EXISTENCE AND REGULARITY OF THE REFLECTOR SURFACES IN $\mathbb{R}^{n+1}$

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Abstract. In this paper we study the problem of constructing reflector surfaces from the near field data. The light is transmitted as a collinear beam and the reflected rays illuminate a given domain on the fixed receiver surface. We consider two types of weak solutions and prove their equivalence under some convexity assumptions on the target domain. The regularity of weak solutions is a very delicate problem and the positive answer depends on a number of conditions characterizing the geometric positioning of the reflector and receiver. In fact, we show that there is a domain $\mathcal{D}$ in the ambient space such that the weak solution is smooth if and only if its graph lies in $\mathcal{D}$.

1. Introduction

We are given a smooth surface Σ in $\mathbb{R}^{n+1}$, a pair of bounded regular domains $\mathcal{U} \subset \Pi = \{X \in \mathbb{R}^{n+1} : X^{n+1} = 0\}$ and $\mathcal{V} \subset \Sigma$ and a pair of nonnegative, integrable functions $f : \mathcal{U} \longrightarrow \mathbb{R}$ and $g : \mathcal{V} \longrightarrow \mathbb{R}$. For $x \in \mathcal{U}$ we issue a ray parallel to e_{n+1} that after reflection from the unknown surface Γ_u strikes $\mathcal{V}$ at some point $Z \in \mathcal{V}$, see Figure 1. Denote by $\mathscr{R}_u : x \mapsto Z$ the reflector mapping. The main problem that we study in this paper is formulated as follows:

Find a function $u : \mathcal{U} \longrightarrow \mathbb{R}$ such that the reflector mapping $\mathscr{R}_u$ verifies the following two conditions:

$$(\mathbf{P}) \qquad \mathscr{R}_u(\mathcal{U}) = \mathcal{V} \qquad \text{and} \qquad \int\limits_{\mathcal{U}'} f = \int\limits_{\mathscr{R}_u(\mathcal{U}')} g \qquad \text{for any measurable } \mathcal{U}' \subset \mathcal{U}.$$

The first equation $\mathscr{R}_u(\mathcal{U}) = \mathcal{V}$ expresses the boundary condition, namely that after reflection the rays strike the whole target domain $\mathcal{V}$. For the perfect reflector the integral identity manifests the local form of conservation of energy. The full energy balance condition demands that the pairs $(f, \mathcal{U})$ and $(g, \mathcal{V})$ verify the following identity

$$\int_{\mathcal{U}} f(x)dx = \int_{\mathcal{V}} g d\mathcal{H}^n_{\Sigma}. \tag{1.1}$$

Notice that, both conditions in $(\mathbf{P})$ are formal because in general the surface Γ_u may not be smooth and some extra care will be necessary to formulate $(\mathbf{P})$ in a suitable weak sense.

For $u \in C^2(\mathcal{U})$ we denote the reflector mapping by $Z_u(x)$. Let Y be the unit direction of the reflected ray and γ be the normal at M. By Snell's law γ, Y and e_{n+1} are coplanar and γ forms equal angles with $-e_{n+1}$ and Y. As a result we obtain the identity

$$Y = e_{n+1} - 2\gamma\langle e_{n+1}, \gamma\rangle \tag{1.2}$$

where $\langle , \rangle$ denotes the inner product in $\mathbb{R}^{n+1}$. In order to derive the differential equation for u we employ the method of stretch function introduced in [8, 9]. Utilizing the local energy balance condition and computing the Jacobian of Z_u we find that u solves the Monge-Ampère type equation

$$\frac{|\nabla\psi|}{|\langle\nabla\psi, Y\rangle|}\left|\det\left[-\mathrm{Id} - \frac{2t}{1+|Du|^2}D^2u\right]\right| = \frac{f}{g} \tag{1.3}$$

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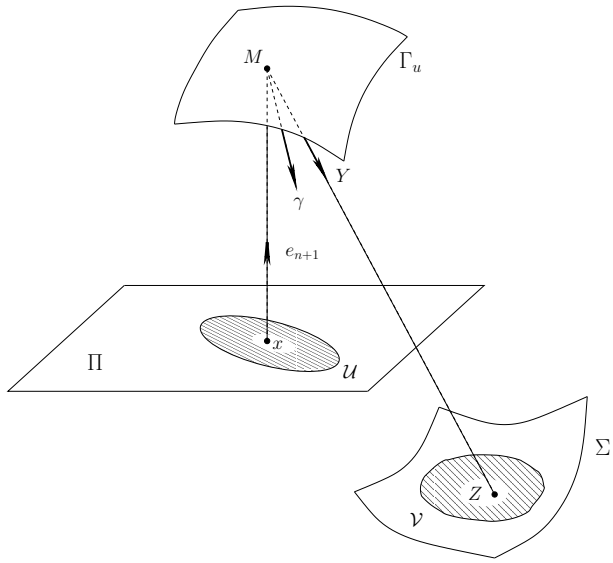

FIGURE 1. The reflector problem.

where $t > 0$ is *the stretch function*, Y is the unit direction of reflected ray and ψ is the defining function of Σ, i.e. $\Sigma = \{X \in \mathbb{R}^{n+1} : \psi(X) = 0\}$. Here the stretch function $t \geq 0$ is determined from the implicit equation $\psi(x + ue_{n+1} + tY) = 0$ and hence it depends on x, u and Du.

For $u \in C^2(\mathcal{U})$ we consider the symmetric matrix

$$\mathcal{W}(u) = -\frac{1 + |Du|^2}{2t}\mathrm{Id} - D^2u. \tag{1.4}$$

In general, this matrix may be indefinite. Note that if $|\langle \nabla\psi, Y\rangle| \neq 0$, say $\langle \nabla\psi, Y\rangle > 0$, and $\mathcal{W}(u) \geq 0$ then (1.3) can be written in an equivalent form

$$\frac{|\nabla\psi|}{\langle \nabla\psi, Y\rangle}\left[\frac{2t}{1 + |Du|^2}\right]^n \det\left[-\frac{1 + |Du|^2}{2t}\mathrm{Id} - D^2u\right] = \frac{f}{g}. \tag{1.5}$$

In this paper we study the solution of (1.5) for which $\mathcal{W}(u) \geq 0$. For such $u \in C^2(\mathcal{U})$ the equation (1.5) is degenerate elliptic. Thus the inequality $\mathcal{W}(u) \geq 0$ defines the class of C^2 admissible function for which the equation (1.5) is of *elliptic* type. Our first step is to introduce a suitable notion of weak solution for the equation (1.3) such that the condition $\mathcal{W}(u) \geq 0$ still holds for non-smooth solutions u in a.e. sense. In fact, we consider two such notions called respectively A-type and B-type weak solutions. Let us define the class of upper-admissible functions $\mathbb{W}^+(\mathcal{U}, \mathcal{V})$ consisting of all $w : \mathcal{U} \longrightarrow \mathbb{R}$ such that for each point x there is a paraboloid of revolution $P(\cdot, \sigma, Z)$ (regarded as a concave graph over $\Pi = \{X \in \mathbb{R}^{n+1} : x^{n+1} = 0\}$) with focal axis parallel to e_{n+1}, focal parameter σ and focus $Z \in \mathcal{V} \subset \Sigma$ that touches Γ_u at $M = (x, u(x))$ from above. Then we say that P is a *supporting paraboloid* of u at x. For $u \in \mathbb{W}^+(\mathcal{U}, \mathcal{V})$ we define the mapping

$$\mathscr{S}_u(Z) = \{x \in \mathcal{U} \text{ such that } P(\cdot, \sigma, Z) \text{ is a supporting paraboloid at } x\}.$$

Since $u \in \mathbb{W}^+(\mathcal{U}, \mathcal{V})$ is concave in usual sense then it follows from Aleksandrov's theorem that $\mathscr{S}_u$ is one-to-one modulo a set of vanishing measure. Subsequently $\mathscr{S}_u$ generates the set function $\beta_{u,f}(E) = \int_{\mathscr{S}_u(E)} f(x)dx$, defined

for each Borel $E \subset \mathcal{V}$. Then we say that $u \in \mathbb{W}^+(\mathcal{U}, \mathcal{V})$ is a B-type weak solution of (1.5) if $\int_E g d\mathcal{H}^n_\Sigma = \beta_{u,f}(E)$ for any Borel $E \subset \mathcal{V}$.

The B-type weak solutions are easy to construct since the measure $\beta_{u,f}$ defined via the mapping $\mathscr{S}_u : \mathcal{V} \longrightarrow \mathcal{U}$, see Section 8, is countably additive thanks to Aleksandrov's theorem, see Lemma 8.2. No additional assumptions are imposed on $f, g, \mathcal{U}$ and $\mathcal{V}$.

The construction of A-type weak solutions is more delicate and we require stronger assumptions on the data. Namely, we suppose that the following conditions hold

$$f, g > 0, \tag{1.6}$$

$$\operatorname{dist}(\mathcal{U}, \mathcal{V}) > 0, \tag{1.7}$$

$$\langle Y, \nabla \psi \rangle > 0, \tag{1.8}$$

$$\mathcal{V} \text{ is } R\text{-convex with respect to } \mathcal{U}, \tag{1.9}$$

$$-\frac{2t}{1+|Du|^2}\mathrm{II} + \mathrm{Id}\cos\theta < 0, \tag{1.10}$$

where Y is the unit direction of reflected ray at $x, u(x))$, $\theta \in [0, \pi]$ is the angle between e_{n+1} and the normal $\frac{\nabla\psi}{|\nabla\psi|}$ of $\Sigma = \{Z \in \mathbb{R}^{n+1} : \psi(Z) = 0\}$ and II is the second quadratic form of Σ.

Before explaining the meaning of these conditions it is convenient to describe the idea behind the construction of A-type weak solutions. First we define the mapping

$$\mathscr{R}_u(x) = \{Z \in \mathcal{V} \text{ such that } P(\cdot, \sigma, Z) \text{ is a supporting paraboloid at } x\}.$$

One of our tasks will be to prove that under conditions (1.7), (1.8)

$$\alpha_{u,g}(\omega) = \int_{\mathscr{R}_u(\omega)} g d\mathcal{H}^n_\Sigma, \quad \omega \subset \mathcal{U}$$

is a countably additive measure defined on Borel subsets $\omega \subset \mathcal{U}$.

Unlike the B-type weak solutions we don't get the countable additivity for $\alpha_{u,g}$ directly. Recall that for the classical Monge-Ampère equation there are two approaches to prove the countable additivity of curvature measure: one is by Fubini's theorem, see [1], Theorem 4, page 190 for $n = 2$, [7] for $n \geq 3$, and the other by Legendre's transformation, see [18]. Note that the Legenedre transformation also works for a more general class of problems, including optimal mass transport, where the corresponding matrix $\mathcal{W}(u)$ is invariant with respect to translations, i.e. $\mathcal{W}(u + c) = \mathcal{W}(u), c \in \mathbb{R}$.

Unfortunately, (1.4) is not translation invariant with respect to u due to the fact that (1.5) is not of variational form. This means that $-\frac{1+|Du|^2}{2t}\mathrm{Id}$ cannot be written as $c_{x_i x_j}(x, Z)$ for some cost function $c : \mathcal{U} \times \mathcal{V} \longrightarrow \mathbb{R}$. In the context of optimal transport theory the classical Legendre transformation corresponds to the cost function $c(x, Z) = \langle x, Z \rangle$, and for general cost c one can define the c-transform which is obtained directly from Kantorovitch's duality argument, see [20].

In order to prove that $\alpha_{u,g}$ is countably additive we first examine the focal parameters of supporting paraboloids. From a geometric argument describing the confocal expansion of paraboloids we express the focal parameter of a supporting paraboloid as a function of Z, x and u and observe that for each fixed Z there is a unique σ such that $P(\cdot, \sigma, Z)$ is a supporting paraboloid of u at some $x \in \mathcal{U}$. That done, we can proceed to define one of the main novel tools introduced in this paper, a Legendre-like transformation for upper admissible functions $u \in \mathbb{W}^+(\mathcal{U}, \mathcal{V})$

such that the transformed function $u^\star$, called $R-$transform of u, is semi-convex. Moreover, if u is admissible then the mapping $\mathscr{R}_u$ is one-to-one modulo a set of vanishing surface measure, see Proposition 10.5. In the definition of Legendre-like transformation we use the fact that $u^\star(Z)$ can be seen as the smallest focal parameter of $P(x,\sigma,Z)$ touching u from above. The set where $\mathscr{R}_u$ is not one-to-one is a subset of the points where $u^\star$ is not differentiable. Hence the semiconvexity of $u^\star(Z)$ implies that the surface measure of this set on Σ vanishes. This, in turn, implies that the set function $\alpha_{u,g}(E)=\int_{\mathscr{R}_u(E)} g d\mathcal{H}^n_\Sigma$ is well-defined and countably additive, see Section 10.

Note that, (1.7) is necessary in order to infer that $u^\star$ has C^1 smooth lower supporting functions and hence $u^\star$ is semiconvex, (1.6) helps to construct A-type weak solutions for target domains $\mathcal{V}$ which are not $R-$convex in the sense of Definition 10.7. It should be remarked here that for general $\mathcal{V}$ the resulted α measure is obtained indirectly via approximation by $R-$convex domains and weak convergence, see Section 11.

Our existence result for A and B type solutions is contained in

Theorem 1. *Let $\mathcal{U}\subset\Pi=\{X\in\mathbb{R}^{n+1}: x^{n+1}=0\}$ and $\mathcal{V}\subset\Sigma$ be bounded domains. Suppose $f:\mathcal{U}\to\mathbb{R}, g:\Sigma\to\mathbb{R}$ are nonnegative and integrable so that the energy balance condition (1.1) holds.*

- **a)** *Then there exists a B-type weak solution to (**P**).*
- **b)** *If, in addition, the conditions (1.7), (1.8) are satisfied then the measure $\alpha_{u,g}$ is countably additive.*
- **c)** *Finally, if $g>0$ then under the same conditions as in part* **b**)*, there is a A-type weak solution of (**P**) in the sense of Definition 10.7.*

If $\mathcal{V}$ is not $R-$convex then we can still show that A-type weak solution exists however one must require that $f>0$, see Section 11 proof of Theorem 1 **b**). This is due to the condition in (10.11), see Definition 10.7. The third part of Theorem 1 is proven indirectly. Namely, we show that the B-type weak solution, constructed by an approximation method, is also of A-type provided that $\mathcal{V}$ is $R-$convex and (1.7)-(1.8) are satisfied. Finally, assuming that $f,g>0$ we can remove the $R-$convexity assumption on $\mathcal{V}$ in order to establish the existence of A-type weak solution via an approximation of $\mathcal{V}$ by $R-$convex domains.

Next, we focus on the problem of C^2 regularity of weak solutions. The first step is to study Dirichlet's problem for the equation (1.5). We use Perron's method and suitable barrier construction to establish the existence of A-type weak solutions to Dirichlet's problem, see Section 12. A crucial step towards proving the higher regularity of A-type weak solutions is the *a priori* estimate of $C^{1,1}$ norm of smooth solution, obtained in Sections 5 and 13. That done, we can employ the continuity method to conclude the existence of smooth solution in a small ball. This method is well-known for the classical Monge-Ampère equation, see Pogorelov [17], [15].

It is worthwhile to point out that the higher regularity is expected for the A-type weak solutions, under suitable assumptions on data. This is because the equation (1.5) is the (local) energy balance condition for $\mathscr{R}_u$ generated by a reflector surface Γ_u, regarded as a graph over $\mathcal{U}$ and u solves the Monge-Ampère type equation (1.3). In other words, we can think of u as a potential that gives rise to the mapping $\mathscr{R}_u$ with Jacobian that satisfies the equation (1.5).

If we try to derive a similar equation for the mapping $\mathscr{S}_u$ then the equation will involve the function $\sigma(Z)$-the focal parameter of supporting paraboloids of B-type solution regarded as a function of Z. The study of this problem will appear elsewhere.

Note that, in the proof of Lemma 11.2 (with the aid of which we are able to prove Theorem 1, **c**)) we exploit the countable additivity of measure $\alpha_{u,g}$, which is an indispensable property of A-type weak solutions.

Throughout the paper we tacitly assume that the target domain $\mathcal{V}$ is a subset of *larger* smooth receiver Σ. However in some arguments we take $\mathcal{V} = \Sigma$ if there is no confusion.

To formulate our regularity result it is convenient to define the regularity domain $\mathcal{D}$ where all four conditions (1.7)-(1.10) are satisfied.

The domain $\mathcal{D}$ plays a crucial role in the regularity theory. In fact, one can show that if one of the conditions (1.7)-(1.10) is not satisfied then the weak solution may not be C^1, see Remark 11.2. The construction of counterexamples is similar to that of [9, 10] where a the authors have considered the point source of light and exploits the approximation of a two-point target $\mathcal{V}$ via smooth R-convex sets. The corresponding solutions converge to two-paraboloid configuration and hence the C^1 regularity breaks down as the approximation proceeds. We refer the reader to [9, 10] for more details. Notice that the problem studied in [9, 10] can be formulated as an optimisation problem, see [12].

The class of receivers satisfying (1.10) is considerably large, in particular for any plane (1.10) holds true. More examples are in Section 4.4. In Section 9 we will see that under condition (1.10) it follows that the local supporting paraboloid is also global. With the aid of these facts we can prove our main regularity

Theorem 2. *Let $f, g > 0$ be C^2 smooth functions and the conditions of Theorem 1* **b**) *and (1.7)-(1.10) are satisfied. Then A-type weak solutions of (**P**) are locally C^2 regular in $\mathcal{U}$.*

The proof uses Pogorelov's method of comparing the weak solution with upper and lower smooth barriers in a small ball. We also remark here that to do this we need to consider the (weak) Dirichlet problem for the equation (1.3) which is done in Section 12.

Recall that in optimal transfer theory one deals with the following Monge-Ampère type equation

$$\det[c_{x^i x^j}(x,y) - D^2 u(x)] = h(x)$$

where $c(x,y)$ is the cost function and h is determined from the data. In [14] the A3 condition was introduced which allows to employ a Pogorelov-type estimate for the second order derivatives of the smooth solution u. In this context the A3 condition takes the from

$$\partial^2_{p_k p_l} c_{x^i x^j}(x,y)\xi_i\xi_j\eta_k\eta_l \geq c_0|\xi|^2|\eta|^2$$

where c_0 is a positive constant, $\xi \perp \eta \in \mathbb{R}^n$ and $y = y(x,p)$ is the transport mapping (here p is the dummy variable denoting Du). For our reflector problem (**P**) the A3 condition takes the following form

$$\frac{1+|Du|^2}{t}\nabla^2\psi Z_{p_k} Z_{p_l} + 2\psi_{n+1}\delta_{kl} < 0. \tag{1.11}$$

We remark that (1.11) is equivalent to (1.10), see Section 4.5.

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2. Notations

$C, C_0, C_n, \cdots$	generic constants,
$\overline{\mathcal{U}}$	closure of a set $\mathcal{U}$,
$\partial\mathcal{U}$	boundary of a set $\mathcal{U}$,
$\widehat{X}$	$\widehat{X} = (x^1, \dots, x^n, 0)$ projection of $X = (x^1, \dots, x^{n+1}) \in \mathbb{R}^{n+1}$,
$\langle \cdot, \cdot \rangle$	inner product in $\mathbb{R}^{n+1}$,
$B_r(x)$	$\{y \in \mathbb{R}^n : \lvert y - x\rvert < r\}$, open ball centered at y,
B_r	$B_r(0)$,
Γ_u	graph of function u,
$\partial_i u, D_i u, Du$	$\partial_i u = D_i u = \frac{\partial u}{\partial x^i}$ and $Du = (D_1 u, \dots, D_n u)$,
ψ	defining function of receiver $\Sigma = \{Z \in \mathbb{R}^{n+1} : \psi(Z) = 0\}$,
$\nabla\psi$	$(n+1)$-dimensional gradient of receiver $\psi : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$,
$\widehat{\nabla}\psi$	$(\psi_1, \dots \psi_n, 0)$, projection of $\nabla\psi$,
Π	hyperplane $\{X \in \mathbb{R}^{n+1} : x^{n+1} = 0\}$,
$\mathbb{S}^{n+1}$	units sphere in $\mathbb{R}^{n+1}$,
$\lvert E\rvert$	$n-$dimensional Lebesgue measure of $E \subset \Pi$,
$\mathcal{H}^n_\Sigma$	$n-$dimensional Hausdorff measure restricted on Σ,
$\mathbb{P}_L(\mathcal{U}, \mathcal{V})$	see (6.5),
$\mathbb{W}^+(\mathcal{U}, \mathcal{V}), \mathbb{W}^+_0(\mathcal{U}, \mathcal{V})$	see Definitions 6.1 and 6.2,
$\mathcal{AS}^+(\mathcal{U}, \Sigma)$	see Definition 12.1

3. Main Equation

3.1. **Preliminaries.** In this subsection we gather some useful facts to be used along the proof of Proposition 3.3.

Lemma 3.1. *If $\mu = \mathrm{Id} + \alpha\xi \otimes \eta$,$\alpha \in \mathbb{R}$ and $\xi, \eta \in \mathbb{R}^n$, then we have*

$$\begin{aligned} \det \mu &= 1 + \alpha\langle \xi, \eta\rangle, \\ \mu^{-1} &= \mathrm{Id} - \frac{\alpha\xi \otimes \eta}{1 + \alpha\langle \xi, \eta\rangle}. \end{aligned}$$

Here and henceforth Id *is the identity matrix.*

Proof. To prove the first equality we assume, without loss of generality, that $\xi = e_1$. Then the formula follows as the matrix μ has triangular form.

As for the second formula we compute

$$\begin{aligned} \mu\left[\mathrm{Id}-\frac{\alpha\xi\otimes\eta}{1+\alpha\langle\xi,\eta\rangle}\right] &= [\mathrm{Id}+\alpha\xi\otimes\eta]\left[\mathrm{Id}-\frac{\alpha\xi\otimes\eta}{1+\alpha\langle\xi,\eta\rangle}\right] \\ &= \mathrm{Id}+\alpha\xi\otimes\eta-\frac{\alpha\xi\otimes\eta}{1+\alpha\langle\xi,\eta\rangle}(\alpha\langle\xi,\eta\rangle+1) \\ &= \mathrm{Id}. \end{aligned} \tag{3.1}$$

■

If we write down the energy balance condition utilizing the change of variables formula then the resulted Jacobian matrix is of $(n+1)\times(n+1)$ dimensions. Our next step is to reduce it to $n\times n$ and write the resulted equation in $\mathcal{U}\subset\mathbb{R}^n$.

We follow the approach introduced in [9]. Notice that in our definition of stretch function (see Proposition 3.3 below) $t>0$ whereas in [9] the stretch function may change its sign. Let $Z:\mathcal{U}\longrightarrow\mathcal{V}$ be C^2 smooth. Then for any $i, 1\le i\le n$ the vectors $\partial_i Z(x)\in\mathbb{T}_Z\Sigma$, where $\mathbb{T}_Z\Sigma$ is the tangent space of the receiver Σ at $Z\in\Sigma$. Moreover, the volume of the $n+1$ dimensional parallelepiped spanned by $(\partial_1 Z,\partial_2 Z,\dots,\partial_n Z,\widetilde{\gamma})$ is

$$\begin{vmatrix} Z_1^1 & \cdots & Z_n^1 & \widetilde{\gamma}_1 \\ \vdots & \ddots & \vdots & \vdots \\ Z_1^n & \cdots & Z_n^n & \widetilde{\gamma}_n \\ Z_1^{n+1} & \cdots & Z_n^{n+1} & \widetilde{\gamma}_{n+1} \end{vmatrix}.$$

Here $\widetilde{\gamma}$ is the normal of Σ at Z. We use this observation to prove the following

Lemma 3.2. *Let us denote $Z(x)=(z(x),Z^{n+1}(x))$ and assume that the receiver $\Sigma=\{X\in\mathbb{R}^{n+1}:\psi(X)=0\}$ for a given smooth function $\psi:\mathbb{R}^{n+1}\longrightarrow\mathbb{R}$. Then the following formula is true*

$$\begin{aligned} J=\frac{dS_{\mathcal{V}}}{dS_{\mathcal{U}}} &= \begin{vmatrix} Z_1^1 & \cdots & Z_n^1 & \widetilde{\gamma}_1 \\ \vdots & \ddots & \vdots & \vdots \\ Z_1^n & \cdots & Z_n^n & \widetilde{\gamma}_n \\ Z_1^{n+1} & \cdots & Z_n^{n+1} & \widetilde{\gamma}_{n+1} \end{vmatrix} \\ &= -\frac{|\nabla\psi|}{\psi_{n+1}}\det Dz. \end{aligned} \tag{3.2}$$

Here $S_{\mathcal{U}}$ (resp. $S_{\mathcal{V}}$) is the surface area on $\mathcal{U}$ (resp. $\mathcal{V}\subset\Sigma$).

Proof. Because of the observation above about the $n+1$ dimensional parallelepipeds we only need to prove the last equality (3.2). Differentiating the equality $\psi(Z)=0$ by x^i we find that $\partial_i Z^{n+1}=-\frac{1}{\partial_{n+1}\psi}\sum_{k=1}^{n}\partial_i z^k\partial_{z_k}\psi$.

Using this identity we multiply jth row by $\partial_{z_j}\psi$ and subtract it from $(n+1)$st row we get

$$\det\begin{vmatrix} Z_1^1 & \cdots & Z_n^1 & \widetilde{\gamma}_1 \\ \vdots & \ddots & \vdots & \vdots \\ Z_1^n & \cdots & Z_n^n & \widetilde{\gamma}_n \\ Z_1^{n+1} & \cdots & Z_n^{n+1} & \widetilde{\gamma}_{n+1} \end{vmatrix} = -\frac{1}{\psi_{n+1}}\det\begin{vmatrix} Z_1^1 & \cdots & Z_n^1 & \widetilde{\gamma}_1 \\ \vdots & \ddots & \vdots & \vdots \\ Z_1^n & \cdots & Z_n^n & \widetilde{\gamma}_n \\ \sum\limits_{k=1}^n \partial_1 z^k \partial_{z_k}\psi & \cdots & \sum\limits_{k=1}^n \partial_n z^k \partial_{z_k}\psi & -\psi_{n+1}\widetilde{\gamma}_{n+1} \end{vmatrix}$$

$$= -\frac{1}{\psi_{n+1}}\det\begin{vmatrix} Z_1^1 & \cdots & Z_n^1 & \widetilde{\gamma}_1 \\ \vdots & \ddots & \vdots & \vdots \\ Z_1^n & \cdots & Z_n^n & \widetilde{\gamma}_n \\ 0 & \cdots & 0 & -\sum\limits_{k=1}^{n+1}\psi_k\widetilde{\gamma}_k \end{vmatrix}$$

and the result follows if we note that $\widetilde{\gamma} = \frac{\nabla\psi}{|\nabla\psi|}$. ■

3.2. **Main formulae.** Now we are ready to derive the main equations manifesting the conservation of energy.

Proposition 3.3. *Let $u \in C^2$ then*

$$Y(x) = \frac{1}{1+|Du|^2}(2Du, |Du|^2 - 1), \tag{3.3}$$

$$Z(x) = x + u(x)e_{n+1} + tY(x), \tag{3.4}$$

$$J = \frac{|\nabla\psi|}{\langle\nabla\psi, Y\rangle}\det\left[\mathrm{Id} + \frac{2t}{1+|Du|^2}D^2u\right], \tag{3.5}$$

where Y is the unit direction of the reflected ray emanated from x and $t = t(x,u,Du)$ is the stretch function determined from the implicit relation

$$\psi(x + u(x)e_{n+1} + tY(x)) = 0. \tag{3.6}$$

Proof. Let Y be the unit direction of the reflected ray. According to the reflection law

$$-e_{n+1} + Y = 2\gamma\langle -e_{n+1}, \gamma\rangle \tag{3.7}$$

where $\gamma = \frac{1}{\sqrt{1+|Du|^2}}(Du, -1)$ is the unit normal of Γ_u at $x \in \mathcal{U}$. Thus we have

$$\gamma = \left(\frac{D_1u}{\sqrt{1+|Du|^2}}, \frac{D_2u}{\sqrt{1+|Du|^2}}, \dots, \frac{D_nu}{\sqrt{1+|Du|^2}}, -\frac{1}{\sqrt{1+|Du|^2}}\right)$$

and hence from (3.7)

$$Y = \left(\frac{2D_1u}{1+|Du|^2}, \frac{2D_2u}{1+|Du|^2}, \dots, \frac{2D_nu}{1+|Du|^2}, \frac{|Du|^2-1}{1+|Du|^2}\right). \tag{3.8}$$

Thus for $y = \widehat{Y}$, the projection of Y onto Π, we obtain

$$\begin{aligned} D_jy^i &= \frac{2D_{ij}u}{1+|Du|^2} - 4\frac{D_iuD_muD_{mj}u}{(1+Du|^2)^2} = \\ &= \frac{2}{1+|Du|^2}\left\{\delta_{im} - 2\frac{D_iuD_mu}{1+|Du|^2}\right\}D_{mj}u. \end{aligned} \tag{3.9}$$

In order to prove (3.5) we use Lemma 3.2 and (3.9). Thus we want to compute the determinant of $n\times n$ matrix Dz, where $Z = (z, Z^{n+1})$ and Z is given by (3.4).

Taking the x^j derivative of z we get

$$\begin{aligned} z^i_j &= (x^i + ty^i)_{x^j} \\ &= \delta_{ij} + t_j y^i + t y^i_j. \end{aligned} \tag{3.10}$$

Next, we want to express t_j in terms of t, u, Du. Differentiating $\psi(x + ty, u + ty^{n+1}) = 0$ with respect to x_j we obtain

$$\sum_{k=1}^{n} \psi_k(\delta_{kj} + t_j y^k + t y^k_j) + \psi_{n+1}(u_j + t_j y^{n+1} + t y^{n+1}_j) = 0. \tag{3.11}$$

Using the fact that $[y^{n+1}]^2 = 1 - |y|^2$, we infer $y^{n+1}_j = -\frac{y^l y^l_j}{y^{n+1}}$, which together with (3.11) yields

$$\begin{aligned} t_j &= -\frac{1}{\langle \nabla\psi, Y\rangle}\left\{\psi_k\delta_{kj} + \psi_{n+1}u_j + t(\psi_k y^k_j + \psi_{n+1}y^{n+1}_j)\right\} \\ &= -\frac{1}{\langle \nabla\psi, Y\rangle}\left\{\psi_k\delta_{kj} + \psi_{n+1}u_j + t\left(\psi_k y^k_j - \psi_{n+1}\frac{y^l y^l_j}{y^{n+1}}\right)\right\} \\ &= -\frac{1}{\langle \nabla\psi, Y\rangle}\left\{\psi_k\delta_{kj} + \psi_{n+1}u_j + t\left(\psi_k - \psi_{n+1}\frac{y^k}{y^{n+1}}\right)y^k_j\right\}. \end{aligned} \tag{3.12}$$

Combining (3.10) and (3.12) we see that

$$\begin{aligned} z^i_j &= \delta_{ij} - \frac{1}{\langle \nabla\psi, Y\rangle}\left\{\psi_k\delta_{kj} + \psi_{n+1}u_j + t\left(\psi_k - \psi_{n+1}\frac{y^k}{y^{n+1}}\right)y^k_j\right\}y^i + ty^i_j \\ &= \underbrace{\delta_{ij} - \frac{1}{\langle \nabla\psi, Y\rangle}y^i\left[\psi_k\delta_{kj} + \psi_{n+1}u_j\right]}_{\beta_1} + t\underbrace{\left[\delta_{ik} - \frac{1}{\langle \nabla\psi, Y\rangle}y^i(\psi_k - \psi_{n+1}\frac{y^k}{y^{n+1}})\right]}_{\beta_2}y^k_j. \end{aligned}$$

The matrix on the right hand side can be further simplified. Using the notation $\widehat{\nabla}\psi = (\partial_1\psi, \dots, \partial_n\psi, 0)$ we have the following intrinsic form for the matrix $\beta_2 Dy$,

$$\begin{aligned} \beta_2 Dy &= \left[\mathrm{Id} - \frac{1}{\langle \nabla\psi, Y\rangle}y \otimes (\widehat{\nabla}\psi - \psi_{n+1}\frac{y}{y^{n+1}})\right]\frac{2}{1+|Du|^2}\left[\mathrm{Id} - 2\frac{Du\otimes Du}{1+|Du|^2}\right]D^2u \\ &= \frac{2}{1+|Du|^2}\mu D^2 u \end{aligned}$$

where μ is the matrix

$$\begin{aligned} \mu &= \beta_2\left[\mathrm{Id} - 2\frac{Du\otimes Du}{1+|Du|^2}\right] \\ &= \left[\mathrm{Id} - \frac{1}{\langle \nabla\psi, Y\rangle}y \otimes (\partial\psi - \psi_{n+1}\frac{y}{y^{n+1}})\right][\mathrm{Id} - y\otimes Du] \\ &= \mathrm{Id} - \frac{1}{\langle \nabla\psi, Y\rangle}y\otimes\left[\widehat{\nabla}\psi - \frac{\psi_{n+1}}{y^{n+1}}y + (\nabla\psi\cdot Y)Du - (\partial\psi\cdot y - \frac{\psi_{n+1}}{y^{n+1}}|y|^2)Du\right] \\ &= \mathrm{Id} - \frac{1}{\langle \nabla\psi, Y\rangle}y\otimes\left[\widehat{\nabla}\psi - \frac{\psi_{n+1}}{y^{n+1}}y + \psi_{n+1}y^{n+1}Du + \frac{\psi_{n+1}}{y^{n+1}}|y|^2 Du\right] \\ &= \mathrm{Id} - \frac{1}{\langle \nabla\psi, Y\rangle}y\otimes\left[\widehat{\nabla}\psi - \frac{\psi_{n+1}}{y^{n+1}}y + \frac{\psi_{n+1}}{y^{n+1}}Du\right] \\ &= \mathrm{Id} - \frac{1}{\langle \nabla\psi, Y\rangle}y\otimes\left[\widehat{\nabla}\psi + \psi_{n+1}Du\right] \\ &= \beta_1. \end{aligned}$$

Returning to Dz we get

$$\begin{aligned} Dz &= \mu\left[\mathrm{Id}+\frac{2t}{1+|Du|^2}D^2u\right],\\ \mu &= \mathrm{Id}-\frac{1}{\langle\nabla\psi,Y\rangle}y\otimes\left[\widehat{\nabla}\psi+\psi_{n+1}Du\right]. \end{aligned}$$

By Lemma 3.1

$$\det\mu = 1-\frac{\langle y,\widehat{\nabla}\psi\rangle+\psi_{n+1}\langle y,Du\rangle}{\langle\nabla\psi,Y\rangle}=\psi_{n+1}\frac{y^{n+1}-\langle y,Du\rangle}{\langle\nabla\psi,Y\rangle}.$$

From (3.3) we conclude that $y^{n+1}-\langle y,Du\rangle=-1$. Thus $\det\mu=-\frac{\psi_{n+1}}{\langle\nabla\psi,Y\rangle}$ because $\langle y,Du\rangle=\frac{2|Du|^2}{1+|Du|^2}$ and hence from (3.2) we obtain

$$\begin{aligned} J &= -\frac{|\nabla\psi|}{\psi_{n+1}}\det\mu\det\left[\mathrm{Id}+\frac{2t}{1+|Du|^2}D^2u\right]\\ &= \frac{|\nabla\psi|}{\langle\nabla\psi,Y\rangle}\det\left[\mathrm{Id}+\frac{2t}{1+|Du|^2}D^2u\right]. \end{aligned}$$

Now the proof is complete. ■

4. Convexity of G

4.1. **Non-Degeneracy.** In this section we examine the equation (3.5) manifesting the energy balance condition for a perfect reflector, see (**P**). Hence, making use of change of variables formula and computing the Jacobian, see Lemma 3.2, we infer

$$1=\frac{gdS_{\mathcal{V}}}{fdS_{\mathcal{U}}}=\frac{g}{f}\frac{|\nabla\psi|}{|\langle\nabla\psi,Y\rangle|}\left|\det\left[\mathrm{Id}+\frac{2t}{1+|Du|^2}D^2u\right]\right|$$

or equivalently

$$\left|\det\left[\mathrm{Id}+\frac{2t}{1+|Du|^2}D^2u\right]\right|=\frac{f}{g}\frac{|\langle\nabla\psi,Y\rangle|}{|\nabla\psi|}. \tag{4.1}$$

Note that the matrix

$$\mathcal{W}=-\mathrm{Id}\frac{1+|Du|^2}{2t}-D^2u \tag{4.2}$$

is identically zero for any paraboloid $P\in\mathbb{P}_L(\mathcal{U},\mathcal{V})$, see Section 4.4. Thus, for admissible $u\in C^2$ we have $\mathcal{W}(u)\geq 0$. Hence (4.1) is degenerate elliptic.

Further, we impose the following non-degeneracy condition

$$\langle\nabla\psi,Y\rangle\neq 0, \tag{4.3}$$

say, $\langle\nabla\psi,Y\rangle>0$ (see (1.8)). In particular this condition implies that $\nabla\psi\neq 0$. Note that $|\langle\nabla\psi,Y\rangle|\neq 0$ has a simple geometric meaning, namely it prevents the reflected rays from approaching Σ tangentially which would make impossible to detect the scattered data on Σ. We recall the definition of regularity domain $\mathcal{D}$, see (1.7)-(1.10).

Thus if (1.8) holds true then we can write (4.1) as

$$\det\mathcal{W}=\frac{f}{\eta g\circ Z}, \tag{4.4}$$

where

$$\eta=\left[\frac{2t}{a}\right]^n\frac{\langle\nabla\psi,Y\rangle}{|\nabla\psi|},\quad a=|Du|^2+1. \tag{4.5}$$

4.2. **Convexity of G.** In this section we formulate a necessary condition for the regularity of weak solution. It is called the A3 condition and was introduced in [14] with regard to the optimal mass transport problems. Recall that the if u is the potential function in Kontorovich's formulation then formally u solves the equation

$$\det[c_{x^ix^j}(x,y) - D^2u(x)] = h(x)$$

where $c(x,y)$ is the cost function and h is determined from the data, see [20]. The A3 condition, in this context, takes the from

$$\partial_{p_kp_l}c_{x^ix^j}(x,y)\xi_i\xi_j\eta_k\eta_l \geq c_0|\xi|^2|\eta|^2$$

where c_0 is a positive constant, $\xi \perp \eta \in \mathbb{R}^n$ and $y = y(x,p)$ is the transport mapping. Our equation is not of variational form (see Introduction) and cannot be formulated as a mass transport problem. However this condition still plays a crucial role in the regularity theory of weak A-type solutions of (**P**).

4.3. **The general case.** Let u be a C^2 solution to

$$\det\left[-\frac{1}{2}G\mathrm{Id} - D^2u\right] = h(x,u,Du), \qquad G(x,u(x),Du(x)) = \frac{a}{t} \tag{4.6}$$

where $a = |Du|^2+1$, $h = \frac{f}{\eta g\circ Z}$ (see (4.5)) and t is the stretch function. In what follows we write $G = \dfrac{a}{t}$ for short and use the dummy variable $p = Du$ for the gradient of u. We start with computing the first and second order partial derivatives of G as follows

$$\begin{aligned}
\partial_{p_k}G &= -\frac{t_{p_k}}{t^2}a + \frac{1}{t}a_{p_k},\\
\partial_{p_kp_l}G &= \left(\frac{2t_{p_k}t_{p_l}}{t^3} - \frac{t_{p_kp_l}}{t^2}\right)a -\\
&\quad -\frac{t_{p_k}}{t^2}a_{p_l} - \frac{t_{p_l}}{t^2}a_{p_k}\\
&\quad +\frac{1}{t}a_{p_lp_k}.
\end{aligned}$$

Next, we compute the partial derivatives of t. Recall that by (3.4) $Z(x) = x + ue_{n+1} + tY$, where Y is the unit direction of the reflected ray. Let $\psi : \mathbb{R}^{n+1} \to \mathbb{R}$ be the defining function of Σ, i.e. $\Sigma = \{X \in \mathbb{R}^{n+1} : \psi(X) = 0\}$. Since $Z(x) \in \Sigma$ it follows that $\psi(Z(x)) = 0$. Differentiating $\psi(Z(x)) = 0$ with respect to p_k we get

$$\frac{t_{p_k}}{t} = -\frac{\sum_{i=1}^{n+1}\psi_{z_i}Y^i_{p_k}}{\langle\nabla\psi, Y\rangle}. \tag{4.7}$$

After differentiating (4.7) by p_l we obtain

$$\begin{aligned}
\partial_{p_l}\left[\frac{t_{p_k}}{t}\right] &= \frac{t_{p_kp_l}}{t} - \frac{t_{p_k}t_{p_l}}{t^2}\\
&= -\frac{1}{\langle\nabla\psi, Y\rangle}\left[\sum_{i,j=1}^{n+1}\psi_{z^iz^j}Z^j_{p_l}Y^i_{p_k} + \sum_{i=1}^{n+1}\psi_{z^i}Y^i_{p_kp_l}\right]\\
&\quad + \frac{\sum_{i=1}^{n+1}\psi_{z^i}Y^i_{p_k}}{\langle\nabla\psi, Y\rangle^2}\left(\sum_{i,j=1}^{n+1}\psi_{z^iz^j}Z^j_{p_l}Y^i + \sum_{i=1}^{n+1}\psi_{z^i}Y^i_{p_l}\right)\\
&= -\frac{1}{\langle\nabla\psi, Y\rangle}\left[\sum_{i,j=1}^{n+1}\psi_{z^iz^j}Z^j_{p_l}Y^i_{p_k} + \frac{t_{p_k}}{t}\sum_{i,j=1}^{n+1}\psi_{z^iz^j}Z^j_{p_l}Y^i\right]\\
&\quad - \frac{1}{\langle\nabla\psi, Y\rangle}\sum_{i=1}^{n+1}\psi_{z^i}Y^i_{p_kp_l} + \frac{t_{p_k}t_{p_l}}{t^2}
\end{aligned}$$

where to get the last line we used (4.7). Therefore we obtain

$$\frac{2t_{p_k}t_{p_l}}{t^3} - \frac{t_{p_kp_l}}{t^2} = \frac{1}{t\langle\nabla\psi, Y\rangle}\left[\frac{1}{t}\nabla^2\psi Z_{p_k}Z_{p_l} + \nabla\psi Y_{p_kp_l}\right].$$

Returning to $\partial_{p_kp_l}G$ we get

$$\begin{aligned}
\partial_{p_kp_l}G &= \frac{a}{t\langle\nabla\psi, Y\rangle}\left[\frac{1}{t}\nabla^2\psi Z_{p_k}Z_{p_l} + \nabla\psi Y_{p_kp_l}\right]\\
&\quad - \frac{t_{p_k}}{t^2}a_{p_l} - \frac{t_{p_l}}{t^2}a_{p_k}\\
&\quad + \frac{1}{t}a_{p_lp_k}.
\end{aligned} \tag{4.8}$$

In order to simplify further this identity we utilize (3.3) and rewrite it as $aY = (2p, a-2)$. Hence, we have

$$\begin{aligned}
a_{p_k}Y + aY_{p_k} &= 2e_k + e_{n+1}a_{p_k}\\
a_{p_kp_l}Y + a_{p_k}Y_{p_l} + a_{p_l}Y_{p_k} + aY_{p_kp_l} &= e_{n+1}a_{p_kp_l}.
\end{aligned}$$

Taking the inner product with $\nabla\psi$ we obtain

$$a_{p_kp_l}\langle\nabla\psi, Y\rangle + a_{p_k}\langle\nabla\psi, Y_{p_l}\rangle + a_{p_l}\langle\nabla\psi, Y_{p_k}\rangle + a\langle\nabla\psi, Y_{p_kp_l}\rangle = \psi_{n+1}a_{p_kp_l}$$

and this in view of (4.7) yields

$$a_{p_kp_l} - a_{p_k}\frac{t_{p_l}}{t} - a_{p_l}\frac{t_{p_k}}{t} = \frac{\psi_{n+1}a_{p_kp_l}}{\langle\nabla\psi, Y\rangle} - \frac{a\langle\nabla\psi, Y_{p_kp_l}\rangle}{\langle\nabla\psi, Y\rangle}. \tag{4.9}$$

Substituting the last identity into (4.8) we see that

$$\partial_{p_kp_l}G = \frac{a}{t^2\langle\nabla\psi, Y\rangle}\nabla^2\psi Z_{p_k}Z_{p_l} + \frac{\psi_{n+1}a_{p_kp_l}}{t\langle\nabla\psi, Y\rangle} \tag{4.10}$$

and after recalling that by definition $a = |Du|^2 + 1$ we finally obtain

$$\partial_{p_kp_l}G = \frac{a}{t^2\langle\nabla\psi, Y\rangle}\nabla^2\psi Z_{p_k}Z_{p_l} + \frac{2\psi_{n+1}}{t\langle\nabla\psi, Y\rangle}\delta_{kl}. \tag{4.11}$$

In what follows we require

$$\partial_{p_k p_l} G \xi_k \xi_l \leq -c_0 |\xi|^2, \qquad c_0 > 0 \tag{4.12}$$

which is the analogous of (1.11) for the reflector problem (**P**).

4.4. **The case of planar receiver** $\psi(Z) = \langle Z, n_0 \rangle - d_0$**.** If Σ is the hyperplane $\langle Z, n_0 \rangle = d_0$ then one can readily verify that the (1.11) is satisfied. In this spacial case, we have from (3.4)

$$t = \frac{d_0 - \langle x + u e_{n+1}, n_0 \rangle}{\langle Y, n_0 \rangle},$$

so for $u(x) = P(x) = \frac{\sigma}{2} + Z^{n+1} - \frac{1}{2\sigma}|x - z|^2$ to show that $\mathcal{W} = 0$ it is enough to check that $\sigma(1 + |DP|^2) - 2t = 0$.

From $Z \in \Sigma$ it follows that $d_0 = \langle Z, n_0 \rangle$ and hence we have that

$$t = \frac{\langle Z, n_0 \rangle - \langle x + P(x) e_{n+1}, n_0 \rangle}{\langle Y(x), n_0 \rangle}. \tag{4.13}$$

On the other hand

$$\begin{aligned} Z - x - P(x) e_{n+1} &= Z - x - e_{n+1} \left(\frac{\sigma}{2} + Z^{n+1} - \frac{1}{2\sigma} |x - z|^2 \right) \\ &= z - x - e_{n+1} \frac{\sigma}{2} \left(1 - \frac{1}{\sigma^2} |x - z|^2 \right) \\ &= z - x - e_{n+1} \frac{\sigma}{2} \left(1 - |DP|^2 \right). \end{aligned} \tag{4.14}$$

Furthermore, we have from (3.3)

$$\begin{aligned} Y &= \frac{1}{1 + |DP|^2} \left[-\frac{2}{\sigma}(x - z) + e_{n+1} \left(\frac{|x - z|^2}{\sigma^2} - 1 \right) \right] \\ &= \frac{1}{1 + |DP|^2} \frac{2}{\sigma} \left[(z - x) - e_{n+1} \frac{\sigma}{2} \left(1 - |DP|^2 \right) \right]. \end{aligned} \tag{4.15}$$

Plugging in (4.14) and (4.15) into (4.13) yields

$$t = \frac{\sigma(1 + |DP|^2)}{2}. \tag{4.16}$$

Next, we verify the condition (1.11) for the linear ψ

$$\partial_{p_k p_l} G = \frac{2 n_0^{n+1}}{t \langle n_0, Y \rangle} \delta_{kl}.$$

In particular if $n_0 = e_{n+1}$ and $\psi = z^{n+1} - c, c \in \mathbb{R}$ then $y^{n+1} \leq 0$ (see 3.3 and Figure 2) and hence for the horizontal planar receiver the condition (1.11) does hold.

Another example of receiver is the sheet of hyperboloid of revolution $\varphi(z) = \ell_0 + \frac{a}{b}\sqrt{b^2 + |z|^2}$ with $a, b > 0$. Then $\nabla^2 \varphi = \frac{a}{b\sqrt{b^2 + |z|^2}} \left(\mathrm{Id} + \frac{x \otimes x}{b^2 + |x|^2} \right)$ and hence (1.10) is satisfied.

Remark 4.1. *The stretch function in the paper differs from that of introduced in* [9, 10]*, namely in this paper* $t > 0$ *whereas in* [9, 10] *the stretch function may change its sign. The present derivation of equation is shorter and simpler than in the early version of the paper* [8].

4.5. **Refined (4.12) condition.** The condition $\partial_{p_k p_l} G < 0$ in (4.12) can be reformulated in more geometric way if one uses the second fundamental form of Σ. Note that it is enough to consider

$$G^*_{lk} = \left[\frac{a}{t}\nabla^2\psi Z_{p_k} Z_{p_l} + 2\psi_{n+1}\delta_{kl}\right] \tag{4.17}$$

since

$$\partial_{p_k p_l} G = \frac{a}{t^2\langle\nabla\psi, Y\rangle}\nabla^2\psi Z_{p_k} Z_{p_l} + \frac{2\psi_{n+1}}{t\langle\nabla\psi, Y\rangle}\delta_{kl}. = \frac{1}{t(\nabla\psi\cdot\xi)} G^*_{kl}.$$

Let us fix $Z_0 \in \Sigma$ and $\mathbb{T}_{Z_0}\Sigma$ denote the tangent space of Σ at Z_0. If $x_0 \in \mathcal{U}$ and $Z(x_0) = Z_0$ then $Z_{p_k}(x_0) \in \mathbb{T}_{Z_0}\Sigma$ since $\langle\nabla\psi(Z_0), Z_{p_k}(x_0)\rangle = t\langle\nabla\psi, Y_{p_k}\rangle + t_{p_k}\langle\nabla\psi, Y\rangle = 0$ thanks to (4.7).

Next, we want to show that $Y, Y_{p_1}, \dots, Y_{p_n}$ are mutually orthogonal. From (3.3) we have $Y = \frac{1}{a}(2p, a-2), |Y| = 1$ where $a = p^2 + 1$. Thus $Y \perp Y_{p_k}, k = 1, \dots, n$. Moreover

$$\begin{aligned} Y_{p_k} &= -\frac{2p_k}{a^2}(2p, a-2) + \frac{2}{a}(e_k + p_k e_{n+1}) = \\ &= -\frac{2p_k}{a}Y + \frac{2}{a}(e_k + p_k e_{n+1}). \end{aligned} \tag{4.18}$$

Therefore

$$\begin{aligned} \langle Y_{p_k}, Y_{p_l}\rangle &= \left(\frac{2}{a}\right)^2 \langle -p_p Y + e_k + p_k e_{n+1}, -p_l Y + e_l + p_l e_{n+1}\rangle \\ &= \left(\frac{2}{a}\right)^2 \left[-\frac{2p_k p_l}{a} + \delta_{kl} - p_k p_l \frac{a-2}{a} + p_k p_l\right] = \\ &= \left(\frac{2}{a}\right)^2 \delta_{kl}. \end{aligned} \tag{4.19}$$

In particular, from (4.19) we get $|Y_{p_k}| = \frac{2}{a}$.

To compute the second derivatives of ψ, we consider a new coordinate system $\widehat{x}_1, \dots, \widehat{x}_n, \widehat{x}_{n+1}$ near Z_0, with x_{n+1} having direction $-Y$. Suppose that near Z_0, in $\widehat{x}_1, \dots, \widehat{x}_n, \widehat{x}_{n+1}$ coordinate system Σ admits a representation of the form $\widehat{x}_{n+1} = \varphi(\widehat{x}_1, \dots, \widehat{x}_n)$. Recall that the second fundamental form of Σ is

$$\mathrm{II} = \frac{\partial^2_{\widehat{x}_i, \widehat{x}_j}\varphi}{\sqrt{1 + |D\varphi|^2}}, \qquad i, j = 1, \dots, n \tag{4.20}$$

if we choose the normal of Σ at Z_0 to be $\frac{(-D_{\widehat{x}_1}\varphi, \dots, -D_{\widehat{x}_n}\varphi, 1)}{\sqrt{1+|D\varphi|^2}}, D\varphi = (D_{\widehat{x}_1}\varphi, \dots, D_{\widehat{x}_n}\varphi, 0)$.

From now on we denote $\widetilde{\psi}(Z) = Z^{n+1} - \varphi(z)$ and assume that near Z_0, Σ is given by the equation $\widetilde{\psi} = 0$. Then

$$\nabla^2\widetilde{\psi} = -\begin{vmatrix} \varphi_{11} & \cdots & \varphi_{1n} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ \varphi_{n1} & \cdots & \varphi_{nn} & 0 \\ 0 & \cdots & 0 & 0 \end{vmatrix}. \tag{4.21}$$

Hence for $Z = x + ue_{n+1} + tY$ we have

$$\begin{aligned} \nabla^2\widetilde{\psi} Z_{p_k} Z_{p_l} &= -\nabla^2\varphi(tY_{p_k} + t_{p_k}Y)(tY_{p_l} + t_{p_l}Y) \\ &= -t^2\nabla^2\varphi Y_{p_k} Y_{p_l} \\ &= -\left(\frac{2t}{a}\right)^2 \partial^2_{p_k p_l}\varphi \\ &= -\left(\frac{2t}{a}\right)^2 \sqrt{1 + |D\varphi|^2}\ \mathrm{II}. \end{aligned} \tag{4.22}$$

Since Y is a unit vector and (4.19) holds true we may assume that Y_{p_k} has the direction of $\widehat{x}_k, k = 1, 2, \dots n$.

Combining these formulae and noting that the second fundamental form of Σ is $\mathrm{II} = \frac{1}{\sqrt{1+|D\varphi|^2}}\partial_k\partial_l\varphi$ we arrive at

$$G^*_{lk} = 2\sqrt{1+|D\varphi|^2}\left[-\frac{2t}{a}\mathrm{II} + \delta_{kl}\cos\theta\right] \tag{4.23}$$

where θ is the angle between e_{n+1} and $\nabla\psi$. Summarizing we see that (4.12) is equivalent to (1.10) condition in the definition of regularity domain $\mathcal{D}$.

Remark 4.2. *As one can see from Figure 2 the reflected rays may converge to focus F from either side of the focal plane. The inequalities $y^{n+1} > 0$ or $y^{n+1} < 0$ determine the side of the approach to F. Consequently we may have that for a chosen orientation of Σ the reflected rays strike the target domain from either side making it harder to verify the condition (1.10) (recall that (1.10) was derived under assumption $y^{n+1} < 0$ and for fixed orientation of Σ). The mixed striking can be ruled out if we assume that Σ is visible from any supporting paraboloid's focal plane. In particular this is true if Σ is a graph over Π.*

5. Local C^2 estimates

The proof of C^2 a priori estimate is similar to that of far-field problem with point source, see [9, 10]. The main idea goes back to [21] and [14], where a general method was introduced to prove such estimates for the smooth solution. We give a concise proof here for the sake of completeness.

Proposition 5.1. *Let $u \in C^4$ be a classical solution of (1.5) and matrix $\mathcal{W} > 0$. Assume that right hand side of (4.4) is $C^{1,1}$ regular and strictly positive. Then under assumption (1.11) we have C^2 a priori local estimate for the second order derivatives, i.e. for any subdomain $\mathcal{U}' \subset\subset \mathcal{U}$ there is $C > 0$ depending of dist$(\mathcal{U},\mathcal{U}')$ such that*

$$\sup_{\mathcal{U}'} |D^2u| \le C.$$

Proof. Denote $w = -u$ and $\mathcal{W} = D^2w - \frac{1}{2}G\delta_{ij}$. Let $F[\mathcal{W}] = [\det\mathcal{W}]^{\frac{1}{n}}$, $\mathcal{W} = \{\mathcal{W}_{ij}\}$ and let $\bar{h} = h^{\frac{1}{n}}$, where $h = f/(\eta g \circ Z)$ and η is given by (4.5). Then (4.4) takes the following form

$$F[\mathcal{W}] = \bar{h}.$$

Let us differentiate this equation with respect to x_k twice. Denoting $F^{ij} = \frac{\partial F}{\partial \mathcal{W}_{ij}}$ we obtain

$$F^{ij}\mathcal{W}_{ij,kk} = -\frac{\partial^2 \log\det\mathcal{W}}{\partial\mathcal{W}_{ij}\partial\mathcal{W}_{rs}}\mathcal{W}_{ij,k}\mathcal{W}_{rs,k} + D_{kk}\bar{h} \ge D_{kk}\bar{h}, \qquad k = 1, \dots, n \tag{5.1}$$

where to get the last inequality we used the concavity of $\log\det\mathcal{W}$. Note that

$$F^{ij} = \frac{\partial F}{\partial\mathcal{W}_{ij}} = \frac{\mathrm{cof}\mathcal{W}}{\det\mathcal{W}} = [\mathcal{W}]^{-1} \tag{5.2}$$

where cof$\mathcal{W}$ is the cofactor matrix of $\mathcal{W}$.

For $\xi \in \mathbb{S}^n$ consider the auxiliary function $z(x,\xi) = \rho^2 \sum\limits_{ij=1}^{n} \xi_i\xi_j\mathcal{W}_{ij}, x \in B_1(0)$ where ρ is the standard cut off function of $B_{1/2}(0)$. Assume that $\sup\limits_{\mathbb{S}^n\times B_1(0)} z(x,\xi)$ is attained at $\bar{x}$ and $\xi = e_1$. By a rotation of the coordinate

axis we may assume that $\{\mathcal{W}_{ij}\}$ is diagonal at $\bar{x}$ and $\mathcal{W}_{11} \geq \mathcal{W}_{22} \geq \cdots \geq \mathcal{W}_{nn}$. At $\bar{x}$ we have

$$(\log z)_i = 2\frac{\rho_i}{\rho} + \frac{\mathcal{W}_{11,i}}{\mathcal{W}_{11}} = 0, \tag{5.3}$$

$$\begin{aligned}(\log z)_{ij} &= 2\frac{\rho_{ij}}{\rho} - 2\frac{\rho_i\rho_j}{\rho^2} + \frac{\mathcal{W}_{11,ij}}{\mathcal{W}_{11}} - \frac{\mathcal{W}_{11,i}\mathcal{W}_{11,j}}{(\mathcal{W}_{11})^2} \\ &= 2\frac{\rho_{ij}}{\rho} - 6\frac{\rho_i\rho_j}{\rho^2} + \frac{\mathcal{W}_{11,ij}}{\mathcal{W}_{11}} \leq 0.\end{aligned} \tag{5.4}$$

Next we have that

$$\begin{aligned}\mathcal{W}_{11,i} &= \partial_{x_i}\mathcal{W}_{11} = w_{11i} - \frac{1}{2}(G_{x_i} + G_u u_i + G_{p_k}u_{ki}), \\ \mathcal{W}_{11,ii} &= w_{11ii} - \frac{1}{2}G_{p_ip_i}w_{ii}^2 - \frac{1}{2}G_{p_k}w_{kii} + O(1+w_{11}).\end{aligned} \tag{5.5}$$

Consequently we find that

$$\mathcal{W}_{ii,11} = w_{11ii} - \frac{1}{2}(G_{p_1p_1}w_{11}^2 + G_{p_k}w_{k11}) + O(1+w_{11}). \tag{5.6}$$

At $\bar{x}$ $\mathcal{W}$ is diagonal, in particular so is D^2w, thus from (5.5) and (5.6) we infer

$$\begin{aligned}F^{ii}\mathcal{W}_{11,ii} - F^{ii}\mathcal{W}_{ii,11} &= F^{ii}w_{11ii} - \frac{1}{2}F^{ii}G_{p_ip_i}w_{ii}^2 - \frac{1}{2}F^{ii}G_{p_k}w_{kii} - \\ &\quad -F^{ii}w_{11ii} + \frac{1}{2}F^{ii}G_{p_1p_1}w_{11}^2 + \frac{1}{2}F^{ii}G_{p_k}w_{k11} + \\ &\quad +F^{ii}O(1+w_{11}) = \\ &= \frac{1}{2}F^{ii}G_{p_1p_1}w_{11}^2 - \frac{1}{2}F^{ii}G_{p_ip_i}w_{ii}^2 + \\ &\quad +\frac{1}{2}F^{ii}G_{p_k}w_{k11} - \frac{1}{2}F^{ii}G_{p_k}w_{kii} + \\ &\quad +F^{ii}O(1+w_{11}).\end{aligned} \tag{5.7}$$

It follows from the identity (5.3) that $|w_{11k}| \leq C(1+\mathcal{W}_{11})/\rho$ at $\bar{x}$ therefore

$$F^{ii}G_{p_k}w_{k11} \leq F^{ii}G_{p_k}C\frac{1+\mathcal{W}_{11}}{\rho} \leq C\mathrm{Tr}F^{ij}\frac{1+\mathcal{W}_{11}}{\rho}. \tag{5.8}$$

As for the the quadratic term we estimate

$$\begin{aligned}F^{ii}w_{ii}^2 &= F^{ii}(\mathcal{W}_{ii} + \frac{1}{2}G)^2 \\ &= F^{ii}\left((\mathcal{W}_{ii})^2 + \mathcal{W}_{ii}G + \frac{1}{4}G^2\right) \\ &= \frac{1}{\det\mathcal{W}}\frac{\partial\det\mathcal{W}}{\partial\mathcal{W}_{ii}}\left((\mathcal{W}_{ii})^2 + \mathcal{W}_{ii}G + \frac{1}{4}G^2\right) \\ &= O(1 + \mathrm{Tr}F^{ij} + \mathrm{Tr}\mathcal{W}_{ij})\end{aligned} \tag{5.9}$$

where to get the last line we used (5.2).

Utilizing the estimates (5.8) and (5.9) we get from (5.7)

$$\begin{aligned} F^{ii}\mathcal{W}_{11,ii} \geq\ & F^{ii}\mathcal{W}_{ii,11} + \frac{F^{ii}}{2}G_{p_1p_1}w_{11}^2 \\ & -O(1+\mathrm{Tr}F^{ij}+\mathrm{Tr}\mathcal{W}_{ij}) - C\mathrm{Tr}F^{ij}\frac{1+\mathcal{W}_{11}}{\rho} + F^{ii}O(1+w_{11}) \\ =\ & F^{ii}\mathcal{W}_{ii,11} + \frac{F^{ii}}{2}G_{p_1p_1}w_{11}^2 + O\left(\frac{1+(\mathrm{Tr}F^{ij})(\mathrm{Tr}\mathcal{W}_{ij})}{\rho}\right). \end{aligned} \tag{5.10}$$

By (5.4) we have at $\bar{x}$

$$\begin{aligned} 0 \geq\ & F^{ii}(\log z)_{ii} = \\ =\ & \frac{F^{ii}\mathcal{W}_{ii,11}}{\mathcal{W}_{11}} - O\left(\frac{\mathrm{Tr}F^{ij}}{\rho}\right). \end{aligned}$$

This in conjunction with (5.10) and (5.1) yields

$$\begin{aligned} O\left(\frac{\mathrm{Tr}F^{ij}}{\rho}\right) \geq\ & \frac{1}{\mathcal{W}_{11}}\left[D_{11}\bar{h} + \frac{\mathrm{Tr}F^{ij}}{2}G_{p_1p_1}w_{11}^2 + O\left(\frac{1+(\mathrm{Tr}F^{ij})(\mathrm{Tr}\mathcal{W}_{ij})}{\rho}\right)\right] \\ \geq\ & \mathrm{Tr}F^{ij}\frac{c_0}{2}\mathcal{W}_{11} + \frac{\bar{h}_{11}}{\mathcal{W}_{11}} + \frac{1}{\mathcal{W}_{11}}O\left(\frac{1+(\mathrm{Tr}F^{ij})(\mathrm{Tr}\mathcal{W}_{ij})}{\rho}\right). \end{aligned}$$

Here c_0 is the constant from (4.12).

It remains to estimate $\bar{h}_{11}$. We have $\bar{h}_{11} = \bar{h}_{p_sp_r}w_{r1}w_{s1} + \bar{h}_{p_s}w_{s11} + O(1)$ and utilizing (5.3) we conclude $\bar{h}_{11} \geq -C\mathcal{W}_{11}(1+\mathcal{W}_{11}) + O(1)$.

Now if $\mathcal{W}_{11}$ is sufficiently large then $\mathrm{Tr}F^{ij} \gg 1$ at $\bar{x}$ because by (5.2) at $\bar{x}$ we have $F^{ij} = \mathrm{diag}[\mathcal{W}_{11}^{-1}, \ldots, \mathcal{W}_{nn}^{-1}]$. Therefore

$$O\left(\frac{1}{\rho}\right) + \frac{C(1+\mathcal{W}_{11})}{\mathrm{Tr}F^{ij}} + O\left(\frac{1+\mathrm{Tr}F^{ij}}{\rho\mathrm{Tr}F^{ij}}\right) \geq \frac{c_0}{2}\mathcal{W}_{11}$$

implying the estimate $\mathcal{W}_{11} \leq C_1$ and the result follows. ■

6. $R-$concave or admissible functions

6.1. **Paraboloids of revolution.** Let Z be a given point in $\mathbb{R}^{n+1}$ and $\sigma > 0$. A paraboloid of revolution with focus Z, focal parameter σ and focal axis parallel to x^{n+1} axis is denoted by

$$P(x,\sigma,Z) = h - m|x-z|^2 \qquad \text{with} \qquad z = \widehat{Z}. \tag{6.1}$$

Constants h and m can be expressed in terms of σ and Z as follows (see Figure 2); the height of the paraboloid measured from the hyperplane $\Pi = \{X \in \mathbb{R}^{n+1} : x^{n+1} = 0\}$ is equal to h, hence

$$h = \frac{\sigma}{2} + Z^{n+1}. \tag{6.2}$$

To determine m we first notice that if $P(x_0,\sigma,Z) = 0$ at $x_0 \in \Pi$, i.e. paraboloid intersects the hyperplane Π, then $m = h/|x_0 - z|^2$. By definition x_0 is equidistant from the directrix and the focus Z. Thus from the Pythagorean theorem

$$|x_0 - z|^2 = (\sigma + Z^{n+1})^2 - (Z^{n+1})^2$$

implying

$$\begin{aligned} m =\ & \frac{h}{|x_0-z|^2} = \frac{\frac{\sigma}{2} + Z^{n+1}}{\sigma^2 + 2\sigma Z^{n+1}} \\ =\ & \frac{1}{2\sigma}. \end{aligned} \tag{6.3}$$

In what follows we write $P(x)$ instead of

$$P(x,\sigma,Z)=\frac{\sigma}{2}+Z^{n+1}-\frac{1}{2\sigma}|x-z|^2 \tag{6.4}$$

if there is no ambiguity. For $L>0$

$$\mathbb{P}_L(\mathcal{U},\Sigma)=\{P(x,\sigma,Z):P(x,\sigma,Z)>L\}. \tag{6.5}$$

denotes the class of paraboloids of revolution that lie above the hyperplane $\{X\in\mathbb{R}^{n+1}:x^{n+1}>L\}$ in $\mathcal{U}$.

Definition 6.1. *Let u be a nonnegative continuous function defined in $\mathcal{U}$.*

1) *Let $x_0\in\mathcal{U}$. Then a paraboloid of revolution $P(x)=P(x,\sigma,Z)\in\mathbb{P}_L(\mathcal{U},\mathcal{V})$ is said to be an upper supporting paraboloid of u at x_0, if*

$$\begin{aligned} P(x_0) &= u(x_0)\\ P(x) &\geq u(x),\ \forall x\in\mathcal{U}. \end{aligned}$$

2) *A function u is said to be upper admissible or $R-$concave with respect to $\mathcal{V}$ if for any $x\in\mathcal{U}$ there exist $Z\in\mathcal{V}$ and a supporting paraboloid $P(x,\sigma,Z)\in\mathbb{P}_L(\mathcal{U},\mathcal{V})$ at x.*

3) *The class of all upper admissible functions is denoted by $\mathbb{W}^+(\mathcal{U},\mathcal{V})$.*

For instance, any paraboloid in $\mathbb{P}_L(\mathcal{U},\mathcal{V})$ is admissible. Furthermore it is easy to see that if $u_1(x)$ and $u_2(x)$ are $R-$concave then so is $\min(u_1(x),u_2(x))$. In fact, if $u_i, i=1,\dots,N$ are $R-$concave then so is $u=\min\limits_{1\leq i\leq N}(u_1,\dots,u_N)$. In particular, if $u_i\in\mathbb{P}_L(\mathcal{U},\mathcal{V})$ then $u=\min\limits_{1\leq i\leq N}(u_1,\dots,u_N)$ is called $R-$concave polyhedron or $R-$polyhedron for short. The graph of $R-$polyhedron is a finite union of pieces of paraboloids $P(x,\sigma,Z)\in\mathbb{P}_L(\mathcal{U},\mathcal{V})$.

Definition 6.2. *The class of $R-$polyhedrons is denoted by $\mathbb{W}_0^+(\mathcal{U},\mathcal{V})$.*

Remark 6.1. *It is easy to see that upper admissible functions are concave in the classical sense and hence locally Lipschitz continuous.*

Next, we prove that $R-$concave functions can be approximated via $R-$concave polyhedrons.

Lemma 6.3. *Let $u\in C^0(\overline{\mathcal{U}})$ be an $R-$concave function. Then there is a sequence of $R-$concave polyhedrons u_k such that $u_k\to u$ uniformly in $\mathcal{U}$.*

Proof. Let $\mathbb{Q}^n$ be the set of points of Π with rational coordinates. Denote $E=\mathbb{Q}^n\cap\mathcal{U}$. Since $\mathbb{Q}^n$ is countable we have $E=\bigcup_{k=1}^{\infty}E_k$ where $E_k=\{m_1,m_2,\dots,m_k\}, m_i\in\mathbb{Q}^n\cap E, i=1,\dots,k$. Because u is $R-$concave, there are supporting paraboloids $P_i(x)$ at the points $m_i\in E_k$. Then $u_k(x)=\min(P_1(x),\dots,P_k(x))$ is an $R-$polyhedron and $u\leq u_k$. Let us show that u_k converges to u uniformly in $\mathcal{U}$.

Take any $\varepsilon>0$ and fix a compact set $K\subset\mathcal{U}$. Suppose that there is a sequence $x_k\in K$ such that $u_k(x_k)-u(x_k)>\varepsilon$. Since K is compact then there is a subsequence $\{x_{k_j}\}\subset\{x_k\}$ such that $x_{k_j}\to x_0\in K$. Let $\delta>0$ be a small positive number to be fixed below. By choosing j large enough we get $|z_{k_j}-x_0|<\delta$ for some $z_{k_j}\in E_{k_j}\subset\mathbb{Q}^n$. This implies $|x_{k_j}-z_{k_j}|<2\delta$ if k_j is sufficiently large. Therefore we get

$$\begin{aligned} \varepsilon &< u_{k_j}(x_{k_j})-u(x_{k_j})\\ &\leq u_{k_j}(x_{k_j})-u(z_{k_j})+|u(z_{k_j})-u(x_{k_j})|. \end{aligned}$$

It follows from Remark 6.1 that $u \in C^{0,1}(K)$ and hence $|u(z_{k_j}) - u(x_{k_j})| < C|z_{k_j} - x_{k_j}| < 2\delta C$. On the other hand it follows from Lemma 7.2 that $|u_{k_j}(x_{k_j}) - u(z_{k_j})| = |u_{k_j}(x_{k_j}) - u_{k_j}(z_{k_j})| \leq 2\delta C$, with $C > 0$ independent of j. Combining these estimates we conclude that $\varepsilon < 4C\delta$ which gives a contradiction if we take $\delta < \frac{\varepsilon}{4C}$. ■

Lemma 6.4. *If the sequence of admissible functions u_k uniformly converges to a function u and the sequence $X_k \in \Gamma_{u_k}$ converges to $X_0 \in \Gamma_u$ then the limit of any converging sequence of supporting paraboloids of u_k at $X_k \in \Gamma_{u_k}$ converges to a supporting paraboloid of u at X_0.*

Proof. Let $P_k(x)$ be an upper supporting paraboloid of $\Gamma_k = \Gamma_{u_k}$ at $X_k \in \Gamma_k$. Then Γ_{u_k} is a subset of the solid $\{X \in \mathbb{R}^{n+1} : X^{n+1} \leq P_k(x)\}$. Therefore if P_k converges to a paraboloid P then $\Gamma_u \subset \{X \in \mathbb{R}^{n+1} : X^{n+1} \leq P(x)\}$. On the other hand $P(x_0) = u(x_0)$ implying that P is an upper supporting paraboloid of Γ_u at X_0 where $\widehat{X}_0 = x_0$. ■

Remark 6.2. *The lower admissible function can be defined accordingly. Notice that lower admissible functions are only semiconvex. Moreover, their graphs may contain saddle points and hence one cannot expect to obtain full regularity results.*

7. Paraboloids of revolution

7.1. **Properties of $P(x, \sigma, Z)$.** The following property of $P(x, \sigma, Z)$ is well-known: all rays issued from Π parallel to e_{n+1} and lying in the epigraph of $P(x, \sigma, Z)$ after reflection converge to the focus Z. Thus $P(x, \sigma, Z)$ is a solution to our problem (**P**) when the receiver Σ consists of one point i.e. $\Sigma = \{Z\}$.

If the rays emanate from $\mathcal{U} \subset \Pi$ in the direction of e_{n+1} then we require $P(x, \sigma, Z) \geq 0$ for all $x \in \mathcal{U}$. This is a natural condition stating that $P(x, \sigma, Z)$ is visible from each point of $\mathcal{U}$ in $e_{n+1}-$direction. If we demand a stronger condition, $P(x, \sigma, Z) \geq L$ for some positive L, then it will imply a lower bound for the focal parameter σ. Indeed, if $P(x, \sigma, Z) \geq L$ then we have from (6.2) that $\frac{\sigma}{2} + Z^{n+1} - m|x - \widehat{Z}|^2 \geq L$ or equivalently

$$\frac{\sigma}{2} \geq L - Z^{n+1} \geq L - \sup_{Z \in \mathcal{V}} |Z^{n+1}| \geq \frac{L}{2} \tag{7.1}$$

Thus the lower estimate for σ follows if we choose L large enough.

Notice that for fixed $Z \in \Sigma$ the curvature of $P(x, \sigma, Z)$ decreases as $\sigma \to \infty$ because $D^2 P = m\mathrm{Id} = \frac{1}{2\sigma}\mathrm{Id}$, where Id is the identity matrix. Thus the paraboloids become flatter as σ increases.

7.2. **Continuous expansion of confocal paraboloids.** Let Z be fixed then $P(x, \sigma, Z)$ is a one parameter family of surfaces with respect to σ. If σ increases then $P(x, \sigma, Z)$ moves away from $\Pi = \{X \in \mathbb{R}^{n+1} : X^{n+1} = 0\}$.

We want to introduce the pointwise intensity at fixed $Z \in \Sigma$ and determine its dependence from σ.

Let $w \in \mathbb{W}_0^+(\mathcal{U}, \mathcal{V})$ be an $R-$concave polyhedron and $\Omega_i \subset \Gamma_w$ be a piece of a paraboloid of revolution $P(x, \sigma_i, Z_i), Z_i \in \mathcal{V}$. Let $\mathcal{U}_i \subset \mathcal{U}$ be the projection of Ω_i onto Π. For each ray ℓ_x emitted from $x \in \Omega_i$ in the direction of e_{n+1} let Y be the unit direction of ℓ_x's reflection from Ω_i. Let S_i be the set of all unit directions Y on the unit sphere centered at Z_i, corresponding to ℓ_x with $x \in \Omega_i$. The reflection will give rise the atomic measure $c_i \delta_{Z_i}$ with $c_i > 0$. By energy balance condition and the formula (3.9) we get

$$c_i = \int_{S_i} g_i(Y) dS = \int_{\widehat{S}_i} g_i(Y) \frac{dy}{-Y^{n+1}} = \int_{\Omega_i} f(x) dx \geq 0. \tag{7.2}$$

implying

$$g(Y) \det Dy = -y^{n+1} f(x). \tag{7.3}$$

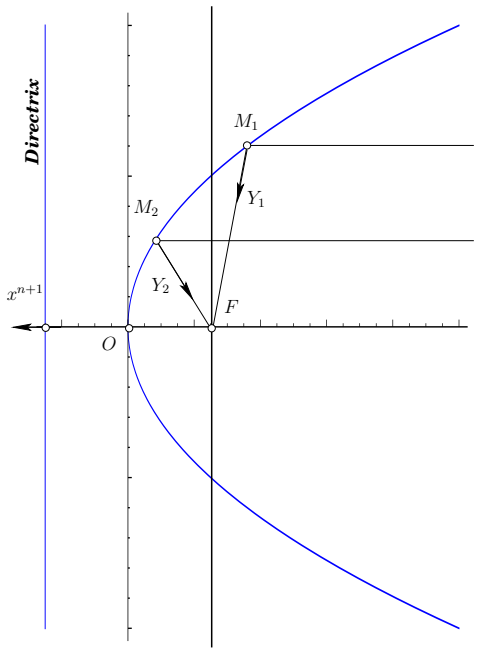

FIGURE 2. The reflection property of paraboloids of revolution.

Let Ω_i be the graph of $u_i(x) = A_i - B_i|x - z_i|^2, x \in \mathcal{U}_i$. Then

$$\begin{aligned} \det Dy &= \frac{2^n}{(1+|Du|^2)^n}\left[1 - \frac{2|Du|^2}{1+|Du|^2}\right]|\det D^2u| \\ &= -y^{n+1}\frac{2^n}{(1+|Du|^2)^n}|\det D^2u| \end{aligned} \tag{7.4}$$

hence from (7.3) and (3.8) we infer

$$\begin{aligned} g_i(Y) &= \frac{f(x)}{\frac{2^n}{(1+|Du|^2)^n}|\det D^2u|} \\ &= \frac{f(x)}{\frac{2^n}{(1+4B_i^2|x-z_i|^2)^n}(2B_i)^n} \\ &= f(x)\left[\frac{1+4B_i^2|x-z_i|^2}{4B_i}\right]^n. \end{aligned} \tag{7.5}$$

Recall that $B_i = \frac{1}{2\sigma}$, and hence

$$g_i(Y(x)) = f(x)\left[\frac{\sigma}{2} + \frac{1}{2\sigma}|x - z_i|^2\right]^n.$$

Differentiating g_i by σ we get

$$\frac{d}{d\sigma}g_i = \frac{n}{2}f(x)\left[\frac{\sigma}{2} + \frac{1}{2\sigma}|x - z_i|^2\right]^{n-1}(1 - \frac{1}{\sigma^2}|x - z_i|^2).$$

Thus g_i is increasing in σ if $|x - z_i| < \sigma$ and decreasing in $|x - z_i| > \sigma$. As the Figure 2 shows Y^{n+1} may have different signs (regarding Y as a vector on the units sphere) depending on whether the point on the reflector is above or below the focal plane passing through F and perpendicular to e_{n+1}. If $F = Z$ is the focus then for M_1 we have $|x - z_i| > \sigma$, whereas for M_2, $|x - z_i| < \sigma$, see Figure 2.

7.3. **Touching poraboloids.** Let $P_1(x) = P(x, \sigma_1, Z_1)$ be a paraboloid of revolution and $0 \neq Z_2$. It is easy to see that there is $P_2(x) = P(x, \sigma_2, Z_2)$ such that P_2 is the upper supporting of P_1 at M where M, Z_1 and Z_2 lie on the same line, see Figure 3. Without loss of generality we assume that $Z_1 = 0$. If $X^{n+1} = d_1 > 0$ is the directrix of the parabola generating P_1 and σ_1 its focal parameter then the distance of M from the directrix is $|MA_1| = |MZ_1|$. Thus, if $X^{n+1} = d_2$ is the directrix of P_2 then $d_2 = d_1 - Z_2^{n+1} + |Z_2|$, hence $\sigma_2 = d_1 - Z_2^{n+1} + |Z_2| = \sigma_1 - Z_2^{n+1} + |Z_2|$. Let us show that $P_2(x) = P(x, \sigma_1 - Z_2^{n+1}, Z_2)$ touches P_1 at M.

Indeed, we have that

$$\begin{aligned} P_2(x) - P_1(x) &= \frac{\sigma_2}{2} + Z_2^{n+1} - \frac{|x - \widehat{Z}_2|^2}{2\sigma_2} - \frac{\sigma_1}{2} + \frac{|x|^2}{2\sigma_1} \\ &= \frac{\sigma_2 - \sigma_1}{2} + Z_2^{n+1} + \frac{|x|^2}{2\sigma_1} - \frac{|x - \widehat{Z}_2|^2}{2\sigma_2} \\ &= \frac{|Z_2| + Z_2^{n+1}}{2} + \frac{|x|^2}{2\sigma_1} - \frac{|x - \widehat{Z}_2|^2}{2\sigma_2} \\ &= \frac{|Z_2| + Z_2^{n+1}}{2} + \frac{(\sigma_2 - \sigma_1)|x|^2 + 2\sigma_1\langle x, \widehat{Z}_2\rangle - \sigma_1|\widehat{Z}_2|^2}{2\sigma_1\sigma_2}. \end{aligned}$$

Note that $\sigma_2 - \sigma_1 = |Z_2| - Z_2^{n+1}$ using which we can transform the last term as follows

$$\begin{aligned} \frac{(\sigma_2 - \sigma_1)|x|^2 + \sigma_1(2\langle x, \widehat{Z}_2\rangle - |\widehat{Z}_2|^2)}{2\sigma_1\sigma_2} &= \frac{|Z_2| - Z_2^{n+1}}{2\sigma_1\sigma_2}\left[\left(x + \sigma_1\frac{\widehat{Z}_2}{|Z_2| - Z_2^{n+1}}\right)^2 - \right. \\ &\quad \left. - \frac{\sigma_1|\widehat{Z}_2|^2}{|Z_2| - Z_2^{n+1}}\left(\frac{\sigma_1}{|Z_2| - Z_2^{n+1}} + 1\right)\right] \\ &= \frac{|Z_2| - Z_2^{n+1}}{2\sigma_1\sigma_2}\left(x + \sigma_1\frac{\widehat{Z}_2}{|Z_2| - Z_2^{n+1}}\right)^2 - \\ &\quad - \frac{|\widehat{Z}_2|^2}{2\sigma_2}\left(\frac{\sigma_1}{|Z_2| - Z_2^{n+1}} + 1\right). \end{aligned}$$

On the other hand

$$\begin{aligned} \frac{|Z_2| + Z_2^{n+1}}{2} - \frac{|\widehat{Z}_2|^2}{2\sigma_2}\left(\frac{\sigma_1}{|Z_2| - Z_2^{n+1}} + 1\right) &= \frac{|Z_2| + Z_2^{n+1}}{2} - \frac{|\widehat{Z}_2|^2}{2\sigma_2}\frac{\sigma_1 + |Z_2| - Z_2^{n+1}}{|Z_2| - Z_2^{n+1}} \\ &= \frac{|Z_2| + Z_2^{n+1}}{2} - \frac{|\widehat{Z}_2|^2}{2}\frac{1}{|Z_2| - Z_2^{n+1}} \\ &= \frac{|\widehat{Z}_2|^2}{2}\frac{1}{|Z_2| - Z_2^{n+1}} - \frac{|\widehat{Z}_2|^2}{2}\frac{1}{|Z_2| - Z_2^{n+1}} \\ &= 0. \end{aligned}$$

Thus we conclude that

$$P_2(x) - P_1(x) = \frac{|Z_2| - Z_2^{n+1}}{2\sigma_1\sigma_2}\left(x + \sigma_1\frac{\widehat{Z}_2}{|Z_2| - Z_2^{n+1}}\right)^2 \geq 0.$$

Note that $x = -\sigma_1\frac{\widehat{Z}_2}{|Z_2| - Z_2^{n+1}}$ is the projection of M onto Π and we have that

$$P_2(x) = P(x, \sigma_1 + |Z_1 Z_2| - (Z_2^{n+1} - Z_1^{n+1}), Z_2) \tag{7.6}$$

is the upper support of $P_1(x) = P(x, \sigma_1, Z_1)$ at $x = -\sigma_1\frac{\widehat{Z_2 - Z_1}}{|Z_2 - Z_1| - (Z_2^{n+1} - Z_1^{n+1})}$.

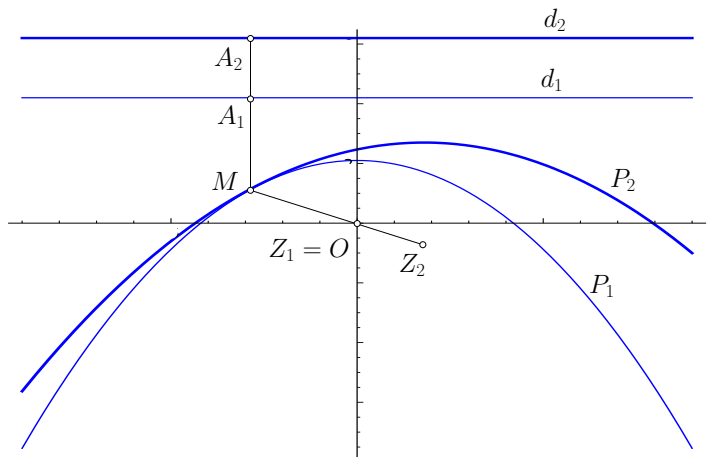

FIGURE 3. The touch of two paraboloids P_1 and P_2 at M.

Remark 7.1. *In Section 12 we will use this argument to show that if $u \in \mathbb{W}^+(\mathcal{U}, \Sigma)$ then $u + \varepsilon \in \mathbb{W}^+(\mathcal{U}, \widetilde{\Sigma})$ for $\varepsilon > 0$, provided that Σ satisfies the visibility condition (12.4), i.e. Σ is visible from any focal plane, see Remark 4.2. Here $\widetilde{\Sigma} = \Sigma - Me_{n+1}, M > 0$ is a downwards translation of Σ in e_{n+1} direction (see also Lemma 12.2).*

7.4. **Uniform estimates.** It is convenient to work with particular classes of paraboloids. Let $L > 0$ and define

$$\mathbb{P}_L(\mathcal{U}, \Sigma) = \{P(x, \sigma, Z) : P(x, \sigma, Z) > L\}. \tag{7.7}$$

Note that $\mathbb{P}_L(\mathcal{U}, \mathcal{V})$ is not empty since for fixed $L > 0$, $Z \in \mathcal{V}$ and sufficiently large σ we have $P(x, \sigma, Z) \in \mathbb{P}_L(\mathcal{U}, \mathcal{V})$, see (7.1).

We will be rather sloppy with the definition of $\mathbb{P}_L(\mathcal{U}, \mathcal{V})$ in Section 8.1 where $\mathbb{P}_L(\mathcal{U}, \mathcal{V})$ is defined as the set of all paraboloids $P(x, \sigma, Z)$ such that (7.1) holds for all $x \in \mathcal{U}$ and $Z \in \mathcal{V}$ with some $L > 0$. Clearly, this slight modification is coherent with the inequality (7.1).

Lemma 7.1. *Let $d_0 = \sup_{Z \in \mathcal{V}} |Z_{n+1}|$ and $d_1 = \sup_{\substack{X \in \mathcal{U} \\ Z \in \mathcal{V}}} |X - Z|$. For every $P(x, \sigma, Z) \in \mathbb{P}_L(\mathcal{U}, \mathcal{V})$ we have*

$$\sup_{\mathcal{U}} P(x, \sigma, Z) \leq \inf_{\mathcal{U}} P(x, \sigma, Z) + d_1 + 2d_0. \tag{7.8}$$

Proof. Let $x^0 \in \mathcal{U}$ be a point where the infimum is realized and $X^0 \in \mathbb{R}^{n+1}$ is the corresponding point on the graph of $P(x, \sigma, Z)$. Then X^0 is equidistant from Z and the directrix. But the distance of X^0 from the directrix is bigger than $\frac{\sigma}{2}$ hence

$$\frac{\sigma}{2} \leq |X^0 - Z|.$$

Notice that $\sup_{\mathcal{U}} P(x, \sigma, Z) \leq |h| = \left|\frac{\sigma}{2} + Z^{n+1}\right|$, thus

$$\begin{aligned}
\sup_{\mathcal{U}} P(x,\sigma,Z) &\leq \frac{\sigma}{2} + |Z^{n+1}| \\
&\leq |X^0 - Z| + |Z^{n+1}| \\
&= \sqrt{|x^0 - z|^2 + \Big[\inf_{\mathcal{U}} P(x,\sigma,Z) - Z^{n+1}\Big]^2} + |Z^{n+1}| \\
&\leq \sup_{\substack{X\in\mathcal{U}\\ Z\in\mathcal{V}}} |X - Z| + \inf_{\mathcal{U}} P(x,\sigma,Z) + 2\sup_{Z\in\mathcal{V}} |Z^{n+1}| \\
&= \inf_{\mathcal{U}} P(x,\sigma,Z) + d_1 + 2d_0.
\end{aligned}$$

■

Next we prove a gradient estimate

Lemma 7.2. *Retain the assumptions of previous lemma. Let $P(x,\sigma,Z) \in \mathbb{P}_L(\mathcal{U},\mathcal{V})$ then*

$$\sup_{x\in\mathcal{U}} |DP| \leq \frac{d_1}{2(L-d_0)}. \tag{7.9}$$

Proof. We have that

$$|DP| = \frac{|x-z|}{\sigma} \leq \frac{d_1}{\sigma}.$$

Now the desired estimate follows from (7.1). ■

8. Weak solutions of B-type: Proof of Theorem 1 **a**)

We develop our approach along the lines of the classical Monge-Ampère equation [1, 16] where in order to construct a weak solution one uses the method of approximation by convex polyhedrons. Since the supporting functions for the reflector problem (**P**) are paraboloids of revolution then one has to consider the "paraboloidal polyhedrons". For the "ellipsoidal" case we refer to [9, 10] (see also [21] and [11]).

Let $u \in \mathbb{W}^+(\mathcal{U},\mathcal{V})$. Consider the mapping

$$\mathscr{S}_u(Z) = \{x \in \mathcal{U} : \ \exists \text{ a supporting paraboloid of } u \text{ at } x \text{ with focus at } Z \in \mathcal{V}\}.$$

For any Borel set $\omega \subset \mathcal{V}$ we put

$$\mathscr{S}_u(\omega) = \bigcup_{Z\in\omega} \mathscr{S}_u(Z). \tag{8.1}$$

Below we establish some properties of $\mathscr{S}_u$. We will also use the notation $\mathscr{S}(E)$ instead of $\mathscr{S}_u(E)$ if there is no ambiguity.

Lemma 8.1. *$\mathscr{S} : \mathcal{V} \longrightarrow \Pi$ maps the closed sets to closed sets.*

Proof. The proof follows from Lemma 6.4. ■

Lemma 8.2. *Let $u \in \mathbb{W}^+(\mathcal{U},\mathcal{V})$. Then*

$$\big|\big\{x \in \Pi : x \in \mathscr{S}(Z_1) \cap \mathscr{S}(Z_2) \text{ for } Z_1 \neq Z_2, \quad Z_i \in \mathcal{V}, i = 1,2\big\}\big| = 0.$$

Proof. Denote $A = \big\{x \in \Pi : x \in \mathscr{S}(Z_1) \cap \mathscr{S}(Z_2) \text{ for } Z_1 \neq Z_2, \quad Z_i \in \mathcal{V}, i = 1,2\big\}$. If $x \in A$ then u cannot be differentiable at x. By Aleksandrov's theorem the concave function u is twice differentiable a.e. Hence $|A| = 0$. ■

Lemma 8.3. *Let $u \in \mathbb{W}^+(\mathcal{U},\mathcal{V})$. Consider $\mathscr{F} = \{E \subset \mathcal{V}$ such that $\mathscr{S}(E)$ is measurable$\}$. Then $\mathscr{F}$ is a $\sigma-$algebra.*

Proof. We want to show that the following three conditions hold

a) $\mathcal{V} \in \mathscr{F}$,
b) if $A \in \mathscr{F}$ then $\mathcal{V} \setminus A \in \mathscr{F}$,
c) if $A_i \in \mathscr{F}$ then $\bigcup_{i=1}^{\infty} A_i \in \mathscr{F}$.

We first prove a). Note that if $A_i \in \mathcal{V}$ is any sequence of subsets of $\mathcal{V}$ then $\mathscr{S}(\cup_{i=1}^{\infty} A_i) = \cup_{i=1}^{\infty} \mathscr{S}(A_i)$. Hence, writing $\mathcal{V} = \cup_{i=1}^{\infty} E_i$, where $E_i \subset \mathcal{V}$ are closed subsets we conclude that $\mathscr{S}(\mathcal{V}) = \mathscr{S}(\cup_{i=1}^{\infty} E_i) = \cup_{i=1}^{\infty} \mathscr{S}(E_i)$. By Lemma 8.1 it follows that $\mathscr{S}(E_i)$ is closed for any i, and hence measurable, implying that $\mathscr{S}(\mathcal{V})$ is measurable.

Let $A \in \mathscr{F}$. We use the following well known identity

$$\mathscr{S}(\mathcal{V} \setminus A) = [\mathscr{S}(\mathcal{V}) \setminus \mathscr{S}(A)] \bigcup [\mathscr{S}(\mathcal{V} \setminus A) \cap \mathscr{S}(A)]. \tag{8.2}$$

From Lemma 8.2 it follows that $|\mathscr{S}(\mathcal{V} \setminus A) \cap \mathscr{S}(A)| = 0$. Therefore $|\mathscr{S}(\mathcal{V} \setminus A)| = |\mathscr{S}(\mathcal{V}) \setminus \mathscr{S}(A)|$ and b) is proven.

It remains to check c). Without loss of generality we assume that A_i's are disjoint, see [2]. Thus, let $A_i \in \mathscr{F}, A_i \cap A_j = \emptyset, i \neq j$. Then

$$\sum_{i=1}^{\infty} |\mathscr{S}(A_i)| \geq |\mathscr{S}(\cup_{i=1}^{\infty} A_i)| \geq \sum_{i=1}^{\infty} |\mathscr{S}(A_i)| - \sum_{ij=1}^{\infty} |\mathscr{S}(A_i) \cap \mathscr{S}(A_j)| \geq$$
$$\geq \sum_{i=1}^{\infty} |\mathscr{S}(A_i)|.$$

■

For $u \in \mathbb{W}^+(\mathcal{U}, \mathcal{V})$ introduce the set function

$$\beta_{u,f}(\omega) = \int_{\mathscr{S}(\omega)} f \tag{8.3}$$

for any Borel subset $\omega \subset \mathcal{V}$. Since $\mathscr{F}$ contains the closed sets (see Lemma 8.1) we infer that $\beta_{u,f}$ is a Borel measure. Moreover, from the proof of Lemma 8.3 we conclude that $\beta_{u,f}$ is countably additive.

Definition 8.4. *A function u (or its graph Γ_u) is said to be a B-type weak solution to (**P**) if $u \in \mathbb{W}^+(\mathcal{U}, \mathcal{V})$ and for any Borel set $\omega \subset \mathcal{V}$ the following identity holds*

$$\beta_{u,f}(\omega) = \int_{\omega} g d\mathcal{H}^n_{\Sigma}, \qquad \mathscr{S}_u(\mathcal{V}) = \mathcal{U}. \tag{8.4}$$

Two classes of receivers are of particular interest to us: vertical, where Σ is a cylinder in e_{n+1} direction, and planar receivers. Verticals are more natural for upper admissible solutions whereas for lower admissible u horizontal plane is more natural since the regularity theory, developed in Section 13 can be applied to establish the smoothness of weak solutions in this case.

8.1. **Existence of weak solutions of B-type.** The measure β, defined in (8.3), enjoys a number of interesting properties, notably it is weakly continuous. We have

Lemma 8.5. *Let u_k be a sequence of B-type weak solutions and β_k is the corresponding measure, defined by (8.3). If $u_k \to u$ uniformly on compact subsets of $\mathcal{U}$ then u is $R-$concave and β_k weakly converges to $\beta_{u,f}$.*

Proof. That u is admissible follows from Lemma 6.4. Recall that the weak convergence is equivalent to the following two inequalities (see [2] Theorem 4.5.1)

1) $\limsup_{k\to\infty} \beta_k(E) \le \beta(E)$ for any compact $E \subset \mathcal{V}$,
2) $\liminf_{k\to\infty} \beta_k(H) \ge \beta(H)$ for any open $H \subset \mathcal{V}$.

Take a closed set E and let E^*_ε be an $\varepsilon-$neighbourhood of the closed set $E^* = \mathscr{S}(E)$, see Lemma 8.1. We claim that for any $\varepsilon > 0$ there is $i_0 \in \mathbb{N}$ such that $\mathscr{S}_i(E) \subset E^*_\varepsilon$ whenever $i > i_0$, where $\mathscr{S}_i$ is the mapping corresponding to u_i. If this fails then there is $\varepsilon > 0$ and a sequence of points $x_i \in \mathscr{S}_i(E)$ such that $x_i \in \complement E^*_\varepsilon$. By definition there is $Z_i \in E$ such that $x_i \in \mathscr{S}_i(Z_i)$. We can assume that $x_i \to x_0$ and $Z_i \to Z_0 \in E$ at least for a subsequence. Thus, $x_0 \in \complement E^*_\varepsilon, x_0 \in \mathscr{S}(Z_0)$ and $Z_0 \in E$ which is a contradiction.

To prove the second inequality, let $H \subset \mathcal{V}$ be an open subset and denote $H^* = \mathscr{S}(H)$. By Lemma 8.3 H^* is measurable, hence there is a closed set H^*_ε such that $H^*_\varepsilon \subset H^*$ and $|H^*| - \varepsilon \le |H^*_\varepsilon| \le |H^*|$ for a small $\varepsilon > 0$. Let N_ε be an open set, $|N_\varepsilon| < \varepsilon$ containing the points where the inverse of $\mathscr{S}$ is not defined. By Lemma 8.2 $\mathscr{S}$ is one-to-one modulo a set of measure zero. We claim that

$$H^*_\varepsilon \setminus N_\varepsilon \subset H^*_k \stackrel{\text{def}}{\equiv} \mathscr{S}_k(H). \tag{8.5}$$

Here $\mathscr{S}_k$ is the mapping generated by u_k. We argue towards a contradiction. If 8.5 fails then there is $x_k \in H^*_\varepsilon \setminus N_\varepsilon$ but $x_k \notin H^*_k$. We can assume that $x_k \to x_0$. Since $H^*_\varepsilon \setminus N_\varepsilon$ is closed it follows that $x_0 \in H^*_\varepsilon \setminus N_\varepsilon$. By definition of N_ε the inverse of $\mathscr{S}$ is one-to-one on $H^*_\varepsilon \setminus N_\varepsilon$. Thus there is unique $Z_0 \in H$ such that $x_0 = \mathscr{S}(Z_0)$. There is an open neighborhood of Z_0 contained in H because H is open. If $P(x, \sigma_k, Z_k)$ is a supporting paraboloid of u_k at x_k it follows from Lemma 6.4 that $x_k \in \mathscr{S}_k(Z_k), Z_k \to Z_0$. Thus for large k, $\{Z_k\}$ is in some neighborhood of $Z_0 \in H$ implying that $x_k \in H^*_k$ which contradicts our supposition. ■

Proposition 8.6. *Let f and g be two nonnegative integrable functions. If $\mathcal{U} \subset \Pi$ and $\mathcal{V} \subset \Sigma$ are bounded domains such that the energy balance condition (1.1) holds then there exists a $B-$type weak solution to the problem (**P**).*

The proof is based on an approximation argument, namely we take $g_N = \sum_{i=1}^N C_i \delta_{Z_i}$ with $C_i \ge 0$ such that $\sum_{i=1}^N C_i = \int_{\mathcal{U}} f(x)dx, Z_i \in \Sigma$ and g_N weakly converges to g. For each g_N we construct a $B-$type solution u_N. Then the existence for general case follows from the compactness argument and weak convergence Lemma 8.5.

8.2. **The case of $\mathcal{V} = \{Z_1\} \cup \{Z_2\}$.** In order to construct a B-type weak solution for the problem (**P**) we use an approximation method that utilizes the weak convergence of β measure, established in Lemma 8.5.

First, we examine the case of two point receiver. Assume that $g = C_1\delta_{Z_1} + C_2\delta_{Z_2}$ is a discrete measure supported at Z_1 and Z_2. Here C_1 and C_2 are two nonnegative constants such that the energy balance condition holds $C_1 + C_2 = \int_{\mathcal{U}} f(x)dx$. Let $P_i(x) = P(x, \sigma_i, Z_i) \in \mathbb{P}_L(\mathcal{U}, \mathcal{V}), i = 1, 2$ and $L > 0$ be fixed. If we choose $\sigma_2 \gg L$ to be sufficiently large it follows that $P_2(x) \ge P_1(x)$. Hence

$$\int_{E_{\sigma_1}} f(x)dx \ge C_1 \qquad \text{and} \qquad \int_{E_{\sigma_2}} f(x)dx \le C_2, \tag{8.6}$$

where $E_{\sigma_i} = \{x \in \mathcal{U} : \min[P_1(x), P_2(x)] = P_i(x)\}$ is the $i-$th visibility set, $i = 1, 2$. We note the following simple property of visibility sets: if P_1 is fixed then

$$E_{\sigma_2+\delta} \subset E_{\sigma_2} \tag{8.7}$$

for any $\delta > 0$. This follows from the confocal expansion of paraboloids, see Section 7.2.

Let's fix σ_1, Z_1, Z_2 and consider the set

$$I = \{\sigma_2 > 0 \text{ such that (8.6) is satisfied}\}.$$

We denote $\widehat{\sigma}_2 = \inf_I \sigma_2$ and claim that

$$u_2(x) = \min\left[P(x,\sigma_1,Z_1), P(x,\widehat{\sigma}_2,Z_2)\right]$$

is a B-type weak solution of the two point receiver problem. Indeed, if it's not true then

$$\int_{E_{\widehat{\sigma}_2}} f(x)dx < C_2.$$

On the other hand, by (8.7), the visibility set can only increase as σ decreases. Hence we see that the function $F(\delta) = \int_{E_{\widehat{\sigma}_2-\delta}} f(x)dx$ is continuous, $F(0) < C_2$ and $F(2L) = C_1 + C_2$. Thus there is $\delta_0 > 0$ such that $F(\delta_0) = C_2$. Therefore $\widehat{\sigma}_2 - \delta_0 \in I$ which is a contradiction.

8.3. **The case $\mathcal{V} = \{Z_1, Z_2, \dots, Z_N\}$.** Let's choose $\sigma_1 > 0$ so that $3L \le P(x,\sigma_1,Z_1) \le \lambda L$ where $\lambda > 0$ is a large but fixed constant. If $\sigma_i > \Lambda L, i = 2,3,\dots,N$, for suitable $\Lambda \gg \lambda$ such that $P(x,\sigma_i,Z_i) \ge \lambda L, i = 2,\dots,N$, then

$$\int_{E_1} f \ge C_1, \qquad \int_{E_i} f \le C_i, \qquad i = 2,3,\dots,N. \tag{8.8}$$

Here $E_i = \{x \in \mathcal{U} : P(x,\sigma_i,Z_i) = u_N(x)\}$ is the $i-th$ visibility set with

$$u_N(x) = \min\left[P(x,\sigma_1,Z_1),\dots,P(x,\sigma_N,Z_N)\right].$$

It is convenient to define the following sets

$$I_k = \{\sigma_k > 0 \text{ such that (8.8) is satisfied}\}, \qquad k = 1,2,\dots,N.$$

We want to check that if $\widehat{\sigma}_k = \inf_{I_k} \sigma_k$ then

$$u_N(x) = \min\left[P(x,\widehat{\sigma}_1,Z_1),\dots,P(x,\widehat{\sigma}_N,Z_N)\right]$$

is the desired solution for $\mathcal{V} = \{Z_1, Z_2, \dots, Z_N\}$. Indeed, if for some $k, 2 \le k \le N$ we have $\int_{E_k} f(x)dx < C_k$ then $F_k(\delta) = \int_{E_{\widehat{\sigma}_k-\delta}} f(x)dx$ is continuous function of δ and at the endpoints $F_k(0) < C_k$ and $F_k(2L) = \sum_{i=1}^{N} C_i$. Applying the intermediate value theorem for continuous functions it follows that there is δ_0 such that $F_k(\delta_0) = C_k$. This implies that $\widehat{\sigma}_k - \delta_0 \in I_k$ which is a contradiction.

Now the proof of Theorem 1 **a**) follows if we take a dense sequence $\{Z_n\}_{n=1}^{\infty} \subset \mathcal{V}$, construct a solution $u_N, 3L \le u_N \le \lambda L$ for each finite collection $\{Z_1, Z_2, \dots, Z_N\}$ and utilizing the weak convergence of measures, Lemma 8.5, pass to the limit as $N \to \infty$. ■

9. Local and global supporting paraboloids

In this section we discuss some of the properties of supporting paraboloids that will be used in the definition of the A-type weak solutions, see Section 10. Throughout this section we assume that the condition in Theorem 1 **b**) are satisfied.

9.1. $R-$convexity of target domain.

- **Reflection cone.** Let $Q_{\mathcal{U}} = \{Z \in \mathbb{R}^{n+1}, \widehat{Z} \in \mathcal{U}\}$ and $q \in Q_{\mathcal{U}}$, then $\mathcal{C}_{q,\gamma_1,\gamma_2}$ denotes the reflection cone at q defined as the set of all $Z \in \mathbb{R}^{n+1}$ such that

$$\frac{Z-q}{|Z-q|} = e_{n+1} - 2\frac{c_1\gamma_1 + c_2\gamma_2}{|c_1\gamma_1 + c_2\gamma_2|}\left\langle \frac{c_1\gamma_1 + c_2\gamma_2}{|c_1\gamma_1 + c_2\gamma_2|}, e_{n+1}\right\rangle \tag{9.1}$$

for a pair of unit vectors γ_1, γ_2 and all constants c_1, c_2. Here $\langle , \rangle$ denotes the scalar product in $\mathbb{R}^{n+1}$. It is easy to see that $\mathcal{C}_{q,\gamma_1,\gamma_2}$ is a convex cone in $\mathbb{R}^{n+1}$. Indeed, if $\gamma_0 \perp \mathrm{Span}\{\gamma_1, \gamma_2\}$ then $\left\langle \frac{Z-q}{|Z-q|}, \gamma_0 \right\rangle = \langle e_{n+1}, \gamma_0 \rangle \stackrel{\text{def}}{\equiv} C_0$.

- $R-$**convexity of $\mathcal{V}$.** We say that $\mathcal{V}$ is $R-$convex with respect to a point $q \in Q_{\mathcal{U}}$, if for any γ_1, γ_2 the intersection $C_{q,\gamma_1,\gamma_2} \cap \mathcal{V}$ is connected. If $\mathcal{V}$ is $R-$convex with respect to any $q \in Q_{\mathcal{U}}$ then $\mathcal{V}$ is said to be $R-$convex with respect to $Q_{\mathcal{U}}$, or simply $R-$convex.

Remark 9.1. *The formula (9.1) has a simple geometric interpretation. Indeed, let us think that $q = (x, u(x))$, then $Y = \frac{Z-q}{|Z-q|}$ where $Z \in \Sigma$, see Figure 1. From the reflection law (1.2) we have that $Y = e_{n+1} - 2\gamma\langle e_{n+1}, \gamma\rangle$. If at q the surface Γ_u is not differentiable and γ_1, γ_2 are the normals of any two supporting planes of Γ_u at q then any unit vector $\frac{t\gamma_1+(1-t)\gamma_2}{|t\gamma_1+(1-t)\gamma_2|}, t \in (0,1)$ is also a normal to some supporting plane of Γ_u at q (recall that u is concave). Hence the $R-$convexity of $\mathcal{V}$ means that $\mathcal{V}$ can capture the reflected rays even for the non-smooth reflector Γ_u.*

9.2. **The behaviour of supporting paraboloids near contact point.** Let $P_0, P_1 \in \mathbb{P}_L(\mathcal{U}, \mathcal{V})$ and consider the contact set

$$\Lambda = \left\{ x \in \mathbb{R}^n : \frac{\sigma_1}{2} + Z_1^{n+1} - \frac{1}{\sigma_1}|x - z_1|^2 = \frac{\sigma_0}{2} + Z_0^{n+1} - \frac{1}{\sigma_0}|x - z_0|^2 \right\}.$$

Here $P_i = \frac{\sigma_i}{2} + Z_i^{n+1} - \frac{1}{2\sigma_i}|x - z_i|^2, i = 0, 1$. We want to show that Λ is either a sphere or plane. Indeed, we have

$$\begin{aligned}
&\sigma_1\sigma_0[\sigma_1 - \sigma_0 + 2(Z_1^{n+1} - Z_0^{n+1})] = \\
&= \left[(\sigma_0 - \sigma_1)|x|^2 - 2\langle x, \sigma_0 z_1 - \sigma_1 z_0\rangle + (\sigma_0|z_1|^2 - \sigma_1|z_0|^2)\right] \\
&= (\sigma_0 - \sigma_1)\left(x - \frac{\sigma_0 z_1 - \sigma_1 z_0}{\sigma_0 - \sigma_1}\right)^2 + (\sigma_0|z_1|^2 - \sigma_1|z_0|^2) - \frac{|\sigma_0 z_1 - \sigma_1 z_0|^2}{(\sigma_0 - \sigma_1)}
\end{aligned}$$

Thus we see that if $\Lambda \neq \emptyset$ then it is ether a sphere (if $\sigma_1 \neq \sigma_2$) or a plane ($\sigma_1 = \sigma_2$). Consequently if $P_1 > P_0$ then

$$\sigma_1\sigma_0[\sigma_1 - \sigma_0 + 2(Z_1^{n+1} - Z_0^{n+1})] > (\sigma_0 - \sigma_1)(x - \widetilde{z})^2 + (\sigma_0|z_1|^2 - \sigma_1|z_0|^2) - \frac{|\sigma_0 z_1 - \sigma_1 z_0|^2}{(\sigma_0 - \sigma_1)}$$

where

$$\widetilde{z} = z_1 + (z_1 - z_0)\frac{\sigma_1}{\sigma_0 - \sigma_1} \tag{9.2}$$

is the centre of contact sphere. Note that $\widetilde{z}$ lies on the line passing through z_0 and z_1.

Lemma 9.1. *The local supporting paraboloid is also global.*

Proof. Let Λ be the contact set of P_0 and P_1, $Z_i \in \Sigma$ is the focus of $P_i, i = 0, 1$ and $x_0 \in \Lambda$. Denote by $\gamma_i, i = 0, 1$ the normal of P_i at x_0. Let $\mathcal{C}_{X_0,\gamma_1,\gamma_2}$ be the reflection cone for $X_0 = (x_0, P_0(x_0))$, see (9.1). Consider $\mathcal{K} = \Sigma \cap \mathcal{C}_{X_0,\gamma_1,\gamma_2}$ be the intersection of $\mathcal{C}_{p,\gamma_1,\gamma_2}$ and Σ. Then for any point $Z \in \mathcal{K}$ between Z_0 and Z_1, there is a unique paraboloid $P_{X_0,Z}$ with focus Z, passing through the point X_0 and tangent to $\Gamma_{P_0} \cap \Gamma_{P_1} \subset \mathbb{R}^{n+1}$.

Since $P_{X_0,Z}$ is tangent to Λ at X_0, we have

$$DP_{X_0,Z}(X_0) = \theta DP_1(X_0) + (1 - \theta)DP_0(X_0)$$

for some $\theta \in (0, 1)$. The correspondence $\theta \mapsto Z$ is one-to-one, so now we can consider $P_{X_0,Z}$ to be a function of θ, i.e. from now on P_θ is the paraboloid with focus $Z \in \mathcal{K}$ and tangential to Λ at X_0. By choosing a suitable coordinate system we can take $X_0 = 0$ so that

$$D(P_1 - P_0) = (0, \cdots, 0, \alpha)$$

for some $\alpha \neq 0$ depending on Z. Note that the matrix $\mathcal{W} \equiv 0$ at any paraboloid in $\mathbb{P}_L(\mathcal{U}, \mathcal{V})$, which yields

$$D^2 P_\theta = -\frac{1}{2} G(P_\theta) \mathrm{Id} \tag{9.3}$$

where $G(P_\theta) = \frac{|DP_\theta|^2+1}{t(\cdot, P_\theta, DP_\theta)}$, see (4.6). Suppose that (1.11) holds. Twice differentiating (9.3) with respect to θ , we obtain

$$\frac{d^2}{d\theta^2} D^2 P_\theta = -\frac{1}{2} \partial_{p_n p_n} G |D(P_1 - P_0)|^2 < 0,$$

therefore

$$D^2 P_\theta(\bar{x}) > \theta D^2 P_1(\bar{x}) + (1-\theta) D^2 P_0(\bar{x})$$

for all $\bar{x} \in \Lambda$ and close to 0. This, in particular, implies that near $x = 0$

$$P_\theta(\bar{x}) < \theta P_1(\bar{x}) + (1-\theta) P_0(\bar{x})$$

for $\bar{x} \in \Lambda, \bar{x} \neq 0$ and $\bar{x}$ is close to the origin. Thus we have

$$P_\theta(\bar{x}) > \min(P_1(\bar{x}), P_0(\bar{x})) \ \text{ for } \bar{x} \in \Lambda \text{ near } 0, \bar{x} \neq 0. \tag{9.4}$$

By Taylor's expansion we can extend (9.4) to some neighbourhood of 0. Hence we obtain

$$P_\theta(x) > \min(P_1(x), P_0(x)) \ \text{ for } x \text{ near } 0, x \neq 0. \tag{9.5}$$

This is Leoper's characterization of the (1.11) condition, see [13].

This leads to the following conclusion: the local supporting paraboloids are global. Indeed, assume that we are given two paraboloids $P_i(x) = c_i^k x^k - c_i^* |x|^2, i = 0, 1$ and $P(x) = c^k x^k - c^* |x|^2$. Since $D(P_1 - P_0) = (0, \cdots, 0, \alpha)$ it follows that

$$c^k = c_0^k = c_1^k, 1 \leq k \leq n-1, \qquad c^n = \theta c_1^n + (1-\theta) c_0^n.$$

By (9.4) we have $-c^* |\bar{x}|^2 > \theta(-c_0^* |\bar{x}|^2) + (1-\theta)(-c_0^* |\bar{x}|^2)$ near the origin and $\bar{x} \in \Lambda$ implying $c^* < \theta c_1^* + (1-\theta) c_0^*$. Thus combining the inequalities for the coefficients of the quadratic polynomials P, P_0 and P_1 we infer that $P_\theta(x) > \min(P_1(x), P_0(x))$ globally in $\mathcal{U}$.

Note that we needed (1.11) or (4.11) only in some neighborhood of x_0. ■

Remark 9.2. *Notice that to derive the inequality $c^* < \theta c_1^* + (1-\theta) c_0^*$ we only need to have a closed subset of Λ near the origin.*

10. Weak solutions of A-type: Proof of Theorem 1 **b)**

For a given $u \in \mathbb{W}^+(\mathcal{U}, \Sigma)$ we define the following multiple valued map $\mathscr{R}_u : \mathcal{U} \to \mathcal{V}$ as follows: for any $x \in \mathcal{U}$ we set

$$\mathscr{R}_u(x) = \{Z \in \Sigma : \ Z \text{ is the focus of an upper supporting paraboloid of } u \text{ at } x \in \mathcal{U}\}.$$

If u is differentiable at $x_0 \in \mathcal{U}$ then $\mathscr{R}_u(x_0) = Z(x_0)$ and $Z(x_0)$ is given by the formula (3.4). For any subset $E \subset \mathcal{U}$ we denote

$$\mathscr{R}_u(E) = \bigcup_{x \in E} \mathscr{R}_u(x).$$

Lemma 10.1. *$\mathscr{R}_u(E)$ is closed for any closed subset $E \subset \mathcal{U}$.*

Proof. Let $Z_k \in \mathscr{R}_u(E)$ and $Z_k \to Z_0$. We wish to show that $Z_0 \in \mathscr{R}_u(E)$. It follows from the definition of $\mathscr{R}_u$ that there is $x_k \in E$ such that $Z_k \in \mathscr{R}_u(x_k)$. Let $P_k(x, \sigma_k, Z_k)$ be a supporting paraboloid of Γ_u at x_k. Applying Lemma 6.4 to $u = u_k$ we conclude that for a subsequence $x_{k_j} \to x_0 \in E$, $P_{k_j}(x, \sigma_{k_j}, Z_{k_j})$ converges to a supporting paraboloid $P(x, \sigma, Z_0)$ of u at $x_0 \in E$. Hence $Z_0 \in \mathscr{R}_u(E)$. ■

For a given nonnegative $g : \Sigma \to \mathbb{R}$ and $u \in C^2(\mathcal{U})$ introduce the set function

$$\alpha_{u,g}(E) = \int_{\mathscr{R}(E)} g d\mathcal{H}^n_\Sigma. \tag{10.1}$$

Note that $\alpha_{u,g}$ is well-defined for polyhedral $u \in \mathbb{W}^+_0(\mathcal{U}, \Sigma)$. In fact, it follows from Lemma 10.1 that $\alpha_{u,g}$ is also well defined for closed subset $E \subset \mathcal{U}$.

Our intention is to establish that any $u \in \mathbb{W}^+(\mathcal{U}, \Sigma)$ defines a Radon measure defined by (10.1). This is done with the help of Proposition 10.5 below.

10.1. **Legendre-like transformation of admissible function.** In this section we consider a Legendre-like transformation for the admissible function u which will be used in the construction of the A-type weak solutions for (**P**).

Definition 10.2. *Suppose that* $\Sigma = \{Z \in \mathbb{R}^{n+1} : Z^{n+1} = \psi(z), z = \widehat{Z}\}$. *Let* $u \in \mathbb{W}^+(\mathcal{U}, \mathcal{V})$. *Then*

$$u^\star(z) = \sup_{x \in \mathcal{U}} \left\{ u(x) - \psi(z) + c(x, z) \right\} \tag{10.2}$$

is called the $R-$*transform of* u. *Here* $c(x, z)$ *is the distance between the points on the surfaces* Γ_u *and* Σ, *given by*

$$c(x, z) = \sqrt{\left|x - z\right|^2 + \left[u(x) - \psi(z)\right]^2}. \tag{10.3}$$

Recall that the paraboloid of revolution is given explicitly by $P(x, \sigma, Z) = \frac{\sigma}{2} + \psi(z) - \frac{1}{2\sigma}|x - z|^2$, where $Z = (z, \psi(z))$ is the focus and $\sigma > 0$ the focal parameter, see (6.4). If z is fixed and $u \in \mathbb{W}^+(\mathcal{U}, \mathcal{V})$ then the focal parameter σ_0 of supporting paraboloid with focus $Z = (z, \psi(z))$ is characterized by the following condition

$$\sigma_0(z) = \inf_{P(x,\sigma,Z) \geq u(x)} \sigma. \tag{10.4}$$

At the point x_0, where P and u touch, we have that $u(x_0) = \frac{\sigma_0}{2} + \psi(z) - \frac{1}{2\sigma_0}|x_0 - z|^2$. Hence by solving the quadratic equation $\sigma_0^2 + 2\sigma_0[\psi(z) - u(x_0)] - |x_0 - z|^2 = 0$ we find that

$$\sigma_0 = u(x_0) - \psi(z) + \sqrt{\left|x_0 - z\right|^2 + \left[u(x_0) - \psi(z)\right]^2} \tag{10.5}$$

is the only nonnegative solution. Thus, for given admissible u we can consider the smallest focal parameter of paraboloid with focus $Z = (z, \psi(z))$, defined by (10.4), as a function of z.

Lemma 10.3. *Let* $\mathscr{L}(y) = u(x_0) - \psi(z) + c(x_0, y)$. *Then* $\mathscr{L}$ *is* C^2 *smooth provided that* $\psi \in C^2$ *and dist*$(\mathcal{U}, \mathcal{V}) > 0$.

Proof. Denote $Q(y) = |y - x_0|^2 + \psi^2(y) - 2\psi(y)u(x_0) + u^2(x_0)$ then $\mathscr{L}(y) = u(x_0) - \psi(y) + \sqrt{Q(y)}$. We compute

$$\begin{aligned}
D_i\mathscr{L} &= -D_i\psi + \frac{D_iQ}{2\sqrt{Q}}, \\
D_{ij}\mathscr{L} &= -D_{ij}\psi + \frac{1}{2\sqrt{Q}}\left(Q_{ij} - \frac{Q_iQ_j}{2Q^2}\right), \\
Q_i(y) &= 2(y - x_0) + 2\psi(y)D\psi(y) - 2D\psi(y)u(x_0), \\
Q_{ij}(y) &= 2\delta_{ij} + 2\psi_i(y)\psi_j(y) + 2\psi(y)\psi_{ij}(y) - 2\psi_{ij}(y)u(x_0).
\end{aligned}$$

Using the formulae above we obtain

$$\begin{aligned} D^2\mathscr{L}(y) &= -D^2\psi + \frac{1}{2\sqrt{Q}}\left(D^2Q - \frac{DQ\otimes DQ}{2Q^2}\right) \\ &= -D^2\psi(y)\left(1+\frac{u(x_0)-\psi(y)}{\sqrt{Q(y)}}\right) + \frac{\mathrm{Id}+D\psi(y)\otimes D\psi(y)}{\sqrt{Q(y)}} - \frac{DQ(y)\otimes DQ(y)}{4Q^{\frac{3}{2}}(y)} \end{aligned}$$

which yields the estimate $|D^2\mathscr{L}| \le C$ with $C>0$. ■

In what follows we call $\mathscr{L}_{x_0}$ the $\psi-$support function of $u^\star$ at z.

Lemma 10.4. *Let $u^\star$ be the $R-$transform of $u \in \mathbb{W}^+(\mathcal{U},\Sigma)$. Then*

- $u^\star(z) = u(x_0) - \psi(z) + c(x_0,z)$ *if* $Z=(z,\psi(z)) \in \mathscr{R}_u(x_0)$,
- $u^\star$ *is semi-convex.*

Proof. First, we observe that by definition $u^\star(z)$ is locally bounded, non-negative, lower semi-continuous function. Let us show that if $Z \in \mathscr{R}(x_0)$ then $u^\star(z) = u(x_0)-\psi(z)+c(x_0,z)$. By definition of $u^\star$ we have $u^\star(z) \ge u(x_0)-\psi(z)+c(x_0,z)$. Suppose that $u^\star(z) > u(x_0)-\psi(z)+c(x_0,z) \overset{\text{def}}{\equiv} \sigma_0$ for $Z \in \mathscr{R}_u(x_0) \subset \Sigma$. From (10.3) we see that $\sigma_0 > 0$. From (10.5) it follows that $P(x,\sigma_0,Z)$ is a supporting paraboloid of u at x_0.

On the other hand if $\sigma_1 \overset{\text{def}}{\equiv} u^\star(z)$ then $\sigma_1 > \sigma_0$ and by (10.2) there is a sequence $\{x_k\} \in \mathcal{U}$ such that $x_k \to x_1 \in \overline{\mathcal{U}}$ and $\sigma_1 = u(x_1)-\psi(z)+c(x_1,z)$. From (10.5) we infer that $P(x,\sigma_1,Z)$ is a supporting paraboloid of u at $x_1 \in \overline{\mathcal{U}}$. Thus we have that $P(x,\sigma_1,Z)$ and $P(x,\sigma_0,Z)$ are supporting paraboloids of u at respectively x_1 and x_0 such that $\sigma_1 > \sigma_0$ implying $P(x_1,\sigma_1,Z) > P(x_1,\sigma_0,Z) \ge u(x_1) = P(x_1,\sigma_1,Z)$ which is a contradiction.

To prove the second statement we let $\mathscr{L}_{x_0}(y) = u(x_0)-\psi(y)+c(x_0,y)$. Then

$$u^\star(y) = \sup_{x\in U}\left\{u(x)-\psi(y)+c(x,y)\right\} \ge u(x_0)-\psi(y)+c(x_0,y)$$

which implies that $u^\star(y) \ge \mathscr{L}_{x_0}(y)$ and $u^\star(z) = \mathscr{L}_{x_0}(z)$, where $Z \in \mathscr{R}_u(x_0)$. We can regard $\mathscr{L}_{x_0}(y)$ as an lower supporting function of $u^\star$ at z. By Lemma 10.3 $\mathscr{L}_{x_0}$ is C^2 smooth hence $u^\star(z)+C|z|^2$ is convex for sufficiently large $C>0$. ■

Proposition 10.5. *Let $\mathscr{R}_u$ be the reflector mapping corresponding to $u \in \mathbb{W}^+(\mathcal{U},\Sigma)$ and set $\mathcal{S} = \{Z \in \Sigma : Z \in \mathscr{R}_u(x_1)\cap\mathscr{R}_u(x_2), x_1 \ne x_2\}$. Then*

- *the surface measure of $\mathcal{S}$ on Σ is zero,*
- *furthermore, $\alpha_{u,g}(E) = \int\limits_{\mathscr{R}_u(E)} g d\mathcal{H}^n_\Sigma$ is Radon measure.*

Proof. Let $u^\star$ be the $R-$transform of u. If $(z,\psi(z)) = Z \in \mathcal{S}$ then there are $x_1,x_2 \in \mathcal{U}$ such that $\mathscr{L}_{x_i}(y) = u(x_i)-\psi(y)+c(x_i,y), i=1,2$ are the $\psi-$support functions of $u^\star(y)$ at z. Let us show that $u^\star(y)$ is not differentiable at z.

Indeed, if $u^\star$ is differentiable at z then we have

$$\begin{aligned} Du^\star(z) &= -D\psi(z) + \frac{(z-x_1)-D\psi(z)(u(x_1)-\psi(z))}{c(x_1,z)}, \\ Du^\star(z) &= -D\psi(z) + \frac{(z-x_2)-D\psi(z)(u(x_2)-\psi(z))}{c(x_2,z)}. \end{aligned}$$

The condition $\mathscr{L}_{x_1}(z) = \mathscr{L}_{x_2}(z)$ implies that $u(x_1)-u(x_2) = c(x_2,z)-c(x_1,z)$. From this identity we deduce

$$\frac{(z-x_1)-D\psi(z)(u(x_1)-\psi(z))}{c(x_1,z)} = \frac{(z-x_2)-D\psi(z)(u(x_2)-\psi(z))}{c(x_2,z)}. \tag{10.6}$$

From the definition of stretch function t it follows that $(z-x, \psi(z)-u(x)) = Yc(x,z)$ where $Y=(y,y^{n+1})$ is the unit direction of the reflected ray. With the aid of this observation we can rewrite (10.6) as follows

$$y_1 + D\psi(z)y_1^{n+1} = y_2 + D\psi(z)y_2^{n+1} \quad \Rightarrow \quad Y_1 + (D\psi(z),-1)y_1^{n+1} = Y_2 + (D\psi(z),-1)y_2^{n+1}.$$

The last identity implies that $Y_1 - Y_2$ is collinear to the normal of Σ at Z. Consequently, from the assumption (1.8) we obtain that this is possible if and only if $Y_1 = Y_2$. From this equality we can infer that $x_1 = x_2$ which will be a contradiction. Indeed, from $Y_1 = Y_2$ we have $y_1 = y_2$ and consequently we conclude that

$$\frac{z-x_1}{c(x_1,z)} = \frac{z-x_2}{c(x_2,z)}. \tag{10.7}$$

Taking the reciprocal of both sides in the last identity and recalling the definition of $c(x,z)$ we see that

$$\frac{u(x_1)-\psi(z)}{|x_1-z|} = \frac{u(x_2)-\psi(z)}{|x_2-z|} \tag{10.8}$$

and this yields

$$\begin{aligned} u(x_1) &= \psi(z) + \frac{|z-x_1|}{|z-x_2|}(u(x_2)-\psi(z)) \\ &= \psi(z) + \frac{c(x_1,z)}{c(x_2,z)}(u(x_2)-\psi(z)). \end{aligned}$$

Now the condition $u(x_1)-u(x_2) = c(x_2,z)-c(x_1,z)$ implies

$$\psi(z)\frac{c(x_2,z)-c(x_1,z)}{c(x_2,z)} - u(x_2)\frac{c(x_2,z)-c(x_1,z)}{c(x_2,z)} = c(x_1,z)-c(x_2,z). \tag{10.9}$$

If $c(x_2,z) \neq c(x_1,z)$ then from the last equality it follows that $u(x_2)-\psi(z) = \sqrt{(u(x_2)-\psi(z))^2+(z-x_2)^2}$. Hence $x_2 = z$ and by (10.7) $x_1 = x_2$ which is a contradiction. Thus we must have $c(x_2,z) = c(x_1,z)$ and in view of (10.7) this implies that $x_1 = x_2$, again contradicting our supposition. Therefore we infer that $u^\star$ cannot be differentiable at z. By Rademacher's theorem $u^\star$ is differentiable a.e. in z. Thus $\mathcal{S}$ has vanishing surface measure.

In order to show that $\alpha_{u,g}$ is Radon measure it suffices to check that $\widetilde{\mathscr{F}} = \{E \subset \mathcal{U} : \mathscr{R}_u(E) \text{ is measurable}\}$ is a $\sigma-$algebra. This can be done exactly in the same way as in the proof of Lemma 8.3. It remains to recall that by Lemma 10.1, $\widetilde{\mathscr{F}}$ contains the closed sets. ■

Remark 10.1. *In the definition of $u^\star$ it was assumed that Σ is the graph $Z^{n+1} = \psi(z)$ and ψ is a smooth function. One can easily amend this definition if, say, $Z^1 = \widetilde{\psi}(\widetilde{z}), \widetilde{z} = (0, Z^2, Z^3, \dots, Z^n, Z^{n+1})$ as follows*

$$u^\star(\widetilde{z}) = \sup_{x\in\mathcal{U}}\left\{u(x) - Z^{n+1} + \sqrt{\sum_{i=2}^{n-1}(x^i-Z^i)^2 + \left[u(x)-Z^{n+1}\right]^2 + \left[x^1-\widetilde{\psi}(\widetilde{z})\right]^2}\right\}.$$

This is particularly useful for the cylindrical receivers with generators perpendicular to Π.

10.2. ***A*-type weak solutions.** Now we are ready to define the A-type weak solutions of the problem (**P**).

Definition 10.6. *A function $u \in \mathbb{W}^+(\mathcal{U},\Sigma)$ (or its graph Γ_u) is said to be an A-type weak solution to the equation (1.3), if $\mathscr{R}_u(\mathcal{U}) \subset \Sigma$ and for any Borel set $E \subset \mathcal{U}$ we have*

$$\alpha_{u,g}(E) = \int_E f(x)dx. \tag{10.10}$$

It is worth pointing out that the notion of A-type weak solution stems from Aleksandrov's concept of generalized solution for the classical Monge-Ampère equation. Here $\mathscr{R}_u$ replaces the normal mapping $\partial^+ w$ of convex function w. Accordingly, the paraboloids replace the supporting planes. In order to show that Aleksandrov's measure, defined as $\mu_w(E) = |\partial^+ w(E)|$, is indeed a Radon measure, it is enough to check that $\mu_w(E)$ is countably additive, see [1],[17]. This property follows once we establish that the normal mapping $\partial^+ w$ of convex function w is one-to-one modulo a set of measure zero. This was shown by Aleksandrov for the classical Monge-Ampère equation, see [1], Chapter 5.2.

Definition 10.7. *A function $u \in \mathbb{W}^+(\mathcal{U}, \mathcal{V})$ is said to be A-type weak solution of (**P**) if u is a A-type weak solution of (10.10) and*

$$\overline{\mathcal{V}} \subset \overline{\mathscr{R}_u(\mathcal{U})}, \qquad |\{x \in \mathcal{U} : \mathscr{R}_u(x) \not\subset \mathcal{V}\}| = 0 \tag{10.11}$$

This definition is natural, stating that the target domain $\mathcal{V}$ is covered by the reflected rays and the endpoints of those rays that after reflection do not strike $\mathcal{V}$ constitute a null set on $\mathcal{U}$.

10.3. **Comparison principle.** An immediate consequence of Lemma 9.1 is the following comparison principle.

Proposition 10.8. *Let u_i be weak solutions of (4.4) in Ω with $f = f_i, i = 1, 2$, where Ω is a smooth, bounded domain in Π. Suppose that $R_{u_1}(\Omega) \subset \overline{\Sigma}$, $f_1 < f_2$ in Ω and $u_1 \leq u_2$ on $\partial\Omega$. If Γ_1, the graph of u_1, lies in the region $\mathcal{D}$ then we have $u_1 \leq u_2$ in Ω.*

Proof. Suppose that $\Omega_1 = \{x \in \Omega : u_1(x) > u_2(x)\}$ is not empty. Let $x_0 \in \Omega_1$ and $P(x, \sigma_0, Z_0), Z_0 \in \Sigma$ is a supporting paraboloid of u_2 at x_0. From the confocal expansion of paraboloids (see subsection 7.2) we infer that $P(x, \sigma_0 + \varepsilon, Z_0)$ is a supporting paraboloid of u_1 at an interior point $x_1 \in \Omega_1$ for some $\varepsilon > 0$. Thus $P(x, \sigma_0 + \varepsilon, Z_0)$ is a local supporting paraboloid of u_1. Since Γ_{u_1} is in the regularity domain $\mathcal{D}$ we can apply Lemma 9.1 to conclude that $P(x, \sigma_0 + \varepsilon, Z_0)$ is also a global supporting paraboloid of u_1. Therefore

$$\mathscr{R}_{u_2}(\Omega_1) \subset \mathscr{R}_{u_1}(\Omega_1)$$

implying

$$\int_{\Omega_1} f_1 dx < \int_{\Omega_1} f_2 dx = \int_{\mathscr{R}_{u_2}(\Omega_1)} g d\mathcal{H}^n_\Sigma \leq \int_{\mathscr{R}_{u_1}(\Omega_1)} g d\mathcal{H}^n_\Sigma = \int_{\Omega_1} f_1 dx$$

which gives a contradiction. Thus $\Omega_1 = \emptyset$. ■

In closing this section we state the weak convergence result for the α-measures.

Lemma 10.9. *Let u_k be a sequence of A-type weak solutions and α_k is the corresponding measure, defined by (10.1). If $u_k \to u$ uniformly on compact subsets of $\mathcal{U}$ then u is $R-$concave and α_k weakly converges to $\alpha_{u,g}$.*

The proof is very similar to that of Lemma 8.5 (modulo minor adjustments) and hence omitted.

11. Comparing A and B type solutions

Let $\varphi : \mathbb{R}^N \to \mathbb{R}^n$ be a Borel mapping and $\mu(\mathbb{R}^N) = \nu(\mathbb{R}^n) < \infty$ with μ, ν being two Radon measures on respectively $\mathbb{R}^N$ and $\mathbb{R}^n$. Then φ induces a (push-forward) measure on $\mathbb{R}^n$ defined by $\varphi_\#\mu(E) = \mu(\varphi^{-1}(E))$ for Borel subsets $E \subset \mathbb{R}^n$. A Borel mapping φ is said to be measure preserving if

$$\varphi_\#\mu(E) = \nu(E) \qquad \text{for any Borel set } E \subset \mathbb{R}^n. \tag{11.1}$$

By the change of variables formula (11.1) can be rewritten in the following equivalent form

$$\int h(\varphi(x))d\mu = \int h(y)d\nu, \qquad \forall h \in C(\mathbb{R}^n). \tag{11.2}$$

The integral identity (11.2) was used by Brenier to give a weak formulation for optimal mass transfer problems [4], [5].

If $u \in \mathbb{W}^+(\mathcal{U}, \Sigma)$ is the B-type solution of (**P**), the existence of which is established in Section 8, then taking $\varphi(Z) = \mathscr{S}_u(Z), N = n+1$, $d\mu = gd\mathcal{H}^n_\Sigma$, and ν being the Lebesgue measure one immediately observes that $\mathscr{S}_u$ is measure preserving in the sense of (11.1) or (11.2).

Lemma 11.1. *If $\mathscr{R}_u(x) \subset \mathcal{V}$ for a.e. $x \in \mathcal{U}$ then $\mathscr{R}_u(E) \subset \mathrm{Hull}(\mathcal{V})$, where $\mathrm{Hull}(\mathcal{V})$ is the R-convex hull of $\mathcal{V}$ defined as the smallest R-convex subset of Σ containing $\mathcal{V}$.*

Proof. We only have to consider the points where u is non-differentiable. Let u be non-differentiable at $x_0 \in \mathcal{U}$. and suppose that γ_1, γ_2 are the normals of two supporting planes of u at x_0. The ray with endpoint x_0 after reflection will lie in the reflector cone $\mathcal{C}_{x_0,\gamma_1,\gamma_2}$ and a fortiori the reflected ray will strike $\mathrm{Hull}(\mathcal{V})$, because $\mathcal{C}_{x_0,\gamma_1,\gamma_2} \cap \mathrm{Hull}(\mathcal{V})$ is connected. Considering all normals of supporting planes at x_0 we obtain the desired result. ■

Proposition 11.2. *Let Σ be $R-$convex with respect to $Q_m = \mathcal{U} \times (0, m), m > 0$ and the densities f, g are positive. Then B-type weak solution is also of A-type.*

Proof. First we show that for any compact $K_1 \subset \mathcal{U}$ there holds $\int_{K_2} gd\mathcal{H}^n_\Sigma \geq \int_{K_1} f(x)dx$ with $K_2 = \mathscr{R}_u(K_1)$. In other words the B-type solution is A-type subsolution. For the proof of this inequality we don't need $\mathcal{V}$ to be $R-$convex. Let $\eta \in C(\Sigma)$ such that $\eta \equiv 1$ on $K_2 \subset \Sigma$ and $0 \leq \eta \leq 1$. Consequently we obtain from (11.2)

$$\int_{\mathcal{V}} \eta g d\mathcal{H}^n_\Sigma = \int_{\mathcal{U}} \eta(\mathscr{R}_u(x))f(x)dx \geq \int_{K_1} f(x)dx.$$

Letting η to decrease to the characteristic function of K_2, $h \downarrow \chi_{K_2}$ we conclude the inequality

$$\int_{K_2} gd\mathcal{H}^n_\Sigma \geq \int_{K_1} f(x)dx. \tag{11.3}$$

In this argument K_1 can be replaced by any Borel subset of $\mathcal{U}$ since by Proposition 10.5 the measure $\alpha_{u,g}$ is Borel regular. As a consequence we infer from (11.3) that

$$\text{if } \mathcal{H}^n_\Sigma(\mathscr{R}_u(E)) = 0 \text{ then } |E| = 0. \tag{11.4}$$

To prove the converse estimate of (11.3) we utilize the $R-$convexity of $\mathcal{V}$. Take any compact $K_1 \in \mathcal{U}$ and apply Proposition 10.5 to conclude $\mathcal{H}^n_\Sigma\big(\mathscr{R}_u(K_1) \cap \mathscr{R}_u(\mathcal{U} \setminus K_1)\big) = 0$. We claim that

$$|\mathscr{R}_u^{-1}(\mathscr{R}_u(K_1)) \setminus K_1| = 0 \tag{11.5}$$

where $\mathscr{R}_u^{-1}(\mathscr{R}_u(K_1))$ is the pre-image of $\mathscr{R}_u(K_1)$. Denote $E = \mathscr{R}_u^{-1}(\mathscr{R}_u(K_1))$ and $G = K_1$. In view of (11.4) it is enough to check that $\mathcal{H}^n_\Sigma(E \setminus G) = 0$. Indeed, form the identity (8.2) it follows that

$$\begin{aligned} |\mathscr{R}_u(E \setminus G)| &= \Big|[\mathscr{R}_u(E) \setminus \mathscr{R}_u(G)] \bigcup [\mathscr{R}_u(E \setminus G) \cap \mathscr{R}_u(G)]\Big| \\ &= |\mathscr{R}_u(E \setminus G) \cap \mathscr{R}_u(G)| \\ &= 0 \end{aligned} \tag{11.6}$$

where to get the last line we used the definitions of E and G in order to obtain $\mathscr{R}_u(E)\setminus\mathscr{R}_u(G) = \mathscr{R}_u(K_1)\setminus\mathscr{R}_u(K_1) = \emptyset$ and Proposition 10.5. Hence (11.4) implies $0 = |E \setminus G| = |\mathscr{R}_u^{-1}(\mathscr{R}_u(K_1)) \setminus K_1|$.

Now we are ready to finish the proof and establish the converse of the inequality (11.3). Let $h \in C(\Sigma)$ such that $0 \le h \le 1$ and $h \ge \chi_{\mathscr{R}_u(K_1)}$. Since $\mathcal{V}$ is $R-$convex it follows that $\mathscr{R}_u(K_1) \subset \mathrm{Hull}\mathcal{V}$, see Lemma 11.1. From the definition of B-type solutions we have

$$\begin{aligned} \int_{\mathcal{U}} \eta(\mathscr{R}_u(x)) f(x) dx &= \int_{\mathcal{V}} \eta g d\mathcal{H}^n_\Sigma \\ &= \int_{\mathrm{Hull}(\mathcal{V})} \eta g d\mathcal{H}^n_\Sigma \\ &\ge \int_{\mathscr{R}_u(K_1)} g d\mathcal{H}^n_\Sigma. \end{aligned}$$

If $\eta \to 0$ on compact subsets of $\mathcal{V}\setminus\mathscr{R}_u(K_1)$ then $\eta(\mathscr{R}_u(x))$ uniformly converges to zero one the compact subsets of $\mathcal{U} \setminus \mathscr{R}_u^{-1}(\mathscr{R}_u(K_1))$. Therefore from (11.5) we infer

$$\int_{\mathscr{R}_u(K_1)} g d\mathcal{H}^n_\Sigma \le \int_{\mathcal{U}} \eta(\mathscr{R}_u(x)) f(x) dx \longrightarrow \int_{\mathscr{R}^{-1}(\mathscr{R}_u(K_1))} f(x) dx = \int_{K_1} f(x) dx$$

which in view of (11.5) implies the desired estimate. It remains to check that u verifies the boundary condition (10.11). Suppose that there is $Z_0 \in \overline{\mathcal{V}}$ such that $Z_0 \notin \overline{\mathscr{R}_u(\mathcal{U})}$. Since u is of B-type, it follows that $\mathscr{S}_u(\mathcal{V}) = \mathcal{U}$ implying $x_0 \in \mathscr{S}_u(Z_0)$ (in other words, there is a supporting paraboloid $P(x, \sigma_0, Z_0)$ at x_0). Thus $Z_0 \in \mathscr{R}_u(x_0)$. Therefore $\overline{\mathcal{V}} \subset \overline{\mathscr{R}_u(\mathcal{U})}$. From energy balance condition we have

$$\int_{\overline{\mathscr{R}_u(\mathcal{U})}} g d\mathcal{H}^n_\Sigma = \int_{\mathcal{U}} f(x) dx = \int_{\mathcal{V}} g d\mathcal{H}^n_\Sigma.$$

This yields $|\{x \in \mathcal{U} : \mathscr{R}_u(x) \not\subset \mathcal{V}\}| = 0$ for $f, g > 0$. ■

Remark 11.1. *Since $\mathcal{V}$ is $R-$convex it follows that $\mathscr{R}_u(\mathcal{U}) \subset \mathcal{V}$. Thus we get the equality $\overline{\mathscr{R}_u(\mathcal{U})} = \overline{\mathcal{V}}$ for R-convex $\mathcal{V}$.*

11.1. **Existence of A-type weak solutions: Proof of Theorem 1 c).** Suppose that $\mathcal{V} \subset \Sigma$ and let $\mathrm{Hull}(\mathcal{V})$ be the $R-$convex hull of $\mathcal{V}$. For small $\varepsilon, \varepsilon' > 0$ we consider

$$g_\varepsilon(Z) = \begin{cases} g(Z) - \varepsilon & \text{if } Z \in \mathcal{V} \\ \varepsilon' & \text{if } Z \in \mathrm{Hull}(\mathcal{V}) \setminus \mathcal{V} \end{cases} \tag{11.7}$$

where we choose $\varepsilon, \varepsilon'$ so that g_ε satisfies the energy balance condition (1.1). By Proposition 8.6 for each g_ε there is a B-type weak solution which according to Proposition 11.2 is also of A-type. Moreover, from Remark 11.1 we infer

$$\overline{\mathscr{R}_{u_\varepsilon}(\mathcal{U})} = \overline{\mathcal{V}}. \tag{11.8}$$

Sending $\varepsilon \to 0$ we obtain from Lemma 10.9 that $u_\varepsilon \to u$ and u is an A-type solution to (10.10) and

$$\overline{\mathcal{V}} \subset \overline{\mathscr{R}_u(\mathcal{U})}. \tag{11.9}$$

Since u is second order differentiable a.e. in $\mathcal{U}$ it follows that $\mathscr{R}_u$ is defined for a.e. $x \in \mathcal{U}$. Finally we want to show that $|S| = 0$ where

$$S = \{x \in \mathcal{U} : \exists Z \in \mathscr{R}_u(x) \text{ such that } Z \in \mathscr{R}_u(\mathcal{U}) \setminus \mathcal{V}\}.$$

Indeed, from energy balance condition (1.1) we have

$$\begin{aligned}\int_S f(x)dx &= \int_{\mathcal{U}} f(x)dx - \int_{\mathcal{U}\setminus S} f(x)dx = \\ &= \int_{\mathcal{U}} f(x)dx - \int_{\mathcal{V}} g d\mathcal{H}^n_\Sigma = 0.\end{aligned}$$

Since $f > 0$ we conclude that $|S| = 0$ and hence (10.11) holds and u is a weak A-type weak solution of (**P**). ■

Remark 11.2. *In the proof of Proposition 11.2 (see also Remark 11.1) we used the fact that if $\mathcal{V}$ is $R-$convex then $S = \emptyset$. Notice that if $S \neq \emptyset$ then u is only Lipschitz continuous. In other words, if $\mathcal{V}$ is not R-convex then u may not be C^1 smooth. Such example can be constructed by approximation of two-point receiver problem via smooth R-convex sets in Σ which in the limit converge to a polyhedral solution formed by two paraboloids, see* [9, 10] *for similar examples with regard to point source far-field problem. It is worthwhile to point out that even if $S = \emptyset$ then u may not be C^1, and hence further assumptions must be imposed to assure the smoothness of u.*

Remark 11.3. *The existence of lower admissible solutions can be established analogously. However for the A-type weak solutions we need to modify (1.11) (or its equivalent (1.10)) and require*

$$-\frac{2t}{1+|Du|^2} II + \mathrm{Id}\cos\theta > 0.$$

12. Dirichlet's problem

In this section we will discuss the existence and uniqueness of solutions to Dirichlet's problem. Notice that in the construction of either types of weak solutions we have not used the explicit form of the equation, which was derived for $u \in C^2(\mathcal{U})$ in Section 3. If $u \in C^2(\mathcal{U})$ then $\mathscr{R}_u(x), x \in \mathcal{U}$ and its Jacobian matrix are well defined.

It is convenient to write the equation (4.4) in the following concise form

$$\mathcal{F}u(x) = \frac{f(x)}{g \circ \mathscr{R}_u(x)}, \qquad x \in \mathcal{U}. \tag{12.1}$$

Definition 12.1. *A function $u \in \mathbb{W}^+(\mathcal{U}, \Sigma)$ is said to be a weak A-subsolution of (12.1) if for any Borel set E*

$$\int_{\mathscr{R}_u(E)} g d\mathcal{H}^n_\Sigma \geq \int_E f(x)dx. \tag{12.2}$$

If $\alpha_{u,g}(E) = \int_E f(x)dx$ then we say that u is a weak A-solution. The class of all generalized A-subsolutions is denoted by $\mathcal{AS}^+(\mathcal{U})$.

Let $D \subset \Sigma$ and φ be a smooth function. Consider the Dirichlet problem with boundary data φ

$$\begin{cases} \mathcal{F}u(x) = \dfrac{f(x)}{g \circ \mathscr{R}_u(x)}, & x \in D, \\ u = \varphi & x \in \partial D. \end{cases} \tag{12.3}$$

We will show the existence of weak A-solution to (12.3) by employing Perron's method.

12.1. **Shifting Σ.** We start from the following observation. Let u be a solution to (12.1) and $\varepsilon > 0$. Suppose that Σ is the plane $\langle Z, \gamma_0\rangle = d_0$ with $\gamma_0 \in \mathbb{S}^{n+1}$. One can easily verify that if $u_\varepsilon = u + \varepsilon$ and u is differentiable at $x \in \mathcal{U}$ then

$$\mathscr{R}_{u_\varepsilon}(x) = \mathscr{R}_u(x) + \varepsilon\left(e_{n+1} - Y(x)\frac{\langle \gamma_0, e_{n+1}\rangle}{\langle \gamma_0, Y(x)\rangle}\right),$$

see (3.3) and (3.4). Hence, u_ε may not be a solution to (12.1). Furthermore, it is not clear whether u_ε is upper admissible in the sense of Definition 6.1.

We address a more general question here: Under what conditions $u(x) + K(r^2 - |x - x_0|^2), K > 0$ is upper admissible in $B_r(x_0)$ (for small $r > 0$)? This question is directly connected to the proof of Theorem 2.

We recall the visibility condition for Σ, namely that Σ must be visible from any focal plane of supporting paraboloid (it was mentioned in Remark 4.2). Consequently with the aid of Remark 7.1 we conclude

$$u \in \mathbb{W}^+(\mathcal{U}, \Sigma) \quad \text{then} \quad u \in \mathbb{W}^+(\mathcal{U}, \widetilde{\Sigma}) \quad \text{where} \quad \widetilde{\Sigma} = \Sigma - Me_{n+1}, M > 0. \tag{12.4}$$

Notice that the condition (4.3) implies (12.4).

Lemma 12.2. *Let $u \in \mathbb{W}^+(D, \Sigma)$ and $B_r \subset D$. The following is true:*

- **1°** *the function $\widetilde{u}_\varepsilon = u_\varepsilon + K(r^2 - |x|^2) \in \mathbb{W}^+(B_r, \widetilde{\Sigma})$ for any $\varepsilon > 0$, where $K > \max\left\{\frac{1}{L}, \frac{2}{|L_0|}\right\}$ and u_ε is a mollification of u.*
- **2°** *$\widetilde{u}_\varepsilon \in \mathcal{AS}^+(B_r, \widetilde{\Sigma})$, i.e. u_ε is a subsolution of (12.1) in B_r.*

Proof. **1°** Let $w = u + K(r^2 - |x|^2)$ and $P(x, \sigma, Z)$ be a supporting function of u at some point $\xi \in B_r$. We have

$$\begin{aligned} P(x,\sigma,Z) + K(r^2 - |x|^2) &= \frac{\sigma}{2} + Z^{n+1} - \frac{1}{2\sigma}|x - z|^2 + K(r^2 - |x|^2) \\ &= \frac{\sigma}{2} + Z^{n+1} + Kr^2 - \frac{|z|^2}{2\sigma} - \left(\frac{1}{2\sigma} + K\right)|x|^2 + \frac{1}{\sigma}\langle x, z\rangle \\ &= \frac{\sigma}{2} + Z^{n+1} + Kr^2 - |z|^2\frac{K}{1 + 2\sigma K} - \frac{1 + 2\sigma K}{2\sigma}\left|x - \frac{z}{1 + 2\sigma K}\right|^2 \\ &= \frac{\widetilde{\sigma}}{2} + \widetilde{Z}^{n+1} - \frac{1}{2\widetilde{\sigma}}|x - \widetilde{z}|^2 \overset{\text{def}}{\equiv} \widetilde{P}(x, \widetilde{\sigma}, \widetilde{Z}) \end{aligned}$$

where

$$\widetilde{\sigma} = \frac{\sigma}{1 + 2\sigma K}, \quad \widetilde{z} = \frac{z}{1 + 2\sigma K}, \quad \widetilde{Z}^{n+1} = \frac{\sigma}{2} + Z^{n+1} + Kr^2 - |z|^2\frac{K}{1 + 2\sigma K} - \frac{\sigma}{2(1 + 2\sigma K)}. \tag{12.5}$$

With the aid of (12.5) and the estimates from Section 7.4 we infer that for $P \in \mathbb{P}_L(\mathcal{U}, \Sigma)$ it follows

$$\begin{aligned} \widetilde{Z}^{n+1} &\geq L - |z|^2\frac{K}{1 + 2\sigma K} - \frac{\sigma}{2(1 + 2\sigma K)} \geq \\ &\geq L - \frac{|z|^2}{2\sigma} - \frac{1}{4K} \geq \frac{3L}{4} - \sup\left[\frac{|z|^2}{2\sigma}\right] \overset{\text{def}}{\equiv} L_0 \end{aligned} \tag{12.6}$$

with some fixed L_0 provided that $K > \frac{1}{L}$.

The mollified function u_ε is concave and therefore $D^2\widetilde{u}_\varepsilon = D^2 u_\varepsilon - 2K\mathrm{Id} \leq -2K\mathrm{Id}$. Consequently $\widetilde{u}_\varepsilon$ is strictly concave and $\widetilde{u}_\varepsilon \in C^\infty(D)$. Therefore for any $x_0 \in B_r$ there is a local supporting paraboloid $P_0(x, \sigma_0, Z_0)$ at x_0. Since the curvature of $\widetilde{u}_\varepsilon$ is uniformly bounded by below it follows that we can choose the local supporting paraboloids P_0 such that $Z_0^{n+1} \geq L_0 - 1$ for small ε. Now take $\widetilde{\Sigma} = \Sigma - (L_0 + 10)e_{n+1}$ then applying the argument from Section 7.3 and Lemma 9.1 we see that $\widetilde{u}_\varepsilon \in \mathbb{W}^+(B_r, \widetilde{\Sigma})$.

2° Since $\widetilde{u}_\varepsilon$ is C^∞ smooth it is enough to show that it is a subslution of (1.5) in classical sense. We have

$$\widetilde{W} = \mathrm{Id} - (D^2 u_\varepsilon - 2K\mathrm{Id})\frac{2t(x,\widetilde{u}_\varepsilon, D\widetilde{u}_\varepsilon)}{1+|D\widetilde{u}_\varepsilon|^2} \geq \mathrm{Id}\left[K\frac{4t(x,\widetilde{u}_\varepsilon, D\widetilde{u}_\varepsilon)}{1+|D\widetilde{u}_\varepsilon|^2} - 1\right]$$

where t is the stretch function corresponding to $\widetilde{\Sigma}$. From (12.6) we see that $t \geq |L_0|$. Moreover $|D\widetilde{u}_\varepsilon| = |Du_\varepsilon - 2Kx| \leq 3$ if $rK \leq 1$ implying

$$\widetilde{W} \geq \mathrm{Id}[K|L_0| - 1] \geq \mathrm{Id}\frac{K|L_0|}{2}$$

provided that $K > \frac{2}{|L_0|}$. Consequently

$$\det \widetilde{W}\frac{|\nabla\psi|}{\langle|\nabla\psi, Y\rangle|} \geq \frac{K^n|L_0|^n}{2^n}\inf\left[\frac{|\nabla\psi|}{|\langle\nabla\psi, Y\rangle|}\right] \geq \frac{f}{g}$$

if K is large enough. ■

12.2. **Discrete Dirichlet problem.** In order to show that the weak solutions to the reflector problem (**P**) are locally smooth we will first establish the smoothness of u in a small ball. This is done via the continuity method and standard mollification argument, see [15]. Then from Proposition 10.8 it follows that the smooth solution, obtained via the continuity method must coincide with the weak solution u in small ball, see Section 13 for more details.

Our first aim here is to construct a weak solution to the discrete Dirichlet problem. To do so we follow the approach of Xu-Jia Wang [21]. Let $\{b_i\} \subset \partial D$ be a sequence of points on the boundary of D and $\{a_i\} \subset D$. For each fixed $N \in \mathbb{N}$ we set $A_N = \{a_1, \ldots, a_N\}$ and $B_N = \{b_1, \ldots, b_N\} \subset \partial D$. Suppose that $\nu_k(x)$ is a measure supported at $a_k, 1 \leq k \leq N$ and consider

$$\mathcal{F}v(x) = \nu_k(x)\frac{f(x)}{g \circ \mathscr{R}_v(x)}. \tag{12.7}$$

Proposition 12.3. *Let $\underline{u} \in \mathbb{W}_0^+(D,\Sigma)$ be a polyhedral subsolution of (12.7), i.e. $\mathcal{F}v(x) \geq \nu_k(x)\frac{f(x)}{g\circ\mathscr{R}_v(x)}, a_k \in A_N$. Then there is a unique A-type weak solution to (12.7) verifying the boundary condition $u = \underline{u}$ on B_N.*

Proof. Denote $u_0 = \underline{u}$. From u_0 we want to execute a new function u_1 such that $u_1 \leq u_0$ in A_N, $u_1(b_i) = u_0(b_i), b_i \in B_N$ and $\alpha_{u_1,g}(a_i) \leq \alpha_{u_0,g}(a_i)$ for $a_i \in A_N$.

Introduce the class of paraboloids

$$\Phi_{0,\varepsilon}(a_1) = \left\{ P \in \mathbb{P}_L(D,\Sigma) : \begin{array}{r} P(a_i) \geq u_0(a_i), i \neq 1, \\ P(a_1) \geq u_0(a_1) - \varepsilon, \\ P(b_j) \geq u_0(b_j), 1 \leq j \leq N \end{array} \right\}$$

for $\varepsilon > 0$ and consider

$$T_1^\varepsilon u_0 = \inf_{P\in\Phi_{0,\varepsilon}(a_1)} P.$$

Let $\varepsilon_1 > 0$ be the largest ε for which $T_1^{\varepsilon_1}u_0$ is a subsolution to (12.7) on A_N. Then we denote $u_{0,1} = T_1^{\varepsilon_1}$. We now consider

$$\Phi_{0,\varepsilon}(a_k) = \left\{ P \in \mathbb{P}_L(D,\Sigma) : \begin{array}{r} P(a_i) \geq u_{0,k-1}(a_i), i \neq k, \\ P(a_k) \geq u_{0,,k-1}(a_k) - \varepsilon, \\ P(b_j) \geq u_{0,,k-1}(b_j), 1 \leq j \leq N \end{array} \right\}$$

and take $T_k^\varepsilon u_0 = \inf_{P\in\Phi_{0,\varepsilon}(a_k)} P$. Thus we can successively construct the functions $u_{0,k} = T_k^{\varepsilon_k}u_{0,k-1}$ where ε_k is the largest number for which $T_k^\varepsilon u_{0,k-1}$ is a subsolution to (12.7) in A_N.

Set $u_2(x) \overset{\text{def}}{\equiv} T_N^{\varepsilon_N} u_{0,N-1}$. Then by construction $\alpha_{u_0,g}(a_i) \le \alpha_{u_1,g}(a_i)$, since the Φ classes may only shrink at a_k as we proceed. Therefore we have a sequence of solutions u_m to the Dirichlet problem in A_N such that

$$\begin{aligned} \alpha_{u_m,g}(a_i) &\le \alpha_{u_{m-1},g}(a_i), \\ u_m(a_i) &\le u_{m-1}(a_i), \\ u_m(b_i) &= u_{m-1}(b_i). \end{aligned}$$

The first two inequalities are obvious. As for the boundary condition we note that $u_0(b_i) \le u_1(b_i)$ by construction. If $u_0(b_i) < u_1(b_i)$ then by taking $\min[P_i(x), u_1(x)]$, where $P_i(x) \in \mathbb{P}_L(D,\Sigma)$ is a supporting paraboloid of u_0 at b_i we see that $\min[P_i(x), u_1(x)]$ belongs to the corresponding Φ class. Thus $u_0(b_i) = u_1(b_i)$.

Let us show that $u = \lim_{m\to\infty} u_m$ is a solution to the discrete problem in A_N with $u(b_i) = \underline{u}(b_i), b_i \in B_N$. Indeed, by employing Lemma 6.4 we conclude that $u \in \mathbb{W}^+(D,\Sigma)$ and in view of Lemma 10.9 $\alpha_{u_m,g} \rightharpoonup \alpha_{u,g}$ weakly. Thus the result follows. ■

12.3. **General case.** Below we use Perron's method to establish the existence for the general Dirichlet problem. We take $\{a_i\}_{i=1}^\infty \subset D$ and $\{b_i\}_{i=1}^\infty \subset \partial D$ to be dense subsets and $A_N = \{a_1,\dots,a_N\} \subset D, B_N = \{b_1,\dots,b_N\} \subset \partial D$.

Proposition 12.4. *Let $\underline{u} \in \mathcal{AS}^+(D,\Sigma)$. Then there exists unique weak solution u to the Dirichlet problem*

$$\begin{cases} \mathcal{F}u = \dfrac{f(x)}{g\circ\mathscr{R}_u(x)} & \text{in } D, \\ u(x) = \underline{u}(x) & \text{on } \partial D. \end{cases} \tag{12.8}$$

Proof. For $\delta > 0$ we denote $D_\delta = \{x \in D : \operatorname{dist}(x,\partial D) > \delta\}$ and take $\eta(x)$ to be a smooth function such that $0 \le \eta(x) \le 1$, $\eta \equiv 1$ in $D_{2\delta}$ and $\eta \equiv 0$ in $D \setminus D_\delta$. Consider the equation

$$\mathcal{F}v(x) = \nu_k(x) H(v(x)) \eta_\delta(x) \frac{f(x)-\delta}{g\circ\mathscr{R}_v(x)} \tag{12.9}$$

where $\nu_k(x)$ is a measure supported at $a_k \in A_N$ and

$$H(t) = \begin{cases} 1 & \text{if} \quad 0 \le t \le \sup\limits_D \underline{u}, \\ \frac{2\sup\limits_D \underline{u} - t}{\sup\limits_D \underline{u}} & \text{if} \quad \sup\limits_D \underline{u} \le t \le 2\sup\limits_D \underline{u}, \\ 0 & \text{if} \quad t > 2\sup\limits_D \underline{u}. \end{cases} \tag{12.10}$$

Consider the class

$$\mathbb{W}^+_{N,\underline{u}} = \left\{ v \in \mathbb{W}^+_0(D,\Sigma) : \mathcal{F}v \ge \nu_k H(v) \eta_\delta \frac{f-\delta}{g\circ\mathscr{R}_u} \text{ and } v \ge \underline{u} \text{ on } B_N \right\}. \tag{12.11}$$

Clearly $\mathbb{W}^+_{N,\underline{u}}$ is not empty since $P(\cdot,\sigma,Z)$ is in this class if $\sigma > 0$ is sufficiently large. Set $v_{N,\delta} = \inf\limits_{\mathbb{W}^+_{N,\underline{u}}} v$. We claim that $v_{N,\delta}$ solves (12.1) in the sense of Definition 12.1 and $v_{N,\delta}(b_i) = \underline{u}(b_i), b_i \in B_N$.

It is easy to see that $\alpha_{v_{N,\delta},g}(a_k) = v_k(a_k) H(v_{N,\delta}) \eta_\delta(a_k)\,(f(a_k)-\delta)$. Indeed, if $v_{N,\delta}$ is a strict subsolution at a_i, i.e. for some a_i we have $\alpha_{v_{N,\delta},g}(a_i) > v_k(a_i) H(v_{N,\delta}) \eta_\delta(a_i)(f(a_i)-\delta)$, then we can push $\Gamma_{v_{N,\delta}}$ down by some $\varepsilon > 0$, decreasing the α measure at a_i, which, however, will be in contradiction with the definition of $v_{N,\delta}$. Thus $v_{N,\delta}$ is a solution of the equation (12.9).

Next, we check the boundary condition. Choose $P_i \in \mathbb{P}_L(\mathcal{U}, \Sigma)$ such that $P_i > v_\delta$ in $\mathcal{U}_\delta$ and passes through $(b_i, \underline{u}(b_i))$. Such P_i exists because by construction $v_{N,\delta}(a_i) \leq \underline{u}(a_i)$ and $\delta > 0$.

For $\widetilde{P}_i = \min[P_i, v_{N,\delta}]$, by construction, we see that $\mathcal{F}\widetilde{P}_i \geq \nu_k H(\widetilde{P}_i)\eta_\delta \frac{f-\delta}{g\circ\mathscr{R}_{\widetilde{P}_i}}$ at a_i. Thus $\widetilde{P}_i \in \mathbb{W}^+_{N,\underline{u}}$. Hence

$$v_{N,\delta}(b_i) = \inf_{P\in\mathbb{W}^+_{N,\underline{u}}} P(b_i) \leq \widetilde{P}_i(b_i) = \underline{u}(b_i).$$

Now the desired solution can be obtained via a standard compactness argument that utilizes the estimates from Section 7.4 and Lemma 10.9. More precisely, for fixed $\delta > 0$ we send $N \to \infty$ and obtain a function v_δ that solves the equation $\mathcal{F}v_\delta = H(v_\delta)\eta_\delta \frac{f}{g\circ\mathscr{R}_{v_\delta}}$. To show that $v_\delta = \underline{u}$ on ∂D we take $x_0 \in \partial D$ and again use the comparison with $\min[P_0, v_\delta]$ for a suitable $P_0 \in \mathbb{P}_L(\mathcal{U}, \Sigma)$ such that $P_0(x_0) = \underline{u}(x_0)$. Thus, from Proposition 10.8 we conclude that $v_\delta \leq \underline{u}$ in D. Finally sending $\delta \downarrow 0$ and employing the estimate (7.8) and Lemmas 7.2 and 10.9 we complete the proof. ■

To control the boundary behaviour for the constructed family of approximations v_δ we used Bakelman's construction and Perron's method, see [3] page 218. For the classical Monge-Ampère equation in two spatial dimensions it was observed there that if the equation's right had side is not localized by the cutoff function η_δ, then the boundary curve γ (given beforehand) may not be the boundary of the limit surface constructed by Perron's method. Thus, it was necessary to multiply the right hand side of the equation by the cut-off function η_δ to gain control near ∂D, see [15] page 31. We also note that $H(v)$ was introduced for technical reasons, namely it absorbs the values of sufficiently large paraboloids used in the construction.

Remark 12.1. *For lower admissible functions the solution to Dirichlet's problem can be constructed analogously. The necessary condition then will be the existence of a lower admissible supersolution $\overline{u} \in \mathbb{W}^-(D, \Sigma)$, i.e. $\mathcal{F}\overline{u} \leq \frac{f}{g\circ\mathscr{R}_{\overline{u}}}$, such that $u = \overline{u}$ on ∂D.*

13. Proof of Theorem 2

In this section we prove our main regularity result Theorem 2. We first establish global a priori $C^{2,\alpha}$ estimates in any small ball contained in $\mathcal{U}$. Then using the continuity method we conclude the existence of locally smooth A-type weak solutions.

Let $u^\pm_\varepsilon$ be the solutions to

$$\begin{cases} \mathcal{F}u^\pm_{\varepsilon,\delta} = \frac{f\pm\delta}{\eta g\circ Z_{u^\pm_{\varepsilon,\delta}}} & \text{in } B_r \\ u^\pm_{\varepsilon,\delta} = \widetilde{u}_\varepsilon & \text{on } \partial B_r \end{cases} \tag{13.1}$$

where $\widetilde{u}_\varepsilon = u_\varepsilon + K(r^2 - |x|^2)$, $K > 0$ and u_ε is a mollification of the weak solution u. By Lemma 12.2 $\widetilde{u}_\varepsilon$ is a subsolution and hence by Proposition 12.4 the solution to Dirichlet problem exists. Note that for the Dirichlet problem we have to consider the modified receiver $\widetilde{\Sigma}$, see Lemma 12.2. Letting $\varepsilon \to 0$ and applying the comparison principle (see Proposition 10.8) we have that $u^-_{0,\delta} \leq u \leq u^+_{0,\delta}$ and $u^\pm_{0,\delta} = u$ on ∂B_r. It follows from the a priori estimates established in Section 5 that $u^\pm_{0,\delta}$ are locally uniformly C^2 in B_r for any small $\delta > 0$. After sending $\delta \to 0$ we will conclude the proof of Theorem 2. Thus the result will follow once we establish the existence of C^2 solutions $u^\pm_{\varepsilon,\delta}$ of (13.1) in $\overline{B_r}$.

Estimates for the Dirichlet problem. Let $\underline{w} \in \mathcal{AS}^+(B_r, \mathcal{V}) \cap C^\infty(B_r)$ and for $t \in [0,1]$ consider the solutions to the Dirichlet problem

$$\begin{cases} \mathcal{F}w^t = t\frac{f}{hg\circ Z_w} + (1-t)\mathcal{F}\underline{w} & \text{in } \ B_r, \\ w^t = \underline{w} & \text{on } \partial B_r. \end{cases} \tag{13.2}$$

Using the implicit function theorem, see [19] Theorem 5.1 we can see that the set of t's for which (13.2) is solvable is open.

To show that it is also closed we need to establish global $C^{1,1}$ a priori estimates in $\overline{B_r}$. Recall that if $\partial\Omega \in C^3, u \in C^4(\Omega) \cap C^3(\overline{\Omega})$ and $\underline{u} \in C^4$ then from global $C^{1,1}$ estimates and the elliptic regularity theory we obtain that $w \in C^{2,\alpha}(\overline{\Omega})$. Therefore the existence of smooth $u^{\pm}_{\varepsilon,\delta}$ will follow once we establish the global $C^{1,1}$ estimate for w. We have

Proposition 13.1. *Let $h, \underline{w} \in C^\infty(B_r(x_0))$ and w solves the Dirichlet problem*

$$\begin{cases} \det\left[D^2w - \dfrac{1+|Dw|^2}{2t}\mathrm{Id}\right] = \dfrac{f}{\eta g} & \textit{in } \ B_r(x_0), \\ w = \varphi & \textit{on } \partial B_r(x_0). \end{cases}$$

Then $\|w\|_{C^2(\overline{B_r(x_0)})} \le C$ where C depends on r, $\|f\|_{C^4(\overline{B_r(x_0)})}$, $\|g\|_{C^4(\overline{\mathcal{V}})}$ and $\|\varphi\|_{C^4(\overline{B_r(x_0)})}$. Here η is defined by (4.5)

Proof. We employ the barrier argument from [6] section 7.

If the maximum of D^2w is realized at interior point then we can apply the estimates from Section 5 (with u replaced by $-w$). Thus without loss of generality we assume that the maximum is realized at some $x_0 \in \partial B_r(x_0)$. In what follows we denote $\Omega = B_r(x_0)$ to be consistent with the notations in [6]. For simplicity we take x_0 to be the origin and e_n being the inner normal at $0 \in \partial\Omega$ where $x_0 = re_n$. Introduce the barrier function

$$v(x) = \frac{1}{2}(B_{\alpha\beta} - \mu\delta_{\alpha\beta})x_\alpha x_\beta + \frac{1}{2}Mx_n^2 - x_n$$

with $\mu > 0$ fixed so small that the matrix $B_{\alpha\beta} - \mu\delta_{\alpha\beta} > 0$. If ε is sufficiently small then

$$v(x) \le -c\varepsilon^2 \text{ on } \ \partial(B_\varepsilon \cap \Omega) \tag{13.3}$$

see [6] (7.25), (7.27) and (7.28). In other words, we facilitate the choice of constants ε, M, μ in [6]. We will see that under the same assumptions $v(x) + K|x - re_n|^2$ works well as a barrier function for our equation provided that $K > 0$ is large enough.

Next, we introduce the tangential operator $T_\alpha = \partial_\alpha + \omega_\alpha\partial_n, \alpha < n$, where $x_n = \omega(x')$ is the defining function of Ω near the origin. It follows that

$$|T_\alpha(w - \varphi)| \le C|x'|^2, \alpha < n, \text{ on } \partial\Omega \cap B_\varepsilon \text{ near the origin} \tag{13.4}$$

see [6] (7.21). On the remaining part of $\partial(\Omega \cap B_\varepsilon)$ we have $|T_\alpha(w - \varphi)| \le C$.

Denoting $h = \frac{f}{\eta g}$, where η is given by (4.5), and $\mathcal{F} = D^2w - \frac{G}{2}\mathrm{Id}, G = \frac{1+|Dw|^2}{2t}$ we differentiate the equation $\det\mathcal{F} = h$ to obtain

$$\begin{aligned} \mathcal{F}^{ij}\left[D_{ij}w_k - \frac{\delta_{ij}}{2}G_{k_l}D_lw_k - \frac{\delta_{ij}}{2}\left\{G_ww_k + G_{x_k}\right\}\right] = \\ = h_{p_l}D_lw_k + h_ww_k + h_{x_k} \end{aligned} \tag{13.5}$$

where $\mathcal{F}^{ij}$ is the cofactor matrix

$$\mathcal{F}^{ij} = \det \mathcal{F}([\mathcal{F}]^{-1})_{ij}. \tag{13.6}$$

Introduce the linear operator

$$\mathcal{L} = \mathcal{F}^{ij}(D_{ij} - \frac{\delta_{ij}}{2} G_{p_l} D_l) - h_{p_l} D_l$$

then from (13.5) we infer

$$\mathcal{L} w_k = O(1 + \mathrm{Tr}\mathcal{F}^{ij}). \tag{13.7}$$

Furthermore, we have that

$$\mathcal{L} T_\alpha w = \mathcal{L} w_\alpha + \mathcal{L}(\omega_\alpha w_n) + O(1 + \mathrm{Tr}\mathcal{F}^{ij}).$$

As for the second term we see that

$$\begin{aligned}
\mathcal{L}(\omega_\alpha w_n) \;=\; & \mathcal{F}^{ij}(\omega_{\alpha ij} w_n + \omega_{\alpha i} w_{nj} + \omega_{\alpha j} w_{ni} + \omega_\alpha w_{nij}) - \\
& - \frac{\mathcal{F}^{ij}}{2} G_{p_l} \delta_{ij} [\omega_{\alpha l} w_n + \omega_\alpha w_{nl}] - \\
& - h_{p_l}(\omega_{\alpha l} w_n + \omega_\alpha w_{nl}) \\
= \; & \omega_\alpha \mathcal{L} w_n - h_{p_l} \omega_{\alpha l} w_n \\
& + \mathcal{F}^{ij} \left(\omega_{\alpha ij} w_n + \omega_{\alpha i} w_{nj} + \omega_{\alpha j} w_{ni} - \frac{\delta_{ij}}{2} G_{p_l} \omega_{\alpha l} w_n \right).
\end{aligned}$$

By (13.6) $\mathcal{F}^{ij} w_{nj} = \delta_{jn} \det \mathcal{F}$, hence

$$\begin{aligned}
\mathcal{L} T_\alpha w = \mathcal{L} w_\alpha + \omega_\alpha \mathcal{L} w_n - h_{p_l} \omega_{\alpha l} w_n \\
+ \mathcal{F}^{ij}(\omega_{\alpha ij} w_n - \frac{\delta_{ij}}{2} G_{p_l} \omega_{\alpha l} w_n) \\
+ \mathcal{F}^{ij} \det \mathcal{F}(\delta_{jn} + \delta_{in}) + \\
+ O(1 + \mathrm{Tr}\mathcal{F}^{ij}).
\end{aligned}$$

Next, applying (13.7) we get

$$\mathcal{L} T_\alpha w = O(1 + \mathrm{Tr}\mathcal{F}^{ij}).$$

Since $\varphi \in C^\infty$ it follows that

$$|\mathcal{L}(T_\alpha(w - \varphi))| \leq C(1 + \mathrm{Tr}\mathcal{F}^{ij}) \tag{13.8}$$

for some $C > 0$ under control.

Next, we compute

$$\begin{aligned}
\mathcal{L} v \;=\; & \mathcal{F}^{\alpha\beta}(B_{\alpha\beta} - \mu \delta_{\alpha\beta}) + M \mathcal{F}^{nn} \\
& - \frac{1}{2} \mathrm{Tr}\mathcal{F}^{ij} G_{p_l} O(|x|) - \frac{1}{2} \mathrm{Tr}\mathcal{F}^{ij} G_{p_n} - \\
& - h_{p_l}[(1 + M) O(|x|) - \delta_{kl}].
\end{aligned} \tag{13.9}$$

Using the inequality

$$\frac{1}{2} \mathcal{F}^{\alpha\beta}(B_{\alpha\beta} - \mu \delta_{\alpha\beta}) + M \mathcal{F}^{nn} \geq c_0 M^{\frac{1}{n}} \tag{13.10}$$

(the proof of this inequality is identical to that of in [6] page 395) we can control the last term in the computation above. Indeed, from (13.9) and (13.10) we see that

$$
\begin{aligned}
\mathcal{L}v \geq\ & c_0 M^{\frac{1}{n}} + \frac{1}{2}\mathcal{F}^{\alpha\beta}(B_{\alpha\beta} - \mu\delta_{\alpha\beta}) + M\mathcal{F}^{nn} - \\
& -\frac{1}{2}\mathrm{Tr}\mathcal{F}^{ij}G_{p_l}O(|x|) - \frac{1}{2}\mathrm{Tr}\mathcal{F}^{ij}G_{p_n} - \\
& -h_{p_l}[(1+M)O(|x|) - \delta_{kl}] \\
\geq\ & c_0 M^{\frac{1}{n}} + O(1+M\varepsilon) + c_1(1+\mathrm{Tr}\mathcal{F}^{ij}) - \frac{1}{2}\mathrm{Tr}\mathcal{F}^{\mathrm{ij}}(G_{p_l}O(|x|) + G_{p_n}).
\end{aligned} \tag{13.11}
$$

Recall that $M\varepsilon \leq 1$, see (7.25) [6].

Let $q(x) = K(|x - re_n|^2 - r^2)$ for some $K > 0$ to be fixed later. Clearly $q(x) < 0$ in $\Omega = B_r(re_n)$ and $q(x)$ is convex. Now we take $v_1(x) = v(x) + q(x)$ for some large $K > 0$. Then (13.11) yields

$$
\mathcal{L}v_1 \geq c_2(1 + \mathrm{Tr}\mathcal{F}^{ij}) + 2K\mathrm{Tr}\mathcal{F}^{ij} - \frac{1}{2}\mathrm{Tr}\mathcal{F}^{ij}G_{p_n}.
$$

Choosing K sufficiently large we conclude

$$
\mathcal{L}v_1 \geq c(1 + \mathrm{Tr}\mathcal{F}^{ij})
$$

and $v_1(x) \leq v(x) \leq -c_4\varepsilon^2$ on $\partial(\Omega \cap B_r)$.

Thus v_1 controls $\pm AT_\alpha(w - \varphi)$ for some constant A as in [6] and hence

$$
|D_{n\alpha}w - D_{n\alpha}\varphi| \leq C \quad \alpha = 1, \ldots, n-1.
$$

The remaining derivative w_{nn} can be directly estimated from the equation. ■

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