Workshop 4

November 5, 2014

Homework Assignment

1. Let $L : C[0,1] \to \mathbf{C}$ denote the linear functional defined by

$$L(f) \;=\; f(0).$$

a) Show $L \in (C[0,1], \|\cdot\|_\infty)^*$.

b) Show $L \notin (C[0,1], \|\cdot\|_2)^*$.

Additional problems for self-study

2. Let $\{u_n\}_{n=1}^\infty$ be an orthonormal family in a Hilbert space H. Show that for $x \in H$,

$$\#\{n : |\langle x, u_n\rangle| > \frac{1}{m}\} \;\leq\; m^2\|x\|^2.$$

3. Let H denote the Hilbert space of almost periodic functions. For $s \in \mathbf{R}$, set $u_s(t) = \mathrm{e}^{ist}$. Show $\{u_s\}_{s\in\mathbf{R}}$ is a maximal orthnornormal family and hence deduce that H is not separable.

Solution: Use Bessel's inequality.

4. Let H be a Hilbert space, M a closed subspace of H and let $P : H \to M$ be the orthogonal projection onto M.

a) Show that P is self-adjoint.

b) Show that P is compact if and only if M is finite dimensional.

Solution:

We do the following calculation. Here Q denote the orthogonal projection onto $M^\perp$.

$$\langle Px, y\rangle = \langle Px, Py\rangle + \langle Px, Qy\rangle = \langle Px, Py\rangle = \langle Px, Py\rangle + \langle Qx, Py\rangle = \langle x, Py\rangle.$$

Hence P is self adjoint.

For the part b) we see that

$$\overline{P(\{x \in \ H, \|x\| < 1\})} = \overline{\{x \in \ M, \|x\| < 1\}} = \{x \in M; \|x\| \leq 1\}.$$

The closed unit ball on a subspace M is compact if and only if dim $M < \infty$ by the Theorem we had in class.

5. Let $H = L^2[0,1]$ and consider $T \in \mathcal{L}(H)$ defined on continuous functions by

$$Tf(s) = \int_0^1 e^{s-t} f(t)dt.$$

a) Show that T is Hilbert-Schmidt.

b) Show that

$$\|T\| = \|T\|_{HS} = \sinh 1.$$

Recall that $\sinh 1 = (e - e^{-1})/2$.

Solution: This is an integral operator of the form $Tf(s) = \int_0^1 K(s,t)f(t)dt$ where $K(s,t) = e^{s-t}$. It has been shown in the class that such operator is HS if

$$\int_0^1 \int_0^1 |K(s,t)|^2 dsdt = \int_0^1 \int_0^1 e^{2s-2t} dsdt = (\sinh 1)^2 < \infty.$$

We have already calculated the HS norm since by the work done in the class

$$\|T\|_{HS} = \left(\int_0^1 \int_0^1 |K(s,t)|^2 dsdt \right)^{1/2} = \sinh 1.$$

We also know that $\|T\| \leq \|T\|_{HS}$ so it suffices to show that $\|T\| \geq \sinh 1$. But

$$\|T\|^2 = \sup_{x \neq 0} \frac{\|Tx\|^2}{\|x\|^2} \geq \frac{\|Te^{-t}\|^2}{\|e^{-t}\|^2} = \frac{\int_0^1 e^{2s} \left(\int_0^1 e^{-2t} dt \right)^2 ds}{\int_0^1 e^{-2t} dt} = (\sinh 1)^2.$$

6. Let $T \in \mathcal{L}(L^2[0,1])$ be an integral operator $Tf(x) = \int_0^1 K(x,y)f(y)dt$.
When $K(x,y) = \begin{cases} 1, & \text{if } 0 \leq y \leq x \leq 1; \\ 0, & \text{if } 0 \leq x < y \leq 1. \end{cases}$
show that T has no eigenvalues.

Solution: From the definition of K it follows that $Tf(x) = \int_0^x f(y)dy = F(x)$, where F is the primitive function of f such that $F(0) = 0$. If follows that $Tf(x) = \lambda f(x)$ implies that $F = \lambda F'$. This ODE has a general solution

$$F(x) = Ce^{x/\lambda}, \quad \text{hence} \quad 0 = F(0) = C \quad \text{or} \quad F = 0.$$

So $f = F' = 0$. Hence there is no nonzero eigenvalue.

7. Let H be the Hilbert space of 2π-periodic functions equipped with the inner product

$$\langle f, g\rangle = \int_0^{2\pi} f(x)\overline{g(x)}dx.$$

For a real number s consider the shift operator T_s defined by

$$T_s f(x) = f(s + x).$$

a) Is T_s a bounded operator on $\mathcal{L}(H)$? If so calculate its norm.
b) Is T_s a Hilbert-Schmidt operator? If so, calculate its Hilbert-Schmidt norm.
c) Show that the adjoint operator T_s^* is just T_{-s}.
d) Use the result of part (c) to show that T_π is a self-adjoint operator.
e) Is T_π a compact operator?

Solution:
a) Since

$$\|T_s f\|^2 = \int_0^{2\pi} |f(s + x)|^2 dx = \int_{-s}^{2\pi-s} |f(x)|^2 dx = \int_0^{2\pi} |f(x)|^2 dx = \|f\|^2,$$

by 2π-periodicity. Hence $\|T\| = 1$. In fact it follows that T is an isometry.

b) T_s is not a Hilbert-Schmidt operator. Using part (a) we see that for any ON basis (e_n) of $L^2(0, 2\pi)$ we have that

$$\sum_{n=1}^{\infty} \|T_s e_n\|^2 = \sum_{n=1}^{\infty} \|e_n\|^2 = \sum_{n=1}^{\infty} 1 = \infty.$$

c) We write

$$\langle T_s f, g\rangle = \int_0^{2\pi} f(s + x)\overline{g(x)}dx = \int_{-s}^{2\pi-s} f(x)\overline{g(x - s)}dx = \int_0^{2\pi} f(x)\overline{g(x - s)}dx,$$

the last equality holds due to 2π-periodicity. Hence

$$\int_0^{2\pi} f(x)\overline{g(x - s)}dx = \langle f, T_{-s} g\rangle.$$

d) Since $T_\pi^* = T_{-\pi}$ and the functions are 2π-periodic we see that $T_{-\pi} = T_\pi$ hence the operator is self-adjoint.

e) No. It's not hard to see that

$$\overline{T_s(\{x \in H;\ \|x\| < 1\})} = \overline{\{x \in H;\ \|x\| < 1\}} = \{x \in H;\ \|x\| \leq 1\}.$$

So, if T_s is compact then the set $\{x \in H;\ \|x\| \leq 1\}$ must be compact. However, a closed unit ball is compact only if the space is finite dimensional and our space is NOT.