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Nonlinear Free Boundary Problems with Singular Source Terms*

By

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Abstract. We prove the existence of solutions to nonlinear free boundary problem with singularities at given points.

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1. Introduction

1.1. Problem setting. Our aim is to prove the existence of solutions (u, Ω) for the free boundary problem

$$\begin{cases} \mathscr{L}u = -\sum c_j \delta_{x^j} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ |\nabla u| = 1 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\{x^j\}_{j=1}^k \subset\subset \Omega$, $c_j > 0$ and δ_{x^j} is the Dirac measure with support at point x^j. Two types of operators will be considered. Operators $\mathscr{L}$ of the first type are quasilinear uniformly elliptic operators defined as $\operatorname{div}(f_p(\nabla u))$, where $f(p) = F(|p|^2)$ and $f_p = \nabla f(p)$ for $p \in \mathbb{R}^n$ with convex function F satisfying to $F(0) = 0$ and $F' > 0$. The case $F(t) = t$, that is when $\mathscr{L}$ is the Laplace operator, was studied in [8] in connection with existence of quadrature surfaces with respect to measure $\mu = c_1\delta_1 + \cdots + c_k\delta_k$, in general a signed measure compactly supported in Ω, which have a specific interest. However, in the present study we only deal with the existence of solutions of the problem (1.1) and regularity of free boundary $\partial\Omega$. The operator of second type is the so-called s-Laplacian defined as

$$\Delta_s u := \operatorname{div}(|\nabla u|^{s-1}\nabla u), \quad s > 1.$$

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The first type of operators are related to a variational free boundary problem, discussed in [2]. The analogous problem for second type operators was recently treated in [3], where similar results are obtained. In what follows we will formulate our results and present proofs for operators of the first type restricting ourselves on giving only sketches of the proofs of corresponding statements for Δ_s.

We will use the existence of solutions to the free boundary problem $\mathscr{L}u_m = 0$ in $\Omega_m \setminus \bigcup D_{m,j}$, $u_m = a_j m$ in $D_{m,j}$, $u_m = 0$ on $\partial\Omega_m$, $|\nabla u_m| = 1\ H^{n-1} - a.e.$ on $\partial\Omega_m$ (see ([2]) where the existence of free boundaries was discussed from the variational point of view) and establish the main properties of the free boundary. Here the domain $D_{m,j}$ is the componenet of the set $\{v > a_j m^{n-2}\}$ containing point x^j and v is the solution $\mathscr{L}v = -c_1\delta_1 - \cdots - c_k\delta_k$ (see Appendix).

We prove that outside of any neighborhood of the points x^j this sequence of solutions has a subsequence converging to a function u. Finally we will show that the function u and domain $\Omega := \{u > 0\}$ solve the above problem in a weak sense.

The problem (1.1) naturally arises in the study of electromagnetic flux confinement and dynamics of quantized gauge fields theory describing the interaction between elementary particles by considering particles to be quantized field. In this context the PDE governs the equilibrium field configuration made by classical charges in inhomogeneous environment. It has a very simple meaning: if u is the potential, $\nabla u = \mathbf{E}$ and $\mathbf{D} = f_p(\mathbf{E})$ is field's displacement vector then div$\mathbf{D}$ is spatial distribution of the total charge (see [1]).

1.2. Notation. Let us introduce some notations. For an open $A \subset \mathbb{R}^n$ $C^{2,1}(A)$ is the space of twice continuously differentiable functions, whose second derivatives are Lipschitz continuous in A. $W^{1,s}(A)$ denotes the Sobolev space of functions which belong together with their derivatives to L^s, $s > 1$. $W_0^{1,s}(A)$ is denoted as the closure, by $W^{1,5}$ norm, of the continuously differentiable functions with compact supports contained in A, $s > 1$. $W_{\mathrm{loc}}^{1,s}(A)$ is the space of functions belonging together with their derivative to $L^s(K)$ for each compact $K \subset A$.

The *reduced boundary* $\partial_{\mathrm{red}}A$ of an open set A is defined (see e.g. [4] 4.5.5) as

$$\partial_{\mathrm{red}}A := \{x \in \mathbb{R}^n;\, |\nu(x)| = 1\},$$

where ν is a unique unit vector such that

$$\int_{B(x,\rho)} |\chi_A - \chi_{\{y;(y-x)\cdot\nu(x) < 0\}}|dH^n = o(\rho^n),$$

if such a vector exists, else $\nu = 0$. Here χ_A is the characteristic function of A, $B(x,\rho) = \{y \in \mathbb{R}^n;\, |x-y| < \rho\}$ and H^n denotes the n-dimensional Hausdorff measure. Recall that if $H^{n-1}(\partial A)$ is finite then A is a set of locally finite perimeter and the (normal) vector is well defined on the reduced boundary $\partial_{\mathrm{red}}A$. Moreover,

$$\overline{\partial_{\mathrm{red}}A} = \partial A$$

and $|\nu| = 1$ pointwise on $\partial_{\mathrm{red}}\Omega$.

1.3. First type operators. Let $F(t)$ be a function in $C^{2,1}[0,\infty)$ satisfying:

$$F(0)=0; \quad c_0 \leqslant F'(t) \leqslant C_0; \quad 0 \leqslant F''(t) \leqslant \frac{C_0}{1+t}, \tag{1.2}$$

where c_0, C_0 are positive constants. For $p\in\mathbb{R}^n$ we set $f(p)=F(|p|^2)$. From (1.2) it is clear that $F(t)\geqslant 0$ and $f(p)$ is convex. Moreover,

$$\beta|\xi|^2 \leqslant \sum \frac{\partial^2 f(p)}{\partial p_i \partial p_j}\xi_i\xi_j \leqslant \beta^{-1}|\xi|^2 \quad \forall \xi\in\mathbb{R}^n, (\xi\neq 0),$$

since

$$\frac{\partial^2 f(p)}{\partial p_i \partial p_j} = 2F'(|p|^2)\delta_{i,j} + 4F''(|p|^2)p_i p_j.$$

Consequently

$$\begin{cases} \beta|p|^2 \leqslant f_p(p)\cdot p \\ \beta|p|^2 \leqslant f(p) \leqslant \beta^{-1}|p|^2, \end{cases} \tag{1.3}$$

for some small $\beta>0$, where $f_p=\nabla_p f$.

In this paper we mainly deal with the elliptic quasilinear operator

$$\mathscr{L}u = \operatorname{div}(f_p(\nabla u)),$$

which can be also written in nondivergence form as follows

$$\mathscr{L}u = 2\Delta u F'(|\nabla u|^2) + 4F''(|\nabla u|^2)\sum u_i u_j u_{i,j}. \tag{1.4}$$

By solution of $\mathscr{L}u=0$ we mean weak solution, i.e. $u\in W^{1,2}(A)$ and

$$\int f_p(\nabla u)\nabla\eta = 0, \quad \forall \eta\in C_0^\infty(A). \tag{1.5}$$

From the definition of the operator $\mathscr{L}$ and using (1.3) it is clear that the operator $\mathscr{L}$ is uniformly elliptic and non-degenerate. Differentiating the equation $\mathscr{L}u=0$ with respect to x_k, $1\leqslant k\leqslant n$, we get for $w_k=\partial_k u$ the equation of the form $\operatorname{div}(a(\cdot)\nabla w_k)=0$, and in view of (1.2) the matrix $a^{i,j}(p)=2F'(p)\delta_{i,j}+4F''(p)p_i p_j$ is uniformly elliptic. Hence by the DeGiorgi-Nash theorem, w_k is a C^α function for some $0<\alpha<1$ and $u\in C^{1,\alpha}$. Also it is worth noting that $u\in C^{2,\alpha}$ by Theorem 15.11 of [5].

1.4. Second type operators. Let $s>1$ and take $F(t)=\frac{1}{s}t^{\frac{s}{2}}$. Then repeating the previous construction we get another operator

$$\Delta_s u = \operatorname{div}(|\nabla u|^{s-2}\nabla u).$$

In contrary to first type operators, Δ_s is degenerate quasilinear elliptic operator, since obviously the conditions (1.2) fail.

The weak solution of $\Delta_s=0$ is a function $u\in W^{1,s}$ satisfying for each $\eta\in C_0^\infty$ to identity

$$\int |\nabla u|^{s-2}\nabla u \nabla\eta = 0. \tag{1.6}$$

For the properties of Δ_s we refer to [10] and the references therein.

2. Main Result

Theorem 2.1. *Let $c_j (j = 1, 2, \dots, k)$ be positive constants and x^j be some points in $\mathbb{R}^n$, $n \geqslant 3$ $(x^j \neq x^i; j \neq i)$. Then there exist a bounded open set $\Omega \supset\supset \{x^1 \cdots x^k\}_{j=1}^k$ and a function $u \in W_0^{1,2}(\Omega)$ satisfying*

$$\begin{cases} \mathscr{L}u = -\sum c_j \delta_{x^j} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ |\nabla u| = 1 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

in the sense of distributions, where the last boundary condition $|\nabla u| = 1$ is understood in a weak sense (see Theorem 1.9 [2] *and Theorem* 2.1 [3]).

Remark 2.2. In case of second type operators Ω and $u \in W_0^{1,s}(\Omega)$, $s > 1$, solve

$$\begin{cases} \Delta_s u = -\sum c_j \delta_{x^j} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ |\nabla u| = 1 & \text{on } \partial\Omega, \end{cases} \tag{2.2}$$

Remark 2.3. We recall here that $|\nabla u| = 1$ on $\partial_{\text{red}}\Omega$ in a strong sense and the rest of the boundary, $\partial\Omega \setminus \partial_{\text{red}}\Omega$ has H^{n-1} measure zero.

2.1. Preliminary notices. We introduce the function φ as a solution

$$\mathscr{L}\varphi = -\delta.$$

Note that φ can be obtained as radial solution to (1.4) satisfying to following ODE

$$\frac{v''(r)}{v'(r)} = -\frac{n-1}{r} \cdot \frac{F'(|v'(r)|^2)}{F'(|v'(r)|^2) + F''(|v'(r)|^2)|v'(r)|^2}, \tag{2.3}$$

for any radial symmetric solution $v = v(r)$, $r = |x|$.

Since this last equation has the form $\psi'(r) = G(r, \psi(r))$, where $\varphi'(r) = \psi(r)$ then the existence of $\varphi(r)$ follows from classical ODE theory. Moreover since $\mathscr{L}\varphi = 0$ in $\mathbb{R}^n \setminus \{0\}$ then by Serrin's argument $\varphi(r)/r^{n-2} \to \sigma > 0$. Without loss of generality we may assume that $\sigma = 1$ otherwise we could consider scaled function $\varphi_\sigma(r) = \frac{\varphi(\sigma r)}{\sigma}$. Furthermore $\nabla\varphi(r) - \nabla r^{2-n} = o(\nabla r^{2-n})$.

Indeed the function $\varphi_\rho(x) = \rho^{n-2}\varphi(\rho x)$ is a solution to $\widetilde{\mathscr{L}} = 0$ for $\widetilde{\mathscr{L}}$ constracted from $\widetilde{f(t)} = f(\rho^{n-1}t)$ and hence the $C^{1,\alpha}$ estimate yields

$$\|\nabla\varphi_\rho\|_{C^\alpha} \leqslant M$$

for any compact subset of the annulus $B_1 \setminus B_{112}$. Noting that $\lim_{\rho\to 0}\varphi_\rho(x) = |x|^{2-n}$ the C^α estimate above yields $\lim_{\rho\to 0}\nabla\varphi_\rho(x) = \nabla(|x|^{2-n})$. Returning to original functions the assertion follows.

To show that $\mathscr{L}\varphi = -\delta$ we choose a η_r such that $\eta_r = 0$ in $|x| < r$, $\eta_r = 1$ in $|x| > 2r$ and $|\nabla\eta_r| \leqslant \frac{c}{r}$. For $\zeta \in C_0^\infty$ computing

$$\begin{aligned} \int f_p(\nabla\varphi)\nabla\zeta &= \lim_{r\to 0}\int f_p(\nabla\varphi)\eta_r\nabla\zeta \\ &= -\lim_{r\to 0}\int f_p(\nabla|x|^{2-n})\nabla\eta_r\zeta = -\zeta(0)\frac{C_0}{C_n}. \end{aligned} \tag{2.4}$$

where $C_n = \frac{1}{(2-n)n\omega_n}$, ω_n is the area of the unit sphere in $\mathbb{R}^n$. Let v be the solution of

$$\mathscr{L}v = -a_1\delta_{x^1} - \cdots - a_k\delta_{x^k}$$

(see Appendix) and denote by $\tilde{v} = (v-M)^+$, where positive constants $a_1, \ldots, a_k$ and M will be specified below. Define the domains $D_{m,j}$ as the component of $\{x|\tilde{v} > a_j m^{n-2}\}$ containing point x^j.

As to constant the M, it should be chosen in the manner that $E_i \cap E_j$ is empty if $i \neq j$, where E_j is a component of the set $\{v > M\}$ containing point x^j and the following inequality is satisfied on ∂E_j

$$|\nabla \tilde{v}| = |\nabla v| > 2 \tag{2.5}$$

Thus $\mathrm{supp}(\tilde{v})$ is a union of disjoint domains containing points x^j.

Also we will observe that $\bigcap_{m=1}^{\infty} D_{m,j} = \{x^j\}$. Let us recall a result stated in [2] for $\mathscr{L}$ and in [3] for Δ_s.

Theorem 2.4. *Let $D_{m,j}$ be defined as in* 2.1. *Then there is a solution (u_m, Ω_m) of the following free boundary problem*

$$\begin{cases} \mathscr{L}u_m = 0 & \text{in } \Omega_m \setminus \bigcup_{j=1}^{k} D_{m,j} \\ u_m = 0 & \text{on } \partial\Omega_m, \\ u_m = a_j m^{n-2} & \text{in } \overline{D_{m,j}}, \\ \lim_{\Omega \ni x \to y} |\nabla u_m| \leqslant 1 & y \in \partial\Omega_m, \end{cases} \tag{2.6}$$

where $a_1, \ldots, a_k$ are arbitrary positive numbers and the last gradient condition holds with equality H^{n-1}-a.e. on $\partial\Omega_m$.

Remark 2.5. The last gradient condition in (2.6) for the operators of the first type should be written as $\Phi(|\nabla u_m|) = 1$, where $\Phi(s) = 2sF'(s) - F(s)$. But since Φ is a monotone and continuous function then $|\nabla u_m| = \Phi^{-1}(1) = \lambda$, so we can assume without loss of generality that $\lambda = 1$ (cf. [2], p. 15). Note that the third condition in (2.6) should be changed into $u_m = a_j m^{\frac{n-s}{s-1}}$ for Δ_s.

Remark 2.6. Taking in previous reasonings instead of $|x|^{2-n}\varphi = C_n|x|^{\frac{s-n}{s-1}}$ and $C_n = \frac{s-1}{(s-n)(n\omega_n)^{\frac{1}{s-1}}}$ and repeating argument above we get

$$\Delta_s\varphi = -\delta.$$

Also on account of Theorem 12 [7], v behaves like $\varphi(|x-x^j|)$ that is

$$\lim_{x \to x^j} \frac{v(x)}{\varphi(x-x^j)} = a_j$$

and again from the argument above we conclude that

$$\nabla v(x) - \nabla(a_j|x-x^j|^{2-n}) = o(\nabla|x-x^j|^{2-n}).$$

3. Uniform Estimates and $W^{1,2}$ Bounds

The proof of Theorem 2.1 will be based on the results of this section. First we prove a lemma ensuring non-degeneracy of domains Ω_m near points x^j.

Lemma 3.1. *Let u_m, Ω_m, $\widetilde{v}$ and $D_{m,j}$ $(j=1,\dots,k)$ be defined as in* 2.1. *Then*

(a) $\operatorname{supp}(\widetilde{v}) \subset \Omega_m$,
(b) $u_m \geqslant \widetilde{v}$ *in* $\mathbb{R}^n \backslash D_{m,j}$ (*by the comparison principle*).

Proof. Let $\varepsilon > 0$ and $\operatorname{supp}(\widetilde{v}) \backslash \Omega_m \neq \emptyset$. Then there is the smallest number $\alpha > 1$ such that $B(x^j, r_m/\alpha) \subset \Omega_m$ (r_m is the supremum of the radii of the balls $B_r \subset \operatorname{supp}(\tilde{v})$).

Taking into account (2.5) and using first a Lavrentiev type argument [8, Lemma 2] and then a comparison principle we arrive at the desired result. Part b) is a consequence of a).

Lemma 3.2. *Let u_m be a solution to the above free boundary problem. Then the following hold*:

(a) $u_m \leqslant C$ *outside a compact neighbourhood of the points* x^j, $j=1,\dots,k$ *for sufficiently large* m, *where* C *does not depend on* m;
(b) $|\nabla u_m|$ *are uniformly bounded in* $L^2(U_t)$, *where* $U_t := \{u_m < t\}$ $(t < m^{n-2} \min_j a_j)$;
(c) $H^{n-1}(\partial \Omega_m) \leqslant C$.

Proof. (a) Fix a large integer N and consider $D_N = D_{N,1} \cup \cdots \cup D_{N,k}$. Since $u_m \leqslant v$ on ∂D_N, by definition then, comparison principle ([5] Theorem 10.1) gives $u_m \leqslant v$ in Ω_m for each m.

(b) We need to prove $|\nabla u_m| \leqslant C$ in $U_t = \{u_m < t\} (t < m^{n-2})$. By (1.3) we have

$$\int_{U_t} |\nabla u_m|^2 \leqslant \frac{1}{\beta} \int_{U_t} f_p(\nabla u_m) \cdot \nabla u_m. \tag{3.1}$$

Since

$$\operatorname{div}(u_m f_p(\nabla u_m)) = u_m \mathscr{L} u_m + f_p(\nabla u_m) \cdot \nabla u_m,$$

by using (3.1) and (2.2) we arrive at

$$\int_{U_t} |\nabla u_m|^2 \leqslant \frac{1}{\beta} \int_{U_t} \operatorname{div}(u_m f_p(\nabla u_m)). \tag{3.2}$$

By the divergence theorem and since $u_m = t$ on ∂U_t, it follows that

$$\int_{U_t} \operatorname{div}(u_m f_p(\nabla u_m)) = t \int_{\partial U_t} f_p(\nabla u_m) \cdot \nu, \tag{3.3}$$

where ν denotes the unit outward normal to ∂U_t.

Since $t < m^{n-2} \min_j a_j$, then from (3.2) and (3.3) and again using divergence theorem we find that

$$\int_{U_t} |\nabla u_m|^2 \leqslant \frac{t}{\beta} \int_{\cup \partial D_{m,j}} f_p(\nabla u_m) \cdot \nu. \tag{3.4}$$

Next, by (1.3)

$$f_p(\nabla u_m) \cdot \nu \leqslant |f_p(\nabla u_m)| \leqslant |F'| |\nabla u_m| \leqslant C_0 |\nabla u_m|, \tag{3.5}$$

where C_0 is a positive constant. Since $\widetilde{v} = u_m$ on $\partial D_{m,j}$ and $\widetilde{v} \leqslant u_m \leqslant a_j m^{n-2}$ (Lemma 3.1) we have $|\nabla u_m| \leqslant |\nabla \widetilde{v}|$ on $\partial D_{m,j}$ for $j = 1, \ldots, k$. Applying this together with (3.4) and (3.5) we obtain

$$\int_{U_t} |\nabla u_m|^2 \leqslant \frac{tC_0}{\beta} \int_{\cup \partial D_{m,j}} |\nabla \widetilde{v}| \leqslant \frac{tC_0}{\beta^2} \int_{\cup \partial D_{m,j}} f_p(\nabla \widetilde{v}) \cdot \frac{\nabla \widetilde{v}}{|\nabla \widetilde{v}|}, \tag{3.6}$$

where the last inequality holds from (1.3).

Since $\widetilde{v} = a_j m^{n-2}$ on $\partial D_{m,j}$, then it is obvious that $\nabla \widetilde{v}_j / |\nabla \widetilde{v}_j| = \nu$, therefore using the divergence theorem again to the right hand side integral in (3.6) we get

$$\int_{U_t} |\nabla u_m|^2 \leqslant -\frac{tC_0}{\beta^2} \int_{\cup D_{m,j}} \mathscr{L} \widetilde{v}_j = \frac{tC_0}{\beta^2} (c_1 + \cdots + c_k).$$

This gives that $|\nabla u_m|$ is bounded in $L^2(U_t)$.

(c) is almost included in the proof of part (b). However, we give some details for proof. Since

$$|\nabla u_m| = 1 \quad \text{a.e. on } \partial \Omega_m$$

then, by (1.3) we get

$$H^{n-1}(\partial \Omega_m) = \int_{\partial \Omega_m} dH^{n-1} = \int_{\partial \Omega_m} |\nabla u_m| \leqslant \frac{1}{\beta} \int_{\partial \Omega_m} f_p(\nabla u_m) \cdot \frac{\nabla u_m}{|\nabla u_m|}$$

Observe that since $u_m > 0$ in Ω_m and $u_m = 0$ on $\partial \Omega_m$, then on $\partial \Omega_m$ we will have $\nabla u_m = -|\nabla u_m| \nu$, where ν is the outward unit normal vector on $\partial \Omega_m$. Applying the divergence theorem and using the fact, that $\mathscr{L} u_m = 0$ in $\Omega_m \backslash D_m$, we get from the last inequality

$$H^{n-1}(\partial \Omega_m) \leqslant -\frac{1}{\beta} \int_{\partial D_m} f_p(\nabla u_m) \cdot \nu \leqslant \frac{1}{\beta} \int_{\cup \partial D_{m,j}} |f_p(\nabla u_m)|.$$

The final result now follows by the same argument as in the proof of part (b), after (3.4).

Lemma 3.3. *The domains Ω_m are uniformly bounded, i.e. $\bigcup \Omega_m$ is bounded.*

Proof. Suppose $x_m \in \partial \Omega_m$ and $u_m(x_m) = 0$. Let B_{r_0} be a ball centered at a regular free boundary point x_m with fix radius r_0. Take in Lemma 2.5 [2] $\kappa = \frac{1}{3}$. Then by this lemma for any $\gamma > 1$ we have

$$\frac{4}{3r_0} \left(⨍_{B_{3r_0/4}(y_m)} u_m^{\gamma}(x) dx \right)^{1/\gamma} \leqslant c_m(1/3)$$

where $|x_m - y_m| = r_0/4$ and y_m is in the complement of Ω_m. It is clear that

$$\int_{B_{3r_0/4}(y_m)} u_m^{\gamma}(x) dx \geqslant \int_{B_{r_0/2}(x_m)} u_m^{\gamma}(x) dx = r_0^{\gamma} |B_{r_0/2}| \int_{B_{1/2}(0)} v_m^{\gamma}(x) dx$$

where $v_m(x) = \frac{u(x_m+r_0x)}{r_0}$. Observe that $v_m(0)=0$ and $|\nabla v_m(0)|=1$. Hence multiplying both sides of the last inequality by $|B_{3r_0/4}|$ and powering by $1/\gamma$ we get that

$$\frac{4}{3r_0}\left(\fint_{B_{3r_0/4}(y_m)} u_m^\gamma(x)dx\right)^{1/\gamma} \geqslant \left(\frac{2}{3}\right)^{n/\gamma}\left(\int_{B_{1/2}(0)} v_m^\gamma(x)dx\right)^{1/\gamma}$$

n is the Euclidean dimension. By Theorem 4.1 [2], v_m are uniformly bounded in $C^{0,1}(B_1)$. Therefore, the Ascoli-Arzela lemma implies that a subsequence v_m converges to some function v uniformly in $B_{\frac{1}{2}}$. And we conclude that $c_m(1/3)$ is bounded from below say by $c_\gamma > 0$. Next fix a large number R such that $u_m \leqslant c_\gamma$ in $\mathbb{R}^n \setminus B_R$. If $\bigcup \Omega_m$ is not bounded then for some k there is a Ω_k such that $x \in \Omega_k \setminus B_R$ and $u_k(x) > 0$. Considering balls $B_r(x) \subset \mathbb{R}^n \setminus B_R$ and using Remark 2.6 stated in [2], we get

$$\frac{1}{r}\left(\fint_{B_r(x)} u_k^\gamma(x)dx\right)^{1/\gamma} \geqslant c_\gamma > 0.$$

Since the functions u_m are uniformly bounded by c_γ, then if r is large, $u_k(x)$ is 0 in $B_{r/3}(x)$. Thus $\bigcup \Omega_m$ is bounded.

Corollary 3.4. *The functions $\{u_m\}$ are uniformly bounded in $W^{1,2}(U_t)$.*

Proof. Since $H^{n-1}(\partial\Omega_m) > 0$ and $\bigcup \Omega_m$ is bounded then applying Poincaré's inequality and Lemma 3.2 b) we arrive at desired result.

Remark 3.5. In view of the comparison principle for Δ_s [10] the Lavrentiev principle still holds and therefore the conclusions of Lemma 3.1 can be carried over for Δ_s. As to Lemma 3.2 we can observe that $W^{1,s}_{\text{loc}}$ estimate can be obtained by taking in identity (1.5) $\eta = \zeta^s u$ with standard cut off function ζ. The proof of parts a) and c) go without changes. Note that in the proof of Lemma 3.3, stated for the case of second type operators, one should use Lemma 4.2 of [3] instead of Lemma 2.5 of [2] and the equivalent statement of Theorem 4.1 of [2] in this case is Theorem 7.1 of [3].

4. Proof of Theorem 2.1

Proof of Theorem 2.1. By Corollary 3.4 it follows that $\{u_m\}$ is uniformly bounded in $W^{1,2}$ outside of any neighborhood of $\{x^1,\dots,x^k\}$. Hence using the Rellich-Kondrachov imbedding theorem (see [5], Theorem 7.10), we get a subsequence (still denoted by u_m), which outside of any neighborhood of $\{x^1,\dots,x^k\}$ converges strongly in L^q, $q < 2n/(n-2)$ to a limit function $u \in W_0^{1,2}$. Moreover, $\nabla u_m \to \nabla u$ weakly. Now there remains only to show that for appropriate choice of the numbers a_j, u is the desired solution, that is

$$\int_\Omega f_p(\nabla u)\nabla\eta = c_1\eta(x^1) + \cdots + c_k\eta(x^k), \quad \forall \eta \in C_0^\infty(\Omega). \tag{4.1}$$

Since $u \geqslant 0$ and $\mathscr{L}u = 0$ in $\Omega \setminus \{x^1,\dots,x^k\}$ then by Remark 2.6 and the same argument as in Subsection 2.1 we conclude that

$$\nabla u - \nabla(a_j\varphi(x-x^j)) = o(\nabla|x-x^j|^{2-n})$$

and the assertion (4.1) follows as in (2.4) with $c_j = a_j\frac{C_0}{C_n}$.

Finally let us show that $H^{n-1}(\partial_{\text{red}}\Omega\setminus\partial\Omega)=0$. In view of $C^{1,\alpha}$ estimates $|\nabla u_m|\leqslant C_{r_0}$ in $\bigcup_{j=1}^k \partial B(x^j,r_0)$ for r_0 small. Also $|\nabla u_m|\leqslant 1$ on $\partial\Omega_m$. Then since $|\nabla u_m|^2$ is $\mathscr{L}$-subsolution (respectively $|\nabla u_m|^s$ for Δ_s), applying the weak maximum principle we get

$$|\nabla u_m|\leqslant \max(C_{r_0},1).$$

Therefore the assertion follows as in proof of (3.11) and Remark 3.7 of [2].

5. Appendix

The following theorem is due to Kichenassamy [6].

Theorem 5.1. *For any positive constants $\gamma_1,\dots,\gamma_k$ there is a unique solution of*

$$\Delta_s v=-\sum_{j=1}^k \gamma_j\delta_{x^j},\quad s>1.$$

The method of proving the theorem can be adapted to show the existence of $v\in C^{1,\alpha}(\mathbb{R}^n\setminus\{x^1,\dots,x^k\})$ solving

$$\mathscr{L}v:=\operatorname{div}(f_p(\nabla v))=-\sum_{j=1}^k \gamma_j\delta_{x^j}$$

It can be done as follows. Defining v_ε to be the solution to the Dirichlet problem

$$\begin{cases}\mathscr{L}v_\varepsilon=\chi_\varepsilon & \text{in } B(0,1/\varepsilon),\\ v_\varepsilon\in W_0^{1,2}(B(0,1/\varepsilon)),\end{cases}$$

where $\chi_\varepsilon=\sum_{j=1}^k \gamma_j(\omega_n\varepsilon^n)^{-1}\chi_{B(x^j,\varepsilon)}$. The family of functions v_ε is bounded in $W^{1,2}_{\text{loc}}(\mathbb{R}^n)$. To see this we can without loss of generality establish only $W^{1,2}$ estimate in some ball B_r such that $\{x^1,\dots,x^k\}\subset\subset\mathbb{R}^n\setminus B_{2r}$. Moreover it suffices to establish only $L^2(B_{2r})$ bound, since considering $\eta=v_\varepsilon\zeta^2$ in (1.6), where ζ is standart cutoff function in B_{2r}, one can estimate the $\|\nabla v_\varepsilon\|_{L^2(B_r)}$ by means of $\|v_\varepsilon\|_{L^2(B_{2r})}$.

Now for φ as in 2.1 let $\varphi_\varepsilon(x)=\varphi(x)$ for $|x|>\varepsilon$ and $\varphi_\varepsilon(x)=c_\varepsilon|x|^2+b_\varepsilon$ for $|x|\leqslant\varepsilon$, where the constants c_ε and b_ε are chosen such that φ_ε is a C^1 function. Since the rearrangement of χ_ε is the multiple of $1/\varepsilon^n$ and by direct computations one can see that

$$v_\varepsilon^\star\leqslant C\varphi_\varepsilon\leqslant C\varphi,\quad \text{in } B(0,1/\varepsilon).$$

(cf. [10] Theorem 1), then the inequality of Hardy-Littlewood for rearranged functions ([10] p. 168) leads to $L^q(B_{2r})$ estimate, for some $q>1$ and ε small. Moreover, using Moser's iteration (e.g. [7], p. 256) we can get $L^2(B_{2r})$ bound for v_ε's. Finally from a priory $C^{1,\alpha}$ estimates of Section 1.2 the results follows.

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