THE STEFAN PROBLEM WITH CONSTANT CONVECTION

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ABSTRACT. We prove the optimal local regularity of the weak solutions $u \in H^1(\mathcal{C}_L)$ to the two phase continuous casting problem

$$\Delta u = \operatorname{div}[\beta(u)\mathbf{v}], \quad \text{in } \mathcal{C}_L = \Omega \times (0, L), L > 0$$

with given convection $\mathbf{v}$, a domain $\Omega \subset \mathbb{R}^{N-1}$ with Lipschitz boundary and enthalpy $\beta(u)$. In order to prove our main result for the one phase problem, i.e. when $u \geq 0$, we use the method of dyadic scaling whilst for the two phase problem the Alt-Caffarelli-Friedman monotonicity formula is employed.

1. Introduction

In this article we prove the optimal regularity for the weak solutions to Stefan problem with convection. The Stefan problem provides a mathematical model for the phase-transition phenomenon. An example of this sort is the continuous casting problem, which models a metal fabrication technique used in the production of ingots [6] page 32. In this case the liquid phase moves with a prescribed constant velocity $\mathbf{v} = \mathbf{e}_N$. For similar problems, for instance describing the thawing or freezing of the water where the liquid part is in motion, and relevant physical background we refer to [2] Chapter 10.7, [3], [6].

Here we focus on a model stationary problem reflecting the basic peculiars that most phase-transition problems with convection share.

2. Problem set up

Given a bounded Lipschitz domain $\Omega \subset \mathbb{R}^{N-1}$. Let $L > 0$ and $\mathcal{C}_L = \Omega \times (0, L)$. The points in $\mathcal{C}_L$ are denoted by $X = (x, z)$, where $x \in \Omega$ and $z \in (0, L)$. In what follows $D_{x_i}u, D_z u, i = 1, \ldots, N-1$ denote the partial derivatives of u.

Keywords: Continuous casting problem, Stefan problem, free boundary, convection.

2010 *Mathematics Subject Classification:* 35R35, 35B65, 80A22.

AK thanks the University of Lisboa for a visiting appointment.

Let $u = u(X,t)$ be the temperature at $X \in \mathcal{C}_L$ at time t. The heat flux, i.e. the net amount of heat flowing across surface $S \subset \mathcal{C}_L$ from one side to another each second, is $F = \int_S \mathbf{q} \cdot \nu d\sigma$, where ν is the outer unit normal to S, $\mathbf{q}$ is the flux vector. According to Fourier's law we have $\mathbf{q} = -k\nabla u$ where k is the thermal conductivity.

We assume that the total energy (or *enthalpy* if the pressure is constant) at time t is the sum of heat and latent heat, i.e. $\beta = c\rho u \mathrm{d}V + L\rho \mathrm{d}V$, where ρ is the density and c is the *specific heat constant* of the material and $\mathrm{d}V$ is an elementary volume. In the integral form the total heat for some subregion $B \subset \mathcal{C}_L$ is $H = \int_B \beta \mathrm{d}V$.

It is possible to have heat generated in the various parts of $\mathcal{C}_L$, called heat sources, or extracted called heat sinks. The source (or sink) with density f contributes the amount of heat $E = \int_B f \mathrm{d}V$.

The energy conservation then, if no convection is present, gives $\frac{d}{dt}H = -F + E$ or in the integral form

$$\frac{d}{dt}\int_B \beta = -\int_S \mathbf{q} \cdot \nu d\sigma + \int_B f = -\int_B \operatorname{div} \mathbf{q} \mathrm{d}V + \int_B f.$$

Here and below ν denotes the outer unit normal to $S = \partial B$.

If the liquid part is moving then after time t a point $X \in B$ goes to $Y = Y(X,t) \in B(t)$, where $\mathbf{v}$ is the velocity of convection and $B(t)$ is the image of B under mapping $Y(\cdot, t)$. Employing change of variables and Euler's formula (see [8] Chapter 2.3) we have $\frac{\partial J}{\partial t} = J \operatorname{div} \mathbf{v}$ with $J = \det DY, \mathrm{d}V = JdX$ we obtain from the energy balance condition

$$\begin{aligned} \frac{d}{dt}\int_{B(t)} \beta &= \int_B \left[\frac{\partial \beta}{\partial t} + \nabla\beta \cdot \mathbf{v} + \beta \operatorname{div} \mathbf{v}\right] J dx \\ &= -\int_{B(t)} \operatorname{div} \mathbf{q} \mathrm{d}V + \int_{B(t)} f. \end{aligned}$$

Thereby we have the partial differential equation

$$\frac{\partial}{\partial t}\beta + \operatorname{div}[\beta \mathbf{v}] = -\operatorname{div} \mathbf{q} + f. \tag{2.1}$$

For the steady state problem the equation (2.1) reduces to

$$\operatorname{div}[\beta \mathbf{v}] = \operatorname{div}(k\nabla u) + f. \tag{2.2}$$

Furthermore, we assume that the thermal conductivity k is constant and the liquid is incompressible, e.g. $\rho = 1$. As for the enthalpy $\beta(u)$ we recall it's common form

$$\beta(s) = \begin{cases} as & \text{if} \quad s < 0, \\ \in [0,1] & \text{if} \quad s = 0, \\ as + 1 & \text{if} \quad s > 0, \end{cases} \tag{2.3}$$

see [5]. Note that $\beta(s) = as + H(s)$ where $H(s)$ is the Heaviside function. It follows from (2.2) that the temperature u solves the following boundary value problem

$$\begin{cases} \Delta u = D_z\beta(u) + f & \text{in} \quad \mathcal{C}_L, \\ u(x,0) = h_0(x) & \text{on} \quad \Omega \times \{0\}, \\ u(x,L) = h_1(x) & \text{on} \quad \Omega \times \{L\}, \\ \dfrac{\partial u}{\partial \nu} = (u - u_0) & \text{on} \quad \partial\Omega \times (0,L), \end{cases} \tag{2.4}$$

where h_0, h_1 are given functions, β is the enthalpy defined by (2.3), ν is the exterior unit normal to $\partial\Omega \times (0,L)$, $a > 0$ is a constant, and u_0 is the phase change temperature which, for simplicity, we take $u_0 = 0$.

Remark 1. *The weak formulation of the problem (2.4), and the existence of bounded weak solutions, can be found in* [5] *Theorem 4.1 page 190 (see also* [7]*).*

In in the first part of Theorem 2 we allow the convection $\mathbf{v} : \mathcal{C}_L \to \mathbb{R}^N$ to be a bounded vector-field whereas in the second part we take $\mathbf{v} = \mathbf{e}_N \stackrel{\text{def}}{\equiv} (0, 0, \dots, 1)$, the unit direction of the $z-$direction, see Section 4.1.

3. Optimal Growth

3.1. **Continuity of solutions.** By Theorem 4.1. [5], u is bounded. Moreover the weak solutions of (2.2) are continuous for u solves the divergence structure equation $\Delta u = \operatorname{div} \mathbf{f}$ in $\mathcal{C}_L$, where one can take $\mathbf{f} = \mathbf{e}_N \beta \in L^\infty(\mathcal{C}_L, \mathbb{R}^N)$ if the convection is constant. Thus the continuity of u follows from DeGiorgi's estimates. In fact one can show that u is α-Hölder continuous for any positive $\alpha < 1$. This means that $\{u > 0\}$ and $\{u < 0\}$ are open sets. Clearly u cannot be more than Lipschitz continuous as the Stefan condition

$$\frac{L}{k} = Du^+\nu^+ - Du^-\nu^-$$

indicates, see [5]. Here $u^\pm$ are the positive and negative parts of $u = u^+ - u^-$ and ν^+, ν^- are the outer normals to $\{u > 0\}$ and $\{u < 0\}$, respectively. Thus the following problem rises: Are the weak solutions u to the equation $\Delta u = \operatorname{div}[\beta(u)\mathbf{v}] + f$ locally Lipschitz continuous?

Theorem to follow answers this question.

Theorem 2. **a)** *Let u be a non-negative bounded weak solution to (2.4). Then the weak solutions are locally Lipschitz continuous in $\mathcal{C}_L$, provided that $\boldsymbol{v} \in L^\infty(\overline{\mathcal{C}_L}, \mathbb{R}^N)$ and $f \in C(\overline{\mathcal{C}_L})$.*

b) *If u is a continuous weak solution of*

$$\Delta u = D_z(\beta(u)) \quad in \quad \mathcal{C}_L$$

and there is no sign restriction on u, then u is locally Lipschitz continuous.

The proof of part **a**) is shorter and we demonstrate it here first.

Proof of Theorem 2 a). As it is pointed out in [4], in order to establish the Lipschitz continuity of u, it is enough to show that for any compact set $K \subset\subset \mathcal{C}_L$ there exists a tame constant C, depending on $\operatorname{dist}(K, \partial\mathcal{C}_L)$ such that

$$\sup_{B_{2^{-k-1}}(X)} u \leq \max(C2^{-k}, \sup_{B_{2^{-k}}(X)} u), \qquad \forall X \in K \cap \partial\{u > 0\}.$$

We argue towards a contradiction: Suppose there exist $k_j \in \mathbb{N}, k_j \uparrow \infty, X_j \in K \cap \partial\{u_j > 0\}$ and weak solutions u_j with free boundary $\Gamma_j = \partial\{u_j > 0\}$, such that $0 \leq u_j \leq M$ (see Remark 1) and

$$S_j \overset{\text{def}}{\equiv} \sup_{B_{2^{-k_j-1}}(X_j)} u_j \geq \max(j2^{-k_j}, \frac{1}{2} \sup_{B_{2^{-k_j}}(X_j)} u_j). \tag{3.1}$$

Introduce $v_j(X) = \dfrac{u_j(X_j + 2^{-k_j}X)}{S_j}$, where $S_j = \sup\limits_{B_{2^{-k_j-1}}(X_j)} u$. It follows from (3.1) that

$$v_j(0) = 0, \qquad \sup_{B_{\frac{1}{2}}} v_j \geq \frac{1}{2}, \qquad 0 \leq v_j(X) \leq 2, \quad X \in B_1. \tag{3.2}$$

Since the functions u_j are bounded, it follows from (3.1) that $M > j2^{-k_j}$ implying that $k_j \to \infty$.

According to (2.2), v_j solves the following equation

$$\begin{aligned} \Delta v_j &= \frac{2^{-2k_j}}{S_j}(\Delta u_j)(X_j + 2^{-k_j}X) \\ &= \frac{2^{-k_j}}{S_j} \operatorname{div}[\beta(v_j)\mathbf{v}(X_j + 2^{-k_j}X)] + \frac{2^{-2k_j}}{S_j} f(X_j + 2^{-k_j}X) \\ &\overset{\text{def}}{\equiv} \operatorname{div} \mathbf{F}_j + f_j, \end{aligned}$$

where

$$\mathbf{F}_j = \frac{2^{-k_j}}{S_j}\beta(v_j)\mathbf{v}(X_j + 2^{-k_j}X), \quad f_j = \frac{2^{-2k_j}}{S_j} f(X_j + 2^{-k_j}X).$$

Since by assumptions $\mathbf{v}$ is bounded we get from (3.1) (the definition of S_j) and the explicit form of β given by (2.3), the decay estimate

$$|\mathbf{F}_j| \le \frac{2^{-k_j}}{S_j}\beta(2)\sup_{\mathcal{C}_L}|\mathbf{v}| \le \frac{M}{j}\beta(2)\sup_{\mathcal{C}_L}|\mathbf{v}| \to 0.$$

Similarly, one can see that $\sup_{B_1}|f_j(X)| \to 0$.

Utilizing (3.2) and DeGiorgi's theorem we obtain that the sequence $\{v_j\}$ is uniformly Hölder continuous in $B_{3/4}$ and from the Caccioppoli inequality it follows that $\{v_j\}$ is also bounded in $H^1(B_{\frac{3}{4}})$. Then using a customary compactness argument and the decay estimate for $\{\mathbf{F}_j\}$ and $\{f_j\}$, we can extract a subsequence $\{v_{j_m}\} \subset \{v_j\}$ locally uniformly converging to v_0 in $B_{\frac{3}{4}}$ and weakly in $H^1(B_{\frac{3}{4}})$. Moreover

$$-\int Dv_0 D\varphi \longleftarrow -\int Dv_{j_m} D\varphi = \int f_{j_m}\varphi - \mathbf{F}_{j_m}\cdot D\varphi \longrightarrow 0, \qquad \forall \varphi \in C_0^\infty(B_{\frac{3}{4}}).$$

Thus, v_0 is a nonnegative continuous harmonic function in $B_{\frac{3}{4}}$. From the uniform convergence $v_{j_m} \to v_0$ we see that (3.2) translates to v_0 and we conclude that $v_0(0) = 0$ and $\sup_{B_{\frac{1}{2}}} v_0 = \frac{1}{2}$. However this contradicts the strong maximum principle and the proof follows. □

4. Proof of Theorem 2 b)

4.1. **Preliminary lemmas.** In this subsection we prove some technical lemmas in order to tackle the optimal local regularity of the solution for the two phase problem, i.e. when there is no sign restriction on u. Throughout this section $\mathbf{v} = \mathbf{e}_N$. We begin with the following useful observation. If $w(X) = e^{-\frac{az}{2}}u(X)$ then

$$\Delta w = \left[\operatorname{div}(\mathbf{v}\beta(u)) - au_z + \frac{a^2}{4}u\right]e^{-\frac{az}{2}}. \tag{4.1}$$

We know that in $\{u > 0\} \cup \{u < 0\}$ u solves the equation $\Delta u = au_z$, see (2.4). Thus the positive and negative parts of w satisfy the equation

$$\Delta w^\pm = e^{-\frac{az}{2}}\frac{a^2}{4}u^\pm = \frac{a^2}{4}w^\pm \ge 0. \tag{4.2}$$

Therefore w^+, w^- are continuous, nonnegative subharmonic functions in $\mathcal{C}_L$.

This observation, together with (4.2) and Lemma 2, will allow us to employ the monotonicity formula, Lemma 5.1 in [1], and show that u is locally Lipschitz continuous in $\mathcal{C}_L$.

First we need some preliminary estimates near the free boundary $\Gamma(u) = \partial\{u > 0\}$.

Lemma 1. *For any compact set $K \subset \mathcal{C}_L$ there exists a positive tame constant $\widehat{C}$, depending on K such that for any $B_\rho(X_0) \subset K$ and $X_0 \in \Gamma \cap K$ the following estimate holds*

$$\left| \int_{B_\rho(X_0)} \Delta w \right| \le \widehat{C}\rho^{N-1}. \tag{4.3}$$

Proof. Integrating (4.1) and using Green's formula we get

$$\begin{aligned} \int_{B_\rho(X_0)} \Delta w &= \int_{B_\rho(X_0)} \left[\operatorname{div}(\mathbf{e}_N \beta(u)) - a u_z + \frac{a^2}{4} u\right] e^{\frac{a}{2}z} \\ &= \int_{\partial B_\rho(X_0)} e^{-\frac{a}{2}z} \left[\beta(u) + au\right] \mathbf{e}_N \cdot \nu \\ &\quad - \int_{B_\rho(X_0)} [\beta(u) - au] \mathbf{e}_N \cdot D e^{-\frac{a}{2}z} \\ &\quad + \int_{B_\rho(X_0)} e^{-\frac{a}{2}z} \frac{a^2}{4} u \\ &\le \widehat{C}\rho^{N-1}. \end{aligned} \tag{4.4}$$

□

Lemma 2. *For any compact set $K \subset \mathcal{C}_L$ there exists a positive number ρ_0 depending only on dist$(K, \partial\mathcal{C}_L), N$ and a positive tame constant $C = C(K)$ such that for any $B_\rho(X_0) \subset K$ and $X_0 \in \Gamma \cap K$ the following estimate holds*

$$\left| \fint_{\partial B_\rho(X_0)} w \right| \le C\rho, \qquad \rho < \rho_0. \tag{4.5}$$

Proof. From Green's representation formula

$$w(X_0) = \int_{\partial B_\rho(X_0)} w(Y) P(Y, X_0) d\mathcal{H}^{N-1} - \int_{B_\rho(X_0)} G(X, X_0) \Delta w(X) dX,$$

where $P(Y, X_0)$ is the Poisson kernel and $G(X, X_0)$ is the Green function corresponding to $B_\rho(X_0)$. At $X_0 \in \Gamma$, $w(X_0) = 0$ implying that

$$\begin{aligned}
\fint_{\partial B_\rho(X_0)} w(Y) d\mathcal{H}^{N-1} &= \int_{B_\rho(X_0)} G(X, X_0)\Delta w(X) dX \\
&= \int_0^\rho G(s) \frac{d}{ds}\left(\int_0^s t^{N-1} \int_{\partial B_1} \Delta w(t\xi) dH^{N-1}(\xi) dt\right) ds \\
&= G(s) \int_{B_s(X_0)} \Delta w \Big|_0^\rho - \int_0^\rho G'(s) \int_{B_s(X_0)} \Delta w.
\end{aligned}$$

Now the result follows from (4.3) and the obvious estimate $|G'(s)| \le C/s^{N-1}$. □

The next crucial step is to use the Alt-Caffarelli-Friedman monotonicity formula from [1]. Recall Lemma 5.1 from [1].

Lemma 3. *Let w^+, w^- be two continuous, nonnegative subharmonic functions in $B_1(Z)$, $w^- w^+ = 0, w^+(Z) = w^-(Z) = 0$. Then*

$$\Phi(R, Z, w^+, w^-) = \frac{1}{R^4} \int_{B_R(Z)} \frac{|\nabla w^+(X)|^2}{|X - Z|^{N-2}} dX \int_{B_R(Z)} \frac{|\nabla w^-(X)|^2}{|X - Z|^{N-2}} dX$$

is a nondecreasing function of R and

$$\Phi(1) \le C\left(1 + \int_{B_1(Z)} (w^+)^2 + \int_{B_1(Z)} (w^-)^2\right).$$

Remark 3. *It follows from (4.2) that $w^\pm$ defined by $w(X) = e^{-\frac{az}{2}} u(X)$ are subharmonic functions with disjoint supports. Moreover it follows from DeGiorgi's theorem, on the Hölder continuity of weak solutions of $\Delta u = \operatorname{div} \boldsymbol{f}$ with bounded $\boldsymbol{f}$, that $w^\pm$ are continuous. Hence Lemma 3 can be applied to the pair w^+, w^-.*

4.2. **Proof of Theorem 2 b).** It is enough to prove that u grows away from the free boundary linearly. To fix the ideas we assume that $B_1(X_0) \subset \mathcal{C}_L$. Assume that $X_0 \in \mathcal{C}_L$ and denote by $X \in \Gamma = \partial\{u > 0\}$ the closest point to X_0. Put $\rho = |X - X_0| = \operatorname{dist}(X, \Gamma)$ and suppose that $w(X_0) \ge M\rho > 0$ for some large M. It follows from Harnack's inequality that $w > c_0 M$ in

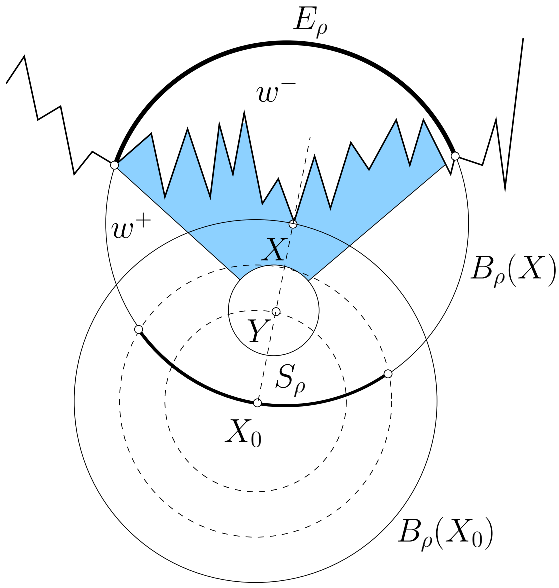

FIGURE 1. The blue domain is $\mathcal{D}$.

$B_{\frac{3\rho}{4}}(X_0) \subset \mathcal{C}_L$, hence

$$\fint_{\partial B_\rho(X)} w^+ \ge c_1 \fint_{S_\rho} w^+ \ge c_0 c_1 M\rho,$$

where $S_\rho = \partial B_\rho(X) \cap B_{\frac{3\rho}{4}}(X_0)$ and c_1 depends only on the dimension N. By Lemma 2

$$\fint_{\partial B_\rho(X)} w^- \ge \fint_{\partial B_\rho(X)} w^+ - C\rho \ge (c_0c_1M - C)\rho > \frac{M}{2}\rho$$

if M is large enough.

Next, let $Y \in B_{\frac{\rho}{2}}(X_0)$ be a point on $\overrightarrow{XX_0}$. Then $w^+ \ge c_0 M$ in $B_{\frac{\rho}{4}}(Y)$. We use polar coordinates (r, ω) about Y. Let E_ρ be the set of $\omega \in \partial B_1(X_0)$ such that if $(\rho, \omega) \in \partial B_\rho(X)$ then $u(\rho, \omega) < 0$. After switching to polar coordinates with the unit direction in $\frac{1}{\rho}E_\rho \subset \partial B_1$ and using Hölder's inequality we get

$$\text{(4.6)}\qquad \begin{aligned} \frac{M}{2} &\le \frac{1}{\rho} \fint_{\partial B_\rho(X)} w^- = \frac{1}{\rho}\int_{\frac{1}{\rho}E_\rho} w^-(\rho,\omega)d\omega \\ &= \frac{1}{\rho}\int_{\frac{1}{\rho}E_\rho}\int_{I_\omega} D_r w^-(r,\omega)drd\omega \\ &= \frac{1}{\rho}(\rho|E_\rho|)^{\frac12}\left(\int_{\frac{1}{\rho}E_\rho}\int_{I_\omega} |D_r w^-(r,\omega)|^2 drd\omega\right)^{\frac12} \\ &\le \frac{1}{\rho}(\rho|E_\rho|)^{\frac12}\left(\int_{B_\rho(X)} |\nabla w^-(Z)|^2 dZ\right)^{\frac12} \end{aligned}$$

Introduce the cone $K = \{t\omega, t > 0, \omega \in E_\rho\}$. Recall that $w \ge c_0 M$ in $B_{\frac{\rho}{2}}(Y)$ by Harnack's inequality. Integrating $D_r w^+$ over $\mathcal{D} = K \cap \Big((B_\rho(X) \setminus B_{\frac{\rho}{4}}(Y)\Big) \cap \{w > 0\}$ (see Figure 1) we obtain

$$\text{(4.7)}\qquad \begin{aligned} |E_\rho| M c_0 \rho^N &\le \int_{\partial B_{\frac{\rho}{4}}(Y)\cap K} w^+\left(\frac{\rho}{4},\omega\right) \\ &\le \iint_{\mathcal{D}} D_r w^+(r,\omega)drd\omega \le \\ &\lesssim (\rho|E_\rho|)^{\frac12}\left(\int_{B_\rho(X)} |Dw^+(Z)|^2 dZ\right)^{\frac12}. \end{aligned}$$

Thus combining (4.6)-(4.7) and applying Lemma 3 and Remark 3 we obtain

$$
\begin{aligned}
M^2 &\leq \frac{2c_2}{c_0\rho^N}\left(\int\limits_{B_\rho(X)}|Dw^+(Z)|^2dZ\right)^{\frac{1}{2}}\left(\int\limits_{B_\rho(X)}|Dw^-(Z)|^2dZ\right)^{\frac{1}{2}} \\
&\leq \frac{2c_2}{c_0\rho^2}\left(\frac{1}{\rho^{N-2}}\int\limits_{B_\rho(X)}|Dw^+(Z)|^2dZ\frac{1}{\rho^{N-2}}\int\limits_{B_\rho(X)}|Dw^-(Z)|^2dZ\right)^{\frac{1}{2}} \\
&= \frac{2c_2}{c_0\rho^2}\left[\rho^4\Phi(\rho)\right]^{\frac{1}{2}} \\
&= \frac{2c_2}{c_0}\left[\Phi(\rho)\right]^{\frac{1}{2}} \\
&\leq \frac{2c_2C}{c_0}\left(1+\int\limits_{B_1(X_0)}(w^+)^2+\int\limits_{B_1(X_0)}(w^-)^2\right)^{\frac{1}{2}}
\end{aligned}
$$

and we conclude the proof of Theorem 2 **b**). □

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