ANALYSIS OF A FREE BOUNDARY AT CONTACT POINTS WITH LIPSCHITZ DATA

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Abstract. In this paper we consider a minimization problem for the functional

$$J(u) = \int_{B_1^+} |\nabla u|^2 + \lambda_+^2 \chi_{\{u>0\}} + \lambda_-^2 \chi_{\{u\leq 0\}},$$

in the upper half ball $B_1^+ \subset \mathbb{R}^n, n \geq 2$ subject to a Lipschitz continuous Dirichlet data on ∂B_1^+. More precisely we assume that $0 \in \partial\{u > 0\}$ and the derivative of the boundary data has a jump discontinuity. If $0 \in \overline{\partial(\{u > 0\} \cap B_1^+)}$ then (for $n = 2$ or $n > 3$ and one-phase case) we prove, among other things, that the free boundary $\partial\{u > 0\}$ approaches the origin along one of the two possible planes given by

$$\gamma x_1 = \pm x_2,$$

where γ is an explicit constant given by the boundary data and $\lambda_\pm$ the constants seen in the definition of $J(u)$. Moreover the speed of the approach to $\gamma x_1 = x_2$ is uniform.

1. Introduction

In this paper we consider the local minimizers of the functional

$$J(u) = \int_{B_1^+} |\nabla u|^2 + \lambda_+^2 \chi_{\{u>0\}} + \lambda_-^2 \chi_{\{u\leq 0\}},$$

where $B_1^+ \subset \mathbb{R}^n, n \geq 2$, is the upper half of the open unit ball, χ_D is the characteristic function of $D \subset \mathbb{R}^n$, $\lambda_\pm$ are given positive constants and $u = f$ on ∂B_1^+ with Lipschitz continuous f. The local regularity of the minimizers u and the free boundary $\partial\{u > 0\}$ were studied in [AC] and [ACF], notably it was shown that u is locally Lipschitz continuous.

The boundary regularity of u with smooth boundary data f such that $|f(x)| \approx o(|x|)$ near the origin was considered in [KKS] where, assuming the origin is a contact point, the authors have proved that close to the origin, the free boundary approaches the plane $\{x_1 = 0\}$ in a tangential fashion.

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The objective of this paper is to consider boundary data that gives rise to non-tangential touch between the free and the fixed boundaries. Such problems appear naturally in the mathematical formulation of the so-called Dam problem for the water reservoirs (see [AG]). Other problems of this kind emerge in wake and cavity formations in stationary Eulerian flows moving through cylindrical domains (see [BZ] Chapters 1.9 and 9.5 for more applications).

Since the formulation of our main results requires some technical definitions, we refrain ourselves of giving an exact account of our main results here. However, in lay terms, one can say that our main result in this paper states that for a boundary data such as $\alpha_+ x_2^+ - \alpha_- x_2^-$, the free boundary $\Gamma(u)$ approaches the fixed one along one of the planes $\gamma x_1 = \pm x_2$, where

$$\gamma = \sqrt{\frac{\lambda_+^2 - \lambda_-^2}{\alpha_+^2 - \alpha_-^2} - 1}.$$

This we only prove for $n = 2$ or $n \geq 3$ and one-phase case. The difficulty for two-phase in higher dimensions comes from the classification of global homogeneous solutions, that is not feasible by our technique.

Contents

1.1. **Plan of the paper.** The plan of this paper is as follows. In this introductory part we give the necessary notations and definitions to formulate the problem. Section 2 contains a heuristic discussion of the optimal regularity of solutions. The key point is the uniform linear growth of minimizers at the origin. We formulate the main results of this paper in Section 3. To deal with the boundary behavior of minimizers one needs to obtain up-to boundary uniform continuity near contact points. The proof of this result as well as a basic compactness theorem for blow up sequences is contained in Section 4 and Appendix. Section 5 takes care of the optimal regularity of minimizers to our functional. In Sections 6-8 we show that homogeneous global solutions in one phase case are two-dimensional, and hence independent of $x_3, x_4, \dots, x_n$. A stability result is given in Section 9. In fact Section 9 contains the proof of the main result of this paper, describing how the free boundary behaves close to the origin. Finally in Section 10 we give an example of a non-homogeneous global solution.

1.2. **Notations.** We will use the following notations throughout the paper.

$C_0, C_n, \dots$	generic constants,
χ_D	the characteristic function of the set $D \subset \mathbb{R}^n,\ n \geq 2$,
$\overline{D}$	the closure of D,
∂D	the boundary of a set D ,
x, x'	$x = (x_1, \dots, x_n), \quad x' = (0, x_2, \cdots, x_n)$,
$\mathbb{R}^n_+, \mathbb{R}^n_-$	$\{x \in \mathbb{R}^n :\ x_1 > 0\};\ \{x \in \mathbb{R}^n :\ x_1 < 0\}$,
Π	$\{x \in \mathbb{R}^n :\ x_1 = 0\}$,
$B_r(x),\ B_r^+(x)$	$\{y \in \mathbb{R}^n : \lvert y - x\rvert < r\},\ B_r(x) \cap \mathbb{R}^n_+$,
$B_r,\ B_r^+$	$B_r(0),\ B_r^+(0)$,
B_r'	$B_r \cap \Pi$,
S_r^+	$\partial B_r \cap \mathbb{R}^n_+$,
$\lambda_\pm,\ \Lambda$	λ_+, λ_- are positive numbers and $\boxed{\Lambda = \lambda_+^2 - \lambda_-^2 \neq 0}$,
$\Gamma_u, \Gamma(u)$	$\partial\{u > 0\}$; the free boundary of u,
$\Omega^+(u), \Omega^-(u)$	$\Omega^+(u) = \{x : u(x) > 0\},\ \Omega^+(u) = \{x : u(x) < 0\}$,
$K_\delta(x_0)$	the open cone $K_\delta = \{x \in \mathbb{R}^n_+ \ :\ \lvert x - x'\rvert > \delta \lvert x - x_0\rvert\}$,
K_δ	the open cone $K_\delta = \{x \in \mathbb{R}^n_+ \ :\ x_1 > \delta \lvert x'\rvert\}$,
$\mathcal{P}_r, \mathcal{P}_\infty, \mathcal{HP}_\infty, \mathcal{P}_r'$	see Definitions 1.3, 1.5 and 1.6,
$v^\pm$	$v^+ = \max(v, 0)$ and $v^- = \max(-v, 0)$. Thus $v = v^+ - v^-$.

1.3. **Problem set-up.** Throughout this paper we assume

$$\boxed{f(x) = \alpha_+ x_2^+ - \alpha_- x_2^- + g(x),} \tag{1.1}$$

where α_+, α_- are nonnegative constants such that $\alpha_+ + \alpha_- > 0$, and $g(x) \in C^{1,\alpha}(\overline{B_1^+}),\ g(x) = o(|x|)$. Typically $g(x) = C|x|^{1+\kappa}$ for positive constants C and κ.

For a fixed domain $D \subset \mathbb{R}^n_+$ we put

$$J(u, D) = \int_D |\nabla u|^2 + \lambda_+^2 \chi_{\{u>0\}} + \lambda_-^2 \chi_{\{u\leq 0\}}.$$

When it is clear for which D the functional J is considered, we just write it as $J(u)$ omitting the explicit dependence on D. The case $D = B_R^+, R > 0$ is of particular interest.

Definition 1.1. *Let $\mathcal{K}_f(D) = \{w : w \in H^1(D),\ w - f \in H_0^1(D)\}$ be the class of admissible functions.*

- *A function u is said to be a local minimizer of $J(u, D)$ if for any function $v \in \mathcal{K}_f(D)$ such that $u = v$ on $\partial D'$, for $D' \subset D$, it follows that*

$$J(u) \leq J(v).$$

- *The class of local minimizers is denoted by $\mathcal{P}(D, n, \lambda_\pm, \alpha_\pm, g)$.*

Remark 1.2. *For $D = B_r^+$ we denote the corresponding class by $\mathcal{P}_r(n, \lambda_\pm, \alpha_\pm, g)$. We also set $\mathcal{P}_r(n, \lambda_\pm, \alpha_\pm) = \mathcal{P}_r(n, \lambda_\pm, \alpha_\pm, 0)$. It is worthwhile to point out that if $u \in \mathcal{P}_r(n, \lambda_\pm, \alpha_\pm, g)$ then $u_r(x) = \frac{u(rx)}{r} \in \mathcal{P}_1(n, \lambda_\pm, \alpha_\pm, \frac{g(rx)}{r})$ by the scale invariance of $J(u, B_r^+)$.*

If D is a bounded domain then, it follows from the definition of $J(u, D)$, that

$$J(u, D) = \lambda_-^2 |D| + \int_D |\nabla u|^2 + \Lambda \chi_{\{u>0\}}, \tag{1.2}$$

where $\Lambda = \lambda_+^2 - \lambda_-^2 > 0$. In what follows we take

$$\boxed{J(u, D) = \int_D |\nabla u|^2 + \Lambda \chi_{\{u>0\}}.} \tag{1.3}$$

Next we introduce a particular class of local minimizers u, such that the free boundary $\partial\{u > 0\}$ satisfies the $\delta-$thickness assumption.

Definition 1.3. *Let $u \in \mathcal{P}_r(n, \lambda_\pm, \alpha_\pm, g)$.*

1° *We say that the free boundary* Γ_u *is* $\delta-$*thick at* $x_0 \in B'_r \cap \Gamma_u$ *if there exists a* $\delta > 0$ *such that*

$$(B^+_{2\rho}(x_0) \setminus B^+_\rho(x_0)) \cap \partial\{u > 0\} \cap K_\delta(x_0) \neq \emptyset, \qquad \forall \rho \in (0, r), \tag{1.4}$$

where $K_\delta(x_0) = \{x \in \mathbb{R}^n_+ : x_1 > \delta|x' - x'_0|\}$.

2° *The class of all local minimizers in* $B^+_R(x_0)$ *with* δ*-thick free boundary is denoted by* $\mathcal{P}_r(x_0, n, \lambda_\pm, \alpha_\pm, g, \delta)$. *When* $x_0 = 0$ *and* $R = 1$ *we often omit the dependence of* $\mathcal{P}_r$ *from* x_0 *and write* $\mathcal{P}_1(n, \lambda_\pm, \alpha_\pm, g, \delta)$ *for brevity.*

One can interpret the condition (1.4) geometrically as follows: There is a free boundary point at each intersection of the cone $K_\delta(x_0)$ with $B^+_{2r}(x_0) \setminus B^+_r(x_0)$ and hence the free boundary does not fall down too quickly towards the plane $x_1 = 0$ as $r \to 0$. The next section contains more discussion on $\delta-$thickness as a necessary condition for linear growth.

Remark 1.4. *The* $\delta-$*thickness assumption can be weakened as follows. Let* $r > 0$, $z \in \partial\{u > 0\}$ *is a non-isolated point of the free boundary and assume that there is a point* $x \in (B_{2r}(z) \setminus B_r(z)) \cap K_\delta$ *such that*

$$|u(x)| \leq Cr \tag{1.5}$$

for some fixed constants δ, C *independent from* r. *Then one can prove that* u *grows linearly from the origin. It should be noted here that the weak thickness assumption is always true for the solutions to one phase problem, see (5.9).*

1.4. **Blow-up limits and Global Solutions.** Let $u_j \in \mathcal{P}_1(n, \lambda_\pm, \alpha_\pm, g)$, $j = 1, 2, \dots$ and x_0 be a contact point, i.e. $x_0 \in \Gamma_{u_j} \cap B'_1$. Typically $x_0 = 0$. For $r_j > 0$ we introduce the *blow-up sequence* of functions at x_0

$$v_j(x) = \frac{u_j(x_0 + r_j x)}{r_j}, \qquad r_j \downarrow 0 \ \text{ as } \ j \to \infty. \tag{1.6}$$

If the sequence v_j is bounded in a suitable space then sending r_j to 0 we obtain a so called *blow-up limit* u_0. One of our main objectives in this paper is to classify the blow-up limits of the sequence v_j in (1.6) as j tends to infinity. It is worth to point out that, in general, the blow-up limit depends on the sequence $\{r_j\}_1^\infty$. Thus in general the blow up limit u_0 is not unique. Hence it is natural to ask how many blow up limits would exist? To answer this question we employ the monotonicity formula

(4.10) and show that the blowup at the contact points is only one of the functions (3.3) (see sections 4.3 and 7.1).

The classification of all possible blow-up limits u_0 is based on geometric properties that these functions share notably the linear growth and the homogeneity.

Definition 1.5. *Let u be a local minimizer in $\mathbb{R}^n_+$.*

1° *We say that u is a **global solution** if $u \in \mathcal{P}_\infty$, where*

$$\mathcal{P}_\infty(C) = \bigcap_{r>0} \left\{u \in \mathcal{P}_r(n, \lambda_\pm, \alpha_\pm) \ : \ u(0, x') = \alpha_+ x_2^+ - \alpha_- x_2^-, \ |u(x)| \le C(x_1 + |x_2|)\right\}$$

for some positive constant C and $\mathcal{P}_r(n, \lambda_\pm, \alpha_\pm) = \mathcal{P}_r(n, \lambda_\pm, \alpha_\pm, 0)$.

2° *The class of all **homogeneous global solutions** is denoted by*

$$\mathcal{HP}_\infty(C) = \{u \in \mathcal{P}_\infty : u(tx) = tu(x), \forall t > 0\}.$$

This definition requires some explanation. First we note that any blow-up limit of linearly growing u is a global solution. Moreover it follows from the monotonicity theorem in Section 4.3 that the blow-up $u_0 \in \mathcal{HP}_\infty$. The linear growth constant C appearing in the definition must be consistent with the constants $\alpha_\pm$ that determine the boundary date. Clearly we must have $C \ge \max(\alpha_+, \alpha_-)$ otherwise at least one of $\alpha_\pm$ must be zero. A posteriori $\mathcal{HP}_\infty$ contains only two functions, by Theorem C (3.3), linking C with constants $\lambda_\pm$ too. In fact if $\frac{\lambda_+^2 - \lambda_-^2}{\alpha_+^2 - \alpha_-^2} - 1 < 0$ then $\mathcal{HP}_\infty$ is empty. Therefore whenever constant C is chosen large enough and $\frac{\Lambda}{\alpha_+^2 - \alpha_-^2} - 1 \ge 0$ the resulted class of global homogeneous solutions is determined uniquely.

Finally we define the extreme global solutions and stability in order to classify the global solutions.

Definition 1.6.

1° *$u \in \mathcal{P}_r(n, \lambda_\pm, \alpha_\pm, g)$ is said to be the smallest (resp. largest) global solution if for any $v \in \mathcal{P}_r(n, \lambda_\pm, \alpha_\pm, g)$ we have $u \le v$ (resp. $u \ge v$).*

2° *The class of all local minimizers that after blow-up coincide with the smallest homogeneous global solution v_S*

$$\mathcal{P}'_r = \left\{u \in \mathcal{P}_r(n, \lambda_\pm, \alpha_\pm, g) : \lim_{r_j \to 0} \frac{u(r_j x)}{r_j} = v_S(x), \quad \text{for some sequence } r_j\right\}. \tag{1.7}$$

*If $u \in \mathcal{P}'_r$ then we say that u is **stable**.*

2. Linear Growth: A Heuristic Discussion

In analyzing the behavior of the free boundary one needs, in general, to start with the growth rate of the solution at free boundary points. Lipschitz regularity, up to the boundary, would be the most desirable property for minimizers of our functional. This property, or at least the linear growth property at the origin, is indispensable for the rest of the theory to follow.

In general, one cannot expect this property to hold, and one is forced to impose conditions to assure this. Indeed, a harmonic function in B_1^+ with merely Lipschitz data on $\{x_1 = 0\}$ is *not* Lipschitz. In such cases the extra logarithmic term enters into the game, and the solution will belong merely to the little-o Zygmund class

$$|u(x)| \leq C|x| \log |x|^{-1}.$$

In one phase case it is possible to obtain linear growth from the origin, provided the origin is a non-isolated free boundary point. In other words if there is a sequence of free boundary points in $\{x_1 > 0\}$ approaching the origin, then we expect linear growth for the solutions. We will state and give a proof of this below. A similar result of this type was proven in [AG]. Observe that if, even in the one phase case, we chose the boundary data large enough, e.g. $\alpha_+^2 > \Lambda$, then one may show that the function u minimizing J is harmonic in the upper half ball, see Section 7.1. Thus, a harmonic function with Lipschitz data can impossibly be Lipschitz up to the boundary.

For the two phase problem the analysis becomes much more complicated, and we could not find any complete theory. Since the Dirichlet data has two signs close to the origin

$$f(x) \approx \alpha_+ x_2^+ - \alpha_- x_2^-,$$

the free boundary $\partial\{u > 0\}$ is always presented in the upper half ball. The problem is that it might approach the fixed boundary $\{x_1 = 0\}$ tangentially, and give rise to a non-lipschitz behavior of the solution. (This argument does not apply to the one-phase case.) The reader may verify that if the free boundary approaches tangentially to the fixed boundary and at the same time the solution is Lipschitz then a blow up limit would result in the fact that one of the phases vanishes but the boundary data is a two-phase data, hence a contradiction would arise. In particular this suggests that for the two phase problem, a natural condition to impose is that the free boundary does not touch the fixed one in a tangential fashion.

It is also not too hard to prove that there are certain Lipschitz boundary data, for which the solution is not Lipschitz and touches the fixed boundary tangentially. For the proof we would need a classification of homogeneous global solution (as in Theorem C). Suppose $n = 2$, then the proof of Theorem C is more or less elementary in this case (see the proof). If we accept this result, for the moment, we see that for $\alpha := \alpha_+ = \alpha_-$, and $\Lambda > 0$ one may conclude that the solution cannot be Lipschitz. Otherwise, if this was the case, then a blow-up of the solution would result in a global solution, with linear growth. Hence the classification theorem, Theorem C, would then suggest that the solution is $u = \alpha x_2$, but then the free boundary condition $|\nabla u^+|^2 - |\nabla u^-|^2 = \Lambda > 0$ fails.

From the representation (3.3), we also see that if $\alpha_+^2 - \alpha_-^2 > \Lambda$, then again an up to the boundary Lipschitz continuous solution cannot exist.

The question that remain is: What are the optimal conditions assuring a linear growth for the minimizers from the origin? We have answered this question in Theorems A and B, below under mild conditions.

3. Main Results

In this section we state the main results of this paper. To begin our analysis we need the optimal growth estimate for the local minimizers u near the the contact points. More precisely we have to show that any $u \in \mathcal{P}_1(n, \lambda_\pm, \alpha_\pm, g)$ grows linearly away from $z \in \partial\{u > 0\} \cap \Pi$. Clearly we can assume that $z = 0$.

Theorem A. *Let $u \in \mathcal{P}_1(n, \lambda_\pm, \alpha_\pm, g)$ and either of the following holds:*

- **1°** *$u \in \mathcal{P}_1(n, \lambda_\pm, \alpha_\pm, g, \delta)$, i.e. the condition (1.4) (or its weaker form (1.5)) is satisfied for some $\delta > 0$ and all $r < 1$,*
- **2°** *$\alpha_- = 0$, $g \geq 0$ and the origin is a non-isolated free boundary point.*

Then

$$|u(x)| \leq C|x|, \qquad x \in B^+_{\frac{1}{2}}, \tag{3.1}$$

where C depends on n, $\lambda_\pm, \alpha_\pm, \sup_{B_1} |u|$ and δ, g.

As for part **2°** of Theorem A, let us note that the weak $\delta-$thickness assumption (1.5) is always satisfied for one phase problem, see (5.9).

Our next result is an improvement of Theorem A in the following sense: Let u_0 be a blow-up of u at the origin then $|u_0(x)| \leq C|x|$ in $\mathbb{R}^n_+$ and $u_0(x) = \alpha_+ x_2^+ - \alpha_- x_2^-$ on Π. However these is not enough to conclude that $u_0 \in \mathcal{P}_\infty$ since the estimate $|u_0(x)| \leq C(x_1 + |x_2|)$ in the definition of $\mathcal{P}_\infty$ does not follow immediately. Suppose $T_{i,R}(x) = x + Re_i, i \neq 2$ is the translation in e_i direction by $R \in \mathbb{R}$. Then $u_0(T_{i,R}(x))$ is also a minimizer, but possibly with different constant C in the linear growth estimate. Does the boundary data $\alpha_+ x_2^+ - \alpha_- x_2^-$, depending only on x_2, has any effect? Do we get the same growth for $u_0(T_{i,R}(x)$?

Theorem B. *Let $u \in \mathcal{P}_1(n, \lambda_\pm, \alpha_\pm, g)$ and suppose that for any $z \in \partial\{u > 0\} \cap B_{\frac{1}{2}}$ the estimate*

$$|u(x)| \leq C|x - z| \tag{3.2}$$

holds with some constant C. Then for any blow up limit u_0 of u at the origin we have

$$|u_0(x)| \leq C(x_1 + |x_2|).$$

In particular any blow up limit of u belongs to $\mathcal{P}_\infty(C)$.

Theorem B is used to classify homogeneous global solutions by employing a customary dimension reduction argument. Notably we show that if $u \in \mathcal{HP}_\infty$ then u depends only on x_1 and x_2 variables. Again we note that the growth estimate $u(x) \leq |x - z|$ is true for one phase case. As for the two phase case, one can prove that the uniform δ or weak $\delta-$thickness condition for each contact point $z \in B_{\frac{1}{2}}$ will imply $|u(x)| \leq C|x - z|$ in view of Theorem A.

To set forth the implications of Theorem B we return to the translated solution $u_0(T_{i,R}(x)), i \geq 3$. For arbitrary $R_1 < R_2$ one can show that $\max(u_0(T_{i,R_1}(x)), u_0(T_{i,R_2})(x))$ is a minimizer of $J(u, B_1)$ with boundary values $\max(u_0(T_{i,R_1}(x)), u_0(T_{i,R_2})(x))$ on ∂B_1^+. Moreover by Theorem B the maximum of solutions has exactly the same linear growth as u_0. Thus we can construct a translation invariant maximal global solution. Repeating this argument for all $i \geq 3$ we obtain a maximal global solution depending on x_1 and x_2 only. The minimal solution is constructed analogously. Writing Laplace operator in polar coordinates we get the classification of global homogeneous solutions.

Theorem C. *In $\mathbb{R}^2$, there are only two homogeneous global solutions:*

$$\begin{aligned} v_L &= \alpha_+(\gamma x_1 + x_2)^+ - \alpha_-(\gamma x_1 + x_2)^-, \\ v_S &= \alpha_+(-\gamma x_1 + x_2)^+ - \alpha_-(-\gamma x_1 + x_2)^-, \end{aligned} \tag{3.3}$$

where $\gamma = \sqrt{\dfrac{\Lambda}{\alpha_+^2 - \alpha_-^2} - 1}$. Thus $\mathcal{HP}_\infty = \{v_S, v_L\}$.

This also holds in $\mathbb{R}^n$, for $n > 2$, and for one-phase case, with $\alpha_- = \lambda_- = 0$.

A obvious consequence of this theorem is that for any $u \in \mathcal{P}_r$, the angle of the touch between the free and fixed boundaries is dictated by the behavior of v_S or v_L.

From Theorem C one can deduce that the free boundary approaches to the origin along the plane $\{x \in \mathbb{R}^n : \gamma x_1 = x_2\}$. The approach is uniform for the small solution, but in general not for the large one. For the precise formulation we introduce some notations: Let σ be a modulus of continuity and consider

$$\begin{aligned} K_\sigma^+ &:= \left\{x:\ x_1>0, x_2>0,\ \tfrac{x_2}{\gamma+\sigma(|x|)} < x_1 < \tfrac{x_2}{\gamma-\sigma(|x|)}\right\}, \\ K_\sigma^- &:= \left\{x:\ x_1>0, x_2<0,\ \tfrac{-x_2}{\gamma+\sigma(|x|)} < x_1 < \tfrac{-x_2}{\gamma-\sigma(|x|)}\right\}, \end{aligned} \tag{3.4}$$

The free boundaries of maximal and minimal solutions are hyperplanes

$$\Gamma(v_S) = \{x \in \mathbb{R}^n : x_2 = \gamma x_1\}, \qquad \Gamma(v_L) = \{x \in \mathbb{R}^n : x_2 = -\gamma x_1\}.$$

Theorem D. *Let $u \in \mathcal{P}_r$ (see Section 1.2), and v_S, v_L be defined by (3.3). Suppose further $n = 2$ or $n > 2$ and one-phase case. Then, close to the origin, $\Gamma(u)$ tangentially touches one of the hyperplanes $\Gamma(v_S) = \{x \in \mathbb{R}^n : x_2 = \gamma x_1\}$ or $\Gamma(v_L) = \{x \in \mathbb{R}^n : x_2 = -\gamma x_1\}$. More precisely there exists a modulus of continuity $\sigma(r) = \sigma(u, r)$ and $r_0 \in (0, 1)$ such that for any $r \in (0, r_0)$ either*

$$\Gamma(u) \subset B_r^+ \bigcap K_\sigma^+ \qquad \text{or} \qquad \Gamma(u) \subset B_r^+ \bigcap K_\sigma^-.$$

If u touches the hyperplane $\Gamma(v_S)$ (i.e. $u \in \mathcal{P}_1'$), then $\sigma(r)$ and r_0 are independent of u, and thus the neighborhood B_{r_0} is uniform.

It follows from the definition of $\mathcal{P}_\infty$, and by Theorem B, that for $u \in \mathcal{P}_r$, the limit $u_j(x) = \frac{u(r_j x)}{r_j}, r_j \downarrow 0$ is a global solution. Furthermore, from Weiss' formula [W], we have that the limit has to be a homogeneous function of degree one. Thus the blow up limits belong to $\mathcal{HP}_\infty$. However the class of global solutions $\mathcal{P}_\infty$ may contain non-homogeneous solutions, as our last theorem shows.

Theorem E. *There exists a non-homogeneous global solution with boundary values $\alpha_+ x_2^+$.*

A consequence of Theorem E is a kind of instability of the angle of touch, which amounts to the fact that if a free boundary is asymptotically close to v_S, then by slight perturbation of the boundary data the free boundary may come close to v_L, asymptotically. This constitutes the idea in the construction of global non-homogeneous solutions in Theorem E.

Theorem E exhibits the structure of the class of global solutions, namely the fact that there exist non-homogeneous functions in $\mathcal{P}_\infty$. This is due to the following: If $u_j \in \mathcal{P}_1(n, \lambda_\pm, \alpha^j_\pm)$ then the blow-up sequence $v_j = \frac{u_j(r_j x)}{r_j}$ converges to a global solution $v_\infty \in \mathcal{P} - \infty(n, \lambda_\pm, \alpha^\infty_\pm)$ where $\alpha^\infty_\pm = \lim_{j\to\infty} \alpha^j_\pm$. But it does not necessarily imply that v_∞ is homogeneous. If $u = u_j$ and $\alpha_\pm = \alpha^j_\pm$ then from Weiss monotonicity theorem it follows that v_∞ is homogeneous, see Section 4.3.

4. Technicalities

In this section we gather a number of useful properties that all local minimizers share. Some of these properties are of local nature and some are true globally e.g. Hölder continuity up to the fixed boundary. Although the boundary extensions follow from standard techniques we have supplied the proofs for the reader's convenience.

4.1. Uniform Hölder continuity for $u \in \mathcal{P}_1(n, \lambda_\pm, \alpha_\pm, g)$.

We begin with recalling some well-known facts, which can be found can be found in [ACF].

Proposition 4.1. *Let u be a local minimizer of $J(u)$ in B_1^+ and $\Lambda = \lambda_+^2 - \lambda_-^2 > 0$. Then*

1° *u is a bounded subharmonic function in B_1^+, Theorem 2.3* [ACF],

2° *u is harmonic in the interior of $B_1^+ \setminus \{u = 0\}$, Theorem 2.4* [ACF],

3° *u^+ is non-degenerate, Corollary 3.2* [ACF],

4° *if* $\operatorname{meas}\{u = 0\} = 0$ *then* $|\nabla u^+|^2 - |\nabla u^-|^2 = \Lambda$ *across the free boundary Γ_u in some weak sense, Theorem 2.4* [ACF].

The starting point in our study is the uniform Hölder continuity of local minimizers. It will allow us to translate some of the well-known local properties of u into boundary case.

Lemma 4.2. *Let $u \in \mathcal{P}_1(n, \lambda_\pm, \alpha_\pm, g)$. Then u is bounded in $B^+_{\frac{1}{2}}$.*

Proof: By Proposition 4.1 u is harmonic in $\{u \neq 0\}$, hence u^+ is subharmonic. Indeed, if $x \in \Omega^+(u)$ then $\fint_{B_r(x)} u^+ \geq u(x)$ for each $r < r_0$ such that $B_{r_0}(x) \subset \Omega^+(u)$, otherwise $\fint_{B_r(x)} u^+ \geq 0 = u^+(x)$ for $x \notin \Omega^+(u)$. Thus the mean value property is satisfied locally. Moreover by Theorem 2.1 [ACF] u is continuous in each subdomain $D \subset\subset B_1^+$. Thus u^+ is subharmonic.

Let v be the harmonic lifting of u, i.e. $\Delta v = 0, v|_{\partial B_1^+} = u^+$. From maximum principle $u^+ \leq v$ and $\int_{B_1^+} |\nabla v|^2 \leq \int_{B_1^+} |\nabla u^+|^2$. In particular $\|v\|_{H^1(B_1^+)} \leq C\|u\|_{H^1(B_1^+)}$ with some tame constant C. This yields that $v \in C^1(\overline{B^+_{\frac{1}{2}}})$. Hence u^+ is bounded.

By a similar argument one can show that u^- is bounded, $\square$

Next theorem is more general and can be applied to families of local minimizers.

Proposition 4.3. *Let* $u \in \mathcal{P}_{R_0}(n, \lambda_\pm, \alpha_\pm, g)$ *and*

$$\sup_{B^+_{2R}} |u| + \alpha_+ + \alpha_- + \Lambda + \|f\|_{C^{0,1}} \leq M, \qquad 2R < R_0.$$

Then there are positive constants $\beta = \beta(n, R, M)$ *and* $C = C(n, R, M)$ *such that* $u \in C^\beta(\overline{B^+_R})$ *and* $\|u\|_{C^\beta(B^+_R)} + \|u\|_{H^1(B^+_R)} \leq C$.

Proof: Let w be the harmonic lifting of u in B^+_{2R}. Because $u - w \in H^1_0(B^+_{2R})$ then it follows

$$\int_{B^+_{2R}} |\nabla u|^2 - |\nabla w|^2 = \int_{B^+_{2R}} |\nabla(u-w)|^2 + \int_{B^+_{2R}} 2\nabla w \cdot \nabla(u-w) = \int_{B^+_{2R}} |\nabla(u-w)|^2$$

Then from $J(u, B^+_{2R}) \leq J(w, B^+_{2R})$ and the equality above we obtain

$$\begin{aligned} \int_{B^+_{2R}} |\nabla(u-w)|^2 &= \int_{B^+_{2R}} |\nabla u|^2 - |\nabla w|^2 \leq \int_{B^+_{2R}} \Lambda \left[\chi_{\{w>0\}} - \chi_{\{u>0\}}\right] \\ &\leq \Lambda |B_1| (2R)^n. \end{aligned} \tag{4.1}$$

Take $\eta \in C^\infty_0(B_{2R}), \eta \equiv 1$ in B_R, $0 \leq \eta \leq 1$ and $|\nabla \eta| \leq \frac{C}{R}$ for some dimensional constant C. Obviously $(w-f)\eta^2 \in H^1_0(B^+_{2R})$ can be used as a test function in the weak formulation of $\Delta w = 0$

$$\int_{B^+_{2R}} \eta^2 |\nabla w|^2 = \int_{B^+_{2R}} \nabla w \left[\nabla f \eta^2 - 2\eta \nabla \eta (w-f)\right].$$

Applying Cauchy-Schwarz inequality and the estimate $|\nabla \eta| \leq \frac{C}{R}$ we obtain Caccioppolli's inequality

$$\int_{B^+_R} |\nabla w|^2 \leq 8 \int_{B^+_{2R}} \left[|\nabla f|^2 + \frac{4C^2}{R^2}(w-f)^2\right] \leq C_1 \tag{4.2}$$

where $C_1 = 4|B_2| M^2 \left[(2R)^n + 16C^2 (2R)^{n-2}\right]$.

Since $u - f = 0$ in B'_{2R} we can apply Poincarè's inequality to conclude $\int_{B^+_R} (u-f)^2 \leq \frac{c_0}{R^2} \int_{B^+_R} |\nabla(u-f)|^2$ depends on the dimension n and $\mathcal{H}^{n-1}(B'_R)-$ the $n-1$ dimensional Hausdorff measure of B'_R.

Combining inequalities (4.1), (4.2) and Poincarè's inequality we get

$$\begin{aligned} \int_{B^+_R} |\nabla u|^2 &\leq 2\left(\int_{B^+_R} |\nabla w|^2 + \int_{B^+_R} |\nabla(w-u)|^2\right) \\ &= 2(C_1 + \Lambda |B_1| (2R)^n) \equiv C_2 \end{aligned} \tag{4.3}$$

thereby

$$\begin{aligned} \int_{B_R^+} u^2 &\le 2\int_{B_R^+} f^2 + 2\int_{B_R^+}(u-f)^2 \le 2\left(M^2\frac{|B_1|}{2}R^n + \frac{c_0}{R^2}\int_{B_R^+}|\nabla(u-f)|^2\right) \\ &\le M^2|B_1|R^n + \frac{4c_0}{R^2}\left(\int_{B_R^+}|\nabla u|^2 + \int_{B_R^+}|\nabla f|^2\right) \\ &\le M^2|B_1|R^n + \frac{4c_0}{R^2}(C_2 + M^2\frac{|B_1|}{2}R^n) \equiv C_3 \end{aligned} \tag{4.4}$$

implying that $\|u\|_{H^1(B_R^+)} \le \sqrt{C_2 + C_3} \equiv C_4$.

As for Hölder continuity let us note that in view of Theorem 7.19 of [GT] it is enough to show that for $B_r^+(z) \subset B_{2R}^+, z \in B_R', r < \frac{1}{2}$ we have

$$\int_{B_r^+(z)} |\nabla u| \le C_5 r^{n-1+\beta}. \tag{4.5}$$

for some $\beta > 0$ and C_5 depending on M, n and R. Indeed if $z \in B_R^+$ and $|z - z'| > \frac{1}{4}$ we get that $B_{\frac{1}{8}}(z) \in B_1^+$ and by local continuity Theorem 2.1 [ACF] u is uniformly continuous with some $\beta > 0$ depending only on $\|u\|_{H^1(B_R^+)}, n$ and M. Whilst for $r < \frac{1}{2}$ either $|z - z'| \le r$ and $B_r^+(z) \subset B_{2r}^+(z')$ or $r < |z - z'| < \frac{1}{2}$.

First we deal with the case $z \in B_R'$ and $B_{4r}^+(z) \subset B_R^+$. Let v be the harmonic lifting of u in $B_{4r}^+(z)$, i.e. $\Delta v = 0$ in B_{4r}^+ and $v - u \in H_0^1(B_{4r}^+(z))$. Since $J(u, B_{4r}^+) \le J(v, B_{4r}^+)$ it follows that

$$\int_{B_{4r}^+(z)} |\nabla u|^2 + \Lambda\chi_{\{u>0\}} \le \int_{B_{4r}^+(z)} |\nabla v|^2 + \Lambda\chi_{\{v>0\}}.$$

Thereby

$$\begin{aligned} \int_{B_{4r}^+(z)} |\nabla u|^2 - |\nabla v|^2 &= \int_{B_{4r}^+} (\nabla u - \nabla v)(\nabla u + \nabla v) \\ &= \int_{B_{4r}^+(z)} |\nabla(u-v)|^2 \\ &\le \int_{B_{4r}^+(z)} \Lambda\chi_{\{v>0\}} - \Lambda\chi_{\{u>0\}} \\ &\le M|B_4|r^n. \end{aligned} \tag{4.6}$$

From triangle inequality we get

$$\begin{aligned} \int_{B_r^+(z)} |\nabla u| &\le \int_{B_r^+(z)} |\nabla(u-v)| + \int_{B_r^+(z)} |\nabla v| \\ &\le M|B_4|r^n + \int_{B_r^+(z)} |\nabla v|, \end{aligned} \tag{4.7}$$

where the last line follows from (4.6) and Cauchy-Schwarz inequality.

It remains to show that that there are constants $\beta \in (0,1), C_6$ depending on M, R and n such that

$$\int_{B_r^+(z)} |\nabla v|^2 \leq C_6 r^{n-2+2\beta}. \tag{4.8}$$

To see this take $\eta \in C_0^\infty(B_{4r})$ such that $\eta \equiv 1$ in B_r, $0 \leq \eta \leq 1, |\nabla \eta| \leq \frac{C}{r}$, C is a dimensional constant, then $\eta^2(v-f) = 0$ on ∂B_{4r}^+ and we have from the weak formulation of harmonicity of v

$$\int_{B_{4r}^+} \nabla v[2\eta \nabla \eta(v-f) + \eta^2(\nabla v - \nabla f)] = 0.$$

Rearranging the terms and applying Hölder inequality we get

$$\begin{aligned}\int_{B_{4r}^+} \eta^2 |\nabla v|^2 &= -\int_{B_{4r}^+} \nabla v \eta [2\nabla \eta(v-f) - \eta \nabla f] \\ &\leq \varepsilon \int_{B_{4r}^+} \eta^2 |\nabla v|^2 + \frac{1}{\varepsilon} \int_{B_{4r}^+} [2\nabla \eta(v-f) - \eta \nabla f]^2.\end{aligned}$$

Choosing ε suitably small and recalling that $\eta \equiv 1$ in B_r we get the estimate

$$\int_{B_r^+} |\nabla v|^2 \leq \frac{C}{\varepsilon} \int_{B_{4r}^+} [2\nabla \eta(v-f) - \eta \nabla f]^2. \tag{4.9}$$

According to Lemma 1.2.4 in [K] v is Hölder continuous with some exponent $\gamma = \gamma(n, M, R) \in (0,1)$, because $|v| \leq M, \|v\|_{H^1(B_{4r}^+)} \leq M + \|u\|_{H^1(B_{4r}^+)}$. Thus the left hand side of (4.9) can be estimated as follows

$$\int_{B_{4r}^+} [2\nabla \eta(v-f) - \eta \nabla f]^2 \leq C_7 \sup_{B_{4r}^+} |v-f| r^{n-1} + C_7 \sup_{B_{4r}^+} |\nabla f| r^n \leq C_8 r^{n-1+\gamma}$$

where C_8 depends only on n, M, R and to get the first inequality we used the estimate $|\nabla \eta| \leq \frac{C}{r}$. Thus choosing $\beta = \frac{1+\gamma}{2}$ the result follows. Notice that β depends only on n, M and R.

Finally it remains to show (4.5) for $B_r^+(z)$ with $z \in B_R^+$ and $r \leq |z - z'| \leq \frac{1}{2}$. Notice that (4.7) and (4.9) still hold for this case. As for the estimate (4.8), it follows from Poisson representation and the bound $|v| \leq M$. □

Remark 4.4. *One can apply Lemma 4.3 to a countable family of $\mathcal{P}_{R_j}(n, \lambda_\pm^j, \alpha_\pm^j, g_j)$ as $R_j \to \infty$.*

4.2. **Implications of linear growth.** The standard regularity result for free boundary problems states that the free boundary is smooth away from an ineluctable singular set of smaller co-dimension. The genus of regular points is characterized by flatness.

Mathematically the blow-up consists of scaling u in small balls centered on the free boundary: for $u \in \mathcal{P}_1(n, \lambda_\pm, \alpha_\pm, g)$ with linear growth at the origin, the scaled functions $v_j(x) = \frac{u(r_j x)}{r_j}$ are uniformly bounded as $r_j \searrow 0$. Since $f(0) = 0$, one readily verifies that $v_j \in \mathcal{P}_{1/r_j}(n, \alpha_\pm, \lambda_\pm, g_j)$, where $g_j(x) = \frac{g(r_j x)}{r_j}$. Clearly v_j is defined in $B^+_{\frac{1}{r_j}}$ and provides better picture of the free boundary at the origin. Thus by scaling we obtain a sequence of function v_j and a sequence of corresponding free boundaries $\Gamma_j = \Gamma(v_j)$. One expects that the convergence $v_j \to v_0$ implies $\Gamma_j \to \Gamma_0 = \Gamma(v_0)$ in Hausdorff distance, which will follow immediately from a compactness of v_j in a suitable class of functions. For the reader's convenience we recall Theorem 3.1 from [KKS].

Proposition 4.5. *(*[KKS]*) Let v_j be a blow up sequence of u_j, as in (1.6), with $u_j \in \mathcal{P}_1(n, \lambda_\pm, \alpha_\pm, g)$ and $x_0 = 0$. Further assume that u_j have uniform linear growth. Then, after passing to a subsequence, there exists $v \in \mathcal{P}_\infty$ so that*

1° *$v_j \to v$ uniformly on compact subsets of $\mathbb{R}^n_+$ and in $C^\beta(E), 0 < \beta < 1$, for each $E \subset\subset \mathbb{R}^n_+$,*
2° *for each M, $v_j \rightharpoonup v$ weakly in $H^1(B^+_M)$,*
3° *for each M, $\chi\{v_j > 0\} \to \chi\{v > 0\}$ in $L^1(B^+_M)$,*
4° *$\nabla v_j(x) \to \nabla v(x)$ for a.e. x,*
5° *For each $\delta > 0$, $E \subset B^+_M$, $dist(E, \Pi) \geq \delta$, $0 < r < \delta/4$, for j large*

$$\partial\{v_j > 0\} \cap E \subset \bigcup_{x \in \{v>0\} \cap E_{\delta/2}} B_r(x),$$

and

$$\partial\{v > 0\} \cap E \subset \bigcup_{x \in \{v_j>0\} \cap E_{\delta/2}} B_r(x),$$

where $E_{\delta/2}$ is a $\delta/2$-neighborhood of E.

4.3. **Weiss' energy.** It follows from [W] that for any $u \in \mathcal{P}_r(n, \lambda_\pm, \alpha_\pm, 0)$

$$W(R, u, x_0) = W(R) = \frac{1}{R^n} \int_{B^+_R(x_0)} |\nabla u|^2 + \Lambda \chi_{\{u>0\}} - \frac{1}{R^{n+1}} \int_{S^+_R(x_0)} u^2, \tag{4.10}$$

is non-decreasing function of R, with $x_0 \in \Gamma(u), B_R(x_0) \subset B_r^+$, and

$$\frac{dW}{dR} = \frac{1}{R^n} \int_{\partial S_R^+} \left(\nabla u \cdot \nu - \frac{u}{R} \right)^2 .$$

$W(R, u, x_0)$ is called Weiss' energy at x_0. Notice that $\nabla u \cdot x - u = 0$ if and only if u is homogeneous functions of degree 1.

Proposition 4.6. *Let $u \in \mathcal{P}_1(n, \lambda_\pm, \alpha_\pm, g)$ and $g = C|x|^{1+\kappa}$. If u has linear growth then $W(R, u, 0)$ is non-decreasing function of R and*

$$\frac{dW}{dR} \geq \frac{1}{R^n} \int_{\partial S_R^+} \left(\nabla u \cdot \nu - \frac{u}{R} \right)^2 .$$

In particular any blow-up limit of u at the origin is homogeneous function of degree one.

Proof: If $g \neq 0$ and $u \in \mathcal{P}_r(n, \alpha_\pm, \lambda_\pm, g)$ then some extra care is needed to prove the estimate from below for the derivative $W'(R, u, 0)$. See Lemma 11.1 in Appendix for the proof.

It remains to show that $W(r, u, 0)$ is bounded when r tends to zero. If v is the harmonic lifting of u in B_{4r}^+ and u has linear growth at 0, i.e. $\sup_{B_{4r}^+} |u| \leq Cr$, then by maximum principle $\sup_{B_{4r}^+} |v| \leq Cr$. From Caccioppolli's inequality (4.9) we have

$$\int_{B_r^+} |\nabla v|^2 \leq Cr^n.$$

Hence

$$\int_{B_r^+} |\nabla u|^2 \leq 2 \int_{B_r^+} |\nabla v|^2 + 2 \int_{B_r^+} |\nabla (u - v)|^2$$

which, in view of (4.6), implies that W is bounded for small r, whenever $u \in \mathcal{P}_1(n, \lambda_\pm, \alpha_\pm, g)$ is linearly growing solution. □

5. Proof of Theorem A

The proof of Theorem A consists of two parts. The first one deals with the two phase problem. Our method is based on dyadic scaling argument. If the statement of Theorem A fails then it allows us to construct a linearly growing, non-degenerate harmonic function v_0 in $\mathbb{R}_+^n$ vanishing on $\partial \mathbb{R}_+^n$ and at some interior point of $\mathbb{R}_+^n$. The latter is due to $\delta-$thickness condition. Thus, in view of the Liouville theorem, v_0 is zero, which contradicts the non-degeneracy of v_0.

5.1. **Two-phase case.** Set

$$S(j,u) := \sup_{B^+_{2^{-j}}} |u|.$$

It suffices to show

$$S(j+1,u) \le \max\left\{\frac{c2^{-j}}{2}, \frac{S(j,u)}{2}, \dots, \frac{S(0,u)}{2^{j+1}}\right\} \tag{5.1}$$

for some positive constant c. Let us suppose that (5.1) is not true. Then there exists a sequence of minimizers $u_j \in \mathcal{P}_1(n,\lambda_\pm,\alpha_\pm,g)$ and a sequence of integers k_j so that

$$S(k_j+1,u_j) > \max\left\{\frac{j2^{-k_j}}{2}, \frac{S(k_j,u_j)}{2}, \dots, \frac{S(k_j-m,u_j)}{2^{m+1}}, \dots, \frac{S(0,u_j)}{2^{k_j+1}}\right\}. \tag{5.2}$$

Observe that from Lemma 4.2 $|u_j| \le M$ hence $k_j \to \infty$. Put

$$v_j(x) = \frac{u_j(2^{-k_j}x)}{S(k_j+1,u_j)}.$$

We wish to show that (5.2) implies uniform up-to-boundary estimates for the sequence v_j. In fact there are positive constants α and C depending on R but independent of j such that the following estimates hold

$$\|v_j\|_{C^\alpha(B^+_R)} \le C(R), \tag{5.3}$$

$$\|v_j\|_{H^1(B^+_R)} \le C(R). \tag{5.4}$$

For brevity we denote

$$\epsilon_j = \frac{2^{-k_j}}{S(k_j+1,u_j)}, \qquad f_j(x) = \epsilon_j(\alpha_+ x_2^+ - \alpha_- x_2^-) + g_j(x), \tag{5.5}$$

where $g_j(x) = \frac{g(2^{-k_j}x)}{S(k_j+1,u_j)}$. Recall that by (5.2)

$$\epsilon_j = \frac{2^{-k_j}}{S(k_j+1,u_j)} \le \frac{1}{j} \longrightarrow 0 \tag{5.6}$$

thereby $f_j \to 0$ when $j \to \infty$, for $g(x) = o(|x|)$.

Consider the *scaled* functional

$$\widetilde{J}_j(v, B^+_R) = \int_{B^+_R} |\nabla v|^2 + \epsilon_j^2 \Lambda \chi_{\{v>0\}}. \tag{5.7}$$

If $u_j \in \mathcal{P}_1(n, \lambda_\pm, \alpha_\pm, g)$ then $v_j \in \mathcal{P}_{2^{k_j}}(n, \epsilon_j\alpha_\pm, \epsilon_j\lambda_\pm, g_j)$ provided $R < 2^{k_j}$. Indeed by a simple calculation we have

$$\begin{aligned}\widetilde{J}_j(v_j, B_R^+) &= \int_{B_R^+} |\nabla v_j|^2 + \epsilon_j^2 \Lambda \chi_{\{v_j>0\}} \\ &= \epsilon_j^2 2^{k_j n} \int_{B^+_{R/2^{k_j}}} |\nabla u_j|^2 + \Lambda \chi_{\{u_j>0\}} \\ &= \epsilon_j^2 2^{k_j n} J(u_j, B^+_{R/2^{k_j}}).\end{aligned}$$

Furthermore for fixed $R = 2^m$ we infer from (5.2) that

- $\sup_{B^+_{\frac{1}{2}}} |v_j| = 1$,
- $\sup_{B^+_{2m}} |v_j| \leq C2^m, R = 2^m < 2^k_j$, m is fixed.

Now we can apply Proposition 4.3 with $\sup_{2R} |v_j| + \epsilon_j(\alpha_+ + \alpha_- + \lambda_+ + \lambda_-) + \|f_j\|_{C^{0,1}} \leq M$ with $M = 2^{m+1}$ and the estimates (5.3) and (5.4) follow.

Thereby we can extract a subsequence v_{j_k} which converges to some function v_0 such that the following holds: for any fixed $R > 0$

$$(5.8)\qquad \begin{cases} (i) \quad v_{j_k} \to v_0 \text{ in } C^\beta(\overline{B_R^+}), \qquad v_{j_k} \rightharpoonup v_0 \text{ weakly in } H^1_{loc}(\mathbb{R}^n_+), \\ (ii) \quad \sup_{B^+_{1/2}} |v_0| = 1, \ v_0(x) = 0, x \in \Pi \qquad \text{by } C^\beta \text{ regularity}, \\ (iii) \ \Delta v_0 = 0 \text{ in } x_1 > 0, \\ (iv) \quad v_0 \text{ has linear growth}, \\ (v) \quad v_0(y_0) = 0 \text{ for some interior point } y_0 \text{ (by (1.4))}. \end{cases}$$

Once all claims in (5.8) are proven we may use Liouville's theorem for harmonic functions in $\mathbb{R}^n_+$ (utilizing (iii) and (iv)) to conclude $v_0(x) = ax_1$ for some constant $a \neq 0$. But then (ii), (v) and (vi) are in direct contradiction, and hence our supposition (5.2) is false.

Now we proceed by proving (5.8). The first claim follows from standard compactness arguments. The second one follows from (5.5) and the convergence of the traces of v_j in view of Hölder continuity.

Let us prove the third claim. Let $D \subset \overline{B_R^+}$ be a domain and $R > 0$ is fixed. Then $v_j \in \mathcal{P}_{2^{k_j}}(n, \epsilon_j\alpha_\pm, \epsilon_j\lambda_\pm, g_j)$ for the *scaled* functional $\widetilde{J}_j$, defined by (5.7). Observe that for each $\psi \in C_0^\infty(D)$

$$\widetilde{J}_j(\psi, D) \to \int_D |\nabla\psi|^2 \qquad \text{as} \qquad j \to \infty.$$

By (5.3) and (5.4), v_0 exists and $\int_D |\nabla v_0|^2 \leq \liminf_{k\to 0} \int_D |\nabla v_{j_k}|^2$. According to (5.5) $f_0 = v_0 = 0$ on $\Pi = \{x : x_1 = 0\}$, where $f_0 = \lim_{j\to\infty} f_j$ uniformly.

Now let us take $\psi \in H^1_0(D)$, then

$$\widetilde{J}_j(v_j, D) \leq \widetilde{J}_j(v_j + \psi, D)$$

or equivalently

$$\int_D \epsilon_j^2 \Lambda \chi_{\{v_j>0\}} \leq \int_D -2\nabla v_j \nabla\psi + |\nabla\psi|^2 + \epsilon_j^2 \Lambda \chi_{\{v_j+\psi>0\}}.$$

Thereby sending j_k to ∞ and utilizing the weak convergence of gradients $\nabla v_{j_k} \rightharpoonup \nabla v_0$ in $L^2(\overline{B_R^+})$, we conclude

$$0 \leq \int_D -2\nabla v_0 \nabla\psi + |\nabla\psi|^2$$

and upon adding $\int_D |\nabla v_0|^2$ to both sides we infer

$$\int_D |\nabla v_0|^2 \leq \int_D |\nabla(v_0 - \psi)|^2.$$

Since $C^\infty_0(D)$ is dense in $H^1_0(D)$ we conclude the proof of the third claim in (5.8).

The fourth claim follows from (5.2) as indicated above. Hence it remains to prove the fifth claim. By our assumption (1.4) (resp. (1.5)) there exists $x_j \in B^+_{2^{-k_j}} \cap K_\delta$ such that $u_j(x_j) = 0$ (resp. $|u(x_j)| \leq C|x_j|$). Thereby

$$\frac{1}{2} \leq \frac{|x_j|}{2^{k_j}} \leq 1.$$

If we set $y_j = \frac{x_j}{2^{k_j}}$, then one can easily verified that $y_j \in (B_1 \setminus B_{1/2}) \cap K_\delta$ and $y_j \to y_0$, for some $y_0 \in (B^+_1 \setminus B^+_{1/2}) \cap K_\delta$. Clearly $v_0(y_0) = 0$ by (5.5) and Hölder continuity.

Now the proof of (3.1) for two phase case is complete.

5.2. **One-phase case.** To prove (3.1) in Theorem A **2**°, we need to work out the condition (v) in (5.8), because the others follow as above. For two-phase case, (v) was justified by assumption (1.4) whilst for one-phase case, (1.4) is replaced by the condition that the origin is a non-isolated free boundary point. Indeed, this would be enough to force through a similar condition as that in (v) of (5.8). However, the analysis is slightly more delicate and needs care.

Suppose for a sequence $k_j \uparrow \infty$ we have $\{u = 0\} \cap (B_{\rho_{k_j}} \setminus B_{\rho_{k_j+1}}) \cap K_\delta \neq \emptyset$, where $\rho_{k_j} = \frac{1}{2^{k_j}}$. We roll $B_{d_j}(\frac{3e_1}{4})$ with $d_j = \frac{1}{4}\rho_{k_j}$, over $\partial B_{\rho_{k_j+1}}$ keeping it within $B_{\rho_{k_j}} \setminus B_{\rho_{k_j+1}}$ until it touches the free boundary $\Gamma(u)$ for the first time. Let $B_{d_j}(z)$ be the ball touching the free boundary at z_0. Clearly u

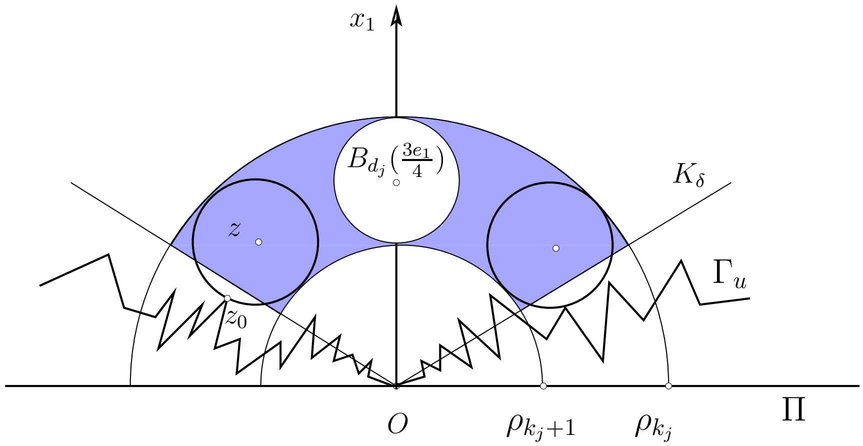

FIGURE 1. δ-thickness for one-phase problem.

is positive and harmonic inside $B_{d_j}(z)$ and attains its minimum at z_0, therefore we can apply Lemma 11.19 from [CS] to get the estimate

$$u(z) \leq C d_j \frac{\partial u}{\partial \nu}(z_0) \tag{5.9}$$

where ν is the inner normal to $B_{d_j}(z)$ at z_0. Then by Theorem 6.3 in [AC] we have $|\nabla u(z_0)| \leq \lambda_+$, which in conjunction with Harnack's inequality implies $\sup_{B_{\frac{d_j}{2}}(z)} u \leq C_0 u(z) \leq C_0 C \lambda_+ d_j = \frac{C_0 C \lambda_+}{4} \rho_{k_j}$. Hence

$$\sup_{B_{\frac{d_j}{2}}(z)} u \leq \frac{C_0 C \lambda_+}{4} \rho_{k_j}. \tag{5.10}$$

For scaled functions $v_j(x) = \frac{u(\rho_{k_j} x)}{S(k_j+1, u_j)}$ it follows from (5.10), that there exists a ball $B_{\frac{1}{4}}(y_0) \subset B_1^+ \setminus B_{1/2}^+$ such that

$$\sup_{B_{\frac{1}{4}}(y_0)} v_j \leq C \frac{\rho_{k_j}}{S(k_j+1, u_j)} = C\epsilon_j \to 0$$

by (5.6) which gives (v) in (5.8) for one phase case.

The proofs of the remaining claims of (5.8) are the same as for the two-phase case and one will have the final contradictory conclusion. $\square$

6. Proof of Theorem B

It follows from the proof of Theorem A **2**°, that $u \geq 0$ grows linearly away from the origin, provided the origin is a non-isolated free boundary point. We can replace the origin by any non-isolated free boundary point z near the origin and apply the same argument to show that for $u \in \mathcal{P}_1(n, \lambda_\pm, \alpha_\pm, g)$ there exists a tame constant C such that the growth estimate

$$0 \leq |u(x)| \leq C|x - z| \tag{6.1}$$

holds for any $z \in B'_r \cap \partial\Omega^+(u)$ for some $r > 0$.

In order to conclude (6.1) for the two phase solutions we further require the $\delta-$thickness to be satisfied in some neighborhood of the origin. Notice that in the two phase case, by the Hölder continuity of u, the origin is automatically a non-isolated free boundary point.

Our goal is to prove that the free boundary Γ_u remains within a cone $\mathcal{C}_{\delta_0} = \{x : x_1 \geq \delta_0|x_2|\}$ in some neighborhood of the origin. This will be enough to prove Theorem B, because for the free boundary of the blow-up it implies $\Gamma(u_0) \subset \mathcal{C}_{\delta_0}$. Thus the uniform $\delta-$thickness condition will be satisfied for u_0, with $\delta_0 = \delta$ and the result will follow from Theorem A via a standard scaling argument.

Lemma 6.1. *Let $u \in \mathcal{P}_1(n, \lambda_\pm, \alpha_\pm, g)$. If the thickness assumption (1.4) is satisfied for any free boundary point $x_0 \in B'_{\frac{1}{2}}$ then there exists a tame constant δ_0 such that*

$$\frac{|z_2|}{z_1} \leq \delta_0, \qquad \forall z \in \Gamma_u \cap B^+_{1/2}. \tag{6.2}$$

In particular for any blow-up limit u_0 the inclusion $\Gamma(u_0) \subset \mathcal{C}_{\delta_0}$ is true.

Proof: It follows from the uniform $\delta-$thickness condition and the discussion above that (6.1) is true. Suppose (6.2) fails, then there exists a sequence $z^k \in B^+_{1/2} \cap \Gamma(u_k)$ of free boundary points of $u_k \in \mathcal{P}_1(n, \lambda_\pm, \alpha_\pm, g)$, such that

$$|z_2^k| \geq k z_1^k, \ \ k > k_0,$$

for sufficiently large $k_0 \in \mathbb{N}$. Setting $d_k = z_1^k, f_0(x) = \alpha_+ x_2^+ - \alpha_- x_2^-$ we have

$$|z_2^k| \geq k d_k, \qquad |f_0(p^k)| = \begin{cases} \alpha_+|z_2^k| \text{ if } z_2^k > 0 \\ \alpha_-|z_2^k| \text{ if } z_2^k < 0 \end{cases},$$

where p^k is the projection of z^k onto Π. In any case we get that $|f(p^k)| \geq k \min(\alpha_+, \alpha_-) d_k$. Put $r_k = |z^k - \xi^k|$, where $\xi^k = (0, 0, z_3^k, \dots, z_n^k)$ is the projection of z^k onto $\Pi \cap \{x_2 = 0\}$. Then from

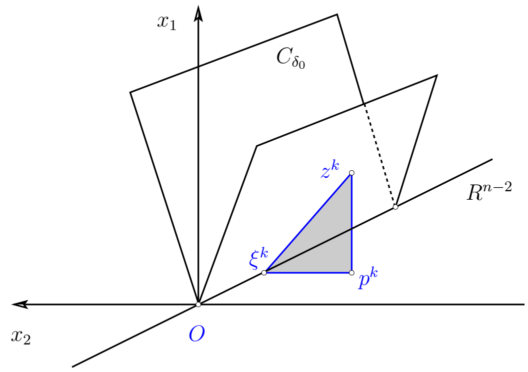

FIGURE 2. The cone $\mathcal{C}_{\delta_0}$.

triangle inequality $r_k = |z^k - \xi^k| \geq |z_2^k| - z_1^k > (k-1)d_k$ implying $\frac{d_k}{r_k} \leq \frac{1}{k-1}$. In particular

$$1 \geq |y_2^k| = \frac{|z_2^k|}{r_k} \geq 1 - \frac{d_k}{r_k} \geq 1 - \frac{1}{k-1}. \tag{6.3}$$

Now introduce the scaled functions

$$v_k(y) = \frac{u_k(\xi^k + r_k y)}{r_k}, \qquad y \in B_2^+.$$

The points $y^k = \dfrac{z^k - \xi^k}{r_k}$ are on the half sphere S_1^+ and from (6.3) we get

$$y_1^k = \frac{d_k}{r_k} < \frac{1}{k-1}.$$

From (6.1) we have $|u_k(x)| \leq C|x - \xi^k|$, since $\xi^k \in B'_{\frac{1}{2}} \cap \Gamma(u_k)$. Therefore it follows that $|v_k(y)| \leq C|y|$, with constant C independent of k. Furthermore v_k is a local minimizer of $J(\cdot, B_2^+)$ because

$$\int_{B_1^+} |\nabla v_k|^2 + \Lambda \chi_{\{v_k>0\}} = \frac{1}{r_k^n} \int_{B_{r_k}} |\nabla u|^2 + \Lambda \chi_{\{u>0\}}.$$

Thus $v_k \in \mathcal{P}_2(n, \lambda_\pm, \alpha_\pm, g_k)$ where $g_k(x) = \frac{g(\xi^k + r_k y)}{r_k}$ and $\lim g_k = 0$ uniformly.

By Proposition 4.3 it follows, that v_k is bounded in $C^\beta(\overline{B_{3/2}^+}) \cap H^1(\overline{B_{3/2}^+})$ for some positive $\beta \in (0,1)$. Then for a subsequence k_j, $v_{k_j} \to v_0$ in $C^\beta(\overline{B_{3/2}^+})$, $\nabla v_{k_j} \rightharpoonup \nabla v_0$ weakly in $L^2(B_{3/2}^+)$ and $y^{k_j} \to y^0$, where y^0 is a free boundary point. From (6.3) $|y_2^k| = \dfrac{|z_2^k|}{r_k} \to 1$ and $y \in S_1^+$. But then $y_1^0 = 0, |y_2^0| = 1$ and this contradicts to $f_0(y^0) = \alpha_\pm \neq 0$. $\square$

Let $v(x) = \frac{u_0(\xi + Rx)}{R}$ then (6.2) translates to the free boundary of the blow-up function u_0 implying that $\Gamma(u_0) \subset \mathcal{C}_{\delta_0}$. Hence we have uniform $\delta-$thickness condition for each $z \in \Gamma(u_0) \cap \Pi$. From Theorem

A we have $|v(x)| \le C|x|$. Returning to u we conclude $|u(x)| \le C|z-\xi| \le C(x_1+|x_2|)$ and this finishes the proof of Theorem B.

7. Largest and Smallest global solutions

Before embarking into the details we briefly go over the main steps of the proof. First we notice that the global solutions enjoy ordering. This implies that there are smallest and largest global homogeneous solutions which we denote respectively by v_S and v_L. It follows from the scale and translation invariance that v_S and v_L depend only on x_1 and x_2. Hence we can explicitly compute them. Moreover v_S has larger W-energy implying that the free boundary of any global homogeneous solution, distinct from v_L and v_S, cannot touch Γ_{v_S} or Γ_{v_L} tangentially.

Thus if there is third global homogeneous solution u then we can construct a new one which is symmetric in $x_3, x_4, \dots, x_n$ variables and neither of the functions v_S, v_L coincides with u. Thus without loss of generality we may assume that u is symmetric in $x_3, x_4, \dots, x_n$ variables. Then a dimension reduction argument will finish the proof since the only 2D solutions are v_S and v_L.

7.1. **Largest and smallest solutions in $\mathcal{P}_\infty$.** We recall (1.3)

$$J(u, B_R^+) = \int_{B_R^+} |\nabla u|^2 + \Lambda \chi_{\{u>0\}}.$$

Let v_1, v_2 be two minimizers of $J(u, B_R^+)$ and $v_1 \le v_2$ (resp. $v_1 \ge v_2$) on ∂B_R^+. Then it is easy to see that $\max(v_1, v_2)$ (resp. $(\min(v_1, v_2))$ is a minimizer of $J(u, B_R^+)$ with boundary values v_2 (resp. v_1).

Indeed testing $\max(v_1, v_2)$ against v_2 in B_R^+ and $\min(v_1, v_2)$ against v_1 we get

$$\begin{aligned} J(v_2, B_R^+) &\le J(\max(v_1, v_2), B_R^+), \\ J(v_1, B_R^+) &\le J(\min(v_1, v_2), B_R^+). \end{aligned} \tag{7.1}$$

Clearly

$$\begin{aligned} J(\max(v_1, v_2), B_R^+) &= \int_{B_R^+ \cap \{v_1 > v_2\}} |\nabla v_1|^2 + \Lambda \chi\{v_1 > 0\} + \int_{B_R^+ \cap \{v_1 \le v_2\}} |\nabla v_2|^2 + \Lambda \chi\{v_2 > 0\} \\ J(\min(v_1, v_2), B_R^+) &= \int_{B_R^+ \cap \{v_1 > v_2\}} |\nabla v_2|^2 + \Lambda \chi\{v_2 > 0\} + \int_{B_R^+ \cap \{v_1 \le v_2\}} |\nabla v_1|^2 + \Lambda \chi\{v_1 > 0\} \end{aligned}$$

which gives

$$J(\max(v_1, v_2), B_R^+) + J(\min(v_1, v_2), B_R^+) = J(v_1, B_R^+) + J(v_2, B_R^+). \tag{7.2}$$

Hence (7.1) in conjunction with (7.2) implies

$$J(v_1, B_R^+) = J(\min(v_1, v_2), B_R^+),$$
$$J(v_2, B_R^+) = J(\max(v_1, v_2), B_R^+).$$

Upon applying this observation to finite number of minimizers we obtain

Lemma 7.1. *If $v_1, \dots, v_N$ are minimizers on B_R^+ and $v_1 \le v_2 \le \cdots \le v_N$ on ∂B_R^+ (resp. $v_1 \ge v_2 \ge \cdots \ge v_N$) then $v_L^R = \max(v_1, \dots, v_N)$ (resp. $v_S^R = \min(v_1, \dots, v_N)$) is a minimizer of J^R with boundary values v_N on ∂B_R^+.*

Employing a compactness argument it follows that there exists a largest and a smallest minimizer denoted respectively by v_L^R and v_S^R.

By definition, for any $u \in \mathcal{P}_R(n, \lambda_\pm, \alpha_\pm) \cap \mathcal{P}_\infty$ we have

$$v_S^R(x) \le u(x) \le v_L^R(x), \qquad x \in B_R^+.$$

Moreover by Definition 1.5, v_S^R and v_L^R have uniform linear growth, i.e. $|v_S^R|, |v_L^R| \le C(x_1 + |x_2|)$ for some tame constant C independent of R. Sending $R \to \infty$ and utilizing the linear growth Proposition 4.3 we infer that $v_L^R \to v_L$ uniformly and weakly in H^1_{loc}. Furthermore $v_L \in \mathcal{P}_\infty$.

Indeed let $\varphi \in C_0^\infty(B_\rho^+)$, ρ is fixed and $\rho < R$ then v_L^R is a minimizer and we have

$$J(v_L^R, B_\rho^+) \le J(v_L^R + \varphi, B_\rho^+), \qquad \forall B_\rho^+ \subset \mathbb{R}^n_+.$$

More explicitly it can be rewritten as $\int_{B_\rho^+} \Lambda \chi_{\{v_L^R > 0\}} \le \int_{B_\rho^+} 2\nabla v_L^R \cdot \nabla\varphi + |\nabla\varphi|^2 + \Lambda\chi_{\{v_L^R + \varphi > 0\}}$.

By a customary compactness argument and weak convergence of gradients we get

$$J(v_L, B_\rho^+) \le J(v_L + \varphi, B_\rho^+), \qquad \forall \varphi \in C_0^\infty(B_\rho^+).$$

The same argument leads to the existence of v_S−the smallest global homogeneous solution. Thus

$$v_S \le u \le v_L, \qquad \forall u \in \mathcal{P}_\infty.$$

Since the class $\mathcal{P}_\infty$ is scale and $e_3, \dots, e_n$ translation invariant it follows that v_S, v_L are homogeneous and depend only on x_1 and x_2 variables.

Now let us explicitly compute v_L and v_S. For this we write the Laplacian in in polar coordinates

$$\Delta w = \frac{1}{r}\left[\frac{\partial(rw_r)}{\partial r} + \frac{\partial}{\partial\varphi}\left(\frac{w_\varphi}{r}\right)\right] = \frac{1}{r}\left[g(\varphi) + g''(\varphi)\right],$$

where $w = rg(\varphi)$. Recall that v_S, v_L are harmonic outside of the zero set by Proposition 4.1. This implies that g is a linear combination of $\sin\varphi$ and $\cos\varphi$. Therefore the largest and smallest solutions are linear combinations of x_1 and x_2.

Assume that

$$v^+ = ax_1 + bx_2, \text{ in } \Omega^+(v), \qquad v^- = Ax_1 + Bx_2, \text{ in } \Omega^-(v),$$

where v^+ and v^- are respectively the positive and negative parts of v and a, b, A, B are constants to be determined. The boundary condition $v = \alpha_+ x_2^+ - \alpha_- x_2^-$ on Π implies $b = \alpha_+, B = \alpha_-$.

Let us assume that the free boundary $\Gamma(v)$ is given by

$$x_1 = x_2 \tan\theta.$$

Both v^+ and v^- must vanish on $\Gamma(v)$. Hence

$$0 = ax_1 + bx_2 = ax_2\tan\theta + \alpha_+ x_2 = x_2(a\tan\theta + \alpha_+)$$

and we easily find that $a = -\dfrac{\alpha_+}{\tan\theta} = -\alpha_+\cot\theta$. Similarly

$$A = -\frac{\alpha_-}{\tan\theta} = -\alpha_-\cot\theta.$$

Summarizing we have

$$v^+ = \alpha_+(-x_1\cot\theta + x_2), \qquad v^- = \alpha_-(-x_1\cot\theta + x_2).$$

Note that $\cot\theta$ takes only two values, positive and negative, corresponding respectively to large and small solutions. To evaluate $\cot\theta$ we need to use the gradient jump condition $|\nabla v^+|^2 - |\nabla v^-|^2 = \Lambda$, which is now satisfied in classical sense, see Proposition 4.1. Substitution of v into this identity gives

$$\alpha_+^2(1 + \cot^2\theta) - \alpha_-^2(1 + \cot^2\theta) = \Lambda$$

or equivalently

$$\cot\theta = \pm\sqrt{\frac{\Lambda}{\alpha_+^2 - \alpha_-^2} - 1}.$$

Note that if $\Lambda \le \alpha_+^2 - \alpha_-^2$ then there is no free boundary. Summarizing we get that

$$
\begin{aligned}
v_L &= \alpha_+(\gamma x_1 + x_2)^+ - \alpha_-(\gamma x_1 + x_2)^-, \\
v_S &= \alpha_+(-\gamma x_1 + x_2)^+ - \alpha_-(-\gamma x_1 + x_2)^-, \\
\gamma &= \sqrt{\frac{\Lambda}{\alpha_+^2 - \alpha_-^2} - 1}.
\end{aligned} \tag{7.3}
$$

The above discussion is summarized in the following proposition.

Proposition 7.2. *The largest and smallest solutions v_L, v_S are given by (7.3) and these are the only two dimensional homogeneous global solutions.*

7.2. **Comparison of W-energy.** The aim of this section is to show that v_S has bigger W-energy than v_L. For all values of $\alpha_\pm$ for which $v_S \neq v_L$ we have

$$
W(1, v_S, 0) > W(1, v_L, 0), \tag{7.4}
$$

As a consequence we get that the largest solution is stable in the following sense:

Proposition 7.3. *Let $u \in \mathcal{P}_1(n, \lambda_\pm, \alpha_\pm, g)$ and suppose there is $R_0 \in (0,1)$ such that $W(R_0, u, 0) < W(1, v_S, 0)$ then any blow-up limit u_0 of u coincides with v_L.*

Proof: To check this we recall the monotonicity of W, to infer that $W(0^+, u, 0) = W(1, u_0, 0) < W(1, v_S, 0)$, which in view of Theorem C and $W(1, v_S, 0) \ge W(1, v_L, 0)$ implies that $u_0 = v_L$.

Now it remains to show (7.4). If v is a homogeneous solution, then W is constant hence it suffices to compute $W(1, \cdot, 0)$. By Green's formula

$$
\int_{B_1^+} |\nabla v|^2 = \int_{\partial B_1^+} v \frac{\partial v}{\partial \nu}.
$$

We can easily compute

$$
\begin{aligned}
W(1, v) &= \int_{\partial B_1^+} v \frac{\partial v}{\partial \nu} + \int_{B_1^+} \Lambda \chi\{v > 0\} - \int_{S_1^+} v^2 \\
&= \int_{B_1'} v \frac{\partial v}{\partial \nu} + \int_{B_1^+} \Lambda \chi\{v > 0\},
\end{aligned}
$$

where the last equality follows from $v(x) = x \cdot \nabla v(x)$ on $S_1^+ = \partial B_1^+ \cap \mathbb{R}^n_+$. In particular one can take v to be v_L or v_S.

Now let $\theta \in (0, \pi/2)$ be determined from

$$\cot\theta = \sqrt{\frac{\Lambda}{\alpha_+^2 - \alpha_-^2} - 1}.$$

Utilizing the explicit form of v_S one can readily verify that

$$\int_{B_1^+} \Lambda\chi\{v_S > 0\} = \int_0^1 \int_{S_\rho^+} \Lambda\chi\{v_S > 0\} = \frac{1}{n}\int_{S_1^+} \Lambda\chi\{v_S > 0\} = \frac{\Lambda\theta}{2\pi}\omega_n,$$

where ω_n is the volume of n-dimensional unit ball. Similarly

$$\int_{B_1^+} \Lambda\chi\{v_L > 0\} = \frac{\Lambda(\pi - \theta)}{2\pi}\omega_n.$$

Next we notice that $\frac{\partial v}{\partial \nu} = -\frac{\partial v}{\partial x_1}$, on B_1', therefore we have

$$\begin{aligned}
-\int_{B_1'} v_S \frac{\partial v_S}{\partial x_1} &= -\int_{B_1' \cap \{x_2 > 0\}} \alpha_+ x_2(-\gamma\alpha_+) - \int_{B_1' \cap \{x_2 < 0\}} \alpha_- x_2(-\gamma\alpha_-) \\
&= \gamma\alpha_+^2 \int_{B_1' \cap \{x_2 > 0\}} x_2 + \gamma\alpha_-^2 \int_{B_1' \cap \{x_2 < 0\}} x_2 \\
&= \gamma(\alpha_+^2 - \alpha_-^2) \int_{B_1' \cap \{x_2 > 0\}} x_2 \\
&= \frac{\omega_{n-2}}{n}\gamma(\alpha_+^2 - \alpha_-^2),
\end{aligned}$$

where $\gamma = \cot\theta = \sqrt{\frac{\Lambda}{\alpha_+^2 - \alpha_-^2} - 1}$. Hence

$$W(1, v_S, 0) = \gamma(\alpha_+^2 - \alpha_-^2)\frac{\omega_{n-2}}{n} + \Lambda\frac{\theta\omega_n}{2\pi},$$

and similarly one can see that

$$W(1, v_L, 0) = -\gamma(\alpha_+^2 - \alpha_-^2)\frac{\omega_{n-2}}{n} + \Lambda\frac{(\pi - \theta)\omega_n}{2\pi}.$$

Summarizing we have that

$$W(1, v_S, 0) - W(1, v_L, 0) = \Lambda\frac{\omega_{n-2}}{n}\left[2\gamma\frac{\alpha_+^2 - \alpha_-^2}{\Lambda} + \frac{(2\theta - \pi)}{2\pi}\frac{n\omega_n}{\omega_{n-2}}\right].$$

Using the explicit computation for ω_n we we obtain

$$n\frac{\omega_n}{\omega_{n-2}} = 2\pi.$$

Finally we observe that $\sin^2\theta = \dfrac{\alpha_+^2 - \alpha_-^2}{\Lambda}$ hence

$$\begin{aligned} 2\gamma\frac{\alpha_+^2-\alpha_-^2}{\Lambda} + \frac{(2\theta-\pi)}{2\pi}\frac{n\omega_n}{\omega_{n-2}} &= 2\gamma\frac{\alpha_+^2-\alpha_-^2}{\Lambda} + (2\theta-\pi) \\ &= 2\left[\cot\theta\sin^2\theta - (\theta - \frac{\pi}{2})\right] \\ &= \sin(2\theta) - 2\theta + \pi \\ &\geq 0\ . \end{aligned}$$

Therefore

$$W(1, v_S, 0) \geq W(1, v_L, 0)$$

and equality holds if and only if $\theta = \pi/2$. □

8. Proof of Theorem C

8.1. **Free boundary as generalized minimal surface.** The aim of this section is to classify homogeneous global solutions. For $n = 2$ this was done in Proposition 7.2. Therefore from now on we shall assume $n > 2$, $\alpha_- = \lambda_- = 0$ (i.e. the one phase case). Notice that the condition $\lambda_- = 0$ can be dropped due to the formulas (1.2) and (1.3). We recall that if u is a global solution, and hence local minimizer, of J for one phase problem then

$$\sup_{B_r(x_0)} |\nabla u|^2 \leq \Lambda + C(x_0) r^\alpha \tag{8.1}$$

for any $x_0 \in \Gamma(u)$, $B_r(x_0) \subset \mathbb{R}^n_+$ and $C(x_0)$ depends on $\operatorname{dist}(x_0, \Pi)$, see Theorem 6.3 [AC]. As a result we obtain that for any free boundary point x_0 the estimate holds

$$\limsup_{\substack{x\in\Omega^+(u)\\ x\to x_0}} |\nabla u(x)|^2 \leq \Lambda. \tag{8.2}$$

Our first task is to show that the estimate (8.2) holds in $\operatorname{supp} u$.

Lemma 8.1. *Let u be a global homogeneous solution. Then for $z^0 \in \Gamma \cup \{x_1 = 0, x_2 > 0\}$*

$$|\nabla u(z^0)|^2 \leq \Lambda.$$

Proof. To see this let $z_0 \in \Pi$ and $u(z^0) > 0$. Then there is $r > 0$ such that $u \in C^1(\overline{B_r^+(z^0)})$. Thus the tangential derivatives are controlled by $\alpha_+ \leq \Lambda$. As for the normal derivative we notice that from the definition of v_S and v_L we have that (since $\alpha_- = 0$)

$$|\nabla v_S| = |\nabla v_L| = \sqrt{\Lambda}.$$

But $v_S \leq u \leq v_L$ and $v_S = u = v_L$ on Π, hence it is enough to estimate the x_1- derivative. Indeed, from the estimate $v_S \leq u \leq v_L$ and $v_S(z^0) = v_L(z^0) = u(z^0)$ we get

$$\frac{\partial v_S(z^0)}{\partial x_1} \leq \frac{\partial u(z^0)}{\partial x_1} \leq \frac{\partial v_L(z^0)}{\partial x_1}.$$

Therefore $|\nabla u(z^0)|^2 \leq \Lambda$.

It is also apparent by the free boundary condition (8.1) that $|\nabla u|^2 \leq \Lambda$ on the free boundary. □

Lemma 8.2. *Let u be a global homogeneous solution. Then*

1° *the following estimate is true*

$$\sup_{x \in \mathbb{R}^n_+ \cap \{u>0\}} |\nabla u(x)|^2 \leq \Lambda. \tag{8.3}$$

2° *In particular $\Gamma(u)$ is a generalized surface of non-positive outward mean curvature.*

It should be remarked that the estimate (8.2) does not hold for non-homogeneous global solutions; see 10.1.

Proof of Lemma 8.2: Suppose the statement of the lemma fails, then there is a maximizing sequence x^j with the property that $|\nabla u(x^j)|^2 \to \Lambda + \epsilon_0 > \Lambda$. By zero-degree homogeneity of $|\nabla u|^2$ we may assume x^j are on the unit sphere. Also by sub-harmonicity of $|\nabla u|^2$ we assume that x^j tend to the boundary of $\{u > 0\} \cap \{x_1 > 0\}$. By Lemma 8.1 the sequence x^j cannot converge to either of the boundaries (free or fixed). Hence it converges to the "corner"-points $\{x_1 = x_2 = 0,\ |x| = 1\}$.

Let $r_j = \text{dist}(x^j, \Gamma \cup \Pi)$, then we have three different possibilities:

$$\text{Case 1}: \ r_j \approx \text{dist}(x^j, \Gamma) \approx x_1^j, \qquad \text{Case 2}: \ r_j = o(x_1^j), \qquad \text{Case 3}: \ r_j = o(\text{dist}(x^j, \Gamma)).$$

We shall see that all these cases will result into a contradiction.

Case 1: Let $\tilde{x}^j$ be the closest corner point on the $n-2$ dimensional unit sphere, i.e. $\tilde{x}^j \in \{x_1 = x_2 = 0,\ |x| = 1\} = \mathbb{S}^{n-2}$, in first case, and in the other two cases the closest point on the boundary to x^j (we again assume this close point is on the unit sphere).

Now let $d_j = |x^j - \tilde{x}^j|$ and scale u at $\tilde{x}^j$ with d_j,

$$u_j(x) = \frac{u(\tilde{x}^j + d_j x)}{d_j}.$$

Note that $d_j \approx r_j \approx x_1^j$ translates to u_j as follows; there is $y^j \in \mathbb{S}^n, y_3^j = \cdots = y_n^j = 0$ such that $y_1^j \approx \operatorname{dist}(y^j, \Gamma(u_j)) \approx 1$ and

$$\lim_{j\to\infty} |\nabla u_j(y^j)|^2 = \Lambda + \varepsilon_0. \tag{8.4}$$

Clearly u_j should be considered in a new domain, which is a scaled version of the support of u at $\tilde{x}^j$ and it contains $\operatorname{supp} v_S$. In the two other cases below the support of u_j converges to a half space.

Next we see that in all cases u_j converges to a limit function u_0 (at least for a subsequence) with further property that $|\nabla u_0(y^0)|^2 = \Lambda + \varepsilon_0$ (here $y^0 = \lim_{j\to\infty} y^j$, again for a subsequence). In particular, and by construction, $|\nabla u_0(x)|^2$ takes maximum at y^0, an interior point to the support of u_0. Hence by the strong maximum principle it must be constant, and therefore $|\nabla u_0(x)|^2 = \Lambda + \varepsilon_0$ in the support of u_0. This in turn implies u_0 is linear. But u_0 is a global minimizer, hence $|\nabla u_0|^2 = \Lambda$ in $\operatorname{supp} u_0$ which in contradiction with (8.4).

Case 2: Let $u_j(x) = \frac{u(\tilde{x}^j + r_j x)}{r_j}$. We proceed as in Case 1 and extract a subsequence for which $u_j \to u_0$ and u_0 is global minimizer. Furthermore (8.4) holds with $y^0 = \lim y^j$ but in this case $y^0 \in \Gamma(u_0), y_3^0 = \cdots = y_n^0 = 0$. This implies that $|\nabla u_0(y^0)|^2 = \Lambda + \varepsilon_0$ which is in contradiction with (8.1).

Now, in the first two cases, the free boundary is present (due to the length of scale r_j). In first case, we obtain a global minimizer in $\mathbb{R}^n_+$ with boundary data as before. At the same time we have u_0 is linear, which results into the fact that u_0 is one of the functions v_L, v_S. But then this contradicts the fact that $|\nabla u_0|^2 = \Lambda + \varepsilon_0$.

Case 3: Now the last case gives us scaling with center at the fixed boundary. Here we use both the small and the large solutions to bound the scaled function. Indeed, for $\tilde{x}^j$ being the projection of x^j onto Π, we have

$$u_j(x) = \frac{u(r_j x + \tilde{x}^j) - u(\tilde{x}^j)}{r_j} = \frac{u(r_j x + \tilde{x}^j) - \alpha_+ \tilde{x}_2^j}{r_j}$$

and hence the scaled versions of v_S and v_L at $\tilde{x}^j$ satisfy

$$(v_S)_j \le u_j \le (v_L)_j.$$

Hence the blow-up limits keep the order

$$v_S = (v_S)_0 \le u_0 \le (v_L)_0 = v_L. \tag{8.5}$$

Now as before we have $|\nabla u_0|^2 = \Lambda + \varepsilon_0$, and this is impossible due to (8.5), and the fact that $|\nabla v_L|^2 = |\nabla v_S|^2 = \Lambda$.

Now we turn to the proof of the second statement of Lemma 8.2, namely that $\Gamma(u_0)$ is a generalised surface of nonpositive outward mean curvature. Let $S \subset \partial_{\mathrm{red}}\{u > 0\}$ be a portion of free boundary of u and S' a small perturbation of S such that $S' \subset \{u > 0\}$ and $\partial S = \partial S'$. Then

$$\mathcal{H}^{n-1}(S) \le \mathcal{H}^{n-1}(S')$$

i.e. $\partial_{\mathrm{red}}\{u > 0\}$ is a generalized surface of non-positive outer mean curvature. Notice that by Lemma 12.3 $\partial\{u > 0\}$ has finite perimeter in B_1. Thus $\mathcal{H}^{n-1}(S) < \infty$.

To prove this we take the domains G, G_0 such that $\partial G = S \cup S'$ and $\overline{G} \subset G_0 \subset \mathbb{R}^n_+$. Then we have

$$0 = \int_G \Delta u = \int_S \partial_\nu u + \int_{S'} \partial_\nu u. \tag{8.6}$$

On S we have that $\partial_\nu u(x) = |\nabla u(x)| = \sqrt{\Lambda}$, for $\mathcal{H}^{n-1}$-a.e. $x \in \Gamma(u) \cap G_0$ [AC], whereas on S', $|\nabla u| \le \sqrt{\Lambda}$ by (8.2). Comparing the integrals over S and S' we get that

$$\sqrt{\Lambda}\mathcal{H}^{n-1}(S) = \int_S \partial_\nu u = -\int_{S'} \partial_\nu u \le \sqrt{\Lambda}\mathcal{H}^{n-1}(S').$$

After canceling $\sqrt{\Lambda}$ the result follows.

□

8.2. **Preliminary Lemmas.** Suppose that u is a third global homogeneous solution, which by Section 7.1 satisfies $v_S \le u \le v_L$. In particular the free boundary Γ_u lies in between the planes Γ_S, and Γ_L.

We first need a lemma that shows that free boundary is locally a graph.

Proposition 8.3. *Let u be a global homogeneous minimizer and $\Gamma(u)$ touches tangentially the free boundary of v_L, at some point $x^0 = (0, 0, x^0_3, \cdots, x^0_n)$ with $|x^0| = 1$. Then in a small neighborhood of x^0 the free boundary $\Gamma(u)$ is a C^1 graph in the direction normal to Γ_L in the upper half plane. Moreover the normal vector to $\Gamma(u)$ is continuous up to the point x^0, and hence by homogeneity this holds on the axis tx^0 ($t > 0$).*

Let $\Pi_0 = \{x \in \mathbb{R}^n : x_1 = x_2 = 0\}$ and $x^0 \in \Pi_0 \setminus \{0\}$ be any given free boundary point close enough to Π_0. Let further $\tilde{x}^0$ be the projection of x^0 onto Γ_L. Then by tangential touch between the free boundary $\Gamma(u)$ and Γ_L (which is a flat plane) one has that $|x^0 - \tilde{x}^0| = o(x_1^0)$. In particular for $r_0 = x_1^0$, sufficiently small, we have that, in the ball $B_{r_0}(\tilde{x}_0)$, the free boundary $\Gamma(u)$ is flat enough to satisfy the hypothesis of Theorem 8.1 in [AC]. In particular, in the direction of the plane Γ_L the free boundary is a C^1 graph locally in $B_{\frac{r_0}{4}}(\tilde{x}_0)$. From here it follows that $\Gamma(u)$, seen from the plane Γ_L, is C^1 graph over $\Gamma_L \cap B_{\frac{r_0}{8}}(\tilde{x}_0)$.

It is now elementary to show that the normal of $\Gamma(u)$ is continuous up to the point x^0. Indeed, if this fails, then there is a sequence x^j on the free boundary with normal ν^j staying uniformly away from the normal ν^L of Γ_L, $|\nu^j - \nu^L| > \varepsilon_0 > 0$. Scaling u at x^j with $r_j = \operatorname{dist}(x^j, \Gamma_L)$ we have a limit global minimizer in $\mathbb{R}^n$ (observe that this is due to tangential touch). On the other hand the free boundary will then become a plane, on one side of a scaled version of the plane Γ_L, but with the normal at the origin being ν^0, with $|\nu^0 - \nu^L| > \varepsilon_0 > 0$. This is impossible. □

Lemma 8.4. *Let $u \in \mathcal{HP}_\infty, \Gamma_u = \partial\{u > 0\}$. If $u \geq 0, \alpha_- = 0$ then Γ_u does not touch Γ_L tangentially.*

Proof: We argue towards a contradiction. Let x^0 be a point where the free boundaries touch each other. We consider two possible locations: in first $x_1^0 > 0$ and then $x_1^0 = 0$.

Case 1: Let us suppose that Γ_u touches Γ_L at x^0 and $x_1^0 > 0$. To conclude that this is a contradiction we use the free boundary condition and Hopf lemma. Notice that in order to use Hopf's lemma, we need (at least $C^{1,Dini}$) regularity of Γ_u near x^0.

It follows from the one side flatness, and classical regularity result of Theorem 8.1. in [AC]. Then we can apply Hopf's maximum principle to infer

$$\frac{\partial(u - v_L)}{\partial\nu}(x^0) > 0.$$

which is a contradiction in view of the tangential touch condition.

Case 2: We first choose a new coordinates system such that in new coordinates $y = (y_1, y_2, \dots, y_n)$ we have $\Gamma_L = \{y \in \mathbb{R}^n : y_n = 0, y_{n-1} > 0\}$ and $\{u > 0\} \subset \{y \in \mathbb{R}^n : y_n < 0\}$. Now let us assume that Γ_u touches the free boundary of the larger solution v_L at $y^0 \neq 0$ and $y_1^0 = 0$. Then by Proposition 8.3 the free boundary is locally a smooth graph, seen from the plane Γ_L. In particular near y^0, the free boundary can be represented as $y_n = h(y')$, $y' = (y_1, y_2, \dots, y_{n-1})$ and that h is a subsolution to the minimal surface equation in the weak sense.

Indeed, let $\mathbb{R}^{n-1}_+ = \{y \in \mathbb{R}^n : y_n = 0, y_{n-1} > 0\}$ and $\widetilde{B} \subset \mathbb{R}^{n-1}_+$ be a ball touching $\partial\mathbb{R}^{n-1}_+$ at y^0. Then by Lemma 8.2, **2°** the surface area functional will increase, if we replace h by $h_\varepsilon = h + \varepsilon\varphi$ for any $\varphi \in C_0^\infty(\widetilde{B}), \varphi \le 0$ and $\varepsilon > 0$ is small. This comparison yields

$$\mathfrak{M}h(y') = \operatorname{div}\left(\frac{Dh(y')}{\sqrt{1+|\nabla h(y')|^2}}\right) \ge 0 \qquad \text{weakly in } \widetilde{B}.$$

Thus we have

$$\begin{cases} \mathfrak{M}h(y') \ge 0 & \text{in } \widetilde{B}, \\ h(y') \le 0 & \text{in } \widetilde{B}, \\ h(y^0) = 0 & y^0 \in \partial\widetilde{B}. \end{cases}$$

By Hopf's principle

$$\frac{\partial h(y^0)}{\partial y_{n-1}} > 0$$

which is in contradiction with the tangential touch of Γ_u and $\mathbb{R}^{n-1}_+$. □

From Lemma 8.4 we know that $\Gamma(u)$ cannot touch Γ_L. Using this observation we can construct yet another global minimizer $\widetilde{u}$ such that it is two dimensional and distinct from v_L and v_S. This, however, will contradict Proposition 7.2, and the proof of Theorem C will finish.

Thus to complete the proof of Theorem C we need to construct $\widetilde{u}$. This is done by the next lemma.

Lemma 8.5. *Let $\mathbb{V}_\epsilon = \{x \in \mathbb{R}^n_+ : x_2 < -(\gamma - \epsilon)x_1\}$ for small $\epsilon > 0$. If $\mathbb{V}_\epsilon \subset \{u = 0\}$ then there is a two dimensional global solution $\widetilde{u}$ which is distinct from v_L and v_S.*

Proof: Suppose $\mathbb{V}_\epsilon \subset \{u = 0\}$ for some $\epsilon > 0$. Then we can construct a global solution $\widetilde{u}$ such that $\widetilde{u} \ge u$, $\widetilde{u}$ is two dimensional and $\Gamma(\widetilde{u}) \subset \mathbb{R}^n_+ \setminus \mathbb{V}_\epsilon$.

For $r > 0$ fixed and $x \in B_r^+$, we put $g_r(x) = \sup\{u(x + \ell T), T \in \mathbb{R}, \ell \in \mathbb{S}^{n-2}\}$ where

$$\mathbb{S}^{n-2} = \{\ell = (0, 0, \ell_3, \dots, \ell_n), \ell_3^2 + \ell_4^2 + \dots \ell_n^2 = 1\}.$$

Let $w \in \mathcal{P}_r(n, \lambda_\pm, \alpha_+, 0, g_r)$, i.e. w is a local minimizer of $J(\cdot, B_r^+)$ with $w = g_r$ on ∂B_r^+ see Remark 1.2. From Lemma 7.1 we infer that $\widetilde{u}_r = \sup w$ is a local minimizer and $\widetilde{u}_r \ge w$ for any $w \in \mathcal{P}_r(n, \lambda_\pm, \alpha_+, 0, g_r)$. In particular $\widetilde{u}_r \ge u$ in B_r^+.

Taking $r_j \to \infty$, we have from Proposition 4.3, that there is a subsequence r_{j_k} such that $\widetilde{u}_{r_{k_j}} \to \widetilde{u}_0$ locally in H^1 and C^0 and $\widetilde{u}_0 \in \mathcal{P}_\infty$. Because $\mathcal{P}_r(n, \lambda_\pm, \alpha_+, 0, g_r)$ is translation invariant for each

$\ell \in \mathbb{S}^{n-1}$, it follows that $\widetilde{u}_0$ is two dimensional solution. The condition $\mathbb{V}_\epsilon \subset \{u = 0\}$ translates to $\widetilde{u}_0$ and we get that $\Gamma(\widetilde{u}_0) \subset \mathbb{R}^n_+ \setminus \mathbb{V}_\epsilon$. Furthermore

$$\widetilde{u}_0 \geq u.$$

Since v_L and v_S are the only two dimensional homogeneous global solutions, we conclude that $\epsilon = 0$, see Proposition 7.2. □

Remark 8.6. *It is noteworthy that the classification of global homogeneous solutions for the two-phase case would have been available if one already knew that the free boundary is regular. Indeed, if we a priori know that the free boundary is regular, then one can apply maximum principle to $|\nabla u^+|^2$ in the set $\{u > 0\}$, and find out that the maximum must be on the boundary (either free or fixed). Actually, an argument similar to that of the proof of Lemma 8.2, would then result in the fact that maximum is exactly on the boundary.*

Suppose now the maximum is on the free boundary. Then at such a maximum point x^0 (which is a maximum point for both $|\nabla u^+|^2$ due to Bernoulli boundary condition $|\nabla u^+|^2 = \Lambda + |\nabla u^-|^2$) one gets that $\partial_\nu |\nabla u^+(x^0)|^2 < 0$, where ν is the unit normal pointing inwards support of u^+. From here along with a possible regularity of free boundary it follows that $2u^+_\nu u^+_{\nu\nu}(x^0) < 0$, which along with $u^+_\nu(x^0) > 0$ gives that $u^+_{\nu\nu}(x^0) < 0$. By representation of Laplacian on the free boundary we get $0 = \Delta u^+ = \Delta_S u^+ + H u^+_\nu + u^+_{\nu\nu}$, and since $\Delta_S u^+ = 0$, $u^+_\nu(x^0) > 0$, and $u^+_{\nu\nu}(x^0) < 0$ we arrive at $H(x^0) > 0$. A similar argument applied to u^- gives us the converse $H(x^0) < 0$, and we shall have a contradiction, unless $|\nabla u|$ is constant.

Next suppose the maximum for $|\nabla u^+|^2$ is on the fixed boundary $x^0 \in \{x_1 = 0\}$. Then we have by a similar argument $u_1(x^0)u_{11}(x^0) < 0$. Now with a representation of the Laplacian on $\{x_1 = 0\}$ along with linearity of the boundary data we have $0 = Hu_1 + u_{11}$. Since the fixed boundary is a flat surface we have $H = 0$, and hence $u_{11} = 0$ on the fixed boundary. This contradicts $u_1(x^0)u_{11}(x^0) < 0$.

9. Proof of Theorem D

Now we are ready to produce the proof of Theorem D, exhibiting the non-tangential behavior of the free boundary.

Non-uniform approach. Take $u \in \mathcal{P}_1(n, \lambda_\pm, \alpha_+, 0, g)$ and let u_0 be a blow-up of u at the origin. Then by Propositions 4.5 and 4.6 $u_0 \in \mathcal{HP}_\infty$. From Theorem C, u_0 is either v_S or v_L. Suppose that $u_0 = v_S$. Let us consider the cone

$$K_\sigma^+ := \left\{ x \in \mathbb{R}^n_+, x_2 > 0, \frac{x_2}{\gamma + \sigma} < x_1 < \frac{x_2}{\gamma - \sigma} \right\},$$

for small $\sigma > 0$ (cf. (3.4)). Then we claim that for each $\sigma > 0$ there exist a $r_\sigma > 0$ such that for any $r \in (0, r_\sigma)$, the following holds

$$\Gamma(u) \subset B_r^+ \bigcap K_\sigma^+. \tag{9.1}$$

This would suffice to conclude the tangential touch, since the modulus of continuity can be constructed by inverting the relation $\sigma \ \to \ r_\sigma$.

Suppose (9.1) fails. Then there is a sequence of free boundary points $x^j \in \Gamma(u), |x^j| \to 0, u \in \mathcal{P}_1(n, \lambda_\pm, \alpha_+, 0, g)$ such that $x^j \notin K_\sigma^+$ for some fixed $\sigma > 0$.

Set $r_j = |x^j|$ and consider the limit of the sequence $u_j(x) = \frac{u(r_j x)}{r_j}$. In view of Theorem A, u_j's are bounded and therefore by Proposition 4.1 and Theorem B for a subsequence $u_{j_m} \to u_0 \in \mathcal{HP}_\infty$. Moreover the sequence of points $y^j = x^j/|x^j| \in \partial B_1^+$ is such that $y^j \notin K_\sigma^+$ and again by compactness this leads to the existence of $y^0 \in \partial B_1^+ \setminus K_\sigma^+$ such that $u_0(y^0) = 0$.

From monotonicity formula of Weiss, Proposition 4.6 one can also show that $u_0 \in \mathcal{HP}_\infty$ (see Section 4.3) and hence we can invoke Theorem C to conclude that u_0 is v_S. This contradicts the fact that $y^0 \in \partial B_1^+ \setminus K_\sigma^+$, and the proof of the first part is completed. The case when $u_0 = v_L$ is treated analogously.

Uniform approach. To show the uniformity in the second statement of Theorem D, we shall argue in the same way as above, but let u change during the scaling. In other words we define $v_j(x) = \frac{u_j(r_j x)}{r_j}$ with $u_j \in \mathcal{P}'_1(n, \lambda_\pm, \alpha_+, 0, g)$, i.e. $\lim_{r \to 0} W(r, u_j, 0) = W(1, v_S, 0)$. As above the scaled functions will converge to a global solution v_0, but v_0 is not necessarily homogeneous, and this is the only difference between the two cases.

Nevertheless, the assumption that $\lim_{r \to 0} W(r, u_j, 0) = W(0^+, u_j, 0) = W(1, v_S, 0)$ for fixed j implies that $W(tr_j, u_j, 0) = W(1, \frac{u_j(tr_j x)}{tr_j}, 0) = W(t, v_j, 0) \geq W(1, v_S, 0)$ by monotonicity of W (see Proposition 4.6) and after having sent t to zero. This yields

$$W(t, v_0, 0) = \lim_{r_j \to 0} W(tr_j, u_j, 0) = \lim_{r_j \to 0} W(t, v_j, 0) \geq W(1, v_S, 0),$$

where v_0 is the global limit of a subsequence of v_j. The first inequality follows from strong convergence of ∇v_j in L^2, since ∇v_j is a bounded sequence in L^∞ and hence we can apply Theorem 1 from [Z] and Proposition 4.5 to a suitable subsequence $\{r'_j\} \subset \{r_j\}$.

Next, the blow-down of v_0, at infinity, i.e. consider the scaling $v_0(rx)/r$ with $r \to \infty$ which results in a new homogeneous global solutions v_{00}. From monotonicity formula, Proposition 4.6, we have

$$W(1, v_S, 0) \le W(t, v_0, 0) \le W(\infty, v_0, 0) = W(1, v_{00}, 0).$$

Since v_{00} is homogeneous, we can apply Theorem C to conclude v_{00} is either v_L, or v_S. By the energy comparison (7.4) we should then have $v_{00} = v_S$. Therefore $W(t, v_0, 0) = W(1, v_S, 0)$ for any $t > 0$ hence v_0 is homogeneous by Proposition 4.6. Now, as in the previous case, contradiction comes from the fact that $y^0 \in \partial B_1^+ \setminus K_\sigma^+$. □

10. Proof of Theorem E

10.1. **Instability.** The problem studied in this paper is highly unstable in the sense that changing the boundary data, no matter how small, may result in a different behavior of the touch between the free and the fixed boundary. This behavior was already alluded in Theorem D, where we could not prove uniform behavior for solutions that touch tangentially $\Gamma(v_L)$, at the same time that the uniformity worked well for the class $\mathcal{P}'_r(n, \lambda_\pm, \alpha_\pm, g)$.

To illustrate this phenomenon, take $\alpha_- = 0$ and consider the largest homogeneous global solution v_L as in Theorem C. Consider now the minimization problem in the upper half ball using the restriction of suitably scaled v_L on the boundary of B_1^+ as boundary data. Now we know that the function itself is a minimizer. Next let us decrease the data on the plane Π to $(\alpha_+ - \varepsilon)x_2^+$. A minimizer u^ε of the functional with this boundary value on Π will exists, say take the smallest minimizer with boundary values $u^\varepsilon \le v_L$ on S_1^+, so that $u^\varepsilon \le v_L$. In particular this means that the free boundary for this minimizer will not touch the origin. Indeed, if it touches the origin then we can blow up u^ε at the origin, since by Theorem A u^ε has linear growth at the origin, and obtain a global minimizer u_0^ε, with data $(\alpha_+ - \varepsilon)x_2$ on Π.

Now from the classification of the homogeneous global solutions, Theorem C, we must have that $u_0^\varepsilon < v_L$, and thus $u_0^\varepsilon = v_S^\varepsilon$ in B_1^+.

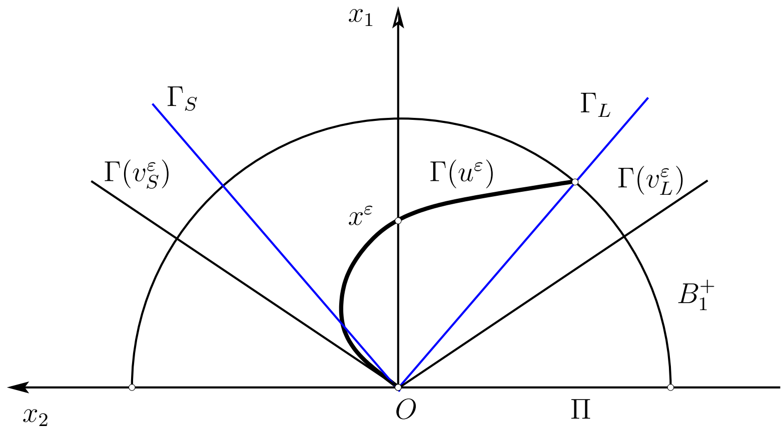

FIGURE 3. The free boundary of u^ε

This means that the free boundary cannot touch the origin, for any $\epsilon > 0$. In particular, by Theorem 5.1 in [KKS], we must have that it touches the fixed boundary tangentially at some point x^0 with $x_2^0 < 0$.

10.2. **Non-homogeneous global solutions.** In this section we show the existence of a global solution which is non-homogeneous. We follow a perturbation method used in [AS].

Let $\alpha_- = 0, 0 < \alpha_+ < 1, \Lambda = 1$ and set $f^\varepsilon = (\alpha_+ - \varepsilon)x_2^+$. Now consider a minimizer of our functional in B_1^+, with admissible functions having boundary data f^ε on Π and $(\alpha_+ - \varepsilon)(-\gamma x_1 + x_2)^+$ on S_1^+ where $\gamma = \sqrt{\frac{1}{\alpha_+^2} - 1}$.

Let $\gamma_\varepsilon = \sqrt{\frac{1}{(\alpha_+ - \varepsilon)^2} - 1}$ then from Theorem C $v_L^\varepsilon = (\alpha_+ - \varepsilon)(-\gamma_\varepsilon x_2 + x_1)^+$ is the largest global homogeneous solution with boundary values f^ε on Π. Notice that $\gamma_\varepsilon > \gamma, f^\varepsilon \leq \alpha_+ x_2^+$ implying that $v_L \geq (\alpha_+ - \varepsilon)(-\gamma x_2 + x_1)^+$ on S_1^+. Consider the class of local minimizers

$$\mathcal{K}_\varepsilon = \{u \in H^1(B_1^+), u = (\alpha_+ - \varepsilon)(-\gamma x_1 + x_2)^+ \text{ on } \partial S_1^+, u \text{ is a local minimizer of } J\}.$$

Then from the results of Section 7 $u^\varepsilon = \inf_{\mathcal{K}_\varepsilon} u$ is a minimizer. Furthermore $u^\varepsilon \leq \min(u^\varepsilon, v_L) \leq v_L$.

For ε fixed, any blow-up of u^ε at origin is a homogeneous global solution u_0^ε, which in view of the inequality $u^\varepsilon \leq v_L$, implies $u_0^\varepsilon \leq v_L$. Now u_0^ε is a global homogeneous solution with boundary data f^ε, and hence it must equal to one of the functions $(\alpha_+ - \varepsilon)(\pm\gamma_\varepsilon x_2 + x_1)^+$. The only way for u_0^ε to be as above and satisfy

$$u_0^\varepsilon \leq v_L = \alpha_+(-\gamma x_2 + x_1)^+$$

is that $u_0^\varepsilon = (\alpha_+ - \varepsilon)(\gamma_\varepsilon x_2 - x_1)^+ = v_S^\varepsilon$. This in turn suggests that the free boundary $\Gamma(u^\varepsilon)$ starts at the origin with a tangential touch to $\Gamma_{v_S^\varepsilon}$, the smallest global solution with boundary data f^ε. Since the free boundary divides B_1^+ into two parts, it has to end on S_1^+, see Figure 3. In particular $\Gamma(u^\varepsilon)$ cuts the x_1-axis at some point $x^\varepsilon = (r_\varepsilon, 0)$. Now we consider the blow up of u^ε with respect to r_ε. Observe that $r_\varepsilon \to 0$ and thereby, utilizing Proposition 4.5 and choosing a suitable subsequence, we obtain a global solution u_0 with boundary data $\alpha_+ x_2$ and with $\Gamma(u_0) \ni (1,0) = \lim_{\varepsilon\to 0} x^\varepsilon / r_\varepsilon$. it follows from Theorem C that this solution cannot be homogeneous.

Remark 10.1. *It should be remarked that in the above example of non-homogeneous global solutions, we have $|\nabla u|^2 \not\leq \Lambda$. Indeed, if this was true, then one may apply maximum principle to $|\nabla u|^2$ in $\{u > 0\}$ and obtain a maximum on the free boundary (the free boundary is regular in 2-space dimension). Hence, by Hopf's lemma one obtains $\partial_\nu |\nabla u|^2 > 0$, where ν is the unit normal on the free boundary pointing outside the support of u. In particular $u_\nu u_{\nu\nu} > 0$. Since $u_\nu = |\nabla u| = \sqrt{\Lambda}$ we will have $u_{\nu\nu} > 0$ on the free boundary. Using representation of Laplacian on the free boundary $\Delta u = \Delta_S u + H u_\nu + u_{\nu\nu}$, where H is the mean curvature, we conclude the convexity of the free boundary. This contradict the geometry of the example above.*

11. Appendix 1

In this section we prove that any blow up limit of $u \in \mathcal{P}_r$ is homogeneous function of degree one. The case when $g = 0$ immediately follows from [W], Section 2. When $g \neq 0$ some extra care is needed, because the comparison of u with its homogeneous extension $u_t(x) = \frac{|x|}{t} u(t\frac{x}{|x|})$ in B_t^+ fails on the flat portion of the boundary, i.e. $u(x) \neq u_t(x)$ when $x \in \Pi \cap B_t$.

To fix the ideas we consider the model case $g(x) = C|x|^{1+\kappa}$ with $\kappa > 0$ and $C = const$. Since $\rho^{-1} g(\rho x) \to 0$ as $\rho \downarrow 0$ it follows that u and $v = u - g$ have the same blowups at the origin.

Lemma 11.1. *Let $u \in \mathcal{P}_r(n, \lambda_\pm, \alpha_\pm, g)$. Set $v = u - g$ where $g(x) = C|x|^{1+\kappa}, \kappa > 0$. Then*

$$\widetilde{W}(t) = \frac{1}{t^n}\int_{B_t^+} |\nabla v|^2 + \Lambda \chi_{\{v > -g\}} - \frac{1}{t^{n+1}}\int_{\partial B_t^+} v^2 + \frac{C_1}{\kappa} t^\kappa$$

is nondecresing function of t. Furthermore

$$\frac{d}{dt}\left\{\frac{1}{t^n}\int_{B_t^+} |\nabla v|^2 + \Lambda \chi_{\{v > -g\}} - \frac{1}{t^{n+1}}\int_{\partial B_t^+} v^2 + \frac{C_1}{\kappa} t^\kappa\right\} \geq \frac{1}{t^n}\int_{\partial B_t^+}\left(\nabla v \cdot \nu - \frac{v}{t}\right)^2. \tag{11.1}$$

Proof: Let $\varphi \in H^1_0(B^+_r), r \in (0,1)$ and let us define $v = u - g, v_\varphi = u + \varphi - g$ in B^+_1. Then $J(u) \le J(u+\varphi)$ transforms into $J(v+g) \le J(v_\varphi + g)$. Employing Green's identity we obtain

$$\begin{aligned}\int_{B_r^+} |\nabla u|^2 &= \int_{B_r^+} |\nabla v|^2 + 2\int_{B_r^+} \nabla v \cdot \nabla g + \int_{B_r^+} |\nabla g|^2 \\ &= \int_{B_r^+} |\nabla v|^2 - v(2\Delta g) + 2\int_{\partial B_r^+} v(\nabla g \cdot \nu) + \int_{B_r^+} |\nabla g|^2.\end{aligned}$$

Utilizing this computation and the fact $v_\varphi - v = H^1_0(B^+_r)$ we see that if u is a minimizer of $J(u, B^+_r)$, subject to its own boundary values on ∂B^+_r, then v is a minimizer of

$$\widetilde{J}(v) = \int_{B_r^+} |\nabla v|^2 - v(2\Delta g) + \Lambda \chi_{\{v > -g\}}, \tag{11.2}$$

because $2\int_{\partial B_r^+} w(\nabla g \cdot \nu) + \int_{B_r^+} |\nabla g|^2$ is constant for any $w \in H^1(B^+_r), w|_{\partial B^+_r} = v|_{\partial B^+_r}$.

Thus it remains to prove that any blow up limit of v at the origin is a homogeneous function of degree one.

Let $t > 0$ be small and take $v_t(x) = \dfrac{|x|}{t} v(t\dfrac{x}{|x|})$, then on ∂B_t v_t agrees with v and it follows $\widetilde{J}(v) \le \widetilde{J}(v_t)$. Using the homogeneity of v_t and the identities

$$\begin{aligned}\nabla v_t(x) &= \frac{x}{t|x|} v(t\frac{x}{|x|}) + \nabla v(t\frac{x}{|x|}) - \nabla v(t\frac{x}{|x|}) \cdot \frac{x}{|x|}\frac{x}{|x|}, \\ |\nabla v_t|^2 &= t^{-2} v^2(t\frac{x}{|x|}) + \left|\nabla v(t\frac{x}{|x|})\right|^2 - \left(\nabla v(t\frac{x}{|x|}) \cdot \frac{x}{|x|}\right)^2\end{aligned}$$

one can easily compute

$$\begin{aligned}\int_{B_t^+} |\nabla v_t|^2 + \Lambda\chi_{\{v_t > -g\}} &= \int_{B_t^+} \left[\frac{x}{t|x|} v(r\frac{x}{|x|}) + \nabla v(t\frac{x}{|x|}) - \nabla v(t\frac{x}{|x|}) \cdot \frac{x}{|x|}\frac{x}{|x|}\right]^2 + \Lambda\chi_{\{v_t > -g\}} \\ &= \int_0^t \int_{\partial B_\rho^+} \left[t^{-2} v^2(t\frac{x}{|x|}) + \left|\nabla v(t\frac{x}{|x|})\right|^2 - \left(\nabla v(t\frac{x}{|x|}) \cdot \frac{x}{|x|}\right)^2\right] + \Lambda\chi_{\{v_t > -g\}} \\ &= \frac{t}{n}\int_{\partial B_t^+} |\nabla v|^2 + \frac{t}{n}\int_{\partial B_t^+} \left[\frac{v^2}{t^2} - (\nabla v \cdot \nu)^2\right] + \int_{B_t^+} \Lambda\chi_{\{v_t > -g\}}.\end{aligned}$$

To deal with the last integral, we first notice that $\{v_t(x) > -g(x)\} \subset \{v(t\frac{x}{|x|}) > -Ct^{1+\kappa}\}$. Indeed if $x \in \{v_t(x) > -g(x)\}$ then $\frac{|x|}{t} v(t\frac{x}{|x|}) > -C|x|^{1+\kappa}$, or equivalently $v(t\frac{x}{|x|}) > -Ct|x|^\kappa$. But $|x| \le t$ since $x \in B^+_t$. Thus $-Ct|x|^\kappa \ge -Ct^{1+\kappa} = -g(t)$. In particular we get that $\int_{B_t^+} \Lambda\chi_{\{v_t > -g\}} \le$

$\int_{B_t^+} \Lambda\chi_{\{v(t\frac{x}{|x|})>-g(t)\}}$ which, after applying Fubini's theorem, yields

$$\int_{B_t^+} \Lambda\chi_{\{v(t\frac{x}{|x|})>-g(t)\}} = \frac{t}{n}\int_{\partial B_t^+} \Lambda\chi_{\{v>-g\}}.$$

Next we notice that if $w \in H^1(B_t^+)$ and $|w(x)| \le C|x|, x \in B_t^+$ then $\left|\int_{B_t^+} w(2\Delta g)\right| \le C_1 t^{n+\kappa}$ with some tame constant C_1. Therefore comparing the $\widetilde{J}$ energies in B_t^+, we get

$$\begin{aligned}
0 &\le \widetilde{J}(v_t) - \widetilde{J}(v) \\
&\le \frac{t}{n}\int_{\partial B_t^+} |\nabla v|^2 + \Lambda\chi_{\{v>-g\}} + \frac{t}{n}\int_{\partial B_t^+}\left[\frac{v^2}{t^2} - (\nabla v\cdot\nu)^2\right] \\
&\quad - \int_{B_t^+} |\nabla v|^2 + \Lambda\chi_{\{v>-g\}} + C_1 t^{n+\kappa} \\
&\le \frac{t}{n}\int_{\partial B_t^+} |\nabla v|^2 + \Lambda\chi_{\{v>-g\}} - \int_{B_t^+} |\nabla v|^2 + \Lambda\chi_{\{v>-g\}} \\
&\quad + \frac{t}{n}\int_{\partial B_t^+}\left[\frac{v^2}{t^2} - (\nabla v\cdot\nu)^2\right] + C_1 t^{n+\kappa} \\
&= \frac{t^{n+1}}{n}\frac{d}{dt}\left\{\frac{1}{t^n}\int_{B_t^+} |\nabla v|^2 + \Lambda\chi_{\{v>-g\}}\right\} \\
&\quad - \frac{t}{n}\int_{\partial B_t^+}\left(\nabla v\cdot\nu - \frac{v}{t}\right)^2 - \frac{2t}{n}\int_{\partial B_t^+}\frac{v}{t^2}\left[\nabla v\cdot\nu - \frac{v}{t}\right] + C_1 t^{n+\kappa}.
\end{aligned} \tag{11.3}$$

Multiplying both sides by nt^{-n-1} we conclude

$$\begin{aligned}
\frac{d}{dt}\left\{\frac{1}{t^n}\int_{B_t^+} |\nabla v|^2 + \Lambda\chi_{\{v>-g\}} - \frac{1}{t^{n+1}}\int_{\partial B_t^+} v^2 + \frac{C_1}{\kappa}t^{\kappa}\right\} \\
\ge \frac{1}{t^n}\int_{\partial B_t^+}\left(\nabla v\cdot\nu - \frac{v}{t}\right)^2.
\end{aligned}$$

□

Remark 11.2. *This argument shows that g can be replaced by any homogeneous polynomial or function of degree $m > 1$.*

Corollary 11.3. *Let v be as in Lemma 11.1. Then any blow up limit of v at the origin is homogeneous of defree one. In particular any blow up of u is homogeneous of degree one.*

Proof: The first statement follows exactly as in [W], Section 2. To show that the blow up of u is homogeneous we need to notice that $g(rx)r^{-1} \to 0$ uniformly as $r \to 0$. Hence the blow ups of u and v coincide.

12. Appendix 2

We shall discuss the rectifiablity of the free boundary in B_1.

Lemma 12.1. *Let u be a global homogeneous minimizer and $\Gamma(u)$ touches tangentially the free boundary of v_L, then u is nondegenerate, i.e. there is a tame constant $c > 0$ such that for any $x \in \Gamma(u)$ the following estimate is true*

$$\sup_{B_r^+(x)} u \geq cr, \qquad \forall B_r(x) \subset \mathbb{R}^n. \tag{12.1}$$

Remark 12.2. *In* [AC]*, a different form of nondegeneracy is proven (see Lemma 3.4 in* [AC]*), namely*

$$\fint_{\partial B_r(x)} u \geq cr, \qquad x \in \partial\{u > 0\}.$$

This integral inequality implies that there is $y \in \partial B_r(x)$ such that $u(y) \geq cr$. But u is subharmonic, therefore $\sup_{B_r(x)} u \geq cr$.

Proof: Let $x \in \partial\{u > 0\}$ and set $\delta(x) = \operatorname{dist}(x, \Pi)$. If $\delta(x) \geq r$ then $B_r(x) \subset \mathbb{R}^n_+$. Taking $u_r(y) = \frac{u(x+ry)}{r}, y \in B_1$ and employing Lemma 8.2 **2°** we see that u_r is a local minimizer. Hence from remark 12.2 we obtain $\sup_{B_{\frac{1}{2}}} u_r \geq c$, which after scaling back implies the desired result.

Now assume that $\delta(x) < r$. We consider two possible scenarios:

Case a) $\frac{r}{1000} \leq \delta(x)$. Then using Remark 12.2 in $B_{\delta(x)}(x)$ we get

$$\sup_{B_r^+(x)} u \geq \sup_{B_{\delta(x)}} u \geq \frac{c\delta(x)}{2} \geq \frac{cr}{2000}.$$

Case b) $\delta(x) < \frac{r}{1000}$. Let $R(x) = \operatorname{dist}(x, \Pi_0)$, where $\Pi_0 = \{x \in \mathbb{R}^n : x_1 = x_2 = 0.\}$ and take $x_0 \in \Pi_0$ such that $R(x) = |x - x_0|$. Notice that $R(x) \sim \delta(x)$, because $\Gamma(u)$ touches Γ_L tangentially. This means that there are two positive constants a, b such that $aR(x) \leq \delta(x) \leq bR(x)$ if x is close to Π (see definitions of the cones K_σ). We have $r > 1000\delta(x) \geq a1000R(x)$ yielding $R(x) \leq \frac{r}{a1000}$. In particular

$$\rho = r - R(x) \geq r - \frac{r}{a1000} \geq \frac{r}{100}.$$

Observing that $B^+_\rho(x_0) \subset B^+_r(x)$ we get

$$\sup_{B^+_r(x)} u \geq \sup_{B^+_\rho(x_0)} u \geq \sup_{B^+_\rho(x_0)} u_S \geq c\rho \geq \frac{cr}{1000}.$$

□

Lemma 12.3. *Let u be as in Lemma 12.1. Then*

$$\mathcal{H}^{n-1}(B_1 \cap \partial\{u > 0\}) < \infty.$$

Proof: For each open ball $B_r(x) \subset \mathbb{R}^n$ let $B^+_r(x) = B_r(x) \cap \mathbb{R}^n_+$. Introduce the measure $\mu = \Delta u$. Clearly μ is nonnegative Radon measure, because $\int_{B^+_r(x)} \mu = \int_{\partial B^+_r(x)} \nabla u \cdot \nu \leq Cr^{n-1}$. Hence for any compact $D \subset \mathbb{R}^n$ we can cover $\overline{D \cap \mathbb{R}^n_+}$ by a finite number of balls, which yields $\mu(\overline{D \cap \mathbb{R}^n_+}) < \infty$.

Next we want to show that there is a positive constant c_0 such that for each $x \in \overline{B^+_1 \cap \Gamma(u)}$ we have

$$\int_{B^+_r(x)} \mu \geq c_0 r^{n-1} \qquad \text{if } r > 0 \text{ is small.} \tag{12.2}$$

From (12.2) one can conclude the proof of Lemma by employing a standard covering argument.

First we note that by Lemma 12.1 u is nondegenerate, that is there is a constant $c > 0$ such that

$$\sup_{B^+_r(x)} u \geq cr, \qquad \forall x \in B^+_1 \cap \Gamma(u) \tag{12.3}$$

for small $r > 0$.

Now suppose that (12.2) fails. Then there is a sequence of free boundary points $x_j \in \Gamma(u)$ and a sequence of positive numbers $r_j > 0$ such that

$$\fint_{B_{r_j}(x_j)} \mu \leq \frac{r_j^{n-1}}{j}. \tag{12.4}$$

First, let us suppose that there is a subsequence $r_{j(m)}$ such that $B_{r_{j(m)}} \cap \Pi_0 \neq \emptyset$. Let $x^0_j \in \Pi_0$ and $\mathrm{dist}(x_j, \Pi_0) = |x_j - x^0_j|$. Then consider $v_m(x) = \frac{u(x^0_j + r_{j(m)}x)}{r_{j(m)}}, x \in B^+_2$. From Proposition 4.5 and Lemma 8.2 we get $v_{m_k} \to v_0, \mu_{m_k} \rightharpoonup \mu_0$ where $\mu_{m_k} = \Delta v_{m_k}$ and $\Delta v_0 = \mu_0$, at least for a subsequence m_k, and v_0 is a local minimizer. Moreover (12.4) translates to

$$\fint_{B^+_1(y^0)} \mu_0 = 0$$

for some $y^0 \in \Gamma_L$, i.e. v_0 is harmonic in $B^+_1(y^0)$. From the strong maximum principle we conclude $v_0 = 0$ which is in contradiction with nongedeneracy of v_{m_k} and v_0.

Finally let us assume that $B_{r_j}(x_j) \cap \Pi_0 = \emptyset$ for any j. Denote $\delta_j = \mathrm{dist}\,(x_j, \Pi)$. From tangential touch of $\Gamma(u)$ and Γ_L it follows that $aR_j \leq \delta_j \leq bR_j$, where $R_j = \mathrm{dist}(x_j, \Pi_0)$. Thus we have $r_j < R_j$. If, moreover, $r_j \geq \delta_j$ then applying Theorem 4.3 [AC] to $\frac{u(x_j+r_jx)}{r_j}$ we will conclude a contradiction if j is large enough.

Thus without loss of generality we may assume that $\delta_j < r_j < R_j$. Introduce $w_j(y) = \frac{u(x_j+\delta_j)}{\delta_j}, y \in B_1$ then

$$\fint_{B_{\delta_j}(x_j)} \mu \leq \fint_{B_{r_j^+}(x_j)} \mu \leq \frac{r_j^{n-1}}{j} \leq \frac{\delta_j^{n-1}}{ja^{n-1}}.$$

Hence for $\Delta w_j = \mu_j$ we have $\int_{B_1} \mu_j \leq \frac{1}{ja^{n-1}}$. On the other hand $\sup_{B_{\frac{1}{2}}} w_j \geq c$. Extracting a subsequence for which $w_j \to w_0, \Delta w_j \rightharpoonup \Delta w_0$ in B_1 at least for a subsequence, where w_0 is a local minimizer in B_1, see Proposition 4.5. But $\fint_{B_1} \Delta w_j \leq \frac{1}{ja^{n-1}} \to 0$. Thus $w_0 \geq 0$ is harmonic and nondegenerate in B_1 and $w_0(0) = 0$. Hence by strong maximum principle $w_0 = 0$ which is in contradiction with $\sup_{B_{\frac{1}{2}}} w_0 \geq c$. ☐

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