Local Solutions of Optimal Power Flow

Waqquas A. Bukhsh, Andreas Grothey, Ken McKinnon, and Paul Trodden
School of Mathematics
University of Edinburgh
James Clerk Maxwell Building
Edinburgh EH9 3JZ
Scotland
E-mail: w.bukhsh@sms.ed.ac.uk,{a.grothey, k.mckinnon, paul.trodden}@ed.ac.uk

***Abstract*—The existence of locally optimal solutions to optimal power flow (OPF) problems has been a question of interest for decades. This paper presents examples of local optima on very simple power system networks. Usually local optima occur when the network is stressed, and the resulting voltage profiles are infeasible in practice; however, we present examples where the voltage is within practical limits. Standard local optimization techniques are shown to converge to these local optima if started close enough to them. We also give examples of local solutions in loop networks. Finally we use a two-bus example with local optima to illustrate the behaviour of a recent dual semi-definite programming approach that aims to find the global solution.**

I. Introduction

Optimal power flow is a well studied optimization problem in power systems. This problem was first introduced by Carpentier [1] in 1962. The objective of OPF is to find a steady state operating point that minimizes the cost of electric power generation while satisfying operating constraints and meeting demand. The problem can be formulated as a nonlinear programming (NLP) problem, in which the constraints and possibly the objective are nonlinear functions. Nonlinearity in the usual real and reactive power formulation of OPF is primarily due to Kirchhoff's voltage law (KVL), which describes the flow of power in transmission lines, and is a nonlinear function of bus voltages and phase angles. The presence of these nonlinear equality constraints results in nonconvexity of the feasible region.

In 1968, Dommel and Tinney [2] proposed a simplified gradient descend method to solve OPF for the first time. Since then numerous NLP methods have been proposed. A good literature survey of classical optimization techniques as applied to OPF over the last 30 years is given in [3], [4]. None of these methods is guaranteed to find the global minimum if a local one exists.

The issue of the possible existence of local optima to the OPF problem is an important one, but one that is not well covered in the literature. We have found mention of local optima but have not been able to find well documented examples. A recent literature survey [5] of OPF covers classical local nonlinear techniques as well as evolutionary algorithms. The evolutionary algorithms are global optimization techniques, and although they cannot guarantee to find the global minimum, they are less likely to get trapped in local solutions if these exist. In [6], authors discuss the role of metaheuristic techniques to solve OPF and give the convergence to local solutions as a major drawback of conventional OPF techniques. Global techniques are usually much slower than local ones so should only be used where local optima do exist. However, none of these references gives examples of local optima of OPF.

An interesting novel approach to OPF is taken by [7], [8]. Here, the authors reformulate the problem as a semi-definite program (SDP), and show that under appropriate assumptions on the network topology there is no duality gap between the original problem and the SDP dual. Hence, it should be possible to recover a global optimal solution to the OPF from the SDP dual. It is unclear, however, if their assumptions are satisfied in practice and if the recovery strategy always works.

In this paper, we give examples of local solutions of OPF on very simple power system networks. These examples show that local optima of OPF can exist and contradict the "common belief" that OPF solution in the feasible region is unique [9]. We also point out that the SDP approach of [7], [8] can sometimes fail to recover a feasible OPF solution.

The layout of the paper is as follows. In Section II the load flow problem is described. In Section III the OPF problem is introduced and its relation to the load flow problem described. Section IV gives results for a range of different examples. Section V illustrate the behaviour of the SDP global optimization method on a 2 bus example, and conclusions are given in Section VI.

II. Load Flow Problem

The flows in a network must satisfy Kirchhoff's current laws, (1) and (2), and Kirchhoff's voltage laws, (3) and (4):

$$\sum_{g\in\mathcal{G}_b} p_g^{\mathrm{G}} = \sum_{d\in\mathcal{D}_b} P_d^{\mathrm{D}} + \sum_{b'\in\mathcal{B}_b} p_{bb'}^{\mathrm{L}} \quad (1)$$

$$\sum_{g\in\mathcal{G}_b} q_g^{\mathrm{G}} = \sum_{d\in\mathcal{D}_b} Q_d^{\mathrm{D}} + \sum_{b'\in\mathcal{B}_b} q_{bb'}^{\mathrm{L}} \quad (2)$$

$$p_{bb'}^{\mathrm{L}} = G_{bb'}v_b(v_b - v_{b'}\cos\delta_{bb'}) - B_{bb'}v_bv_{b'}\sin\delta_{bb'} \quad (3)$$

$$q_{bb'}^{\mathrm{L}} = B_{bb'}v_b(v_{b'}\cos\delta_{bb'} - v_b) - G_{bb'}v_bv_{b'}\sin\delta_{bb'} \quad (4)$$

where p_g^{G} and q_g^{G} are the real and reactive power generated by generator g, P_d^{D} and Q_d^{D} are the real and reactive power

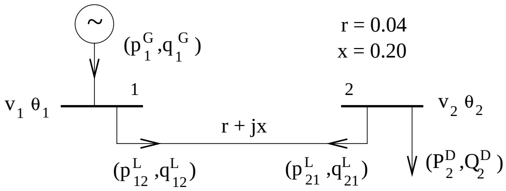


Fig. 1. Two Bus System

consumed by load d, $p^{\mathrm{L}}_{bb'}$ and $q^{\mathrm{L}}_{bb'}$ are the real and reactive power flowing into line bb' from bus b, v_b and θ_b are the voltage and phase angles at bus b and $\delta_{bb'} = \theta_b - \theta_{b'}$. $\mathcal{G}_b$ is the set of generators and $\mathcal{D}_b$ the set of demands at bus b, and $\mathcal{B}_b$ is the set of buses connected to bus b by a line. $B_{bb'}$ and $G_{bb'}$ are, respectively, the susceptance and conductance of a line bb'; we assume $B_{bb'} = B_{b'b}$ and $G_{bb'} = G_{b'b}$. These are defined in terms of the line resistance r and reactance x by

$$G = \frac{r}{r^2 + x^2}, \quad B = -\frac{x}{r^2 + x^2}$$

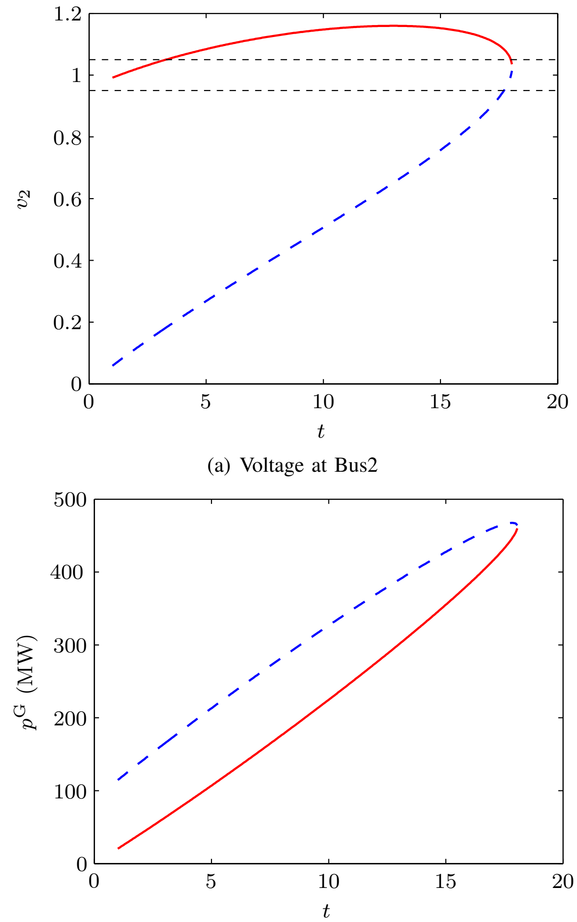
In the load flow problem [10] all loads are fixed and all generator outputs are fixed except at one bus, which is called the slack bus. At the slack bus the generator outputs can vary but the voltage and phase angle are fixed.

It is well known that the load flow problem can have multiple solutions [11]–[13]. Generally, the study of multiple solutions of load flow problem is associated with the voltage stability problem [14]. The load flow problem and OPF problem are related in the sense that every feasible point of OPF must satisfy the load flow equations (1) to (4).

Let us consider a two bus network as shown in Figure 1. Bus 2 is the load bus, and Bus 1 is the generator bus and slack bus, and for it we set $v_1 = 0.96$ and $\theta_1 = 0$. If we know the load at Bus 2 then it is possible to find the remaining variables $(p_g, q_g, v_2, \theta_2)$. There are at most four possible load flow solutions, but two of these have a negative value of v_2 so are not feasible. The other two have positive values of v_2. The solution space of this system is discussed in detail in [15]. Figure 2 shows the alternate solutions when the load at Bus 2 is $(P^{\mathrm{D}}_2, Q^{\mathrm{D}}_2) = t(20, -20.3121)$ for $0 \leq t \leq 18$. Note the load here is capacitative and so reactive power is *injected* from the load into Bus 2. (Underground cables are sources of reactive power – in a larger network a capacitative load could be induced by an underground cable connected at this bus).

Figure 2 shows the two alternative solutions. The red (solid) branch corresponds to lower real power generation (the better case) and to higher voltage at Bus 2, and the blue (dotted) branch corresponds to higher real power generation and lower voltage. Figure 2 shows that for fixed value of load (*i.e.* fixed t) it is possible to get a nonconvex and disconnected feasible region for load flow.

Moreover, from Figure 2, we can see that as the load increases the two solutions get closer and eventually coalesce at a point. For higher loads than this there is no solution. In this example the solutions come together at a voltage that is feasible, but if the load is changed from capacitative to reactive the voltage at the coalescing point drops to an infeasibly low value.



(a) Voltage at Bus2

(b) Real Power Generation

Fig. 2. Alternate load flow solutions as the demand parameter, t, varies.

Typical real system voltage limits of $0.95 \leq v_2 \leq 1.05$ are shown on 2(a). From this diagram it is clear that it is only possible for the two solutions both to be within this voltage range if the load on the line is close to its limit. This is not a point at which it is safe to operate the network, and occurs because the network is stressed.

III. Optimal Power Flow (OPF)

The OPF problem is the problem of minimizing the cost of generation while satisfying Kirchhoff's laws, (1) to (4), and bounds on voltages at buses, phase angle differences and heating limits on lines and outputs of generators. However in this paper the only bounds are those on the voltages at buses. We assume the generating cost for a generator is a linear function of the real power output (and is independent of reactive power output).

The OPF problem is therefore

$$\begin{aligned} \min \quad & \sum_{g \in \mathcal{G}} c_g p^{\mathrm{G}}_g, \\ \text{Subject to} \quad & \text{Equations (1), (2), (3), (4),} \\ & v^{\mathrm{LB}}_b \leq v_b \leq v^{\mathrm{UB}}_b, \end{aligned}$$

where $\mathcal{G}$ is the set of all generators, c_g is the unit cost of generator g, and v_b^{LB} and v_b^{UB} are the lower and upper bounds on voltage at bus b. The variables of the problem are the p_g^{G}, q_g^{G}, $p_{bb'}^{\mathrm{L}}$, $q_{bb'}^{\mathrm{L}}$, v_b, θ_b and $\delta_{bb'}$. There is no longer a slack bus, but to remove redundancy we can arbitrarily fix the phase angle at one bus, and in this paper set $\theta_1 = 0$.

Since the objective function is linear, any local minima that occur can only be due to non-convexity in the feasible region.

Solutions of the load flow problem are feasible points of OPF provided they satisfy the bounds on voltages, line heating limits and generator outputs imposed in OPF.

A. Voltage Constraints

When solving an OPF problem there are normally voltage limits on each bus. These are often 5% or 10% above and below the nominal voltage (which we take to be normalised to 1) [17]. In some of the examples below it will not be possible to meet these voltage limits and in these cases we impose a 5% limit on the generator buses but no limits on the remaining buses. (Usually in practice automatic voltage controllers can maintain the voltage at generator buses within the desired range but there is no direct control of the voltage at the other buses.)

B. Algorithms to solve OPF

To solve the OPF problem we first use MATPOWER [18] with MATLAB's nonlinear solver `fmincon`. To verify the results we also solve the problems using an AMPL model and the nonlinear interior point solver IPOPT [24]. For each solution we have verified that the first order optimality conditions have been satisfied and in each case the algorithms was seen to converge to the solution from starting points distinct from the solution.

IV. Results

A. Analysis of 2 bus case

We start by investigating the local optima in the two bus network as shown in Figure 1. We consider the case when $t = 17.625$, close to the nose of the curve in Figure 1, and so the the real and reactive load at Bus 2 is now fixed at 352.5 MW and -358 MVAr respectively.

The optimization variables in this case are $(p_1^{\mathrm{G}}, q_1^{\mathrm{G}}, v_1, v_2, \theta_2)$. The following two initial points were used for the optimization.

$$I_1 = (400, 100, 1, 1, -65)$$

$$I_2 = (400, 100, 1, 1, -55)$$

The only difference in initial points is the phase angles at Bus 2. The results are given in Table I. The OPF algorithm has converged to the local solution from initial point I_1 and to the global solution from initial point I_2.

Note that almost all OPF solvers recommend using $v = 1$ p.u. for voltages at the buses in the initial iterate. This works well in normal situation but the above example shows that in stressed and loaded system this does not necessarily lead to the global solution.

TABLE I
Two OPF Solutions for the 2 bus problem

	Bus	v (p.u.)	θ (deg)	p^{G} (MW)	q^{G} (MVAr)
S_1	1	0.950	0.00	456.55	162.25
	2	0.985	−65.01		
S_2	1	0.951	0.00	444.08	99.91
	2	1.050	−58.12		

TABLE II
Load Data for the 5 bus problem

Bus	P^{D} (MW)	Q^{D} (MVAr)
2	100	35
3	100	25
4	110	35

In Section II, we fixed the slack voltage to $v_1 = 0.96$ to obtain Figure 2. In OPF problems the voltage at the slack bus is one of the degrees of freedom, and in this one generator example it is the only degree of freedom. Figure 3 shows the effect of changing v_1, the voltage at Bus 1. The loading conditions here are for $t = 17.625$, the same as used above in Table I. On the red (continuous) branch, the optimization (which is to minimize the real power generated) will try to push v_1 as low as possible. With v_1 at its lower bound of 0.95, $v_2 = 0.985$, which lies within its bounds. This gives the local solution, S_1 in Table I. On the blue (dotted) branch in Figure 3, the optimization will try to push v_1 as high as possible. The limit is $v_1 = 0.951$ because for higher than that v_2 exceeds its upper bound of 1.05. This gives the global solution, S_2 in Table I.

B. Local optima on 5, 14 and 30 bus cases

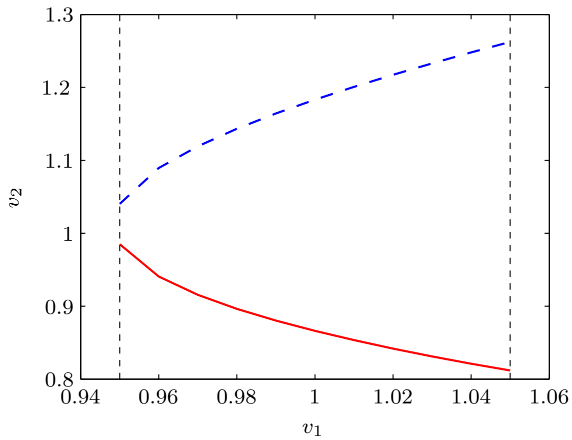
To get a local solution on slightly larger network we have investigated the five bus network shown in Figure 4. There is one generator and three load buses. A synchronous condenser is attached to bus 5 – a synchronous condenser is a synchronous machine which runs without any mechanical load and can vary its reactive power output. In OPF problems, synchronous condensers can be modelled as conventional generators but with an upper bound of zero on their real power output. The load data is given in Table II.

The optimization variables for this example are given by $(p_1^{\mathrm{G}}, q_1^{\mathrm{G}}, v_1, v_2, \theta_2, v_3, \theta_3, v_4, \theta_4, p_5^{\mathrm{G}}, q_5^{\mathrm{G}}, v_5, \theta_5)$. We use the following two initial conditions:

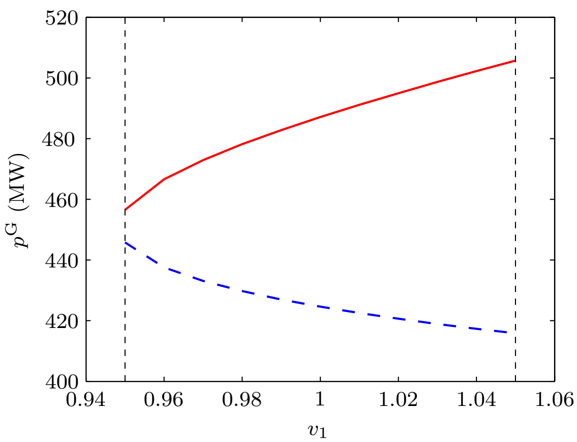
$$I_3 = (300, 100, 1.05, 1, -10, 1, -10, 1, -10, 0, 150, 1, -10)$$

$$I_4 = (506, 445, 1, 0.7, -10, 0.7, -10, 0.8, -50, 0, 394, 1, -50)$$

The results are given in Table III. The global solution S_3 is reached from initial condition I_3, whereas a local solution S_4 is reached from I_4. Again, we find that the local solution has lower bus voltages (well outside the usual operating limits of



(a) Voltage



(b) Real Power

Fig. 3. Effect of Voltage

TABLE III
TWO OPF SOLUTIONS FOR THE 5 BUS PROBLEM

	Bus	v (p.u.)	θ (deg)	p^G (MW)	q^G (MVAr)
	1	1.050	0.00	348.88	121.17
	2	0.900	−20.39		
S_3	3	0.943	−18.84		
	4	0.860	−27.17		
	5	1.028	−25.67	0.00	129.75
	1	0.950	0.00	506.04	445.65
	2	0.526	−50.03		
S_4	3	0.628	−49.11		
	4	0.496	−76.23		
	5	0.950	−76.33	0.00	394.13

$\pm$5–10%) – although even the voltages for the global solutions are outside this range, but not as severely.

Local solutions corresponding to low voltage profiles also exist in (a slight modification of) the IEEE 14 and 30 bus test systems [19]. The voltage and phase angle profiles, together with generator outputs of these solutions, are given in Tables IV–VII. The IEEE 14-bus network has been modified as follows: voltage bounds at load and generator buses are set to $[0, 1.05]$ and $[0.95, 1.05]$ respectively. Only linear cost terms are used. Real and reactive power limits for generators as

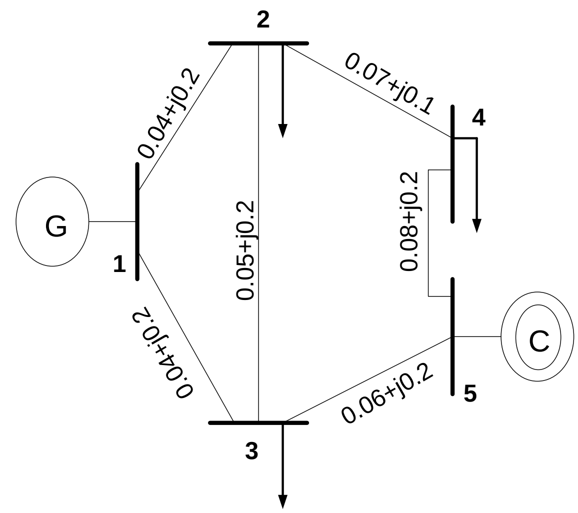


Fig. 4. Five Bus System

TABLE IV
GLOBAL SOLUTION FOR THE (MODIFIED) IEEE 14-BUS SYSTEM

Bus	v (p.u.)	θ (deg)	p^G (MW)	q^G (MVAr)
1	1.050	0.00	43.74	2.82
2	1.050	0.01	224.05	18.95
3	1.016	−8.29	0.00	31.66
4	1.012	−6.23		
5	1.014	−5.06		
6	1.050	−10.52	0.00	7.66
7	1.039	−9.42		
8	1.050	−9.42	0.00	6.58
9	1.036	−11.10		
10	1.031	−11.29		
11	1.037	−11.04		
12	1.035	−11.40		
13	1.030	−11.47		
14	1.015	−12.30		

well as reactive power limits for the synchronous condensers have been removed. The IEEE 30-bus system is unmodified. Using the NLP solvers in MATPOWER and IPOPT the global solution is obtained in both problems when using an initial iterate with $v = 1.0$ and $\delta = 0$ throughout. However when changing the initial iterate to $v_9 < 0.7$, at bus 9, for the 14-bus system or $v_{30} < 0.51$, at bus 30, for the 30-bus system the local solutions are obtained.

It can be noticed that some voltages for the local solution are extremely low and in normal operating conditions would not be an acceptable operating point. The situation changes however when the system is put under stress. Tables VIII and IX give a global and local solution for the same 14-bus system as before but with all real loads multiplied by a factor of 3.5. Here the lowest voltage in the local solution has increased to 0.501, while voltages in the global solution have decreased. This situation is analogous to that of the two bus system, where the two solutions get closer to each other as load is increased (Figure 2). If loads are increased even further in the 14-bus system, the problem becomes infeasible due to line flow limits. It seems, however, conceivable that there

TABLE V
LOCAL (LOW VOLTAGE) SOLUTION FOR THE (MODIFIED) IEEE 14-BUS SYSTEM

Bus	v (p.u.)	θ (deg)	p^{G} (MW)	q^{G} (MVAr)
1	1.007	0.00	101.00	72.91
2	1.002	−0.53	316.71	198.47
3	0.950	−13.02	0.00	123.28
4	0.774	−8.95		
5	0.809	−10.41		
6	0.950	−43.98	0.00	367.98
7	0.508	−14.19		
8	1.050	−14.19	0.00	323.32
9	0.044	−74.71		
10	0.147	−56.88		
11	0.529	−45.41		
12	0.861	−46.25		
13	0.781	−45.64		
14	0.292	−57.44		

TABLE VI
GLOBAL SOLUTION FOR THE IEEE 30-BUS SYSTEM

Bus	v (p.u.)	θ (deg)	p^{G} (MW)	q^{G} (MVAr)
1	0.995	0.00	0.00	0.21
2	0.995	−0.19	0.00	33.24
3	0.989	0.66		
4	0.987	0.85		
5	0.984	−0.06		
6	0.984	1.16		
7	0.974	0.13		
8	0.971	0.69		
9	0.994	6.72		
10	1.003	9.57		
11	0.994	6.72		
12	0.991	2.79		
13	1.032	2.79	0.00	30.37
14	0.980	2.82		
15	0.990	3.66		
16	0.988	5.39		
17	0.993	8.13		
18	0.980	4.97		
19	0.979	5.97		
20	0.984	6.82		
21	1.035	12.51		
22	1.050	13.42	201.36	34.96
23	1.003	5.20	0.00	10.79
24	1.005	8.43		
25	0.993	3.95		
26	0.976	3.50		
27	1.000	1.42	0.00	19.09
28	0.984	1.05		
29	0.980	0.12		
30	0.968	−0.80		

TABLE VII
LOCAL (LOW VOLTAGE) SOLUTION FOR THE IEEE 30-BUS SYSTEM

Bus	v (p.u.)	θ (deg)	p^{G} (MW)	q^{G} (MVAr)
1	1.050	0.00	0.00	13.96
2	1.050	−0.21	0.00	81.15
3	1.020	1.12		
4	1.014	1.41		
5	1.020	0.07		
6	1.001	1.58		
7	1.000	0.47		
8	0.986	0.86		
9	0.985	11.17		
10	0.988	16.25		
11	0.985	11.17		
12	0.993	5.29		
13	1.043	5.29	0.00	37.29
14	0.980	5.48		
15	0.986	6.50		
16	0.981	9.65		
17	0.980	14.09		
18	0.973	9.15		
19	0.969	10.93		
20	0.972	12.22		
21	1.031	21.18		
22	1.050	22.62	335.18	27.71
23	1.002	7.55	0.00	24.91
24	0.966	11.32		
25	0.921	−5.41		
26	0.902	−5.93		
27	0.950	−15.59	0.00	235.57
28	0.981	−0.13		
29	0.498	−18.71		
30	0.048	−65.55		

TABLE VIII
GLOBAL SOLUTION FOR THE (MODIFIED) IEEE 14-BUS SYSTEM WITH SCALED DEMANDS

Bus	v (p.u.)	θ (deg)	p^{G} (MW)	q^{G} (MVAr)
1	1.050	0.0	192.49	47.97
2	1.050	−0.65	861.78	168.11
3	1.029	−36.65	0.00	283.74
4	0.894	−27.13		
5	0.900	−22.04		
6	1.050	−45.73	0.00	121.86
7	0.959	−40.72		
8	1.050	−40.72	0.00	54.08
9	0.947	−47.58		
10	0.948	−48.69		
11	0.990	−47.76		
12	1.004	−49.32		
13	0.986	−49.75		
14	0.910	−53.49		

are complex networks for which there are global and local solutions with acceptable voltage profiles for high demands.

C. Local solutions in loop networks

Generally power systems networks are meshed, *i.e.,* there are more lines than nodes (substations). So it is natural to have loops within the network. Loops also appear because of AC interconnection around seas or deserts. It is possible to have many loops within a network. Moreover the network itself can have a loop like structure like the Icelandic transmission network [23].

For loop networks, it has been shown that there are several dynamically stable solutions to the load flow problem. This non-uniqueness allows the machines to settle to a new dynamically stable point after a transient. One particular dynamically stable solution studied by [21]–[22] corresponds to a 2π phase shift across the loop. The phase shift induces the circulating power p_c in the loop. This phenomenon has been repeatedly observed in power systems [22].

Here we shall show that the solution corresponding to the 2π phase shift is actually a local solution of the OPF problem. We shall not discuss transient stability issues; however, it

TABLE IX
LOCAL (LOW VOLTAGE) SOLUTION FOR THE (MODIFIED) IEEE 14-BUS SYSTEM WITH SCALED DEMAND

Bus	v (p.u.)	θ (deg)	p^{G} (MW)	q^{G} (MVAr)
1	1.050	0.0	135.30	77.42
2	1.050	−0.37	687.78	282.37
3	0.950	−33.53	0.00	244.37
4	0.798	−16.62		
5	0.838	−15.33		
6	0.950	−47.28	0.00	241.74
7	0.501	8.38		
8	0.950	68.71	0.00	378.47
9	0.518	−31.79		
10	0.563	−39.71		
11	0.741	−45.21		
12	0.867	−51.91		
13	0.817	−51.75		
14	0.539	−55.32		

naturally indicates that transiently constrained OPF may have local optima as well.

Consider a loop of n buses as shown in Figure 5. To simplify without loss of generality, we assume that n is even, the loads and generators are symmetrically arranged along the loop and all lines have same impedance of $z = 0.01 + j0.1$. Moreover all loads in the system have fixed, purely real demand of $P = 80$ MW. This symmetrical arrangement implies that the voltage magnitude at alternate ends is same, *i.e.,*

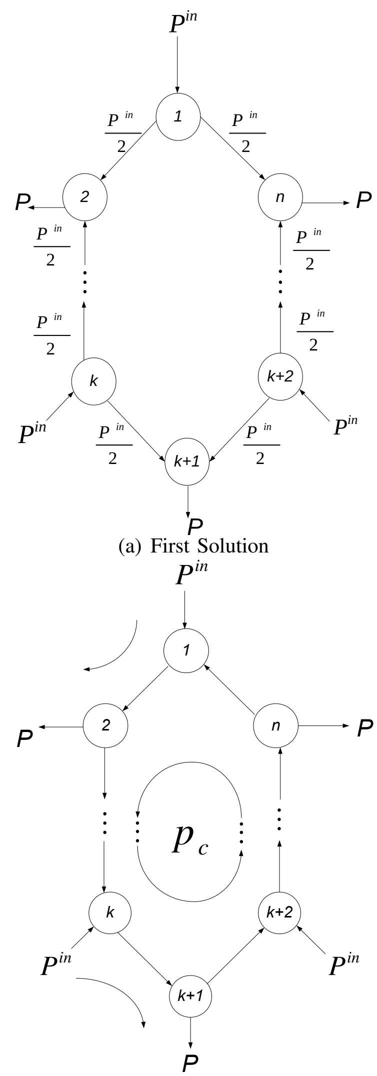
$$v_{2k+1} = v_1 \qquad \text{for} \qquad k = 1, \dots, \frac{n}{2} - 1 \tag{5}$$

$$v_{2k} = v_2 \qquad \text{for} \qquad k = 2, \dots, \frac{n}{2} \tag{6}$$

In case of 2π phase shift the voltage angle is given by

$$\theta_k = \begin{cases} 2\pi - \dfrac{2\pi}{n}k + \theta & k \in 2\mathbb{Z} \\ \\ 2\pi - \dfrac{2\pi}{n}(k-1) & k \in 2\mathbb{Z} + 1 \end{cases} \tag{7}$$

where θ is the voltage angle at the n^{th} bus. Figure 5 shows two OPF solutions. The solution given by 5(b) is local solution, where p_c denotes the circulating power induced by 2π phase difference across the loop. This induced circulating power will result in increased transmission losses as compared to the solution without phase shift and hence leads to a local OPF solution. We note from Figure 6 that the voltages improve as we increase the number of buses in the loop. Figure 7 shows the circulating power and angle θ respectively. We conclude that the circulating power decrease as the number of buses increases in a loop. In the 30 bus loop example the voltages at alternate buses are 0.95 and 0.92, which lie within the acceptable 10% range.



(a) First Solution

(b) Second Solution

Fig. 5. Two Solutions of 6 Bus Loop Network

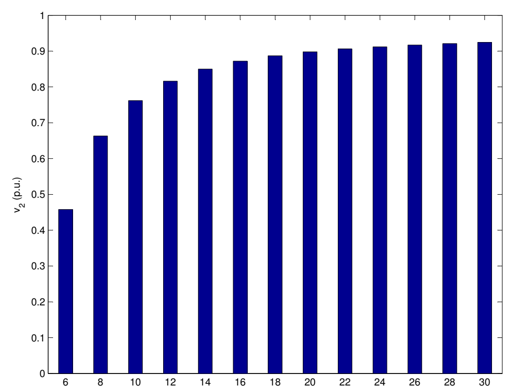
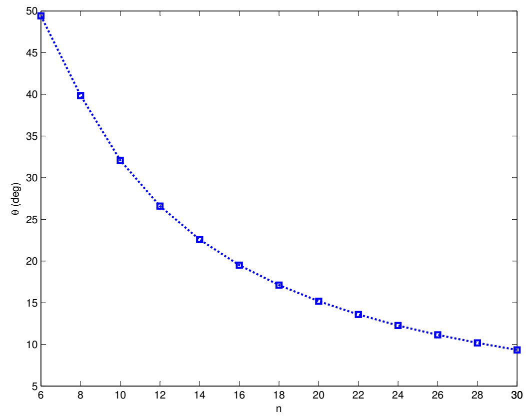


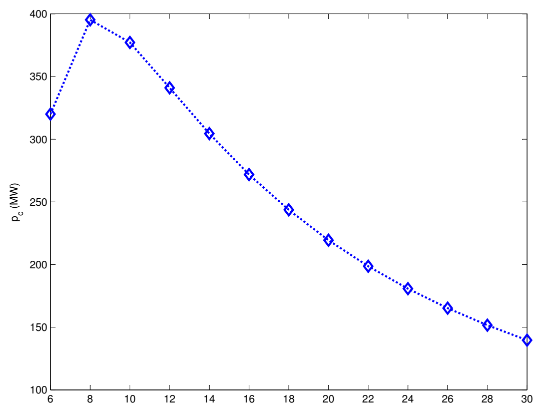
Fig. 6. Voltage as Loop Size Increases

V. RECOVERABILITY OF LOCAL SOLUTIONS BY SDP

The authors of [8] propose a reformulation of the OPF problem as a semi-definite program (SDP), and show that, if a certain sufficient condition is satisfied, there is no duality gap between the original problem and the convex SDP dual, and that the global optimal solution can be recovered from the solution of the SDP dual. The stated sufficient condition however is rather technical; to be more precise, it states that a certain matrix A^{opt} depending on the dual solution must have

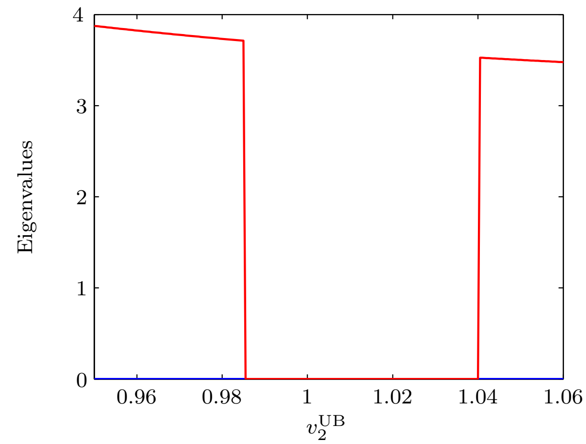


(a) Angle

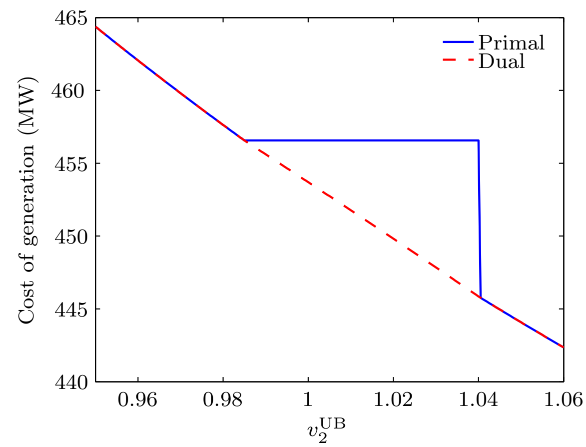


(b) Circulating Power

Fig. 7. Loop Size vs Angle and Circulating Power



(a) Eigenvalues



(b) Primal and Dual Objectives

Fig. 8. Performance of SDP Optimization Method on 2 Bus Problem

exactly two zero eigenvalues. It is unclear when this condition is satisfied in practice. We investigate this issue using the test systems and local solutions that we have found.

Consider again the two bus network of Figure 1 and the corresponding solutions S_1 and S_2 in Table I. When applying the SDP method to solve the problem with the standard voltage bounds of $[0.95, 1.05]$ we recover the global solution S_2 as claimed.

If, however, we restrict the feasible voltage domain to exclude solution S_2, the previously local solution S_1 should become the global solution. Indeed using bounds $0.60 \leq v_2 \leq 0.9851$ the SDP approach recovers solution S_1, confirming its status as the next best local solution. The transition between these two cases is not smooth however.

Let v_2^{UB} denote the upper bound of v_2. Figure 3 shows that as we move v_2^{UB} down below 1.04 we can exclude the entire branch corresponding to the global solution from the feasible region. Figure 8 plots the four eigenvalues of A^{opt} and the primal and dual solutions of the SDP method as a function of v_2^{UB}. For upper bounds in a region between 0.985 and 1.04 all four eigenvalues of A^{opt} become zeros, there is a non-zero duality gap and subsequently the SDP method fails to recover any primal feasible solution. This problematic region for v_2^{UB} coincides with the disconnected region for the v_2-values in Figure 3. This shows that the SDP approach [8] will fail to recover primal feasible solution if the voltage bounds lie within the disconnected region.

VI. Conclusion

In this paper, we have demonstrated the existence of local optima in simple power system networks. In most cases the local solution have some bus voltages lying outside the acceptable range. If an OPF problem were solved with the normal voltage limits as constraints then these local solutions would be infeasible and so would be excluded. We have however shown that the local solutions can lie within normal voltage limits, and in such cases a local OPF method could converge to a local solution. However these examples are hard to find, so this may not be a major problem in practice. It seems that it is when the network is stressed and close to singularity that local solutions are close enough to the global solution for the bus voltages to be within normal voltage limits.

We have also shown the existence of local solutions in loop networks with voltage profiles arbitrarily close to 1 p.u., given long enough loops.

The SDP approach of [8] can recover global solutions in some of our examples, however there are cases which seem non-negligible in which the approach fails to find any primal solution. Further investigations of this issue are needed.

More research needs to be done to see if the reported local solutions are transiently stable and if not then whether stability constraints can help to avoid these solutions on OPF.

REFERENCES

[1] J. Carpentier, "Contribution a l'Etude du Dispacting Economique," Bull. Soc. Francaise des Electriciens, vol. 3, pp. 431-447, Aug 1962.

[2] H. W. Dommel, W. F. Tinney, "Optimal Power Flow solutions," IEEE Transactions on Power Apparatus and Systems, vol. PAS-87, pp. 1866-1876, 1968

[3] J. A. Momoh, M. E. El-Hawary, Ramababu Adapa, "A review of selected optimal power flow literature to 1993 Part 1: Nonlinear and Quadratic Programming Approaches," IEEE Trans. on Power Systems, Feb 1999.

[4] J. A. Momoh, M. E. El-Hawary, Ramababu Adapa, "A review of selected optimal power flow literature to 1993 Part 2: Newton, linear programming and interior point methods," IEEE Trans. on Power Systems, Feb 1999

[5] Zhifeng Qiu, Geert Deconinck, Ronnie Belmans, "A literature survey of optimal power flow problems in electricity market context," IEEE Power systems conference and Exposition, 2009

[6] M. R. AlRashidi, M. E. El-Hawary, "Applications of computational intelligence technique for solving the revived optimal power flow problem," Electric Power System Research, 2009

[7] R. A. Jabr, "Radial Distribution load flow using conic programming," IEEE Trans. on Power Systems, vol. 21, no. 3, pp. 1458-1459, 2006

[8] Javad. Lavaei, Steven H. Low, "Convexification of optimal power flow," Forty eighth annual Allerton conference on communication, control and computing, Monticello, IL, Oct. 2010

[9] J. A. Momoh, R. J. Koessler, M. S. Bond, B. Stott, "Challenges to Optimal power flow," IEEE/PES Winter Meeting, Jan 1996

[10] Hadi Saadat, "Power System Analysis," McGraw-Hill, 1999

[11] Y. Tamura, H. Mori, S. Iwamto, "Relationship between voltage instability and multiple load flow solutions in electric power systems," IEEE Trans. on power system apparatus and systems, vol. 5, pp. 1115-1125, May 1983

[12] Kenji Iba, Hiroshi Suzuki, Masanao Egawa, Tsutomu Watanabe, "A method for finding a pair of multiple load flow solutions in bulk power systems," IEEE Trans. on Power Systems, May 1990

[13] Rosa Castro, Angel Coyto, "A new method to obtain multiple power flow solutions," —

[14] M. Abdel-Akher, M. E. Ahmed, R. N. Mahanty, Khalid M. Nor, "On the relationship between multiple power-flow solutions and the voltage stability problem in unbalanced three-phase networks," 12th International Middle East Power System Conference, March 2008

[15] Thomas J. Overbye, "A power flow measure for unsolvable cases," IEEE Transactions on Power Systems, vol. 9, no. 3, Aug 1994

[16] Jesus R. Santos, Alicia T. Lora, Antonio G. Exposito, Jose Ramos, "Finding Improved Local Minima of Power System Optimization Problems by Interior-Point Methods," IEEE Transactions on Power Systems, vol. 18, No. 1, Feb 2003

[17] William Hung, "Frequency and Voltage Operating Range Working Group Meeting-2: Voltage Operating Range Issues," www.nationalgrid.com, Feb 2010

[18] R. D Zimmerman, C. E Murillo-Sanchez, R. J. Thomas, "MATPOWER's Extensible Optimal Power Flow Architecture," Power and Energy Society General Meeting, 2009 IEEE, pp. 1-7, July 26-30 2009

[19] "IEEE Power Systems Test Archive," College of Engineering University of Washington [Online]. Available: http://www.ee.washington.edu/research/pstca/.

[20] A. Klos, J. Wojcicka, "Physical Aspects of the nonuniqueness of load flow solutions," Electrical Power and Energy Systems, 1991

[21] Andrew J. Korsak, "On the question of uniquesness of stable load flow solutions," IEEE summer Meeting and International Symposium on High Power Testing, July 1971

[22] Noel Janssens, Amadow Kamagate, "Loop Flows in a ring AC power system," 14th PSCC, June 2002

[23] Þ. Guðmundsson, G. Nielsen, N. Reddy, "Dynamic VAR Compensator (DVC) Application in Landsnet Grid, Iceland," Power Systems Conference and Exposition, 2006.

[24] Wächter, L. T. Biegler, "On the Implementation of an Interior-Point Filter Line-Search Algorithm for Large-Scale Nonlinear Programming," Mathematical Programming 106(1), pp. 25-57, 2006