# SUFFICIENT CONDITIONS FOR REGULARITY OF AREA-PRESERVING DEFORMATIONS

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ABSTRACT. We discuss some basic regularity properties of the area-preserving deformations $\mathbf{u} : \Omega \mapsto \mathbb{R}^2$ that have minimal elastic energy

$$\int_\Omega |\nabla \mathbf{u}|^2$$

among a suitable class of addmissible vectorfields defined on a smooth, bounded domain $\Omega \subset \mathbb{R}^2$. Although we restrict ourselves to the quadratic stored energy function and 2-space, most of our results extend to three dimensional setting with convex stored energy function.

## 1. INTRODUCTION

This paper is the sequel of [4], where under very weak assumptions we were able to derive the Euler-Lagrange equation of the minimization problem discussed below. Our purpose here is to further investigate the weak equations and to point out some geometric methods that lead to regularity of local minimizers in Hölder spaces. In the interests of brevity we shall concentrate on the two dimensional problem with quadratic energy, however the majority of our results holds in a more general setting.

To begin with let's assume that an elastic body occupies in a reference configuration the smooth, bounded region $\Omega \subset \mathbb{R}^2$. A deformation of $\Omega$ is a map $\mathbf{u} : \Omega \mapsto \mathbb{R}^2, \mathbf{u} = (u^1, u^2)$ and the deformation gradient is $\nabla \mathbf{u} = \partial_{x_j} u^i$. Then the total elastic energy is defined as

$$E(\mathbf{u}) = \int_\Omega W(\nabla \mathbf{u}),$$

where the stored energy function $W : \nabla \mathbf{u} \mapsto \mathbb{R}$ is assumed to be $C^1$ and bounded below. The explicit form of $W$ is determined experimentally and depends on mechanical and termodynamical properties of the material. For instance for Rivlin-Mooney materials, such as rubber for an automobile tyre, the total elastic energy is

$$E(\mathbf{u}) = \int_\Omega |\nabla \mathbf{u}|^2. \tag{1.1}$$

2000 *Mathematics Subject Classification.* Primary 35J60, 42A40,73C50, 73V25.
*Key words: Energy minimizers, Euler-Lagrange equations, area-preserving maps.*

The rubber is also an example of incompressible material. The incompressibility constaraint is a constitutive restriction on the kinds of deformations that a body can suffer. Mathematically the incompressible deformations $\mathbf{u}:\Omega\mapsto\mathbb{R}^N,\Omega\subset\mathbb{R}^N$ are characterized by the following equation

$$\det\nabla\mathbf{u}=1. \tag{1.2}$$

When $N=2$ we call incompressible deformations *area-preserving.*

The reason to work in $\mathbb{R}^2$, beyond obvious technical simplicity, is motivated by the following: if the deformation $\mathbf{u}:\Omega\mapsto\mathbb{R}^3,\Omega\subset\mathbb{R}^N$ is invariant in some fixed direction, e.g. in the direction $\mathbf{e}_3$ then the third component of the deformation $u^3(x_1,x_2,x_3)=x_3$ and $u^1,u^2$ are independent of $x_3$ variable.

Now the problem, we shall be concerned with, in two dimensional space can be formulated as follows: Find an area-preserving deformation $\mathbf{u}\in W^{1,2}(\Omega,\mathbb{R}^2)$ such that for all incompressible deformations $\mathbf{w}\in W^{1,2}(\Omega,\mathbb{R}^2)$ with $\operatorname{supp}(\mathbf{w}-\mathbf{u})\subset\Omega$ the following holds

$$\int_\Omega|\nabla\mathbf{u}|^2\le\int_\Omega|\nabla\mathbf{w}|^2. \tag{1.3}$$

If (1.3) holds then $\mathbf{u}$ is called a *local minimizer* of $E(\cdot)$.

In his fundamental work [1] J.Ball proved that such minimizer exists. However the regularity of $\mathbf{u}$ is far less clear. The nonlinear nature of (1.2) produces several difficulties, notably when one tries to derive the Euler-Lagrange equation. Formally if one introduces the hydrostatic pressure $p$, as the Lagrange multiplier associated with the material constraint (1.2), then for sufficiently smooth $\mathbf{u},p$ the governing equations are [5]

$$\begin{cases}\operatorname{div}\{\nabla\mathbf{u}-p\operatorname{cof}\nabla\mathbf{u}\}=0 & \text{in }\Omega,\\ \det\nabla\mathbf{u}=1 & \text{a.e. in }\Omega.\end{cases} \tag{1.4}$$

Here $\operatorname{cof}\nabla\mathbf{u}$ is the cofactor matrix of $\nabla\mathbf{u}$. Under relaxed conditions these equations are derived in [4]. We summarize the main results of [4] here:

**Theorem 1.** *Let $\boldsymbol{u}$ be a local minimizer of (1.1) subject to (1.2). Then*

i) *if $|\nabla\boldsymbol{u}|^2$ is locally in $L\log(2+L)$ then there exists a locally integrable function $q$, defined in the image domain $\Omega^\star=\boldsymbol{u}(\Omega)$, such that*

$$\int u^i_m(\boldsymbol{u}^{-1}(y))u^j_m(\boldsymbol{u}^{-1}(y))\psi^i_j(y)dy=\int q(y)\operatorname{div}\psi(y)dy \tag{1.5}$$

*for any $\psi(y)\in C^\infty_0(\Omega^\star,\mathbb{R}^2)$,*

ii) *if $|\nabla\boldsymbol{u}|^2$ is locally in $L\log(2+L)$ and if $p(x)=q\circ\boldsymbol{u}(x)$ then $p$ is locally integrable in $\Omega$,*

iii) *if* $\boldsymbol{u} \in W^{1,2}(\Omega, \mathbb{R}^2) \cap W^{1,3}_{\mathrm{loc}}(\Omega, \mathbb{R}^2)$ *then the first equation in (*1.4*) is satisfied in the weak sense.*

It is worth to point out that Theorem 1 is valid for a more general class of stored energy functions subject to suitable structural conditions [4].

The main purpose of this paper is to initiate a study of the regularity of local minimizers by exploiting the system of PDEs $\operatorname{div}\{\nabla\mathbf{u} - p\operatorname{cof}\nabla\mathbf{u}\} = 0$ from (1.4) or its weak form (1.5) in dual variables. It is of interest to note that this system is not elliptic, which produces additional difficulties as one tries to attack the regularity of $\mathbf{u}$. The advantage of $\operatorname{div}\{\nabla\mathbf{u} - p\operatorname{cof}\nabla\mathbf{u}\} = 0$ is that it is "almost" linear unlike the second equation $\det\nabla\mathbf{u} = 1$ in (1.4). Hence some of the basic tools from linear theory of elliptic systems would work under suitable conditions on $p$. A different approach, utilizing the scale invariance of (1.4), will appear in the forthcoming paper [10].

The structure of the article is as follows. In the next section we introduce the basic notations used throughout the paper. The third section contains the discussion of some exceptional properties of the area-preserving maps. As a consequence it follows that the inverse of the local minimizer is a minimizer for the dual energy functional, hence it satisfies the weak Euler-Lagrange equation (1.5) in "image" domain $\Omega^\star$. The link between the deformation and its inverse allows to translate the regularity properties of the one to the other. Hence the minimizer and its inverse enjoy the same regularity.

In the next section we examine the $L\log(2+L)$ estimate for $|\nabla\mathbf{u}|^2$ through some well-known results from the theory of differentiation of integrals in $\mathbb{R}^2$. The key observation here is that the collection of the images of the squares under mapping $\mathbf{u}$ differentiates the integral of $|\nabla\mathbf{u}|^2$, and thus applying the duality this translates to $|\nabla\mathbf{v}|^2$. The assumption that $|\nabla\mathbf{v}|^2$ has equal integrals over the image of the square and some suitable rectangle in $\Omega$ yields a local $L\log(2+L)$ estimate for $|\nabla\mathbf{u}|^2$.

Using the Riesz transform and a suitable one-sided bound for the pressure associated with the deformation we can conclude local $L\log(2+L)$ integrability of the pressure. This is the content of Section 5.

The last two sections contain the proofs of Hölder and Sobolev-type local estimates for the deformation under suitable conditions on the pressure and its dual.

## 2. Notations

We use the following standard set of notations: $\mathbb{R}^N$ is the $N$-dimensional Euclidean space, we always assume that $N = 2$ if not otherwise stated, $B_R(x_0)$ is the open ball of radius $R$ centered at $x_0$, $Q_\rho(x_0)$-the open cube centered at $x_0$ with side length $2\rho$, $Q(x_0) = Q_1(x_0)$, $L\log(2+L)(D)$ is the Orlicz space of all measurable functions $f$ defined on the measurable set $D$ such that

$$\int_D |f|\log(2+|f|) < \infty,$$

$L^s(D), 1 \leq s \leq \infty$ is the space of all measurable function defined on $D$ with finite $L^s$ norm, $W^{1,s}(D)$ is the usual Sobolev space (see [7] chapter 7), $W^{1,s}(D,\mathbb{R}^N)$ is the vectorial analogue of $W^{1,2}(D)$, $\mathscr{M}$ is the space of all $2\times 2$ real matrices equipped with the standard norm

$$\|A\|_{\mathscr{M}} = \sup_{\xi\in\mathbb{R}^2, |\xi|=1} |A\xi|, \qquad A \in \mathscr{M},$$

the scalar product of two matrices $A, B \in \mathscr{M}$ is defined as $A : B = tr(A^tB)$, where $A^t$ is the transpose of $A$, $tr$ is the trace of the matrix and $|A| = \sqrt{A : A}$. The cofactor matrix of $A \in \mathscr{M}$ is denoted by $\mathrm{cof}A = \dfrac{\partial \det A}{\partial A}$, $C_0^\infty(D,\mathbb{R}^m)$ is the class of all $C^\infty$ maps from $\Omega$ to $\mathbb{R}^m$ with compact support in $D$, if $m = 1$ then it is denoted by $C_0^\infty(D)$.

## 3. Duality

Let $\Omega_0$ be a bounded, open set in $\mathbb{R}^2$ having Lipschitz boundary $\partial\Omega_0$. A deformation of $\Omega_0$ is a map $\mathbf{u} : \Omega_0 \mapsto \mathbb{R}^2$ and $\mathbf{u} \in W^{1,2}(\Omega_0,\mathbb{R}^2)$. If in addition $\det\nabla\mathbf{u} = 1$ a.e. in $\Omega_0$ then $\mathbf{u}$ is called area-preserving (incompressible) deformation. Here $\nabla\mathbf{u}$ is the deformation gradient

$$\nabla\mathbf{u} = \left(\frac{\partial u^i}{\partial x_j}\right) = \begin{pmatrix} u_1^1 & u_2^1 \\ u_1^2 & u_2^2 \end{pmatrix}.$$

Notice that $\det\nabla\mathbf{u} = u_1^1u_2^2 - u_1^2u_2^1$ is integrable in $\Omega_0$.

We denote the class of all area preserving deformations of $\Omega_0$ by $\mathscr{A}(\Omega_0)$ :

$$\mathscr{A}(\Omega_0) = \{\mathbf{u} \in W^{1,2}(\Omega_0,\mathbb{R}^2), \det\nabla\mathbf{u} = 1 \text{ a.e. in } \Omega_0\}.$$

The following properties of the elements of $\mathscr{A}(\Omega_0)$ are worth recording (see [14], see also Remark 3.3 [4]):

- $\mathbf{u} \in \mathscr{A}(\Omega_0)$ has continuous representative,
- $\mathbf{u}$ maps open sets onto open sets, i.e. $\mathbf{u}$ is open,

- if $\Omega$ is of class $C^\infty, \overline{\Omega} \subset \Omega_0$, $\mathbf{u}_0 \in \mathscr{A}(\Omega_0)$ and $\mathbf{u}_0$ is a homeomorphism then one can define the inverse map $\mathbf{v}$ of $\mathbf{u} \in \mathscr{A}(\Omega)$, from $\Omega^\star = \mathbf{u}_0(\Omega)$ to $\Omega$ and $\mathbf{u} = \mathbf{u}_0$ on $\partial\Omega$:

$$\mathbf{v} : \Omega^\star \mapsto \Omega, \qquad \mathbf{v}(\mathbf{u}(x)) = x, \qquad \mathbf{u}(\mathbf{v}(y)) = y, \qquad \forall x \in \Omega, \forall y \in \Omega^\star,$$

- $\mathbf{v}$ has a continuous representative and $\mathbf{v} \in W^{1,2}(\Omega^\star, \mathbb{R}^2)$.

Moreover the usual formulae relating the derivatives of $\mathbf{u}$ and $\mathbf{v}$ are valid. We have for the gradient of the inverse mapping $\mathbf{v}$

$$\nabla\mathbf{v}(y) = \begin{pmatrix} u_2^2(\mathbf{v}(y)) & -u_2^1(\mathbf{v}(y)) \\ -u_1^2(\mathbf{v}(y)) & u_1^1(\mathbf{v}(y)) \end{pmatrix}$$

for $\det \nabla\mathbf{u} = 1$ a.e. in $\Omega$. In particular

$$\begin{cases} \int_\Omega |\nabla\mathbf{u}|^2 = \int_{\Omega^\star} |\nabla\mathbf{v}|^2, \\ \int_{\Omega^\star} |\nabla\mathbf{v}(y)|^2 dy = \int_{\Omega^\star} |\nabla\mathbf{u}(\mathbf{v}(y))|^2 dy, \\ \det \nabla\mathbf{v}(y) = 1 \text{ for } \text{ a.e. } y \in \Omega^\star. \end{cases} \tag{3.1}$$

Next we state the main result of this section.

**Theorem 2.** *Let*

$$J^\star(\boldsymbol{w}) = \int_{\Omega^\star} |\nabla\boldsymbol{w}(y)|^2 dy, \qquad \boldsymbol{w} \in \mathscr{A}(\Omega^\star).$$

*Then $\boldsymbol{v}$, the inverse of $\boldsymbol{u}$, is a local minimizer of $J(\boldsymbol{w})$, i.e.*

$$\int_{\Omega^\star} |\nabla\boldsymbol{v}|^2 \leq \int_{\Omega^\star} |\nabla\boldsymbol{w}|^2,$$

*for all $\boldsymbol{w} \in \mathscr{A}(\Omega^\star)$ with supp$(\boldsymbol{w} - \boldsymbol{v}) \subset \Omega^\star$.*

**Proof.** Let $D$ be a subdomain of $\Omega^\star, \overline{D} \subset \Omega^\star$. We want to show that

$$\int_D |\nabla\mathbf{v}(y)|^2 dy \leq \int_D |\nabla\mathbf{w}(y)|^2 dy$$

for any $\mathbf{w} \in \mathscr{A}(\Omega^\star)$ such that $\operatorname{supp}(\mathbf{v} - \mathbf{w}) \subseteq \overline{D}$. Let $D_0 = \operatorname{supp}(\mathbf{v} - \mathbf{w}) \subset D$. By change of variable formula (3.1) we have

$$\int_{\Omega_0} |\nabla\mathbf{u}(x)|^2 dx = \int_{D_0} |\nabla\mathbf{v}(y)|^2 dy,$$
$$\int_{\Omega_0} |\nabla\mathbf{T}(x)|^2 dx = \int_{D_0} |\nabla\mathbf{w}(y)|^2 dy,$$

where $\mathbf{T}$ is the inverse of $\mathbf{w}$ and we set $\Omega_0 = \mathbf{v}(D_0) = \mathbf{w}(D_0)$. We identify $\mathbf{u}$ and $\mathbf{T}$ with their continuous representatives. Notice that $\Omega_0$ is closed for $\mathbf{v}, \mathbf{w}$ are continuous and $D_0$ is closed. Thus it suffices to prove that

$$\int_{\Omega_0} |\nabla \mathbf{u}|^2 \leq \int_{\Omega_0} |\nabla \mathbf{T}|^2.$$

We claim that $\operatorname{supp}(\mathbf{u} - \mathbf{T}) \subset \Omega_0$. We argue towards a contradiction, then there exists $x_0 \in \Omega \setminus \Omega_0$ such that $y_1 = \mathbf{u}(x_0), y_2 = \mathbf{T}(x_0)$ and

$$y_1 \neq y_2.$$

Clearly $y_1, y_2 \in \Omega^\star \setminus D_0$. Thus $\mathbf{v}(y_1) = \mathbf{w}(y_1) = x_0$ and $\mathbf{v}(y_2) = \mathbf{w}(y_2) = x_0$ implying that $\mathbf{v}(y_1) = \mathbf{v}(y_2)$ which contradicts to $y_1 \neq y_2$. □

As an immidiate consequence we get.

**Corollary 3.** *Let $\mathbf{v}$ be as in Theorem 2, then there exists a locally integrable function $q^\star : \Omega \mapsto \mathbb{R}$ such that (1.5) holds for $\mathbf{v}$ and $q^\star$ in $\Omega$.*

## 4. $L \log(2 + L)$ ESTIMATE FOR $|\nabla \mathbf{u}|^2$

In order to state the main result of this section we need some definitions from the theory of differentiation of integrals. These definitions can be found in de Guzman's book [8] pages 42 and 65.

Let $x \in \mathbb{R}^N$. A collection of bounded measurable sets $\mathcal{B}(x)$ is called a differentiation basis at $x$ if

- all the sets $R \in \mathcal{B}(x)$ have positive measure,
- $x \in B$ for all $R \in \mathcal{B}(x)$,
- there is at least a sequence $\{R_k\}_{k=1}^\infty \subset \mathcal{B}(x)$ such that $\operatorname{diam} R_k \to 0$.

The *differentiation basis* in $\mathbb{R}^N$ is $\mathcal{B} = \{\cup \mathcal{B}(x) : x \in \mathbb{R}^N\}$. Next we define the upper and lower derivatives of $f$ with respect to $\mathcal{B}$ at $x$;

$$\begin{aligned}
\overline{D_{\mathcal{B}}(f, x)} &= \sup \left\{ \limsup_{k \to \infty} ⨍_{R_k} f \ : \ \{R_k\} \subset \mathcal{B}(x), R_k \to x \right\}, \\
\underline{D_{\mathcal{B}}(f, x)} &= \inf \left\{ \liminf_{k \to \infty} ⨍_{R_k} f \ : \ \{R_k\} \subset \mathcal{B}(x), R_k \to x \right\}.
\end{aligned}$$

**Definition 4.** *$\mathcal{B}$ is said to differentiate (the integral) of $f$ if for almost every $x$ we have*

$$\overline{D_{\mathcal{B}}(f, x)} = \underline{D_{\mathcal{B}}(f, x)} = f(x).$$

*If $\mathcal{B}$ differentiates every function $f$ in a class $\Phi$ then we say that $\mathcal{B}$ differentiates $\Phi$.*

For $N = 2, x \in \Omega$ we take $\mathcal{B}(x)$ to be the collection of all open rectangles $R$ containing $x$ and with sides parallel to the given orthogonal frame $[e_1, e_2]$ in $\Omega$. Similarly we let $\mathcal{B}^\star(y), y \in \Omega^\star$ be the collection of all open rectangles $R^\star$ containing $y$ and with sides parallel to (in general different) orthogonal frame $[e_1^\star, e_2^\star]$ in $\Omega^\star$. These frames are arbitrary but fixed. We put $\mathcal{B} = \cup\mathcal{B}(x)$ and $\mathcal{B}^\star = \cup\mathcal{B}^\star(y)$.

The basis of rectangles has a number of interesting properties: it is known that if $\mathcal{B}$ differentiates $f \in L^1(\Omega)$ then locally $f \in L\log(2+L)$ (see for example [6]).

Let $Q$ be an open square containing $x \in \Omega$ then from change of variable formula [14] we have

$$⨍_Q |\nabla \mathbf{u}|^2 = ⨍_{\mathbf{u}(Q)} |\nabla \mathbf{v}|^2, \qquad \mathbf{u} \in \mathscr{A}, \qquad \mathbf{v} = \mathbf{u}^{-1}.$$

Since the basis of the squares differentiates $L^1$, we infer that the collection of the sets $\{\mathbf{u}(Q), x \in \Omega\}$ differentiates $\mathbf{v}(y)$. Similarly by considering the sets $\mathbf{v}(Q^\star)$, where $Q^\star$ is an open rectangle containing $y \in \Omega^\star$, one can conclude that the collection of the sets $\{\mathbf{v}(Q^\star), y \in \Omega\}$ differentiates $\mathbf{u}(x)$ for a.e. $x \in \Omega$. Hence if either of the sets $\{\mathbf{u}(Q)\}$ or $\{\mathbf{v}(Q^\star)\}$ "behaves" as $\mathcal{B}$ or $\mathcal{B}^\star$ one should be able to conclude local $L\log(2+L)$ regularity for $|\nabla \mathbf{u}|^2$. In this direction we have

**Theorem 5.** *Let $\mathcal{B}$ and $\mathcal{B}^\star$ be defined as above.*

- *Let $[e_1^\star, e_2^\star]$ be an orthonormal frame in $\Omega^\star$ and assume that there exists an orthogonal frame $[e_1, e_2]$ in $\Omega$ with the following property: for any $R^\star \in \mathcal{B}^\star(y), y \in \Omega^\star$ there exists an open square $Q$ conatining $\boldsymbol{v}(y)$, $|Q| = |R^\star|$, with sides parallel to $[e_1, e_2]$ such that*
$$\int_{R^\star} |\nabla \boldsymbol{v}|^2 = \int_Q |\nabla \boldsymbol{u}|^2.$$
*Then $|\nabla \boldsymbol{v}|^2$ is locally in $L\log(2+L)$.*
- *Let $R \in \mathcal{B}(x)$ corresponding to some fixed orthogonal frame $[e_1, e_2]$. If there exists an orhogonal frame $[e_1^\star, e_2^\star]$ in $\Omega^\star$ such that for any $R$ there exists a square $Q^\star$, $|Q^\star| = |R|$ containing $\boldsymbol{u}(x)$ with sides parallel to $[e_1^\star, e_2^\star]$ such that*
$$\int_R |\nabla \boldsymbol{u}|^2 = \int_{Q^\star} |\nabla \boldsymbol{v}|^2$$
*then $|\nabla \boldsymbol{u}|^2$ locally is in $L\log(2+L)$.*

**Remark 6.** *The hypotheses in Theorem 5 can be relaxed due to a result of R. Moriyón, [**?**], namely if one takes a subbasis of rectangles $\mathcal{B}_M \subset \mathcal{B}$ in Theorem 5, with an additional condition*

$$D^2 \le d \le D \le 1,$$

*where $d$ is the length of the smaller side and $D$ is the one of the bigger side of $R$, then one can conclude the local $L\log(2+L)$ estimate and this is the best space* [?].

Next we want to show that if $\mathbf{u}$ is differentiable at $x_0$ then the sets $\{\mathbf{u}(Q_\rho(x_0))\}$, i.e. the images of the squares centered at $x_0$, asyptotically behave as parallelograms. At this point we don't known if the area preserving deformations are a.e. differentiable. However $\mathbf{u}$ is weakly differentiable in some weak sense, introduced by Yu. Reshetnjak in [11].

**Definition 7.** *$\boldsymbol{u}$ is said to be weakly differentiable in Reshetnjak's sense if there exists a linear map $\ell$ and*

$$\liminf_{t\to 0^+}\gamma(x,t)=0, \qquad \gamma(t,x)=\sup_{z\in Q_1}\left|\frac{\boldsymbol{u}(x+tz)-\boldsymbol{u}(x)-\ell(tz)}{t}\right|. \tag{4.1}$$

It is known that each $\mathbf{u}\in W^{1,p}(\Omega,\mathbb{R}^N)$ is a.e. weakly differentiable in Reshetnjak's sense provided $p>N-1$ [11]. In our case $N=p=2$ and therefore (4.1) holds for local minimizer $\mathbf{u}$.

Assume that $\mathbf{u}$ is weakly differentiable at $x_0$ in the sence of (4.1). By change of variable formula (see [14]) we have that

$$⨍_{Q_\rho(x_0)}|\mathbf{u}(x)-\mathbf{u}(x_0)|^2dx=⨍_{\mathbf{u}(Q_\rho(x_0))}|\mathbf{v}(y)-\mathbf{v}(y_0)|^2dy. \tag{4.2}$$

Introduce the linear mapping $\ell(x)=\mathbf{u}(x_0)+S(x-x_0)$, where $S=\nabla\mathbf{u}(x_0), \det S=1$. The image of the square $Q_\rho(x_0)$ under linear mapping $\ell(x)$ is a parallelogram $P_\rho(y_0)$ of the same area $\rho^2$ *centered at* $y_0$. For a sequence $t_k\downarrow 0^+$ as in (4.1) we let

$$t_k^*=\sup\{t>0, \ell(Q_t(y_0))\subset\subset\mathbf{u}(Q_{t_k}(y_0))\}.$$

Let $y_1\in\partial P_{t_k^*}(y_0)\cap\mathbf{u}(Q_{t_k}(x_0))$, then there exist $x_1\in\partial Q_{t_k}(x_0)$ and $x_2\in Q_{t_k^*}(x_0)$ such that

$$\mathbf{u}(x_1)=\ell(x_2).$$

By (4.1) we have that $\mathbf{u}(x_1)=\mathbf{u}(x_0)+S(x_1-x_0)+t_k\theta(t_k)$ with $|\theta(t_k)|\le\gamma(x_0,t_k)$. Thus

$$|S(x_1-x_2)|\le t_k\gamma(x_0,t_k).$$

Utilizing $\det S=1$ we estimate $|(x_1-x_2)|=|S^{-1}S(x_1-x_2)|\le\|S^{-1}\|_{\mathcal{M}}|(x_1-x_2)S|$ thereby

$$|t_k-t_k^*|\le C(x_0)t_k\gamma(x_0,t_k)$$

and
$$\frac{|P_{t_k^*}(y_0)|}{|Q_{t_k}(x_0)|}=\left(\frac{t_k^*}{t_k}\right)^2\to 1 \text{ as } t_k\to 0.$$
In particular by applying (4.2)

$$\fint_{P_{t_k^*}(y_0)}|\mathbf{v}(y)-\mathbf{v}(y_0)|^2dy\le\left(\frac{t_k}{t_k^*}\right)^2\fint_{\mathbf{u}(Q_\rho(x_0))}|\mathbf{v}(y)-\mathbf{v}(y_0)|^2dy\longrightarrow 0.$$

**Remark 8.** *Although the discussion of the 3-dimensional problem is beyond the scope of this paper we would like to point out the following: to formulate the analogue of Theorem* 5 *in 3-space one needs to replace the rectangles with the rectangular intervals with two equal sides while considering all rectangular intervals will result that* $|\nabla \boldsymbol{u}|^2+|cof\nabla\boldsymbol{u}|^2$ *is locally in* $L(\log(2+L))^2$.

## 5. $L\log(2+L)$ estimate for pressure $q$

The goal of this section is to show that a suitable one-sided bound for $q$ and the assumption $\sigma_{ij}\in L\log(2+L)$ imply that locally $q\in L\log(2+L)$. It follows from the results of [4] that $q\in L^1_{\text{loc}}(D)$ provided $\sigma_{ij}\in L\log(2+L)(D), D\subset\Omega^\star$. Here $\sigma_{ij}(y)=u^i_m(\mathbf{u}^{-1}(y))u^j_m(\mathbf{u}^{-1}(y)), y\in D$.

From the physical point of view the pressure is essentially non-negative. However it is not clear how one can mathimaticaly justify this statement.

Notice that a necessary condition on $q$ is that it must have a vanishing integral [4]. To formulate our result we recall the definition of the $i$-Riesz transformation of $f$ defined as in [13] page 57
$$R_if(z)=\lim_{\varepsilon\to 0}c_N\int_{|y|>\varepsilon}\frac{y_i}{|y|^{N+1}}f(z-y)dy,\qquad c_N=\frac{\Gamma(\frac{N+1}{2})}{\pi^{\frac{N+1}{2}}}.$$
The computation below also works in higher dimensions, but we keep in mind that $N=2$.

**Lemma 9.** *Let* $\sigma_{ij}\in L\log(2+L)(D), D\subset\subset\Omega^\star$ *and for some function* $b\in L\log(2+L)(D)$ *we have that* $q\ge b$ *(or* $q\le b$*) in* $D$*. Then* $q$ *is locally in* $L\log(2+L)(D)$.

**Remark 10.** *Obviously if the pressure is bounded above or below then one can take* $b=$ *const.*

**Proof.** To fix the ideas we suppose that $q$ assumes a lower bound. Let $b^\varepsilon$ be the mollification of $b$ then $\hat{q}^\varepsilon=q^\varepsilon-b^\varepsilon\ge 0$ satisfies

$$D_j\hat{\sigma}^\varepsilon_{ij}=D_i\hat{q}^\varepsilon, \tag{5.1}$$

where $\hat\sigma^\varepsilon_{ij}=\sigma^\varepsilon_{ij}-\delta_{ij}b^\varepsilon$ and $\sigma_{ij}$ is defined as above. Let $\eta$ be a cut-off function, supp$\eta\subset D$. Multiplying (5.1) by $\eta$ and using partial integration we get

$$\begin{aligned}0&=\int_{B_R(y_0)}\eta D_j(\hat\sigma^\varepsilon_{ij}-\delta_{ij}\hat q^\varepsilon)\\&=\int_{\partial B_R(y_0)}\eta(\hat\sigma^\varepsilon_{ij}-\delta_{ij}\hat q^\varepsilon)\frac{y^j-y_0^j}{R}-\int_{B_R(y_0)}D_j\eta(\hat\sigma^\varepsilon_{ij}-\delta_{ij}\hat q^\varepsilon).\end{aligned}$$

Dividing by $R^N$ and integrating over $(0,\rho)\ni R$ we obtain

$$\int_{B_\rho(y_0)}\eta(\hat\sigma^\varepsilon_{ij}-\delta_{ij}\hat q^\varepsilon)\frac{y^j-y_0^j}{|y-y_0|^{N+1}}=\int_0^\rho\left[\int_{B_R(y_0)}D_j\eta(\hat\sigma^\varepsilon_{ij}-\delta_{ij}\hat q^\varepsilon)\right]\frac{dR}{R^N}.$$

Now partial integration with respect to $R$ leads to

$$\begin{aligned}\int_0^\rho\left[\int_{B_R(y_0)}D_j\eta(\hat\sigma^\varepsilon_{ij}-\delta_{ij}\hat q^\varepsilon)\right]\frac{dR}{R^N}&=-\frac{1}{N-1}\left\{\frac{1}{R^{N-1}}\int_{B_R(y_0)}D_j\eta(\hat\sigma^\varepsilon_{ij}-\delta_{ij}\hat q^\varepsilon)\Big|_0^\rho\right\}\\&\quad+\frac{1}{N-1}\int_0^\rho\frac{1}{\rho^{N-1}}\int_{\partial B_\rho(y_0)}D_j\eta(\hat\sigma^\varepsilon_{ij}-\delta_{ij}\hat q^\varepsilon)\\&=-\frac{1}{N-1}\frac{1}{\rho^{N-1}}\int_{B_\rho(y_0)}D_j\eta(\hat\sigma^\varepsilon_{ij}-\delta_{ij}\hat q^\varepsilon)\\&\quad+\frac{1}{N-1}\int_{B_\rho(y_0)}D_j\eta\frac{\hat\sigma^\varepsilon_{ij}-\delta_{ij}\hat q^\varepsilon}{|y-y_0|^{N-1}}.\end{aligned}$$

Letting $\rho\to\infty$ we get

$$\int\eta(\hat\sigma^\varepsilon_{ij}-\delta_{ij}\hat q^\varepsilon)\frac{y^i-y_0^i}{|y-y_0|^{N+1}}dy=\frac{1}{N-1}\int(\hat\sigma^\varepsilon_{ij}-\delta_{ij}\hat q^\varepsilon)\frac{D_j\eta}{|y-y_0|^{N-1}}.\tag{5.2}$$

To recognize the $i$-Riesz transformation we set $y_0-y=\xi$ then (5.2) implies

$$-\int_{\mathbb{R}^N}\eta(y_0-\xi)\left\{\hat\sigma^\varepsilon_{ij}(y_0-\xi)-\delta_{ij}\hat q^\varepsilon(y_0-\xi)\right\}\frac{\xi^i}{|\xi|^{N+1}}d\xi=\frac{1}{N-1}\int(\hat\sigma^\varepsilon_{ij}-\delta_{ij}\hat q^\varepsilon)\frac{D_j\eta}{|y-y_0|^{N-1}}dy$$

or equivalently

$$R_i(\eta\hat q^\varepsilon)(y_0)=\sum_j R_j(\eta\hat\sigma^\varepsilon_{ij})(y_0)+\frac{1}{N-1}\int(\hat\sigma^\varepsilon_{ij}-\delta_{ij}\hat q^\varepsilon)\frac{D_j\eta}{|y-y_0|^{N-1}}.$$

The $L^1$ norm of the integral on the right can be estimated via $L^1$ norm of $\eta[\hat\sigma_{ij}-\delta_{ij}\hat q]\in L^1$ uniformly in $\varepsilon$ [7] Theorem 7.18. Since $\hat q\geq 0$ and $\hat\sigma_{ij}\in L\log(2+L)$ we can apply the result of E. Stein Theorem 3 page 309 [12] to infer that $\eta\hat q^\varepsilon\in L\log(2+L)(D)$ uniformly in $\varepsilon$. A customary compactness argument in Orlicz space $L\log(2+L)$ finishes the proof. The other case $q \le b$ can be treated similarly by considering $\hat{q}^\varepsilon = b^\varepsilon - q^\varepsilon \ge 0$. $\square$

**Corollary 11.** *If the condition $q \ge b$ (or $q \le b$) is not satisfied then*

$$\left\| R_i(\eta q^\varepsilon)(y_0) - \frac{1}{N-1}\int (\sigma_{ij}^\varepsilon - \delta_{ij}\hat{q}^\varepsilon)\frac{D_j\eta}{|y-y_0|^{N-1}} \right\|_{L^1} \le C$$

*with $C$ depending only on $L\log(2+L)$ norm of $\sigma_{ij}$, $L^1$ norm of $q$, $C^1$ norm of $\eta$ and dimension $N$.*

## 6. Hölder estimate

The previous theorem indicates that a one-sided bound for $q$ elevates the regulaity of the pressure upto Orlicz class $L\log(2+L)$. In fact an upper constant bound on the dual pressure $q^\star$ implies local Hölder continuity for $\mathbf{u}$.

**Theorem 12.** *Let $\boldsymbol{u}$ be a local minimizer. Assume that the dual pressure $q^\star$ (i.e. the pressure assosiated with $\boldsymbol{v}$, the inverse of $\boldsymbol{u}$) admits a lower bound, namely*

$$q^\star \le C$$

*for some constant $C$. Then $u \in C^\alpha_{loc}(\Omega)$ for some $\alpha \in (0,1)$.*

**Proof.** In view of (1.5) $q^\star$ is defined modulo a constant hence without loss of generality we may assume that $C = 0$. Thus the local $L\log(2+L)$ estimate from section 5 is valid for $q^\star$. Employing Corollary 3 we deduce that the inverse map $\mathbf{v} = \mathbf{u}^{-1}$ satisfies the Euler-Lagrange equation

$$\int v^i_m(\mathbf{v}^{-1}(x))v^j_m(\mathbf{v}^{-1}(x))\psi^i_j(x)dx = \int q^\star(x)\operatorname{div}\psi(x)dx$$

for any $\psi(x) \in C^\infty_0(\Omega, \mathbb{R}^2)$. Take $\psi(x) = (x^i - x^i_0)\eta$, with $\eta \in C^\infty_0(\Omega), \eta \equiv 1$ in $B_R(x_0)$, supp$\eta \subset B_{2R}(x_0), \eta \ge 0, |D\eta| \le C/R$ in the former equation. Then $\psi^i_j = \delta_{ij}\eta + (x^i - x^i_0)\eta_j$ and

$$\int \left[v^i_m(\mathbf{v}^{-1}(x))\right]^2\eta(x) + v^i_m(\mathbf{v}^{-1}(x))v^j_m(\mathbf{v}^{-1}(x))(x^i - x^i_0)\eta_j = \int q^\star[2\eta + (x^i - x^i_0)\eta_j].$$

Thus rearranging the terms we get

$$\begin{aligned}\int_{B_{2R}(x_0)} \left[|\nabla\mathbf{v}(\mathbf{v}^{-1}(x))|^2 - 2q^\star\right]\eta &= \int_{B_{2R}(x_0)\setminus B_R(x_0)} \left\{q^\star - (v^i_m(\mathbf{v}^{-1}(x))v^j_m(\mathbf{v}^{-1}(x)))\right\}(x^i - x^i_0)\eta_j \\ &\le C\int_{B_{2R}(x_0)\setminus B_R(x_0)} |q^\star| + |\nabla\mathbf{v}(\mathbf{v}^{-1}(x))|^2.\end{aligned}$$

Utilizing the change of variable formula, (3.1) and the obvious inequality $|\nabla \mathbf{u}|^2 - 2q^\star \geq |\nabla \mathbf{u}|^2 - q^\star \geq 0$ we obtain

$$\int_{B_R(x_0)} |\nabla \mathbf{u}|^2 - q^\star \leq C \int_{B_{2R}(x_0)\setminus B_R(x_0)} |\nabla \mathbf{u}|^2 - q^\star.$$

Adding $C \int_{B_R(x_0)} |\nabla \mathbf{u}|^2 - q^\star$ and dividing by $1 + C$ we get that

$$\int_{B_R(x_0)} \left(|\nabla \mathbf{u}|^2 - q^\star\right) \leq \theta \int_{B_{2R}(x_0)} \left(|\nabla \mathbf{u}|^2 - q^\star\right)$$

with $\theta = C/(C+1) < 1$.

Consequently from DeGiorgi's itaration we have

$$\int_{B_\rho(x_0)} |\nabla \mathbf{u}|^2 - q^\star \leq C\rho^\beta$$

for some $\beta \in (0, 1)$. In particular

$$\frac{1}{|B_\rho|} \int_{B_\rho(x_0)} |\nabla \mathbf{u}|^2 \leq C\rho^{\beta-2} = C\rho^{2(\beta/2-1)}.$$

Taking $\beta/2 = \alpha$ and employing Theorem 7.19 from [7] we complete the proof. □

## 7. $W^{1,s}$ ESTIMATE FOR

If the pressure is constant then in view of (1.5) or (1.4), it follows that $\mathbf{u}$ is harmonic. This observation suggests that if the values of the pressure are close to some constant $p_0$ then one should expect higher regularity for $\mathbf{u}$.

**Theorem 13.** *Let $\boldsymbol{u} \in W^{1,2}(\Omega, \mathbb{R}^2, ) \cap W^{1,3}_{loc}(\Omega, \mathbb{R}^2)$ be a weak solution of (1.4). Then for any $s > 0$ there exists an $\varepsilon = \varepsilon(s)$ such that if $|p - p_0| < \varepsilon$ for some constant $p_0$ then $\boldsymbol{u} \in W^{1,s}_{loc}(\Omega, \mathbb{R}^2)$.*

**Proof.** Let $\mathbf{u}_h$ be the solution to Dirichlet problem

$$\begin{cases} \Delta \mathbf{u}_h = 0 & \text{in } Q, \\ \mathbf{u}_h = \mathbf{u} & \text{on } \partial Q, \end{cases}$$

for any square $Q \subset \Omega$. Notice that $\mathrm{div}(p_0 \mathrm{cof} \nabla \mathbf{u}) = p_0 \, \mathrm{div}(\mathrm{cof} \nabla \mathbf{u}) = 0$ since $p_0$ is a constant and

$$\mathrm{cof} \nabla \mathbf{u} = \begin{pmatrix} u^2_2 & -u^2_1 \\ -u^1_2 & u^1_1 \end{pmatrix}.$$

It is well-known that $\mathbf{u}_h$ exists. Since $\mathbf{u} \in W^{1,3}_{\mathrm{loc}}$ then by Theorem 1 the equation $\mathrm{div}(\nabla \mathbf{u} - p\mathrm{adj}\nabla \mathbf{u}) = 0$ is satisfied in the weak sense. Then

$$\begin{aligned}
\int_Q |\nabla(\mathbf{u}-\mathbf{u}_h)|^2 &= \int_Q (\nabla \mathbf{u} - \nabla \mathbf{u}_h) : (\nabla \mathbf{u} + \nabla \mathbf{u}_h) \\
&= \int_Q \nabla \mathbf{u} : (\nabla \mathbf{u} - \nabla \mathbf{u}_h) \\
&= \int_Q (p-p_0)\mathrm{cof}\nabla \mathbf{u} : \nabla(\mathbf{u}-\mathbf{u}_h) \\
&\leq \left[\int_Q |(p-p_0)\mathrm{cof}\nabla \mathbf{u}|^2\right]^{\frac{1}{2}} \left[\int_Q |\nabla(\mathbf{u}-\mathbf{u}_h)|^2\right]^{\frac{1}{2}}
\end{aligned}$$

thus

$$\begin{aligned}
\int_Q |\nabla(\mathbf{u}-\mathbf{u}_0)|^2 &\leq \int_Q |(p-p_0)\mathrm{cof}\nabla \mathbf{u}|^2 \\
&\leq \varepsilon^2 \int_Q |\nabla \mathbf{u}|^2.
\end{aligned}$$

On the other hand

$$\begin{aligned}
\int_Q |\nabla \mathbf{u}_h|^2 &\leq \int_Q |\nabla \mathbf{u}|^2 \\
&\leq (1+\varepsilon^2)\int_Q |\nabla \mathbf{u}|^2.
\end{aligned}$$

Consequently we have

$$\frac{1}{|Q|}\int_Q |\nabla \mathbf{u}_h|^2 \leq \frac{1+\varepsilon^2}{|Q|}\int_Q |\nabla \mathbf{u}|^2,$$
$$\frac{1}{|Q|}\int_Q |\nabla \mathbf{u} - \nabla \mathbf{u}_h|^2 \leq \frac{\varepsilon^2}{|Q|}\int_Q |\nabla \mathbf{u}|^2.$$

Applying Theorem A [3], page 3, the result follows. □

## References

[1] Ball J.: Convexity conditions and existence theorems in nonlinear elasticity. Arch. Rational Mech. Analysis. 63, 337–403 (1977)
[2] Ball J.: Progress and puzzels in nonlinear elasticity, preprint.
[3] Caffarelli L.A., Peral I.: On $W^{1,p}$ estimates for elliptic equations in divergence form. Comm. in Pure and Appl. Math. vol. **LI** 1–21 (1998)
[4] Chaudhuri N. Karakhanyan A.L.: On derivation of Euler-Lagrange Equations for incompressible energy-minimizers. Cal. of Variations and PDE, in press. Available online at `www.arxiv.org`.
[5] Ciarlet P.: Mathematical elasticity, North Holland (1994)
[6] Cordóba A.: Maximal functions: a proof of a conjecture of A.Zygmund. Bulletin of AMS. vol. **1**, No. 1, 255-257, (1979)
[7] Gilbarg D., Trudinger N.: Elliptic Partial Differential Equations of Second Order, Springer (1998)
[8] de Guzman M.: Differentation of integrals in $\mathbb{R}^N$, Lecture Notes in Mathematics, Springer-Verlag (1975)
[9] M Moriyón R.: On the derivation properties of a class of bases, ibid. Appendix IV

[10] Karakhanyan A.L.: Regularity of energy-minimizing incompressible deformations, in preparation.
[11] Reshetnjak Yu.: Space mapping with bounded distortion. Sibirsk. Matem. Zh. 8 No. 3, 629658 (1967)
[12] Stein E.M.: Note on the class $L \log L$. Studia Mathematica. XXXII, 305–310 (1969)
[13] Stein E.M.: Singular integrals and differentiability properties of function. Princeton University Press (1970)
[14] Šverák V.: Regularity properties of deformations with finite energy. Archive for Rational Mechanics and Analysis. 105–127 (1988)

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