u form that of w, which solves an obstacle like problem. The non-degeneracy of u, established in Section 5,, is crucial in our analysis, especially in the proof of strong monotonicity in $z-$variable, see Proposition 3. The proof of the main regularity result for free boundary is contained in Section 6. Finally, last section contains the proof of comparison principle, Proposition 2.

3. Optimal Growth

By Proposition 1, u is bounded. Moreover the weak solutions of (2.2) are continuous for u solves the divergence form equation $\operatorname{div}(A\nabla u) = \operatorname{div} \mathbf{F} + f$ in $\mathcal{C}_L$, where one can take $\mathbf{f} = \mathbf{v}\beta \in \mathbf{L}^\infty(\mathcal{C}_L, \mathbb{R}^N)$. Thus the continuity of u follows from DeGiorgi's estimates. In fact, from the proof of Proposition 1 one sees that if f is sufficiently regular u is α-Hölder continuous for any positive $\alpha < 1$. This means that $\{u > 0\}$ is open.

3.1. **Proof of Theorem 1.** As it is pointed out in [7], it is enough to show that for any compact set $K \subset\subset \mathcal{C}_L$ there exists a tame constant C, depending on $\operatorname{dist}(K, \partial\mathcal{C}_L)$ such that

$$\sup_{B_{2^{-k-1}}(X)} u \leq \max\left(C2^{-k}, \sup_{B_{2^{-k}}(X)} u\right), \qquad \forall X \in K \cap \partial\{u > 0\}.$$

Assume that this inequality is false. Then there exist a sequence of weak solution u_j such that $0 \leq u_j \leq M$ for some fixed constant $M > 0$, a sequence $\{k_j\} \subset \mathbb{N}, X_j \in K \cap \partial\{u_j > 0\}$ and there holds

$$\sup_{B_{2^{-k_j-1}}(X)} u_j > \max\left(j2^{-k_j}, \frac{1}{2}\sup_{B_{2^{-k_j}}(X_j)} u_j\right). \tag{3.1}$$

Define the scaled functions $v_j(X) = \dfrac{u_j(X_j + 2^{-k_j}X)}{S_j}$, where $S_j = \sup\limits_{B_{2^{-(k_j+1)}}(X_j)} u$. It follows from (3.1) that

$$v_j(0) = 0, \qquad \sup_{B_{\frac{1}{2}}} v_j \geq \frac{1}{2}, \qquad 0 \leq v_j(X) \leq 2, \quad X \in B_1. \tag{3.2}$$

Since the weak solutions u_j are bounded it follows from (3.1) that $M > j2^{-k_j}$ implying that $k_j \to \infty$.

According to (2.2), v_j solves the following equation

$$\begin{aligned}\operatorname{div}(A(X_j+2^{-k_j}X)\nabla v_j) &= \frac{2^{-2k_j}}{S_j}(\mathcal{L}_A u_j)(X_j+2^{-k_j}X)\\ &= \frac{2^{-k_j}}{S_j}\operatorname{div}[\beta(v_j)\mathbf{v}(X_j+2^{-k_j}X)]+f_j\\ &\equiv \operatorname{div}\mathbf{F}_j+f_j,\end{aligned}$$

where $\mathbf{F}_j=\frac{2^{-k_j}}{S_j}\beta(v_j)\mathbf{v}(X_j+2^{-k_j}X), f_j=\frac{2^{-2k_j}}{S_j}f(X_j+2^{-k_j}X)$. From $\mathbf{v}\in\mathbf{L}^\infty(\mathcal{C}_L,\mathbb{R}^N)$ we obtain, using (3.1), definition of S_j and (2.3), the inequality

$$|\mathbf{F}_j|\le\frac{2^{-k_j}}{S_j}\beta(2)\sup|\mathbf{v}|\le\frac{M}{j}\beta(2)\sup|\mathbf{v}|\to 0.$$

Similarly we obtain $\sup\limits_{B_1}|f_j(X)|\to 0$.

From the Caccioppoli inequality it follows that $\{v_j\}$ is bounded in $H^1(B_{\frac{3}{4}})$. Furthermore, utilizing (3.2) and DeGiorgi's theorem for inhomogeneous divergence form elliptic operators, we infer that the sequence $\{v_j\}$ is uniformly Hölder continuous in $B_{3/4}$. Now employing a customary compactness argument and the estimates for $\{\mathbf{F}_j\}$ and $\{f_j\}$, we can extract a subsequences j_m such that $X_{j_m}\to X_0$, $\{v_{j_m}\}\subset\{v_j\}$ which uniformly converges to some v_0 in $B_{\frac{3}{4}}$ and weakly in $H^1(B_{\frac{3}{4}})$. Moreover, it follows that

$$-\int A(X_0)\nabla v_0\nabla\varphi\longleftarrow-\int A(X_j+2^{-k_j}X)\nabla v_{j_m}\nabla\varphi=\int f_{j_m}\varphi-\mathbf{F}_{j_m}\cdot D\varphi\longrightarrow 0,\qquad\forall\varphi\in C_0^\infty(B_{\frac{3}{4}}).$$

Thus $v_0\in H^1(B_{\frac{3}{4}})$ is a nonnegative continuous solution of $\operatorname{div}(A(X_0)\nabla v_0)=0$ in $B_{\frac{3}{4}}$. On the other hand, it follows from uniform convergence $v_{j_m}\to v_0$ that (3.2) translates to v_0 and we have $v_0(0)=0$ and $\sup\limits_{B_{\frac{1}{2}}}v_0=\frac{1}{2}$. However this is in contradiction with the strong maximum principle and the proof follows. $\square$

4. Baiocchi's transformation and its properties

In this section we study the weak solutions u of the continuous casting problems which are monotone in z variable, i.e. $\partial_z u\ge 0$. The monotonicity in z variable can be achieved for a suitable choice of boundary data [3], see (2.7).

We establish the key estimate for weak solutions of (2.2), which will be used in the proof of Theorem 2. Our first lemma is of technical nature linking u with the solution of obstacle problem via Baiocchi's transformation. Recall that Baiocchi's transformation w of u is defined by

$$w=\int_0^z u(x,s)ds\ge 0,\quad \partial_z w=u. \tag{4.1}$$

From definition it follows that w is convex in z variable provided that $\partial_z u\ge 0$.

Lemma 1. *Let $u\in H^1(\mathcal{C}_L)$ be a weak solution of (2.1). Then the Baiocchi transformation w given by (4.1) verifies the equation*

$$\operatorname{div}(A\nabla w)=au+\ell\chi_{\{u>0\}}.$$

Proof. By direct computation we have

$$\begin{aligned}
\mathrm{div}(A\nabla w) &= \sum_{ij=1}^{N} \partial_{x_i}(A_{ij}(x)\partial_{x_j} w(X)) \\
&= \sum_{i=1}^{N-1}\sum_{j=1}^{N} \partial_{x_i}(A_{ij}(x)\partial_{x_j} w(X)) + \sum_{j=1}^{N} A_{Nj}(x) w_{Nj}(X) \\
&= \sum_{ij=1}^{N-1} \partial_{x_i}\left(A_{i,j}(x)\int_0^z \partial_{x_j} u(x,s)ds\right) + \sum_{i=1}^{N-1} A_{iN}(x)u_{x_i}(X) + \sum_{j=1}^{N} A_{Nj}(x) w_{Nj}(X) \\
&= \sum_{i,j=1}^{N-1}\int_0^z \partial_{x_i}\left(A_{ij}(x)\partial_{x_j} u(x,s)ds\right) + \sum_{i=1}^{N-1} A_{iN}(x)u_{x_i}(X) + \sum_{j=1}^{N} A_{Nj}(x) u_{x_j}(X) \\
&= \int_0^z \sum_{ij=1}^{N-1} \partial_{x_i}[A_{ij}(x)\partial_{x_j} u(x,s)]ds + \\
&\quad + \sum_{i=1}^{N-1} A_{iN}(x)u_{x_i}(X) + \sum_{j=1}^{N} A_{Nj}(x) u_{x_j}(X) \\
&= \int_0^z \sum_{ij=1}^{N} \partial_{x_i}[A_{ij}(x)\partial_{x_j} u(x,s)]ds - \\
&\quad - \int_0^z \sum_{i=1}^{N-1} \partial_{x_i}(A_{iN}(x)\partial_{x_N} u(x,s))ds - \int_0^z \sum_{j=1}^{N} \partial_{x_N}(A_{Nj}(x)\partial_{x_j} u(x,s))ds \\
&\quad + \sum_{i=1}^{N-1} A_{iN}(x)u_{x_i}(X) + \sum_{j=1}^{N} A_{Nj}(x) u_{x_j}(X).
\end{aligned}$$

The first term is $\int_0^z \mathcal{L}_A u(x,s)ds = au + \ell\chi_{\{u>0\}}$. It remains to combine the second and fourth line in the computation in order to obtain

$$\begin{aligned}
\mathrm{div}(A(x)\nabla w) &= au + \ell\chi_{\{u>0\}} - \\
&\quad - \int_0^z \left\{ \sum_{i=1}^{N-1} \partial^2_{x_i x_N}(A_{iN}(x)u(x,s)) + \sum_{j=1}^{N} \partial_{x_N}(A_{Nj}(x)\partial_{x_j} u(x,s)) \right\} ds \\
&\quad + \sum_{i=1}^{N-1} A_{iN}(x)u_{x_i}(X) + \sum_{j=1}^{N} A_{Nj}(x) u_{x_j}(X) \\
&= au + \ell\chi_{\{u>0\}},
\end{aligned}$$

where to get the second line we used $\partial_{x_N} A_{ij} = \partial_z A_{ij} = 0$. Now the proof is complete. $\square$

Lemma 2. *Let $D \subset \mathcal{C}_L$ be a fixed subdomain such that dist$(D, \partial\mathcal{C}_L) > 0$ and w be a bounded solution of*

$$\mathrm{div}(A(x)\nabla w) = \beta(u)$$

in $B_R(X_0) \subset D$ with $X_0 \in \partial\{u > 0\}$. Then there is a universal constant C that depends on dist$(D, \partial\mathcal{C}_L)$ and data such that

$$\sup_{B_{2^{-k-1}}(X_0)} w \le \max\left(\frac{C}{2^{2k}}, \frac{1}{4}\sup_{B_{2^{-k}}(X_0)} w\right) \tag{4.2}$$

for all $R < \frac{1}{2^k}$.

Proof. Suppose that (4.2) fails. Then there is a sequence k_j such that

$$\sup_{B_{2^{-k_j-1}}(X_j)} w > \max\left(\frac{j}{2^{2k_j}}, \frac{1}{4}\sup_{B_{2^{-k_j}}(X_j)} w\right). \tag{4.3}$$

Using the same reasoning as in Theorem 1 we conclude from (4.3) that the scaled functions $w_j(X) = \frac{w(X_j+r_jX)}{S(k_j+1)}$ has the properties

$$w_j(0) = 0, \qquad \sup_{B_{\frac{1}{2}}} w_j \geq \frac{1}{4}, \qquad 0 \leq w_j(X) \leq 4, \quad X \in B_1, \tag{4.4}$$

where $r_j = 2^{-k_j}$ and $S(k_j+1) = \sup_{B_{2^{-k_j-1}}(X_j)} w$. Furthermore, w_j solves the equation

$$\mathrm{div}(\widetilde{A}_j(x)\nabla w_j(X)) = \frac{r_j^2}{S(k_j+1)}\beta(u(X_j+r_jX)) \quad \text{in } B_1$$

with scaled matrix $\widetilde{A}_j(X) = A(X_j+r_jX), X \in B_1$. From (4.3) we see that $\frac{r_j^2}{S(k_j+1)} < \frac{1}{j}$. Thanks to (2.3) and Proposition 1 $0 \leq u \leq M$ for some $M > 0$, hence $|\beta(u(X_j+r_jX))| \leq aM + \ell$ which implies that the functions $f_j(X) = \frac{r_j^2}{S(k_j+1)}\beta(u(X_j+r_jX))$ strongly converges to zero in B_1 as $j \to \infty$. Applying the standard Caccioppoli inequality we obtain

$$\int_{B_{\frac{7}{8}}} |\nabla w_j|^2 \leq C\int_{B_1}(w_j^2 + f_j^2)$$

with C depending only on N, λ, Λ. Thus we have the uniform estimate for H^1 norm $\|w_j\|_{H^1(B_{\frac{3}{4}})} \leq \sqrt{C(16+(aM+\ell)^2)}$. Furthermore, from DeGiorgi's estimates it follows that $\|w_j\|_{C^\gamma(B_{\frac{3}{4}})}$ are uniformly bounded for some $\gamma > 0$. Using a customary compactness argument we can extract a subsequence $\{w_{j_m}\} \subset \{w_j\}$ such that

(i) $w_{j_m} \rightharpoonup w_0$ weakly in $H^1(B_{\frac{7}{8}})$ for some function $w_0 \in H^1(B_{\frac{7}{8}})$,
(ii) $w_{j_m} \to w_0$ uniformly in $B_{\frac{3}{4}}$,
(iii) $\widetilde{A}_{j_m} \to \widetilde{A}_0$ uniformly in B_1, where $\widetilde{A}_0$ is a constant uniformly elliptic matrix,
(iv) $\mathrm{div}(\widetilde{A}_0\nabla w_0) = 0$ in $B_{\frac{3}{4}}$.

Recalling (4.4) and utilizing (i)-(iv) we see that that w_0 is a non-negative, non-zero $\widetilde{A}_0$-harmonic function in $B_{3/4}$ such that $w_0(0)$=0, which however is in contradiction with the maximum principle. The proof is complete. □

We close this section by proving the non-degeneracy of w.

Lemma 3. *Let u be a weak solution of (2.2) such that $\partial_z u \geq 0$. Then for any $D \subset \mathbb{R}^N$ there is a positive constant $r_0 = r_0(D) < \min\left(dist(D, \partial\mathcal{C}_L), \frac{\Lambda}{N\|A\|_{C^{0,1}}}\right)$ such that for the Biaocchi transformation there holds*

$$\sup_{B_r(X_0)} w \geq \frac{\ell}{8N\Lambda}, \quad \textit{for any} \ \ X_0 \in \overline{\{w>0\} \cap D}.$$

Proof. From Lemma 1 we know that $\mathcal{L}_A w = aw_z + \ell\chi_{\{w>0\}} = au + \ell\chi_{\{u>0\}}$. Moreover, if $\partial_z u \geq 0$ then the positivity sets of u and w are equal, i.e. $\{X \in \mathcal{C}_L, u(X) > 0\} = \{X \in \mathcal{C}_L, w(X) > 0\}$. Otherwise, if we drop the monotonicity condition $\partial_z u \geq 0$ then the inclusion $\{X \in \mathcal{C}_L, u(X) > 0\} \subset \{X \in \mathcal{C}_L, w(X) > 0\}$ is always true. Hence we conclude that

$$\mathcal{L}_A w = au + \ell\chi_{\{w>0\}}.$$

We want to show that $\sup_{\partial B_r(X_0)} w \geq C_0 R^2$ for $X_0 \in \{w > 0\}$, where $C_0 = \frac{\ell}{8N\Lambda}$. If this inequality fails then $\eta(X) = w(X) - C_0|X - X_0|^2 < 0$ for $X \in \partial B_r(X_0) \cap \{w > 0\}$. On the other hand $\eta \leq 0$ on $B_r \cap \partial\{w > 0\}$. Furthermore, in $B_R(X_0) \cap \{u > 0\}$ we have

$$\begin{aligned}
\mathcal{L}_A \eta &= \mathcal{L}_A w - 2C_0 \mathrm{Tr} A - 2C_0 \partial_i A_{ij}(X^i - X_0^i) \\
&\geq au + \ell - 2C_0 \mathrm{Tr} A - 2C_0 N^2 R \|A\|_{C^{0,1}} \\
&\geq \ell - 2C_0 N \left(\Lambda + NR\|A\|_{C^{0,1}}\right) \\
&\geq \ell - 4C_0 N\Lambda
\end{aligned}$$

provided that $R \leq \frac{\Lambda}{N\|A\|_{C^{0,1}}}$. Thus, recalling $C_0 = \frac{\ell}{8N\Lambda}$ we conclude that $\mathcal{L}_A \eta \geq \frac{\ell}{2} > 0$. Applying the maximum principle to η we get $\eta < 0$ in $B_r(X_0) \cap \{w > 0\}$. From $\eta < 0$ we also see that that $w(X_0) < 0$ which is a contradiction. $\square$

5. Non-degeneracy of u

Now we turn to the non-degeneracy of weak solution u to the continuous casting problem.

Lemma 4. *Let w be as in Lemma 1 such that $\partial_z u \geq 0$. Let $D \subset \mathcal{C}_L$ be a fixed subdomain such that dist$(D, \partial\mathcal{C}_L) > 0$. Then there are two constant $C_1 > 0$ and $R_0 > 0$ that depends only on dist$(D, \partial\mathcal{C}_L)$ and the data such that for any $B_R(X_0) \subset D, X_0 \in \partial\{u > 0\}$ there holds*

$$\sup_{\partial B_R(X_0)} w_z \geq C_1 R. \tag{5.1}$$

Proof. Recall that $\mathcal{L}_A w = au + \ell$ in $\{w > 0\}$ and $\mathcal{L}_A u = a\partial_z u$ in $\{u > 0\}$. Therefore

$$\sup_{B_R(X_0)} w_z = \sup_{B_R(X_0)} u = \sup_{\partial B_R(X_0)} u.$$

Furthermore, $\{u > 0\} = \{w > 0\}$.

The proof of (5.1) is by contradiction. Suppose that for some fixed $D \subset \mathcal{C}_L$ with $\mathrm{dist}(D, \partial\mathcal{C}_L) > 0$ there are $R_j > 0, X_j \in D \cap \partial\{u > 0\}$ such that

$$\sup_{B_{R_j}(X_j)} u = \sup_{B_{R_j}(X_j)} w_z < \frac{R_j}{j}. \tag{5.2}$$

Define

$$w_j(X) = \frac{w(X_j + R_j X)}{R_j^2}, u_j(X) = \frac{u(X_j + R_j X)}{R_j} \quad X \in B_1.$$

It follows that w_j solves the equation $\mathcal{L}_{\widetilde{A}_j} w_j = aR_j u_j(X) + \ell\chi_{\{u_j > 0\}}$ in B_1. Here $\widetilde{A}_j(x) = A(X_j + R_j X)$. Furthermore, w_j has the following properties

$$\sup_{B_{\frac{1}{2}}} w_j \geq \frac{C_0}{4}, \quad \|w_j\|_{C^{1,1}} \leq C, \quad u_j(0) = w_j(0) = 0, \quad \sup_{B_1} u_j \leq \frac{1}{j}, \quad \|u_j\|_{C^{0,1}(B_1)} \leq C \tag{5.3}$$

where C is independent of j and $C_0 = \frac{\ell}{8N\Lambda}$, see Lemma 3. Using a standard compactness argument we can extract a subsequence $\{j_m\}$ such that

(i) $w_{j_m} \rightharpoonup w_0$ weakly in $H^1(B_1)$ for some function $w_0 \in H^1(B_1)$,
(ii) $w_{j_m} \to w_0$ uniformly in B_1,
(iii) $\widetilde{A}_{j_m} \to \widetilde{A}_0$ uniformly in B_1, where $\widetilde{A}_0$ is a constant uniformly elliptic matrix.

We claim that $\operatorname{div}(\widetilde{A}_0\nabla w_0) = 0$ in B_1. Indeed, from (5.2) we infer that $\chi_{\{u_j>0\}} \to 0$ almost everywhere. Thus from Lebesgue's dominating convergence theorem we get that $\lim\limits_{j\to\infty} \int_{B_1} \chi_{\{u_j>0\}}\varphi = 0$ for any $\varphi \in C_0^\infty(B_1)$. Therefore w_0 is a weak solution of $\operatorname{div}(\widetilde{A}_0\nabla w_0) = 0$ in B_1. Moreover, $w_0 \geq 0, \sup_{B_{\frac{1}{2}}} w_0 \geq \frac{C_0}{4}$ and $w_0(0) = 0$, thanks to (5.3). But this is in contradiction with the maximum principle. □

Next, we record some properties of the blow up limits. Recall that the blow up limit of u at X_0 is defined as $v_0(X) = \lim\limits_{r_j\to 0} \frac{u(X_0+r_jX)}{r_j}$ where $X_0 \in \partial\{u > 0\}$ and $\{r_j\}$ is a sequence of positive numbers tending to zero. Notice that the sequence $u_j(X) = \frac{u(X_0+r_jX)}{r_j}$ is Lipschitz continuous in view of Theorem 1, hence by a customary compactness argument one can extract a converging subsequence from $r_j^{-1}u(X_0 + r_jX)$ for any sequence $\{r_j\}, r_j \to 0$. It is worthwhile to point out that v_0 solves the equation $\operatorname{div}(A(X_0)\nabla v_0) = \ell\partial_z(H(v_0))$ in $\mathbb{R}^N$.

Lemma 5. *Let $v_0 \geq 0, v_0(0) = 0$ be a blow up limit of u. Then v_0 is non-degenerate, i.e. for any bounded domain $D \subset \mathbb{R}^N$ there exists $c_D > 0$ such that*

$$\sup_{B_r(X_0)} v_0 \geq c_D r, \qquad \forall X_0 \in \partial\{u > 0\} \cap D, B_r(X_0) \subset D. \tag{5.4}$$

Proof. It is enough to notice that $\sup\limits_{B_s} \frac{u(X_0+r_jX)}{r_j} \geq C_1$ for a fixed $s > 0$ and small r_j. To see this one needs to apply Lemma 4 and use a customary compactness argument. □

Corollary 2. *Let v_0 be as in Lemma 5, then there is a constant C_D such that*

$$\fint_{B_r(X_0)} v_0^2 \geq C_D r^2, \qquad \forall X_0 \in \partial\{u > 0\} \cap D, B_r(X_0) \subset D.$$

Proof. If not then there exist $X_j \in \partial\{v_0 > 0\}\cap D$ and a sequence $0 < r_j \downarrow 0$ such that $\fint_{B_{r_j}(X_j)} v_0^2 \leq \frac{1}{j}$. Set $v_j(X) = r_j^{-1}v_0(X_j + r_jX)$, then clearly

$$\fint_{B_1} v_j^2 \leq \frac{1}{j}.$$

Since $\nabla v_j(X) = \nabla v_0(X_j + r_jX)$ and v_0 is Lipschitz, it follows from Arzelà-Ascoli theorem that there exists a subsequence j_k such that $v_{j_k}(X) \to V(X)$ uniformly in B_1 for some function V. In particular $\fint_{B_1} V^2 = 0$. However this contradicts the non-degeneracy of v_0, Lemma 4, because

$$\sup_{B_1} v_j = \frac{1}{r_j} \sup_{B_{r_j}(X_j)} v_0 \geq c_D > 0.$$

□

Corollary 3. *Let v_0 be as in Lemma 4. Then there exists $C'_D > 0$ such that*

$$\fint_{B_r(X_0)} |\nabla v_0|^2 \geq C'_D > 0, \qquad \forall X_0 \in \partial\{v_0 > 0\} \cap D, \quad B_r(X_0) \subset D.$$

Proof. We argue as in the proof of the previous Corollary. Thus there are $X_j \in \partial\{v_0 > 0\} \cap D$ and $0 < r_j \downarrow 0$ such that $\fint_{B_{r_j}(X_j)} |\nabla v_0|^2 \leq \frac{1}{j}$. Put $v_j(X) = r_j^{-1}v_0(X_j + r_jX)$ then it follows that

$$\fint_{B_1} |\nabla v_j|^2 \leq \frac{1}{j},$$

because $\nabla v_j(X) = (\nabla v_0)(X_j + r_jX)$, thus in particular the sequence $\{v_j\}$ is uniformly Lipschitz continuous in B_1. By a customary compactness argument we can extract a subsequence j_k such that $v_{j_k} \to V$ uniformly in B_1 and $\nabla v_{j_k} \rightharpoonup \nabla V$ weakly in B_1.

By the semicontinuity of Dirichlet integral we get

$$0=\liminf_{k\to\infty}\int_{B_1}|\nabla v_{j_k}|^2\geq\int_{B_1}|\nabla V|^2$$

implying that $V\equiv 0$ in B_1 (recall that $v_0(X_j)=0$ which translates to $v_{j_k}(0)=0$). But this contradicts the non-degeneracy of v_0, see (5.4), because $\frac{1}{r_j}\sup_{B_{r_j}(X_j)}v_0=\sup_{B_1}v_j\geq c_D$ and by uniform convergence this yields $\sup_{B_1}V\geq c_D$. $\square$

We close this section by giving an application of Corollary 3, see [2]. It provides a rough estimate for the measure of a neighbourhood of free boundary and will be used in the proof of strong monotonicity of u in the next section.

Lemma 6. *Let v_0 be as in Lemma 5. Then there is a tame constant $C>0$ such that for any R, and small σ, $0<\sigma<R$ the following inequality holds*

$$|\{0<v_0<\sigma\}\cap B_R|\leq C\sigma R^{N-1}.$$

Proof. Let $\sigma>0$ be fixed and $0<t<\sigma<R$. Let $v_{\sigma,t}=\max\big(\min(v_0,\sigma),t\big)$. Recalling that $\operatorname{div}(A(X_0)\nabla v_0)=\ell\partial_z H(v_0)$ we see that $\operatorname{div}(A(X_0)\nabla v_0)=0$ in $\{v_0>0\}$. Applying Green's formula we get

$$0=\int\limits_{B_R}v_{\sigma,t}\mathcal{L}_{A(X_0)}v_0=\int\limits_{\partial B_R}v_{\sigma,t}(A(X_0)\nabla v_0)\nu-\int\limits_{B_R\cap\{t<v_0<\sigma\}}(A(X_0)\nabla v_0)\nabla v_0.$$

Notice that $t<v_{\sigma,t}\leq\sigma$ and v_0 is Lipschitz continuous, thereby

$$\lambda\int\limits_{B_R\cap\{t<v_0<\sigma\}}|\nabla v_0|^2\leq\left|\int\limits_{\partial B_R}v_{\sigma,t}(A(X_0)\nabla v_0)\nu\right|\leq C\Lambda\sigma R^{N-1}.$$

Sending t to zero we conclude

$$\int\limits_{B_R\cap\{0<v_0<\sigma\}}|\nabla v_0|^2\leq C\sigma R^{N-1}. \tag{5.5}$$

Next, we define the maximal distance of $\{v_0=\sigma\}$ from $\partial\{v_0>0\}$, i.e. $d=\sup\limits_{Z\in\partial\{v_0>0\}}\operatorname{dist}(Z,\{v_0=\sigma\}\cap B_R)$. Let us show that $d\leq C\sigma$. To see this we make a use of the non-degeneracy of v_0 in $B_{d(Z)}(Z)$, where $d(Z)=\operatorname{dist}(Z,\{v_0=\sigma\}), Z\in\partial\{v_0>0\}$. Thus by (5.4) $\sup\limits_{B_{d(Z)}(Z)}v_0\geq c_Dd(Z)$. On the other hand $B_{d(Z)}(Z)\cap\{v_0>0\}\subset\{0<v_0<\sigma\}$, hence $\sup\limits_{B_{d(Z)}(Z)}v_0\leq\sigma$ and $d(Z)\leq\frac{\sigma}{c_D}\equiv C\sigma$ for any $Z\in\partial\{v_0>0\}\cap B_R$.

This, in particular, yields $\{0<v_0<\sigma\}\cap B_R\subset B_{2C\sigma}(\Gamma_0)$ where $\Gamma_0=\partial\{v_0>0\}$ and $B_{2C\sigma}(\Gamma_0)$ is the $2C\sigma$ neighborhood of the free boundary Γ_0. Observe that by Sard's theorem $\{v_0=\sigma\}$ is smooth for almost every $\sigma>0$.

Now let us consider a Besicovitch type covering $\bigcup_i B_{r_i}(Z_i), Z_i\in\partial\{v_0>0\}$ of the free boundary such that the balls have finite overlapping. Applying Corollary 3 we obtain

$$C'_D|\{0<v_0<\sigma\}| \le C'_D\sum_j r_j^N \le \sum_j \int_{B_{r_j}(Z_j)} |Dv_0|^2 \le C(N)\int_{B_{4C\sigma}(\Gamma_0)\bigcap\{v_0>0\}} |Dv_0|^2. \tag{5.6}$$

By Lipschitz continuity, Theorem 1, $v_0(X) \le 4C\sigma\|Dv_0\|_\infty$ for any $X \in B_{4C\sigma}(\partial\{v_0>0\})\bigcap\{v_0>0\}$. Thereby

$$B_{4C\sigma}(\partial\{v_0>0\})\bigcap\{v_0>0\} \subset \{0<v_0<4C\sigma\|Dv_0\|_\infty\}. \tag{5.7}$$

Combining (5.6), (5.7) and (5.5) we get

$$|\{0<v_0<\sigma\}\cap B_R| \le C\sigma R^{N-1}$$

and we arrive at the desired inequality. □

6. Lipschitz regularity of free boundary

Now we are ready to demonstrate the strong monotonicity of u in the $z-$direction.

Proposition 3. *Let u be the weak solution to (2.1) such that (5.4) holds. Then there exist $c_1>0$ such that we have*

$$\inf_{X\in D\cap\Gamma}\left(\liminf_{\substack{Y\to X\in\Gamma\\ Y\in\{u>0\}\cap D}} \partial_z u(Y)\right) \ge c_1 > 0, \qquad D\subset\subset \mathcal{C}_L. \tag{6.1}$$

Proof. The proof is by contradiction. Suppose that (6.1) fails, then there are points $X_j \in D\cap\Gamma$ such that $\liminf_{Y\to X_j}\partial_z u(Y) < \frac{1}{j}$ and there exists $Y_j \in \{u(X)>0\}$ such that

$$0 \le \partial_z u(Y_j) \le \frac{2}{j}, \qquad \operatorname{dist}(Y_j, \partial\{u>0\}) \to 0. \tag{6.2}$$

Let $\widetilde{X}_j \in \partial\{u>0\}$ be such that the distance $\rho_j \stackrel{\text{def}}{\equiv} \operatorname{dist}(Y_j,\partial\{u>0\})$ is realized and $\rho_j = |\widetilde{X}_j - Y_j|$. Introduce $v_j(X) = \frac{u(\widetilde{X}_j+\rho X)}{\rho_j}$ where $\rho_j = |\widetilde{X}_j - X_j|$ and $X \in B_2$.

Clearly $B_1(\widetilde{Y}_j) \subset \{v_j(X)>0\}$, with $\widetilde{Y}_j = \frac{Y_j-\widetilde{X}_j}{\rho_j}$ and it touches the free boundary of v_j at the origin $0\in\partial\{v_j>0\}$, see Figure 1. Moreover, (6.2) implies

$$0 \le \partial_z v_j(\widetilde{Y}_j) = (\partial_z u)(Y_j) \le \frac{2}{j}. \tag{6.3}$$

Notice that $\nabla v_j(X) = (\nabla u)(\widetilde{X}_j+\rho_j X), X\in B_2$ and hence by local Lipschitz continuity of u, Theorem 1, we conclude that the functions $v_j(X)$ are uniformly Lipschitz continuous in B_2.

Next, we claim that v_j is uniformly C^2 continuous in $B_{\frac12}(\widetilde{Y}_j)$. Indeed, letting $\widetilde{v}_j(\xi) = v_j(\widetilde{Y}_j+\xi), \xi\in B_1$ we conclude that $\widetilde{v}_j \ge 0$ in B_1 and $\widetilde{v}_j \in C^{0,1}(B_1)$ uniformly. Moreover in B_1 $\widetilde{v}_j$ solves the equation $\operatorname{div}(A(\widetilde{X}_j+\rho_j X)\nabla\widetilde{v}_j(X)) = a\rho_j\partial_z\widetilde{v}_j(X), a>0$, see (2.3). Thus by (2.5) and Schauder's estimate $\|\widetilde{v}_j\|_{C^2(\overline{B_{\frac12}})}$ is uniformly bounded. Returning to v_j the claim follows.

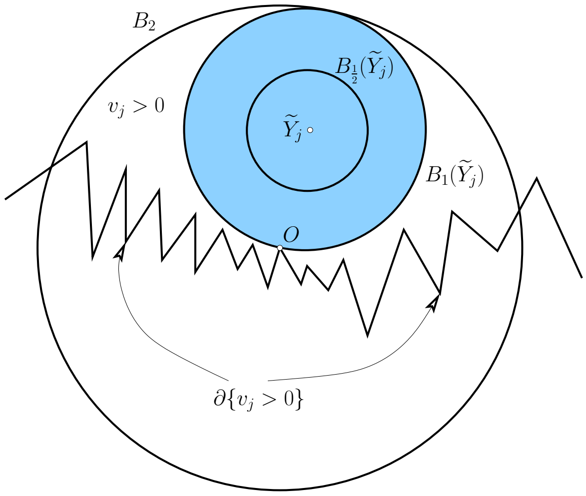

FIGURE 1. The structure of the free boundary of v_j near the origin.

Thus for any $\varepsilon > 0$ there is $\delta > 0$ such that uniformly in j

$$|\partial_z v_j(X) - \partial_z v_j(\widetilde{Y}_j)| < \varepsilon \tag{6.4}$$

whenever $|X - \widetilde{Y}_j| < \delta$. Notice that $|\partial_z v_j(Y_j)| \to 0$ by (6.3).

Since $\widetilde{Y}_j \in B_2$ and by Arzelà-Ascoli theorem there is a subsequence j_k and a function $v_0 \in C^2(B_2)$ such that

$$\begin{aligned}
&\widetilde{Y}_{j_k} \to Y_0 \in B_2,\\
&v_{j_k} \to v_0 \quad \text{uniformly in} \quad C^\alpha(B_2) \cap C^2(\overline{B_{\frac{1}{2}}(Y_0)}), \forall \alpha \in (0,1),\\
&B_1(Y_0) \subset \{v_0(X) > 0\},\\
&\operatorname{div}(A_0 \nabla v_0) = \ell \partial_z H(v_0), \text{in } B_2,\\
&\sup_{B_1} v_0 \geq c_D,
\end{aligned}$$

where A_0 is some constant positive definite matrix (thanks to condition (2.5)), H is the Heaviside function and the last inequality follows from (5.4), the definition of v_j and the uniform convergence of v_{j_k} in B_2. To finish the proof, it remains to establish that $v_0 \equiv 0$ in B_2, since then it will contradict the inequality $\sup_{B_1} v_0 \geq c_D$.

Let $h = \partial_z v_0$, then $h \geq 0$ and harmonic in $B_1(Y_0)$. Moreover by (6.4) $|h(X)| \leq \varepsilon$ whenever $|X - Y_0| \leq \delta$. Thus $h(Y_0) = 0$ and by the strong maximum principle it follows that $h = 0$ wherever A_0-harmonic, i.e. $h = 0$ in $\{v_0(X) > 0\}$ implying that

$$\partial_z v_0(X) = 0, \qquad X \in \{v_0 > 0\}.$$

Since $\operatorname{div}(A_0 \nabla v_0) = \ell \partial_z H(v_0)$ in B_2, then for any $\psi \in C_0^\infty(B_2)$ the following identity holds

$$\begin{aligned}
\int\limits_{B_2 \cap \{v_0(X) > 0\}} \ell \partial_z \psi = \int\limits_{B_2} \ell H(v_0) \partial_z \psi \quad &= \\
&= \int\limits_{B_2} (A_0 \nabla v_0) \nabla \psi.
\end{aligned}$$

Let $\sigma > 0$ be small, fixed number. Then

$$\int\limits_{B_2\cap\{v_0(X)>0\}} \ell\partial_z\psi = \int\limits_{B_2\cap\{v_0(X)\geq\sigma\}} \ell\partial_z\psi + \int\limits_{B_2\cap\{0<v_0(X)<\sigma\}} \ell\partial_z\psi.$$

By Sard's theorem, $\partial\{v_0 > \sigma\}$ is smooth for almost every σ. Thus, if necessary, we can take a slightly different domain $D \subset B_2$ such that $\{u > \sigma\} \cap D$ has Lipschitz continuous boundary. Thereby applying Green's formula

$$\int\limits_{B_2\cap\{v_0(X)\geq\sigma\}} \ell\partial_z\psi = \int\limits_{B_2\cap\partial\{v_0>\sigma\}} \ell\psi\left(\mathbf{e}_N\cdot\frac{\nabla v_0}{|\nabla v_0|}\right) = \int\limits_{B_2\cap\partial\{v_0>\sigma\}} \ell\psi\frac{\partial_z v_0}{|\nabla v_0|} = 0$$

for a.e. σ since $\partial_z v_0 = 0$ in $\{v_0 > 0\}$ and by Sard's $|\nabla v_0| \neq 0$ on $\partial\{v_0 > \sigma\}$ for a.e. $\sigma > 0$.

Finally utilizing Lemma 6 we infer that

$$\int\limits_{B_2\cap\{0<v_0(X)<\sigma\}} \ell\partial_z\psi \to 0$$

as $\sigma \to 0$. Hence $\operatorname{div}(A_0\nabla v_0) = 0$ in B_2. Because $v_0 \geq 0$ and $v_0(0) = 0$ we conclude, again from the maximum principle, that $v_0 \equiv 0$ in B_2 which contradicts to $\sup_{B_1} v_0 \geq c_D$. □

Remark 3. *It is worthwhile to point out that if $\partial_z u(X_0) = 0$ for some $X_0 \in \{u > 0\}$, then $\partial_z u = 0$ in $\{u > 0\}$. This follows from (2.5), the maximum principle for $\partial_z u \geq 0$ and it solves the linear equation $\mathcal{L}_A\partial_z u = a\partial_z(\partial_z u)$ with $a > 0$. Of course, the boundary data and (2.7) prevents this from happening. Thus $\partial_z u$ stays positive away from free boundary.*

6.1. **Proof of Theorem 2.** From Proposition 3 and Theorem 1 we have

$$\begin{aligned} u(x_2, z_2) - u(x_1, z_1) &= [u(x_2, z_2) - u(x_2, z_1)] + u(x_2, z_1) - u(x_1, z_1) \\ &\geq c_1(z_2 - z_1) - C|x_1 - x_2| \geq 0 \end{aligned}$$

provided that $z_2 - z_1 \geq \frac{C}{c_1}|x_1 - x_2|$. Let $h(x) = \inf\{z, u(z, x) > 0\}$ the height function of the free boundary over $x \in \Omega$. Thanks to $\partial_z u \geq c_1 > 0$, h is continuous and the free boundary is a continuous graph over Ω. Then for small $\varepsilon > 0$ we take $z_2 = h(x_1) + \varepsilon + \frac{C}{c_1}|x_1 - x_2|$ and $z_1 = h(x_1) + \varepsilon$. Clearly $z_2 - z_1 = \frac{C}{c_1}|x_1 - x_2|$ and hence $u(x_2, z_2) \geq u(x_1, z_1) > 0$ because $z_1 = h(x_1) + \varepsilon > h(x_1)$. Therefore $h(x_2) < z_2 = h(x_1) + \varepsilon + \frac{C}{c_1}|x_1 - x_2|$ or equivalently $h(x_2) - h(x_1) \leq \varepsilon + \frac{C}{c_1}|x_1 - x_2|$. Swapping x_1 and x_2 and letting $\varepsilon \to 0$ the result follows. □

7. Proof of Proposition 2

The proof is very similar to [3] Lemma 2.1, however there are technical complications due to the heat condition coefficients A_{ij}.

Using $\xi \in C_0^\infty(\mathcal{C}_L), \xi \geq 0$ in the weak formulation of solution u and supersolution $u^\star$ we get

$$\int_{C_L} -(\nabla u^\star - \nabla u)A\nabla\xi + (\eta^\star - \eta)\partial_z\xi \leq 0.$$

After integration by parts we get

$$\int_{C_L} (u^\star - u)\mathcal{L}_A\xi + (\eta^\star - \eta)\partial_z\xi - \int_{\partial C_L} (u^\star - u)(A\nabla\xi)\nu \leq 0. \tag{7.1}$$

First we show that $(A\nabla\xi)\nu \le 0$ on $\mathcal{C}_L$. Take $v(X) = \xi(X_0 + A(X_0)(X - X_0))$ with $X_0 \in \partial\mathcal{C}_L \setminus (\partial\Omega \times \{0\} \cup \partial\Omega \times \{L\})$. Since $v \ge 0$ and $v(X_0) = 0$ it follows

$$\partial_\nu v(X_0) = \lim_{X \to X_0} \frac{-\xi(X_0 - A(X_0)(X - X_0))}{|X - X_0|} \le 0.$$

Notice that $X_0 - A\nu t \in \mathcal{C}_L$ if $t > 0$ is small enough thanks to the ellipticity of A. Thus $\lim\limits_{t\to 0^+} \frac{-\xi(X_0 - A\nu t)}{t} = (\nabla\xi A\nu)(X_0)$ and the claim is proved.

Hence omitting the boundary integral in (7.1) we obtain

$$0 \ge \int_{\mathcal{C}_L} (u^\star - u)\mathcal{L}_A\xi + (\eta^\star - \eta)\partial_z\xi - \int_{\partial\mathcal{C}_L} (u^\star - u)(A\nabla\xi)\nu \ge \int_{\mathcal{C}_L} (\eta^\star - \eta)(\partial_z\xi + \mu\mathcal{L}_A\xi) \le 0 \tag{7.2}$$

where

$$\mu = \begin{cases} \frac{u^\star - u}{\eta^\star - \eta} & \text{if } \eta^\star \ne \eta, \\ 1 & \text{if } \eta^\star = \eta. \end{cases}$$

In order to estimate μ from below we utilize (2.6). If $u^\star(X) > 0$ for some X then $\mu(X) = (u^\star - u)/(au^\star + \ell - \eta) = 1/a$ provided that $u(X) > 0$. If $u(X) = 0$ then $\mu(X) \ge \frac{\rho}{a\rho + \ell}$ where ρ is from the condition (2.6). The estimate for other cases follows by similar reasoning. It also follows that $\mu \le \max(1, \frac{1}{a})$.

Next, for $\varphi \in C_0^\infty(\mathcal{C}_L), \varphi \ge 0$ consider homogeneous boundary value problem

$$\begin{cases} \mu_n\mathcal{L}_A\xi^n + \partial_z\xi^n = -\varphi & \text{in } \mathcal{C}_L \\ \xi^n = 0 & \text{on } \partial\mathcal{C}_L \end{cases}$$

where μ_n is chosen so that $\left\|(\eta^\star - \eta)\frac{\mu_n - \mu}{\sqrt{\mu_n}}\right\|_{L^2(\mathcal{C}_L)} \to 0$ and $\frac{1}{n} \le \mu_n \le 1 + \frac{1}{a}$. It is possible to construct $\{\mu_n\}$ because $u, u^\star \in H^1(\mathcal{C}_L)$ and hence by Sobolev's embedding theorem $\eta^\star - \eta \in L^{2+\varepsilon}(\mathcal{C}_L)$ for some $\varepsilon > 0$.

Moreover,

$$\|\mu_n - \mu\|_{L^2(\mathcal{C}_L)} \le \|\mu\sqrt{\mu_n}(u^\star - u)\|_{L^2(\mathcal{C}_L)} \left\| \frac{1}{\mu} \frac{\mu_n - \mu}{\sqrt{\mu_n}(u^\star - u)} \right\|_{L^2(\mathcal{C}_L)} \to 0$$

thus without loss of generality we assume that $\mu_n \ge \frac{\rho}{a\rho + \ell}$.

Multiplying the equation by $\mathcal{L}_A\xi^n$ we obtain

$$\begin{aligned} \int_{\mathcal{C}_L} \mu_n(\mathcal{L}_A\xi^n)^2 &= -\int_{\mathcal{C}_L} (\varphi + \xi_z^n)\mathcal{L}_A\xi^n \\ &= -\int_{\partial\mathcal{C}_L} \varphi(A\nabla\xi^n)\nu + \int_{\mathcal{C}_L} \nabla\varphi A\nabla\xi^n - \int_{\mathcal{C}_L} \xi_z^n(A\nabla\xi^n)\nu + \int_{\mathcal{C}_L} \nabla\xi_z^n A\nabla\xi^n \\ &= -\int_{\mathcal{C}_L} \xi^n\mathcal{L}_A\varphi + \int_{\mathcal{C}_L} \nabla\xi_z^n A\nabla\xi^n - \int_{\partial\mathcal{C}_L} \xi_z(A\nabla\xi^n)\nu \\ &= I_1 + I_2 + I_3, \end{aligned}$$

where

$$I_1 = -\int_{\mathcal{C}_L} \xi^n\mathcal{L}_A\varphi, \quad I_2 = \int_{\mathcal{C}_L} \nabla\xi_z^n A\nabla\xi^n, \quad I_3 = -\int_{\partial\mathcal{C}_L} \xi_z(A\nabla\xi^n)\nu.$$

Notice that $\xi_z^n = 0$ on $\partial\Omega \times (0, L)$ hence the last integral is $I_3 = -\int_{\Omega\times\{L\}} \xi_z(A\nabla\xi^n)\nu$. Therefore to obtain uniform bound on I_3 it is enough to estimate the normal derivative of ξ^n on $\Omega \times \{L\}$.

As for the remaining two integrals we first set notice that

$$\begin{aligned} I_2=\int_{\mathcal{C}_L}\nabla\xi^n_z A\nabla\xi^n &= \int_{\mathcal{C}_L}\partial_z(\nabla\xi^n A\nabla\xi^n)-\int_{\mathcal{C}_L}\nabla\xi^n\partial_z(A\nabla\xi^n)\\ &= \int_0^L\int_\Omega\partial_z(\nabla\xi^n A\nabla\xi^n)-\int_{\mathcal{C}_L}\nabla\xi^n\partial_z(A\nabla\xi^n). \end{aligned}$$

On the other hand from (2.5) and symmetry of A we conclude that

$$I_2=\int_{\mathcal{C}_L}\nabla\xi^n_z A\nabla\xi^n=\frac{1}{2}\int_\Omega\nabla\xi^n(x,L)A(x)\nabla\xi^n(x,L)dx.$$

And again we see that we only need to estimate the normal derivative of ξ^n on $\Omega\times\{L\}$.

We first prove uniform C^0 bound for ξ^n in order to estimate I_1 and then an estimate for $\partial_\nu\xi^n$ on $\Omega\times\{L\}$.

It is easy to prove that $\xi^n\geq 0$. Indeed, from $\varphi\geq 0$ and the equation $A_{ij}\xi^n_{ij}+\partial_i A_{ij}\xi_j+\frac{\xi_z}{\mu_n}=-\frac{\varphi}{\mu_n}\leq 0$, it follows from maximum principle that $\xi^n\geq\min_{\partial\mathcal{C}_L}\xi^n=0$. In order to prove upper bound we introduce $b=C-e^{Tz}$ for some constants $C,T>0$ to be fixed below. We have

$$\begin{aligned} A_{ij}b_{ij}+\partial_i A_{ij}b^n_j+\frac{b_z}{\mu_n} &= -T^2e^{Tz}A_{NN}-Te^{Tz}\partial_i A_{iN}-\frac{Te^{Tz}}{\mu_n}\\ &= -Te^{Tz}\left(TA_{NN}+\partial_i A_{iN}+\frac{1}{\mu_n}\right)\\ &\leq -\frac{Te^{Tz}}{\mu_n} \end{aligned}$$

provided that $T\geq\frac{N\|A\|_{C^{0,1}}}{\lambda}$ which implies $TA_{NN}+\partial_i A_{iN}\geq 0$. Next if we take $T=\max\left(\lambda^{-1}N\|A\|_{C^{0,1}},\sup_{\mathcal{C}_L}|\varphi|\right)$ it will follow that $\mathcal{L}_A b+\frac{b_z}{\mu_n}\leq\mathcal{L}_A\xi^n+\frac{\xi_z}{\mu_n}$ in $\mathcal{C}_L$. Finally choosing $C=e^{TL}$ we see that $b\geq\xi^n$ on $\partial\mathcal{C}_L$ and hence from comparison principle we infer the estimate $\xi^n\leq b$ for any $n=1,2,\dots$.

It remains to estimate the normal derivative near $\Omega\times\{L\}$. Take $v=\|\varphi\|_\infty\left(1-\left(1-\frac{L-z}{\overline{\rho}}\right)^2\right)$ for some $\overline{\rho}<\rho$ to be fixed below. In $\Omega\times(L-\rho,L)$ we have

$$\begin{aligned} \mathcal{L}_A v+\frac{v_z}{\mu_n} &= -\frac{2\|\varphi\|_\infty}{\overline{\rho}}\left(\frac{1}{\overline{\rho}}A_{NN}+\partial_i A_{iN}\left(1-\frac{L-z}{\overline{\rho}}\right)+\frac{1}{\mu_n}\left(1-\frac{L-z}{\overline{\rho}}\right)\right)\\ &\leq -\frac{2\|\varphi\|_\infty}{\overline{\rho}\mu_n}\left(1-\frac{L-z}{\overline{\rho}}\right) \end{aligned}$$

provided that $\overline{\rho}\leq\min(\frac{\lambda}{N\|A\|_{C^{0,1}}},1)$. Furthermore, if $z\leq\frac{\overline{\rho}}{2}$ then $\mathcal{L}_A v+\frac{v_z}{\mu_n}\leq-\frac{\|\varphi\|_\infty}{\overline{\rho}\mu_n}\leq-\frac{\varphi}{\mu_n}$. Summarizing we see that $\mathcal{L}_A v+\frac{v_z}{\mu_n}\leq\mathcal{L}_A\xi^n+\frac{\xi^n_z}{\mu_n}$ in $\mathscr{S}_{\overline{\rho}}=\Omega\times(L-\frac{\rho}{2},L)$. On the other hand $\xi^n\leq b\leq Cv$ on $\partial\mathscr{S}_{\overline{\rho}}$ for sufficiently large $C>0$ such that $Cv\geq b$ on $\Omega\times\{L-\frac{\overline{\rho}}{2}\}$. Therefore we obtain $|\partial_\nu\xi^n|\leq\frac{2LC}{\overline{\rho}}\|\varphi\|_\infty$ on $\Omega\times\{L\}$. Combining these estimates and bounding the integrals I_1,I_2 and I_3 we obtain the uniform estimate

$$\int_{\mathcal{C}_L}\mu_n(\mathcal{L}_A\xi^n)^2\leq C$$

with some tame constant $C>0$.

Taking $\xi = \xi^n$ in (7.2) we get

$$\begin{aligned} 0 &\geq \int_{\mathcal{C}_L} (\eta^\star - \eta)(\mu \mathcal{L}_A \xi^n + \partial_z \xi^n) = \\ &= -\int_{\mathcal{C}_L} (\eta^\star - \eta)\varphi + \int_{\mathcal{C}_L} (\eta^\star - \eta)(\mu - \mu_n)\mathcal{L}_A \xi^n \\ &\geq -\int_{\mathcal{C}_L} (\eta^\star - \eta)\varphi - \left(\int_{\mathcal{C}_L} \mu_n (\mathcal{L}_A \xi^n)^2 \right)^{\frac{1}{2}} \left\| (\eta^\star - \eta) \frac{\mu_n - \mu}{\sqrt{\mu_n}} \right\|_{L^2(\mathcal{C}_L)} \longrightarrow -\int_{\mathcal{C}_L} (\eta^\star - \eta)\varphi \end{aligned}$$

implying $\eta^\star \geq \eta$ in $\mathcal{C}_L$, and the proof is complete. □

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