Transseries analysis and Stokes phenomena

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The space of (formal) power series is often too small to completely solve problems. This space can be extended to transseries (see below) and that space is big enough to solve most analytic problems. This will involve a deep understanding of the Stokes phenomenon, but also the recently discovered higher order Stokes phenomenon.

To introduce transseries and the Stokes phenomenon for nonlinear ODEs let us have a look at the first Painlevé differential equation which can be converted into the equation

$$u'' + \frac{u'}{z} - \tfrac{3}{2}\left(u^2 - 1\right) - \frac{4u}{25z^2} = 0. \tag{1}$$

This equation has no simple solutions, and there are no special points in the finite complex z plane. For large z it seems that $u^2 \approx 1$ and we can try a power series solution

$$u(z) \sim \sum_{s=0}^{\infty} a_{s0} z^{-s}, \quad \text{as } z \to \infty. \tag{2}$$

The coefficients satisfy a simple recurrence relation, and the only freedom is the choice of $a_{00} = \pm 1$. Once we fix a_{00} all the other coefficients are fixed, and it seems that we have found only 2 solutions, whereas the solution space should be 2 dimensional. Where are the free constants?

When we take for u_0 a solution with this asymptotic expansion then we can try to look for solutions of the form $u(z) = u_0(z) + u_1(z)$ in which $u_1(z)$ is supposed to be exponentially small compared with $u_0(z)$. We substitute this into (1) and obtain the relation

$$u_1'' + \frac{u_1'}{z} - \left(3u_0 + \frac{4}{25z^2}\right) u_1 = \tfrac{3}{2} u_1^2. \tag{3}$$

If u_1 is exponentially small, then the right-hand side of (3) is doubly exponentially small, and this leads naturally to transseries expansions:

$$u(z) \sim \sum_{n=0}^{\infty} C^n u_n(z), \qquad \Longrightarrow \qquad u_n'' + \frac{u_n'}{z} - \left(3u_0 + \frac{4}{25z^2}\right) u_n = \tfrac{3}{2} \sum_{p=1}^{n-1} u_p u_{n-p}. \tag{4}$$

Each u_n has an asymptotic expansion of the form

$$u_n(z) \sim e^{-n\sqrt{3}z} \sum_{s=0}^{\infty} a_{sn} z^{-s-(n/2)}, \quad \text{as } z \to \infty. \tag{5}$$

Again, we can obtain recurrence relations for the coefficients a_{sn}. All the coefficients are fixed, except a_{01}, which is complete free. We take $a_{01} = 1$ and put this freedom in the constant C in the transseries. Hence, the free constant C is multiplying exponentially small terms. One needs exponentially improved asymptotics to give a meaning to these exponentially small terms, but for this introduction we will just focus on the transseries. The transseries expansion above is for $\Re z \to \infty$, when we focus on $\Re z \to -\infty$ a second free constant can be introduced.

The transseries $u(z, C) = \sum_{n=0}^{\infty} C^n u_n(z)$ really lives in a quarter plane, say $-\frac{1}{2}\pi < \arg z < 0$, and when we cross the positive real z axis, then the Stokes phenomenon takes place, that is, in the sector

$0 < \arg z < \frac{1}{2}\pi$ for $u(z,C)$ the transseries representation is $\sum_{n=0}^{\infty} (C+K_1)^n u_n(z)$, in which the constant K_1 is the Stokes multiplier. Normally one can compute these Stokes multipliers only numerically, but the Painlevé differential equation have many nice properties, and one can show that $K_1 = \sqrt{2\sqrt{3}/(5\pi)}$.

The transseries $u(z,C) = \sum_{n=0}^{\infty} C^n u_n(z)$ is a convergent sum of divergent asymptotic series. One can check that for $n > 0$ we have $a_{0n} = n12^{1-n}$. Hence, taking of each $u_n(z)$ only the first term we obtain the sum

$$\sum_{n=1}^{\infty} \frac{nC^n e^{-n\sqrt{3}z}}{z^{n/2}12^{n-1}} = \frac{Ce^{-\sqrt{3}z}/\sqrt{z}}{\left(1 - Ce^{-\sqrt{3}z}/(12\sqrt{z})\right)^2}. \tag{6}$$

The small exponentials become $\mathcal{O}(1)$ when we approach the imaginary axes. Hence, the right-hand side of (6) becomes meaningful when we approach the imaginary axes, and it tells us the type of singularities that we encounter on the boundary of the sector of validity of the transseries. Note that when (6) describes the approximate singularities near the negative imaginary axis, then along the positive imaginary axis C has been replaced by $C + K_1$. This is the impact of the Stokes phenomenon for nonlinear ODEs.

From above it should be obvious that the Stokes phenomenon is the switching on of small exponentials when certain curves are crossed. This phenomenon is well understood. Recently it has been discovered is that there are also curves where the Stokes multiplier itself (above K_1) can change its value. This is the higher order Stokes phenomenon, and a deep understanding is still missing.

In this project you could at the start first focus on transseries analysis and try to obtain all solutions for Painlevé one from their transseries representations. The general local behaviour for solutions of Painlevé one is elliptic (double periodic). What is the link with these local 'periods' and the two free constants in the transseries?

One other starting point can also be the higher order Stokes phenomenon. There are many interesting identities that have been obtained formally, but rigorous proofs are still missing. The ultimate goal for both directions could be rigorous exponential asymptotics for PDEs with a small parameter.

References

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