DIFFERENTIAL EQUATIONS DRIVEN BY ROUGH PATHS: AN APPROACH VIA DISCRETE APPROXIMATION.

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ABSTRACT. A theory of systems of differential equations of the form $dy^i = \sum_j f^i_j(y)dx^i$, where the driving path $x(t)$ is non-differentiable, has recently been developed by Lyons. We develop an alternative approach to this theory, using (modified) Euler approximations, and investigate its applicability to stochastic differential equations driven by Brownian motion. We also give some other examples showing that the main results are reasonably sharp.

1. INTRODUCTION

Lyons [7] has developed a theory of systems of differential equations of the form

$$dy^i = \sum_{j=1}^{d} f^i_j(y)dx^j, \qquad y^i(0) = y^i_0, \quad i = 1, 2, \cdots, n \tag{1}$$

where $x(t)$ is a given continuous (vector-valued) function of t but is not assumed to be differentiable, so the system is not a system of classical differential equations. In [7] $x(t)$ is assumed to have finite p-variation for some positive p. The study of equations driven by such rough paths is motivated by the case of stochastic differential equations driven by Brownian motion, which has finite p-variation only if $p < \frac{1}{2}$. Rough path theory gives an approach to such stochastic equations by viewing them as deterministic equations, for a fixed choice of driving path, which is in contrast to the more classical stochastic approach. Applications of rough path theory to stochastic equations can be found, for example, in [1, 4, 8].

The approach to (1) in [7] mirrors the standard approach to ODE's by writing them as integral equations and using Picard iteration or a contraction mapping argument. So one writes

$$y^i(t) = y^i_0 + \sum \int_0^t f^i_j(y(s))dx^j(s)$$

and the problem then is to interpret the integral on the right. This is fairly straightforward if $p < 2$; if $2 \le p < 3$ then Lyons shows that one can make sense of this integral if one 'knows' integrals of the form $\int x^i(s)dx^j(s)$. His approach is to suppose these latter integrals are given, subject to natural consistency conditions, and then to develop an integration theory which suffices to treat the differential equations. If $p \ge 3$ then it is necessary to assume higher-order iterated integrals of $x(t)$ are given. In this setting Lyons in [7] proves existence and uniqueness of solutions of (1) provided $f \in C^\gamma$ where $\gamma > p$.

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Another way of proving existence and uniqueness theorems for classical ODE's of the form $\dot{y} = f(y)$ is to consider Euler approximations $y_{k+1} = y_k + (t_{k+1} - t_k)f(y_k)$ associated with subdivisions $\{t_0, t_1, \cdots t_K\}$ and show that as the subdivision gets finer these approximations converge to a limit which satisfies the equation. Indeed this is a standard proof for the case where f is continuous but not Lipschitz, so that one gets existence but not uniqueness. In the present paper we use this approach to study the system (1). Using suitable estimates for discrete approximations to (1) we are able to prove their convergence to solutions of (1). When $p < 2$ the simple Euler approximation suffices. When $2 \leq p < 3$ we need to assume the integrals $\int x^i dx^j$ given, and incorporate them into the discrete approximation. We restrict attention to $p < 3$ which is enough to illustrate the basic ideas and covers many of the applications, but avoids the algebra of iterated integrals which is needed for the general case. We show (in sections 2 and 3) existence of solutions when $f \in C^{\gamma-1}$ when $\gamma > p$, and uniqueness when $f \in C^\gamma$ where $\gamma \geq p$. The proofs give, when $f \in C^\gamma$, convergence of the Euler approximation to the solution in the case $p < 2$, and convergence of the modified Euler approximation when $2 \leq p < 3$.

We treat the simpler case $1 \leq p < 2$ first, in section 2, the treatment of $2 \leq p < 3$ in section 3 being similar but with extra terms to handle. This results in some repetition of arguments but it is hoped that treating the simpler case will make the ideas clearer.

In section 4 we consider the application of the theory to the case of equations driven by Brownian motion, which is one of the motivating examples. In this context we investigate the smoothness requirements on f for existence and uniqueness of solutions. In section 5 we give examples to show that the results of sections 2 and 3 are sharp, to the extent that that uniqueness can fail with $f \in C^\gamma$ whenever $1 < \gamma < p < 3$, and existence can fail for $f \in C^{p-1}$ whenever $1 < p < 2$ or $2 < p < 3$. Section 6 treats global existence questions. Finally in Section 7 we show that, under an additional condition, the (unmodified) Euler approximations, with uniform step size, converge to the solution even when $2 \leq p < 3$.

We comment briefly on the relation of our results to those of [7]. When $p \geq 2$ the notion of solution developed in [7] contains more than the solution path $y(t)$ which we obtain. Just as the driving path $x(t)$ has to be accompanied by the associated 'iterated integrals', the approach taken in [7] is that the solution $y(t)$ should also be accompanied by its iterated integrals, and the solution obtained there incorporates these as well as $y(t)$. This leads to a more complete theory at the expense of greater complexity. The results of sections 2 and 3 recover the main existence and uniqueness result of [7] for $\gamma < 3$, in the restricted sense that only the solution path $y(t)$ is obtained, by a different method and somewhat more easily. We get some slight improvement in the regularity requirements, in that existence is shown for $f \in C^{\gamma-1}$ rather than C^γ and uniqueness in the borderline case $\gamma = p$ (for $f \in C^\gamma$). Some more discussion of the relation to [7], including an indication of how the full solution of [7] might be obtained by our method, can be found at the end of section 3. The results of the remaining sections are new to the best of my knowledge.

Notation.

Let n and d be positive integers. For $\rho = N+\alpha$ where N is a positive integer and $0 < \alpha \le 1$ we define C^ρ to be the set of $f = (f^i_j)$ where $i = 1, \cdots, n$ and $j = 1, \cdots, d$, such that each f^i_j is defined on $\mathbb{R}^n$, and has derivatives up to order N, which all satisfy locally a Hölder condition of exponent α (note that this definition of C^N when N is an integer differs from the usual, in that we do not require continuity of the derivatives of order N). We denote by C^ρ_0 the set of $f \in C^\rho$ which vanish outside a bounded set.

We shall generally suppose that either $1 \le p < \gamma \le 2$ or $2 \le p < \gamma \le 3$.

Consider a continuous $\mathbb{R}^d$-valued function $x(t) = (x^1(t), \cdots, x^d(t))$ on an interval $[0,T]$, and suppose $x(t)$ has finite p-variation, in the sense that there is a continuous increasing function ω on $[0,T]$ such that $|x(t) - x(s)|^p \le \omega(t) - \omega(s)$ whenever $0 \le s \le t \le T$. We sometimes write $\omega(s,t)$ for $\omega(t) - \omega(s)$.

We use the summation convention for indices h, q, r, j, where h and q range from 1 to n and r and j from 1 to d. Then (1) can be written

$$dy^i = f^i_j(y)dx^j, \qquad y^i(0) = y^i_0$$

where $y(t) = (y^1(t), \cdots, y^n(t))$. We also write ∂_q for $\partial/\partial y_q$.

We shall occasionally use *dyadic intervals*, by which we mean intervals of the form $[k2^{-m}, (k+1)2^{-m}]$ where k and m are integers.

2. **The case** $p < 2$.

Suppose $1 \le p < \gamma \le 2$, and that $x(t)$ has finite p-variation in the sense defined above. Then we interpret (1) as follows:

Definition 2.1. *We say $y(t)$ is a* solution *of (1) on $[0,T]$ if $y^i(0) = y^i_0$ and there exists a continuous increasing function $\tilde{\omega}$ on $[0,T]$ and a non-negative function θ on $[0,\infty)$ such that $\theta(\delta) = o(\delta)$ as $\delta \to 0$ and such that*

$$|y^i(t) - y^i(s) - f^i_j(y(s))(x^j(t) - x^j(s))| \le \theta(\tilde{\omega}(t) - \tilde{\omega}(s)) \tag{2}$$

for all s, t with $0 \le s < t \le T$.

Note that $\tilde{\omega}$ may *a priori* differ from the function ω that appears in the p-variation condition on x, though we shall in fact see (remark 1 below) that, for any solution y, (2) will actually hold with $\tilde{\omega} = \omega$.

We also consider discrete approximations to a solution: given $0 = t_0 < t_1 < \cdots < t_K$, let $x_k = x(t_k)$ and, given y_0, define y_k by the recurrence relation

$$y^i_{k+1} = y^i_k + f^i_j(y_k)(x^j_{k+1} - x^j_k) \tag{3}$$

Then we can state the following existence result:

Theorem 2.2. *Let $f \in C^{\gamma-1}$ (assuming $1 \le p < \gamma \le 2$) and $y_0 \in \mathbb{R}^n$. Then there exists τ, with $0 < \tau \le T$, and a solution $y(t)$ of (1) for $0 \le t < \tau$, such that if $\tau < T$ then $|y(t)| \to \infty$ as $t \to \tau$.*

Example 3 in section 5 shows that Theorem 2.2 is sharp to the extent that existence can fail for $f \in C^{p-1}$, $1 < p < 2$. (However when $p = 1$, i.e. x has bounded variation, it can be improved; in this case it suffices that f be continuous).

Theorem 2.3. *Let $f \in C^\gamma$ and $y_0 \in \mathbb{R}^n$. Then the solution $y(t)$ of (1) given by Theorem 2.2 is unique in the sense that if $t < \tau$ and $\tilde{y}$ is another solution of (1) on $[0, t]$ then $\tilde{y} = y$ on $[0, t]$.*

Moreover, if $t < \tau$ and $\epsilon > 0$, we can find $\delta > 0$ so that if $0 = t_0 < \cdots < t_K = t$ with $t_k - t_{k-1} < \delta$ for each k, then

$$|y_k - y(t_k)| < \epsilon$$

for each k, where y_k is given by (3).

Example 1 in section 5 shows that, if $\gamma < p$, uniqueness fails for a suitable choice of $f \in C^\gamma$.

These theorems will be proved by analysis of the discrete approximation (3).

Analysis of the discrete problem (3).

Let $x_0, \cdots, x_K \in \mathbb{R}^d$, with $x_k = (x_k^1, \cdots, x_k^d)$, and suppose for $0 \le k \le l \le K$ we are given $\omega_{kl} \ge 0$ such that $w_{km} \ge \omega_{kl} + \omega_{lm}$ when $k \le l \le m$ and $|x_l^r - x_k^r|^p \le \omega_{kl}$.

Note that with $\omega_{kl} = \omega(t_l) - \omega(t_k)$ and $x_k = x(t_k)$ this is just what we get from the problem described in the previous section. However for the purposes of this section we can forget about the variable t.

Given $y_0 \in \mathbb{R}^n$, and $f \in C_0^{\gamma-1}$ we define $y_k \in \mathbb{R}^n$ for $k = 1, 2, \cdots, K$ by the recurrence relation (3). Observe that, if we let $z_k^{iN} = \partial y_k^i / \partial y_0^N$ then

$$z_{k+1}^{iN} = z_k^{iN} + \partial_q f_j^i(y_k) z_k^{qN} (x_{k+1}^j - x_k^j)$$

which is the recurrence relation of the type (3) obtained when f is replaced by $F(y) = (\{f_j^i(y)\}, \{\partial_q f_j^i(y)\})$, taking values in $\mathbb{R}^{n(n+1)} \times \mathbb{R}^d$. This observation enables us to apply results on solutions of (3) to the derivatives z_k^{im}.

Let

$$I_{kl}^i = y_l^i - y_k^i - f_j^i(y_k)(x_l^j - x_k^j)$$

The following is our main technical result:

Lemma 2.4. *(a) There are positive numbers C and M, depending only on $n, d, \gamma, p, \omega_{0K}$ and $\|f\|_{\gamma-1}$, such that if $0 \le k \le l \le K$ then $|I_{kl}^i| \le M \omega_{kl}^{\gamma/p}$ and $|y_l - y_k| \le C \omega_{kl}^{1/p}$.*

(b) Suppose now $f \in C_0^\gamma$. Let $\tilde{y} \in \mathbb{R}^n$ and let $\tilde{y}_k$ be the corresponding solution of the recurrence relation. There is $M' > 0$, depending only on $n, d, \gamma, p, \omega_{0K}$ and $\|f\|_\gamma$, such that if $0 \le k \le K$ then

$$|\tilde{y}_k^i - y_k^i| \le M' \max_i |\tilde{y}_0^i - y_0^i|$$

Proof. We will use B_1, B_2 etc to denote constants depending only on n, d, γ, p and the $C^{\gamma-1}$-norm of f.

If $k \le l \le m$ we have

$$I^i_{km} - I^i_{kl} - I^i_{lm} = (f^i_j(y_l) - f^i_j(y_k))(x^j_m - x^j_l) \tag{4}$$

Claim. If $\delta > 0$ is small enough and L large enough (depending only on n, d, γ, p and the norm of f) then $|I^i_{km}| \le L\omega^{\gamma/p}_{km}$ whenever $\omega_{km} \le \delta$.

Proof of claim. We use induction on $m-k$. Note first that $I_{k,k} = 0$ trivially and $I_{k,k+1} = 0$ by (3). Now suppose k, m chosen with $m - k > 1$ and suppose the claim holds for smaller values of $m - k$.

Let l be the largest integer with $k \le l < m$ satisfying $\omega_{kl} \le \frac{1}{2}\omega_{km}$. Then $\omega_{k,l+1} > \frac{1}{2}\omega_{km}$ so $\omega_{l+1,m} < \frac{1}{2}\omega_{km}$. Then by the inductive hypothesis the claim holds for k, l, i.e. $I_{kl} \le L\omega^{\gamma/p}_{kl}$. Then $|y_l - y_k| \le L\omega^{\gamma/p}_{kl} + B_1\omega^{1/p}_{kl}$. Then provided

$$L\delta^{(\gamma-1)/p} \le B_1 \tag{5}$$

we have

$$|y_l - y_k| \le 2B_1\omega^{1/p}_{kl} \tag{6}$$

Putting these estimates into (4) we find that

$$|I^i_{km}| \le |I^i_{kl}| + |I^i_{lm}| + B_2\omega^{\gamma/p}_{km}$$

In the same way we find that $|I^i_{lm}| \le |I^i_{l,l+1}| + |I^i_{l+1,m}| + B_2\omega^{\gamma/p}_{km}$. Since $I_{l,l+1} = 0$ we get

$$|I^i_{km}| \le |I^i_{kl}| + |I^i_{l+1,m}| + 2B_2\omega^{\gamma/p}_{km} \le L(\omega^{\gamma/p}_{kl} + \omega^{\gamma/p}_{l+1,m}) + 2B_2\omega^{\gamma/p}_{km} \le (2^{1-\gamma/p}L + 2B_2)\omega^{\gamma/p}_{km}$$

and provided

$$(1 - 2^{1-\gamma/p})L \ge 2B_2 \tag{7}$$

we conclude that $|I^i_{km}| \le L\omega^{\gamma/p}_{km}$ which completes the induction, provided we choose L and δ to satisfy (7) and (5). The claim is proved.

For intervals with $\omega_{kl} \le \delta$, part (a) of the lemma now follows from the claim, and the fact that the proof of (6) now holds for any k, l. If $\omega_{kl} > \delta$, we can decompose $k = k_0 < k_1 < \cdots < k_r = l$ where either $\omega_{k_u k_{u+1}} \le \delta$ or $k_{u+1} = k_u + 1$ for each u, and $r \le 1 + 2\delta^{-1}\omega_{kl}$. In either case $|y_{k_{u+1}} - y_{k_u}| \le 2B_1\omega^{1/p}_{k_u k_{u+1}}$. Summing gives $|y_l - y_k| \le (1 + 2\delta^{-1}\omega_{kl})2B_1\omega^{1/p}_{kl}$, and then $|I_{kl}| \le |y_l - y_k| + B_1\omega^{1/p}_{kl} \le \text{const } \omega^{\gamma/p}_{kl}$, using the fact that $\omega_{kl} > \delta$.

To prove (b), we suppose $f \in C^\gamma_0$, and apply (a), using the observation above, to estimate z^{im}_k. We find that, for any choice of y_0, we have $|z^{im}_k| \le \text{const } \omega^{1/p}_{0k} \le 1$ and (b) follows. □

Proof of theorems. We prove theorem 2.2 first. Suppose $f \in C^{\gamma-1}$. Then for $r = 1, 2, \cdots$ we can find $f_{(r)} \in C^{\gamma-1}_0$ with $f_{(r)}(y) = f(y)$ for $|y| \le r$. Now take a sequence of successively finer partitions $\{\mathcal{P}_m : \ m = 1, 2, \cdots\}$ of $[0, T]$ with mesh tending to 0. Let $y^{(m)}_k$, which we also write as $y^{(m)}(t_k)$ be the solution of (3) using the partition $\mathcal{P}_m$. By passing to a subsequence we can assume that $y^{(m)}(s)$ converges to a limit $y(s)$ (possibly $\pm\infty$) for each $s \in \cup_m \mathcal{P}_m$. Let $\tau_r = \sup\{t : \ 0 \le t < T$, there exists m_0 such that $|y^{(m)}(s)| < r$ for all

$m > m_0$ and all $s \in \mathcal{P}_m$ with $0 \le s \le t\}$; by applying Lemma 2.4(a) to $f_{(r)}$ we see that τ_r is well-defined and positive for all $r > |y_0|$. Also from this lemma, if $0 \le t < \tau_r$ we have $|y^{(m)}(s) - y^{(m)}(s')| \le C(r,t)|\omega(s) - \omega(s')|^{1/p}$ for $s, s' \in \mathcal{P}_m$ with $0 \le s, s' \le t$ and m large enough. We then have the same bound for $|y(s) - y(s')|$, and $y(s)$ extends to $[0,t]$ by continuity. By Lemma 2.4(a) again the bound (2) holds on $[0,t]$, for every $t < \tau_r$, and for every r. Now let $\tau = \lim \tau_r$; it follows that y is a solution of (1) on $[0,\tau)$.

Now suppose $\tau < T$. It follows from Lemma 2.4(a) applied to $f_{(r+1)}$ that for each r there is $\sigma_r > 0$ such that if $s \in \mathcal{P}_m$ for some m and if $|y^{(m)}| < r$ then $|y^{(m)}(s)| \le r+1$ for any $s \in \mathcal{P}_m$ satisfying $t < s < t + \sigma_r$. Next, fix r and choose $t \in \cup_m \mathcal{P}_m$ such that $t > \tau - \sigma_r$. Then if $|y(t)| < r$ there exists m_0 such that $|y^{(m)}(t)| < r$ for $m > m_0$, and then $|y^{(m)}(s)| \le r+1$ for all $m > m_0$ and $s \in \mathcal{P}_m$ with $t < s < t + \sigma$, contradicting the definition of τ.

So $|y(t)| \ge r$ for $\tau - \sigma_r < t < \tau$, so $|y(t)| \to \infty$ as $t \to \tau$. This proves Theorem 2.2.

To prove Theorem 2.3, suppose y is a solution of (1) on $[0,t]$ and consider a partition $0 = t_0 < \cdots < t_K = t$. Choose r so that $|y(\tau)| < r$ for $\tau \in [0,t]$. For $l \ge k$ let $z_l^{(k)}$ be the solution of (3), with $f_{(r)}$ in place of f, with initial value $z_k^{(k)} = y(t_k)$. Then by (2) $|z_{k+1}^{(k)} - y_{k+1}| \le \theta(\tilde{\omega}(t_{k+1}) - \tilde{\omega}(t_k)$ and using Lemma 2.4(b) we have for $k < l \le K$ that $|z_l^{(k)} - z_l^{(k+1)}| \le \text{const}\ \theta(\tilde{\omega}(t_{k+1}) - \tilde{\omega}(t_k))$. Summing over k we deduce a bound for $z_l^{(0)} - y_l$ which tends to 0 as the mesh of the partition tends to 0. The conclusions of Theorem 2.3 follow.

Remark 1. The estimates given by Lemma 2.4 show that any solution constructed by the method described in the above proof will satisfy the following stronger form of (2):

$$|y^i(t) - y^i(s) - f^i_j(y(s))(x^j(t) - x^j(s))| = O(\omega(t) - \omega(s))^{\gamma/p} \tag{8}$$

If $f \in C^\gamma$ then the uniqueness shows that any solution will satisfy this stronger inequality. Then we can take $\theta(\delta) = M\delta^{\gamma/p}$ for a suitable constant M, and the proof of Theorem 1 then gives the bound

$$|y_k - y(t_k)| \le C \sum_{j=1}^{k} \omega_{j-1,j}^{\gamma/p} \tag{9}$$

When we only have $f \in C^{\gamma-1}$, then we can still show that any solution satisfies (8), by using a modified form of Lemma 2.4, as follows:

Given a solution y of (1) satisfying (2) on $[0,T]$, and given $\epsilon > 0$, choose $\eta > 0$ so that $\theta(\eta) < \epsilon\eta$, and choose a partition with $\tilde{\omega}_{k,k+1} < \delta$. Then writing $y_k = y(t_k)$, we have

$$|y^i_{k+1} - y^i_k - f^i_j(y_k)(x^j_{k+1} - x^j_k)| \le \epsilon\tilde{\omega}_{k,k+1}$$

and using this instead of (3) we follow the proof of Lemma 2.4; we obtain $|I^i_{km}| \le L\omega_{km}^{\gamma/p} + \epsilon\tilde{\omega}_{km}$ whenever $\omega_{km} \le \delta$ and $\tilde{\omega}_{km} \le \epsilon^{-1/2}\delta$, where now δ should satisfy $\epsilon^{1/2}\delta^{1-1/p} + L\delta^{(\gamma-1)/p} \le B_1$ and L satisfies (7) as before. Letting $\epsilon \to 0$ gives the same estimates as the original form of Lemma 2.4.

Remark 2. If $f \in C^\gamma$, the Euler approximations $y^{(m)}$ used in the proof of Theorem 1 converge to the solution, without the need to pass to a subsequence. This follows from (9), or proved directly as follows.

As in the proof of Theorem 2.3, let $y_k^{(m)} = y^{(m)}(t_k)$ be the solution of (3) corresponding to the partition $\mathcal{P}_m$ given by $0 = t_0 < t_1 < \cdots < t_K$. Let $m' > m$ so that $\mathcal{P}_{m'}$ is a finer partition. Let v_k be the solution of (3) for the partition $\mathcal{P}_{m'}$, at the point t_k (which being a point of $\mathcal{P}_m$ is also a point of $\mathcal{P}_{m'}$. Then for $l \geq k$ let $z_l^{(k)}$ be the solution of (3), for the partition $\mathcal{P}_m$, with initial condition $z_k^{(k)} = v_k$. Then by the bound for I_{kl}^i in Lemma 1(a), applied to the partition $\mathcal{P}_{m'}$, we have $|z_{k+1}^{(k)} - v_{k+1}| \leq c\omega_{k,k+1}^{\gamma/p}$ and the result follows by using Lemma 1(b) and summing over k, as in the last part of the proof of Theorem 2.3.

The above argument may be useful for the generalisation of Theorem 2.2 to an infinite-dimensional setting, where the compactness required for the proof of Theorem 2.3 may fail.

3. The case $2 \leq p < 3$.

We now suppose $2 \leq p < \gamma \leq 3$. In this case (3) does not give a sufficiently good approximation, and we need to include higher-order terms. We can regard (3) as being obtained from (1) by approximating $f_j^i(y)$ by $f_j^i(y_k)$. A better approximation is

$$f_j^i(y) \approx f_j^i(y_k + f_r(y_k)(x^r - x_k^r)) \approx f_j^i(y_k) + \partial_h f_j^i(y_k) f_r^h(y_k)(x^r - x_k^r)$$

To solve (1) using this approximation, we have to integrate $(x^r - x_k^r)dx^j$. With this as motivation, we attempt to define $A^{rj}(s,t)$ for $s \leq t$ by $dA^{rj}(s,t) = \{x^r(t) - x^r(s)\}dx^j(t)$ with $A^{r,j}(s,s) = 0$. We have then the problem of interpreting this equation. The solution adopted in [7] is to make the following assumption:

Assumption 1. We suppose as given the quantities $A^{rj}(s,t)$ for $1 \leq r,j \leq d$ and $0 \leq s \leq t \leq T$ subject to the natural consistency condition

$$A^{rj}(s,u) = A^{rj}(s,t) + A^{rj}(t,u) + (x^r(t) - x^r(s))(x^j(u) - x^j(t))$$

whenever $s \leq t \leq u$. We also assume the bound $|A^{rj}(s,t)|^{p/2} \leq \omega(t) - \omega(s)$ (redefining $\omega(t)$ if necessary).

We remark that, at least if $p > 2$, it is not hard to prove the existence of such $A^{rj}(s,t)$ satisfying the above conditions, for a given choice of $x(t)$. There will be many possible choices of $A^{rj}(s,t)$; given one such, then $\tilde{A}^{rj}(s,t) = A^{rj}(s,t) + \rho(t) - \rho(s)$ will be another, as long as $\rho(t)$ has finite $\frac{p}{2}$-variation. Different choices lead to different interpretations of (1).

We now interpret (1) as follows:

Definition 3.1. *We say $y(t)$ is a solution of (1) on $[0,T]$ if $y^i(0) = y_0^i$ and there exists a continuous increasing function $\tilde{\omega}$ on $[0,T]$ and a non-negative function θ on $[0,\infty)$ such that $\theta(\delta) = o(\delta)$ as $\delta \to 0$ and such that*

$$|y^i(t) - y^i(s) - f_j^i(y(s))(x^j(t) - x^j(s)) - f_r^h(y(s))\partial_h f_j^i(y(s))A^{rj}(s,t)| \leq \theta(\tilde{\omega}(t) - \tilde{\omega}(s)) \tag{10}$$

for all s,t with $0 \leq s < t \leq T$.

As before consider discrete approximations to a solution: given $0 = t_0 < t_1 < \cdots < t_K$, let $x_k = x(t_k)$ and, given y_0, define y_k by the recurrence relation

$$y^i_{k+1} = y^i_k + f^i_j(y_k)(x^j_{k+1} - x^j_k) + f^h_r(y_k)\partial_h f^i_j(y_k)A^{rj}(t_k, t_{k+1}) \tag{11}$$

Then we can state the following:

Theorem 3.2. *Let $f \in C^{\gamma-1}$ and $y_0 \in \mathbb{R}^n$. Then there exists τ, with $0 < \tau \le T$, and a solution $y(t)$ of (1) for $0 \le t < \tau$, such that if $\tau < T$ then $|y(t)| \to \infty$ as $t \to \tau$.*

Theorem 3.3. *Let $f \in C^\gamma$ and $y_0 \in \mathbb{R}^n$. Then the solution $y(t)$ of (1) given by Theorem 3.2 is unique in the sense that if $t < \tau$ and $\tilde{y}$ is another solution of (1) on $[0, t]$ then $\tilde{y} = y$ on $[0, t]$. for $0 \le t < \tau$, such that either $\tau = T$ or $|y(t)| \to \infty$ as $t \to \tau$.*

Moreover, if $t < \tau$ and $\epsilon > 0$, we can find $\delta > 0$ so that if $0 = y_0 < \cdots < t_K = t$ with $t_k - t_{k-1} < \delta$ for each k, then

$$|y_k - y(t_k)| < \epsilon$$

for each k, where y_k is given by (11).

These results will be proved by analysis of the discrete approximation (11).

Analysis of the discrete problem (11).

Let $x_0, \cdots, x_K \in \mathbb{R}^d$, with $x_k = (x^1_k, \cdots, x^d_k)$, and suppose for $0 \le k \le l \le K$ given A^{rj}_{kl} for $1 \le r, j \le d$ such that $A^{rj}_{km} = A^{rj}_{kl} + A^{rj}_{lm} + (x^r_l - x^r_k)(x^j_m - x^j_l)$ whenever $k \le l \le m$. Suppose also given $\omega_{kl} \ge 0$ for $0 \le k \le l \le K$ such that $w_{km} \ge \omega_{kl} + \omega_{lm}$ when $k \le l \le m$ and $|x^r_l - x^r_k|^p \le \omega_{kl}$ and $|A^{rj}_{kl}| \le \omega^{2/p}_{kl}$.

Given $y_0 \in \mathbb{R}^n$, and $f \in C^{\gamma-1}_0$ we define $y_k \in \mathbb{R}^n$ for $k = 1, 2, \cdots, K$ by the recurrence relation (11). Observe that, if we let $z^{iN}_k = \partial y^i_k / \partial y^N_0$ as before, then

$$z^{iN}_{k+1} = z^{iN}_k + \partial_q f^i_j(y_k) z^{qN}_k (x^j_{k+1} - x^j_k) + \{\partial_q f^h_r(y_k) z^{qN}_k \partial_h f^i_j(y_k) + f^h_r(y_k)\partial_{qh} f^i_j(y_k) z^{qN}_k\} A^{rj}$$

which is the recurrence relation of the type (11) obtained when f is replaced by $F(y) = (\{f^i_j(y)\}, \{\partial_q f^i_j(y)\})$, taking values in $\mathbb{R}^{n(n+1)} \times \mathbb{R}^d$.

Let

$$J^i_{kl} = y^i_l - y^i_k - f^i_j(y_k)(x^j_l - x^j_k) - f^h_r(y_k)\partial_h f^i_j(y_k) A^{rj}_{k,l}$$

The following is our main technical result:

Lemma 3.4. *(a) There are positive numbers C and M, depending only on $n, d, \gamma, p, \omega_{0K}$ and $\|f\|_{\gamma-1}$, such that if $0 \le k \le l \le K$ then $|J^i_{kl}| \le M\omega^{\gamma/p}_{kl}$ and $|y_l - y_k| \le C\omega^{1/p}_{kl}$.*

(b) Suppose now $f \in C^\gamma_0$. Let $\tilde{y} \in \mathbb{R}^n$ and let $\tilde{y}_k$ be the corresponding solution of the recurrence relation. There is $M' > 0$, depending only on $n, d, \gamma, p, \omega_{0K}$ and $\|f\|_\gamma$, such that if $0 \le k \le K$ then

$$|\tilde{y}^i_k - y^i_k| \le M' \max_i |\tilde{y}^i_0 - y^i_0|$$

Proof. We will use B_1, B_2 etc to denote constants depending only on n, d, γ, p and the $C^{\gamma-1}$-norm of f. Let I^i_{kl} be as before and let $g^i_{rj}(y) = f^h_r(y)\partial_h f^i_j(y)$ so that

$$J^i_{kl} = I^i_{kl} - g^i_{rj}(y_k)A^{rj}_{kl}$$

Also define

$$R^i_{kl,j} = f^i_j(y_l) - f^i_j(y_k) - \partial_h f^i_j(y_k)(y^h_l - y^h_k)$$

and note that $|R^i_{kl,j}| \leq B_1|y_l - y_k|^{\gamma-1}$. Then if $k \leq l \leq m$ we have

$$J^i_{km} - J^i_{kl} - J^i_{lm} = (R^i_{kl,j} + \partial_h f^i_j(y_k)I^h_{kl})(x^j_m - x^j_l) + \{g^i_{rj}(y_l) - g^i_{rj}(y_k)\}A^{rj}_{lm} \tag{12}$$

Claim. If $\delta > 0$ is small enough and L large enough (depending only on n, d, γ, p and the norm of f) then $|J^i_{km}| \leq L\omega^{\gamma/p}_{km}$ whenever $\omega_{km} \leq \delta$.

Proof of claim. Again we use induction on $m - k$. Note first that $J_{k,k} = 0$ trivially and $J_{k,k+1} = 0$ by (11). Now suppose k, m chosen with $m - k > 1$ and suppose the claim holds for smaller values of $m - k$.

Let l be the largest integer with $k \leq l < m$ satisfying $\omega_{kl} \leq \frac{1}{2}\omega_{km}$. Then $\omega_{k,l+1} > \frac{1}{2}\omega_{km}$ so $\omega_{l+1,m} < \frac{1}{2}\omega_{km}$. Then by the inductive hypothesis the claim holds for k, l, i.e. $J_{kl} \leq L\omega^{\gamma/p}_{kl}$. Then $|I^i_{kl}| \leq L\omega^{\gamma/p}_{kl} + B_2\omega^{2/p}_{kl}$ and $|y_l - y_k| \leq L\omega^{\gamma/p}_{kl} + B_2\omega^{2/p}_{kl} + B_3\omega^{1/p}_{kl}$. Then provided

$$L\delta^{(\gamma-2)/p} \leq B_2 \tag{13}$$

we have

$$|I^i_{kl}| \leq 2B_2\omega^{2/p}_{kl} \quad \text{and} \quad |y_l - y_k| \leq 3B_2\omega^{1/p}_{kl} \tag{14}$$

Putting these estimates into (12) we find that

$$|J^i_{km}| \leq |J^i_{kl}| + |J^i_{lm}| + B_4\omega^{\gamma/p}_{km}$$

In the same way we find that $|J^i_{lm}| \leq |J_{l,l+1}| + |J^i_{l+1,m}| + B_4\omega^{\gamma/p}_{km}$. Since $J_{l,l+1} = 0$ we get

$$|J^i_{km}| \leq |J^i_{kl}| + |J^i_{l+1,m}| + 2B_4\omega^{\gamma/p}_{km} \leq L(\omega^{\gamma/p}_{kl} + \omega^{\gamma/p}_{l+1,m}) + 2B_4\omega^{\gamma/p}_{km} \leq (2^{1-\gamma/p}L + 2B_4)\omega^{\gamma/p}_{km}$$

and provided

$$(1 - 2^{1-\gamma/p})L \geq 2B_4 \tag{15}$$

we conclude that $|J^i_{km}| \leq L\omega^{\gamma/p}_{km}$ which completes the induction, provided we choose L and δ to satisfy (15) and (13), and proves the claim.

As before, part (a) of the lemma follows from the claim.

Part (b) then follows from (a) in exactly the same way as for Lemma 2.4. □

Theorems 3.3 and 3.2 are deduced in exactly the same way as Theorems 2.3 and 2.2 follow from Lemma 2.4.

Remark 3. The same reasoning as in Remark 1 shows that, if $f \in C^{\gamma-1}$, then any solution of (1) on $[0,T]$ satisfies

$$y^i(t) - y^i(s) - f^i_j(y(s))(x^j(t) - x^j(s)) - f^h_r(y(s))\partial_h f^i_j(y(s))A^{rj}(s,t) = O(\omega(t) - \omega(s))^{\gamma/p}$$

and also that (9) holds when $f \in C^\gamma$ in the present situation.

Remark 4. In the same way is in Remark 2, one can prove Theorem 3.2 by showing directly that the discrete approximations converge, without passing to subsequences.

Uniqueness when $\gamma = p$.

Now we prove the slightly more delicate result that (1) has a unique solution when $f \in C^p$, where $2 \le p < 3$ (the proof for $p < 2$ is similar but simpler). We require the following lemma, whose proof is straightforward.

Lemma 3.5. *Suppose $a, b, c, d \in \mathbb{R}^n$ with $|a-b| < \lambda$, $|c-d| < \lambda$, $|a-c| < \epsilon$ and $|a-b-c+d| < \sigma$. Then*

(a) if $F \in C^\gamma_0$, where $1 \le \gamma < 2$, then

$$|F(b) - F(a) - F(d) + F(c)| < C(\lambda^{\gamma-1}\epsilon + \sigma)$$

where C depends only on the C^γ norm of f.

(b) if $F \in C^\gamma_0$, where $2 \le \gamma < 3$, then

$$|F(b) - F(a) - DF(a)(b-a) - F(d) + F(c) + DF(c)(d-c)| < C\lambda(\lambda^{\gamma-2}\epsilon + \sigma)$$

where C depends only on the C^γ norm of f.

Theorem 3.6. *Suppose $f \in C^p$ where $2 \le p < 3$. Then the solution of (1), whose existence is asserted by Theorem 3.2, is unique.*

Proof. Suppose $y(t)$ and $\tilde{y}(t)$ are two solutions; is suffices to prove that $y = \tilde{y}$ on some interval $[0,\tau]$. Suppose on the contrary that no such interval exists. Then for k large enough we can find $t_k > 0$ such that $|y(t_k) - \tilde{y}(t_k)| = 2^{-k}$ but $|y(t) - \tilde{y}(t)| < 2^{-k}$ for $0 < t < t_k$. Then $t_k > t_{k+1} > \cdots$. We shall show that $\omega(t_k, t_{k+1}) >$const.k^{-1}. Since $\sum k^{-1} = \infty$, this will give a contradiction and prove the theorem.

Since the problem is a local one, we can suppose $f \in C^p_0$. Fix γ with $p < \gamma < 3$. We use $C_1, C_2, \cdots$ for constants which depend only on p, γ and the C^γ-norm of f.

We fix k and let L denote the interval $[t_{k+1}, t_k]$. We introduce the notation

$$I^i(s,t) = y^i(t) - y^i(s) - f^i_j(x^j(t) - x^j(s))$$

and

$$J^i(s,t) = I^i(s,t) - f^h_r(y(s))\partial_h f^i_j(y(s))A^{rj}(s,t)$$

We define $\tilde{I}$ and $\tilde{J}$ similarly. Then by Remark 3 we have $|J(s,t)| \le C_1\omega(s,t)^{\gamma/p}$ and then

$$|I(s,t)| \le C_1\omega(s,t)^{2/p} \text{ and } |y(t) - y(s)| \le C_1\omega(s,t)^{1/p} \tag{16}$$

We also introduce the notation $\overline{y}(t) = \tilde{y}(t) - y(t)$, $\overline{I}^i(s,t) = \tilde{I}^i(s,t) - I(s,t)$, etc. Then we have $|\overline{y}(s,t)| \le 2^{-k}$ if $[s,t] \subseteq L$ and $|\overline{J}(s,t)| \le 2C_1\omega(s,t)^{\gamma/p}$.

Now we write

$$R^i_j(s,t) = f^i_j(y(t)) - f^i_j(y(s)) - \partial_h f^i_j(y(s))(y^h(t) - y^h(s))$$

Then

$$\begin{aligned} J^i(s,u) - J^i(s,t) - J^i(t,u) = &\left\{R^i_j(s,t) + \partial_h f^i_j(y(s)) I^h(s,t)\right\}(x^j(u) - x^j(t)) \\ &+ \left\{g^i_{rj}(y(t)) - g^i_{rj}(y(s))\right\} A^{rj}(t,u) \end{aligned}$$

with a similar expression involving $\tilde{J}$. The difference of the two expressions gives

$$\overline{J}^i(s,u) - \overline{J}^i(s,t) - \overline{J}^i(t,u) = W^i_j(s,t)(x^j(u) - x^j(t)) + V^i_{rj}(s,t)A^{rj}(t,u) \tag{17}$$

where $V^i_{rj}(s,t) = g^i_{rj}(\tilde{y}(t)) - g^i_{rj}(y(t)) - g^i_{rj}(\tilde{y}(s)) + g^i_{rj}(y(s))$ and

$$W^i_j(s,t) = \overline{R}^i_j(s,t) + \left\{\partial_h f^i_j(\tilde{y}(s)) - \partial_h f^i_j(y(s))\right\} I^h(s,t) + \partial_h f^i_j(\tilde{y}(s))\overline{I}^h(s,t)$$

For each $n = 1, 2, \cdots$ let K_n be the supremum of $|\overline{J}(s,t)|$ taken over all intervals $[s,t] \subseteq L$ with $\omega(s,t) \le 2^{-n}$. Then for such intervals $[s,t]$ we have the estimates $|\overline{I}(s,t)| \le K_n + C_2 2^{-k-2n/p}$ and

$$|\overline{y}(t) - \overline{y}(s)| \le K_n + C_2 2^{-k-n/p} \tag{18}$$

Using these estimates and Lemma 3.5 we obtain $|V^i_{rj}(s,t)| \le C_2(K_n + 2^{-k-(p-2)n/p})$ and $|\overline{R}^i_j(s,t)| \le C_2 2^{-n/p}(2^{-k-(p-2)n/p} + K_n)$. Now if $[s,u]$ is any interval in L with $\omega(s,u) \le 2^{-n}$, we can find $t \in (s,u)$ with $\omega(s,t) \le 2^{-n-1}$ and $\omega(t,u) \le 2^{-n-1}$. Then putting the above estimates in (16) we obtain $|\overline{J}^i(s,u) - \overline{J}^i(s,t) - \overline{J}^i(t,u)| \le C_3(2^{-n/p}K_{n+1} + 2^{-n-k})$ and so

$$K_n \le (2 + C_3 2^{-n/p})K_{n+1} + C_3 2^{-n-k} \tag{19}$$

From the fact that $|\overline{J}(s,t)| \le 2C_1\omega(s,t)^{\gamma/p}$ we deduce that $K_n \le 2C_1 2^{-k-n}$ if $n > \frac{kp}{\gamma - p}$, and combining this with the recurrence relation (19) we see that $K_n \le C_4 k 2^{-k-n}$ for all n. Now if n is such that $\omega(t_k, t_{k+1}) \le 2^{-n}$ then $2^{-k+1} \le |\overline{y}(t_k) - \overline{y}(t_{k+1})| \le C_5(k2^{-k-n} + 2^{-k-n/p})$ by (18), so $\omega(t_k, t_{k+1}) \ge C_6 k^{-1}$ as required. □

Remark 5. The hypothesis of finite p-variation on $x(t)$ can be weakened slightly in Theorem 3.6 - it suffices to assume that $|x^j(t) - x^j(s)| \le \left(\omega(s,t)\log\log\frac{1}{\omega(s,t)}\right)^{1/p}$ and $|A^{rj}(s,t)| \le \left(\omega(s,t)\log\log\frac{1}{\omega(s,t)}\right)^{2/p}$ for all sufficiently small intervals $[s,t]$.

The proof of Theorem 3.6 can be repeated under these weaker assumptions. To get the same bound for $J(s,t)$ using Remark 3, we can exploit the freedom in the choice of γ to use a slightly bigger p. Then in (16) there is an extra factor of $(\log\log\frac{1}{\omega(s,t})^{2/p}$ in the bound for $I(s,t)$ and an extra factor of $(\log\log\frac{1}{\omega(s,t}^{1/p}$ in the bound for $y(t) - y(s)$. This results in 2^{-n} being replaced by $2^{-n}\log n$ in bounds such as (18), the RHS of which becomes $K_n + C_2 2^{-k-n/p}(\log n)^{1/p}$. Then in place of (19) one has

$$K_n \leq (2 + C2^{-n/p}\log n)K_{n+1} + C2^{-n-k}\log n$$

and deduces that $K_n \leq C'k\log k 2^{-n}$. This gives $\omega(t_k, t_{k+1}) \geq \text{const.}(k\log k)^{-1}$, which, since $\sum(k\log k)^{-1} = \infty$, is sufficient to complete the proof.

It is true that this is a very small improvement on Theorem 3.6, but in the case $p = 2$, it is sufficient to make the theorem applicable to Brownian motion - see Theorem 4.10.

Relation to results of [7]

Here we discuss briefly the relation of our notion of solution of (1) to that given by [7] in the case $2 \leq p < \gamma < 3$. The basic object in [7] is a 'multiplicative functional', which in effect consists of a path $x^i(t)$ together with iterated integrals $A^{ij}(s,t)$ as considered in this paper. The viewpoint of [7] is that the solution should also be a multiplicative functional, so that the solution consists not only of the path $y(t)$ as considered here but also of associated iterated integrals. Indeed the solution as defined in [7] includes the iterated integrals associated with the path $(x(t), y(t))$ which combines the driving path with the solution path. This means, in addition to $A^{ij}(s,t)$ which is given, also $B^{il}(s,t)$, $C^{kj}(s,t)$ and $D^{kl}(s,t)$ being respectively interpretations of $\int_s^t (x^i - x^i(s))dy^l$, $\int_s^t (y^k - y^k(s))dx^j$ and $\int_s^t (y^k - y^k(s))dy^l$.

The inclusion of these additional components in the solution is the main difference between the notion of solution in [7] and ours, apart from the fact that [7] defines solution in terms of an integral equation formulation rather than a difference inequality like (10). The relation between the two notions is as follows. Assume $f \in C^{\gamma-1}$. Then for any solution of (1) in the sense of Definition 4.1.1 of [7], it follows from the estimates in section 3.2 of [7] that the path component y will satisfy (10) and is therefore a solution in our sense. In the other direction, a solution of (2) in the sense of Definition 3.1 does not directly yield a solution in the sense of [7], because of the missing components. But it can be shown that such a solution is obtained from any solution, in the sense of Definition 3.1, of the extended system

$$dx^i = dx^i,\ dy^i = f^i_j(y)dx^j,\ dB^{il} = x^i f^l_j(y)dx^j,\ dC^{kj} = y^k dx^j,\ dD^{kl} = y^k f^l_j(y)dx^j$$

(with the obvious initial conditions for x and y, and arbitrary ones for B, C, D) by setting

$$B^{il}(s,t) = B^{il}(t) - B^{il}(s) - x^i(s)f^l_j(y(s))\{x^j(t) - x^j(s)\}$$

and similarly for the C and D terms.

4. Equations driven by Brownian motion

Stochastic differential equations driven by Brownian motion form one of the main motivating examples for Lyons' theory. See [5] for background on this topic. In this case the driving path is a d-dimensional Brownian motion $W(t) = (W^1(t), \cdots, W^d(t))$ where $W^i(t)$ are independent standard Brownian motions defined on $[0,\infty)$. Then with probability 1, W has finite p-variation for all $p > 2$ on any finite interval, indeed it satisfies a Hölder condition with exponent $1/p$, which means that we can take $\omega(t) = t$ in the definition of p-variation. However W does not have finite 2-variation, so the theory of section 3 is needed.

There are two choices for $A^{rj}(s,t)$ corresponding to Itô and Stratonovich SDEs. For the Itô case we use $A_I^{rj}(s,t) = \int_s^t W^r(s,u)dW^j(u)$, where we use the notation $W^j(s,t) = W^j(t) - W^j(s)$, and the integral is a standard Itô integral. We define $A_S^{rj}(s,t)$ in the same way, but using a Stratonovich integral. The two versions differ only on the diagonal, i.e. $A_S^{rj} = A_I^{rj}$ if $r \neq j$, and we have $A_S^{jj} = A_I^{jj} + (t-s)/2 = \frac{1}{2}W^j(s,t)^2$. For either choice, with probability 1 the $p/2$-variation condition with $\omega(t) = t$ is satisfied on any finite interval for all $p > 2$. We always assume that our Brownian paths satisfy this condition, along with the p-variation condition for W itself.

Then the theory described in section 3 here gives existence and uniqueness of solutions for $f \in C^\gamma$ for $\gamma > 2$ (As we shall see in Theorem 4.10, this can be improved to $f \in C^2$). On the other hand, the well-established theory of SDE's gives existence and uniqueness for any locally Lipschitz f, at least for Itô equations. In this section we attempt to account for this difference in smoothness assumptions. We consider equation (1) where $x(t) = W(t)$ is as above.

In the first place, standard SDE theory regards a solution as a stochastic process, and the uniqueness theorem gives uniqueness of a process rather than uniqueness of a solution for an individual driving path. However, we show in Proposition 4.3 that if $f \in C^\gamma$ for $\gamma > 1$, then with probability 1 the Itô version of equation (1) has a unique solution, in the sense of Definition 3.1). The real reason for the difference in smoothness requirements is that the quantifiers 'for all $f \in C^\gamma$' and 'with probability 1' do not commute. We shall show that, if $\gamma < 2$, the statement 'with probability 1, for all $f \in C^\gamma$, (1) has a unique solution' is false (Theorem 4.8 below). With $\gamma = 2$, it is true (Theorem 4.10).

We start by proving uniqueness of solutions for the Itô version, for given $f \in C^\gamma$ where $\gamma > 1$.

Lemma 4.1. *Consider the Itô equation*

$$dy^i = f^i_j(y)dW^j \tag{20}$$

with $y(0) = y_0$ on the interval $[0,T]$ where $f \in C_0^\gamma$. Let $k \geq 2$. Let $z^i(s,t) = y^i(t) - y^i(s) - f^i_j(y(s))(W^j(t) - W^j(s)) - g^i_{rj}A_I^{rj}(s,t)$ where $g^i_{rj}(y) = f^h_r(y)\partial_h f^i_j(y)$. Then there is a constant C such that

$$\mathbb{E}|z(s,t)|^k \leq C(t-s)^{k(1+\gamma)/2}$$

Proof. We use repeatedly the fact that if $X(t)$ is a stochastic process adapted to the filtration of the Brownian motion, such that $\mathbb{E}|X(t)|^k \leq M$ for all τ in an interval (s,t), then

$$\mathbb{E}\left|\int_s^t X(t)dW^j(\tau)d\tau\right|^k \leq A(t-s)^{k/2}M^k \tag{21}$$

where A is a constant depending on k.

First we have $y^i(t) - y^i(s) = \int_s^t f^i_j(y(\tau))dW^j(\tau)$ so $\mathbb{E}|y(t)-y(s)|^k \leq C_1(t-s)^{k/2}$. Then

$$y^i(t) - y^i(s) - f^i_j(y(s))W^j(s,t) = \int_s^t \{f^i_j(y(\tau) - f^i_j(y(s))\}dW^j(\tau)$$

and so by (21)

$$\begin{aligned}\mathbb{E}|y^i(t)-y^i(s)-f^i_j(y(s))W^j(s,t)|^k &\le C_2(t-s)^{k/2}\max_{s\le\tau\le t}\mathbb{E}|f(y(\tau))-f(y(s))|^k \\ &\le C_3(t-s)^k\end{aligned} \tag{22}$$

Also we have

$$\mathbb{E}|f^i(y(t))-f^i(y(s))-\partial f^i_j(y(s))(y^j(t)-y^j(s))|^k \le C_4\mathbb{E}|y(t)-y(s)|^{\gamma k}\le C_5(t-s)^{\gamma k/2}$$

Combining this with (22) gives

$$\mathbb{E}|f^i_j(y(t))-f^i_j(y(s))-g^i_{rj}(y(s))W^r(s,t)|^k \le C_5(t-s)^{\gamma k/2} \tag{23}$$

Finally

$$z^i(s,t)=\int_s^t\{f^i_j(y(\tau))-f^i_j(y(s))-g^i_{rj}(y(s))W^r(s,\tau)\}dW^j(\tau)$$

and applying (21) and (23) gives the required bound. □

Now we use the fact that, with probability 1, equation (20) has a continuous *solution flow* $(s,t,x)\to F(s,t,x)\in\mathbb{R}^d$, defined for $s<t$ and $x\in\mathbb{R}^d$, such that any choice of s,t,x the solution of (20) with $y(s)=x$ satisfies $y(t)=F(s,t,x)$ with probability 1 (see [6]). Moreover, for any $\beta<\frac{1}{2}$, $F(s,t,x)$ is a C^β function of s and t and a locally Lipschitz function of x, with uniform C^β and Lipschitz bounds on compact sets. We define

$$Z^i(s,t,x)=F^i(s,t,x)-x^i-f^i_j(x)(W^j(t)-W^j(s))-g^i_{rj}A^{rj}(s,t)$$

and deduce that, if $0<\beta<\gamma-1$, then Z is with probability 1 a C^β function of s,t,x.

Now we can prove the following bound.

Lemma 4.2. *Fix $T>0$, $L>0$ and $1<q<\alpha=(1+\gamma)/2$. Then with probability 1 there is a constant C such that $|Z(s,t,y)|\le C(t-s)^q$ for $0\le s<t\le T$ and $|y|<L$.*

Proof. Fix $0<\beta<\min(\frac{1}{2},\gamma-1)$, and then fix k large enough that $k(\alpha-q)>1+q(d+2)\beta^{-1}$. Next, for any positive integer N, let $\eta_N=2^{-Nq/\beta}$ and let Ω_N be a finite set in $\mathbb{R}^d$ such that for any $y\in\mathbb{R}^d$ with $|y|<L$ one can find $y'\in\Omega_N$ with $|y-y'|<\eta_N$, and such that $\#(\Omega_N)\le C_1\eta_N^{-d}$. Also let Λ_N be a finite set in $[0,T]$ with $\#(\Lambda_N)\le T\eta_N^{-1}$ such that for any $t\in[0,T]$ there is $t'\in\Lambda_N$ with $|t-t'|<\eta_N$. Then for any $\lambda>0$ we have, by lemma 4.1, that

$$\begin{aligned}\mathbb{P}(|Z(s,t,y)\ge\lambda 2^{-Nq}\text{ for some }y\in\Omega_N\text{ and }s,t\in\Lambda_N\text{ with }t-s<2^{-N}) &\le C_2\eta_N^{-d-2}\lambda^{-k}2^{-Nk(q-\alpha)} \\ &\le C_2\lambda^{-k}2^{-N}\end{aligned}$$

It follows that with probability 1 there is $\lambda>0$ such that for every choice of $N\in\mathbb{N}$, $y\in\Omega_N$ and $s,t\in\Lambda_N$ with $0<t-s<2^{-N}$ we have $|Z(s,t,y)|\le\lambda 2^{-Nq}$. It also follows from the above-mentioned C^β property of Z that, with probability 1, there exists $B>0$ such that, for any y,y',s,s',t,t' with $|y|<L$, $|y-y'|<\eta_N$, $|s-s'|<\eta_N$ and $|t-t'|<\eta_N$ we have $|Z(s,t,y)-Z(s,t,y')|<B\eta_N^\beta=B2^{-Nq}$. Then given y,s,t with $|y|<L$ and $t-s<2^{-N-1}$, we choose $y' \in \Omega_N$ with $|y - y'| < \eta_N$, and $s', t' \in \Lambda_N$ with $|s - s'| < \eta_N$, $|t - t'| < \eta_N$, and conclude that $|Z(s,t,y)| \le (\lambda + B)2^{-Nq}$, and the required result follows. □

Proposition 4.3. *Suppose $f \in C^\gamma$ where $\gamma > 1$. Then, with probability 1, for any choice of y_0 the Itô equation $dy_i = f^i_j dW^j$ with $y(0) = y_0$ has either a solution in the sense of Definition 3.1 (with $\omega(t) = t$) for all $t \ge 0$ or, for some $T > 0$, a solution on $0 \le t < T$ with $y(t) \to \infty$ as $t \to T$. Moreover the solution is unique in the sense that if $\tilde{y}$ is any solution on $0 \le t < \tau$ in the sense of Definition 3.1 then $\tilde{y} = y$ for $0 \le t < \tau$.*

Proof. For $n = 1, 2, \cdots$ let $f^{(n)} \in C^\gamma_0$ so that $f^{(n)}(y) = f(y)$ for $|y| \le n$. Then we have the associated flow $F^{(n)}(s,t,y)$ and $Z^{(n)}(s,t,y)$ defined as above for $f^{(n)}$ in place of f. By Lemma 4.2, with probability 1 there is a sequence (C_n) such that

$$|Z^{(n)}(s,t,y)| \le C_n (t-s)^q \tag{24}$$

whenever $n \in \mathbb{N}$, $0 \le s < t < n$ and $|y| < n$. Using the Lipschitz property of the flow, we can also require that

$$|F^{(n)}(s,t,x) - F^{(n)}(s,t,y)| \le C_n |x - y| \tag{25}$$

whenever $n \in \mathbb{N}$, $0 \le s < t < n$ and $|x|, |y| < n$. We fix a Brownian path W for which these conditions hold, and prove the required existence and uniqueness for a solution driven by this path.

The existence of a solution y is a consequence of Theorem 2.2. To prove uniqueness, suppose $\tilde{y}$ is a solution on $[0, \tau)$, with $\tau \le T$, which is not identical to y on $[0, \tau)$. Let $\tau_1 = \sup\{t : \ t \ge 0 \text{ and } y(s) = \tilde{y}(s) \text{ for } 0 \le s < t\}$. Then $0 \le \tau_1 < \tau$ and $y(\tau_1) = \tilde{y}(\tau_1)$; we let $y_1 = y(\tau_1)$. Now fix τ' with $\tau_1 < \tau' < \tau$ and choose $n > \tau$ such that $|y(t)| < n$ and $|\tilde{y}(t)| < n$ for all $t \in [\tau_1, \tau']$.

Claim: $y(t) = F^{(n)}(\tau_1, t, y_1)$ for $\tau_1 \le t \le \tau'$.

To prove this claim, fix $t \in [\tau_1, \tau']$ and let $N \in \mathbb{N}$. Let $\tau_1 = t_0 < t_1 < \cdots < t_N = t$ with $t_{k+1} - t_k \le N^{-1}$. Let $v_k = y(t_k)$ and $w_k = F^{(n)}(\tau_k, t, v_k)$. Now from (10) we have

$$|v^i_{k+1} - v^i_k - f^{(n)i}_j(v_k)(W^j_{k+1} - W^j_k) - g^{(n)i}_{rj} A^{rj}(t_k, t_{k+1})| \le \theta(\tilde{\omega}(t_k, t_{k+1}))$$

where θ and $\tilde{\omega}$ are is in (10). Together with (24) this gives

$$|v_{k+1} - u_k| \le \theta(\tilde{\omega}(t_k, t_{k+1})) + C_n N^{-q}$$

where $u_k = F^{(n)}(t_k, t_{k+1}, v_k)$. Then

$$|w_{k+1} - w_k| = |F^{(n)}(t_{k+1}, t, v_{k+1}) - F^{(n)}(t_{k+1}, t, u_k)| \le C_n \left(\theta(\tilde{\omega}(t_k, t_{k+1})) + C_n N^{-q}\right)$$

and so

$$|y(t) - F^{(n)}(\tau_1, t, y_1)| = |w_N - w_0| \le C_n \left(\sum_k \theta(\tilde{\omega}(t_k, t_{k+1})) + C_n N^{1-q} \right)$$

which tends to 0 as $N \to \infty$, so we conclude that $y(t) = F^{(n)}(\tau_1, t, y_1)$ as claimed.

The same argument applies to $\tilde{y}$ and we conclude that $\tilde{y} = y$ on $[\tau_1, \tau']$, contradicting the definition of τ_1. This completes the proof of uniqueness. □

For Stratonovich equations the proof of Proposition 4.3 runs into difficulties because the existence of a solution flow has not been proved in general, and the validity of the Proposition is an open question. In the case when the matrix of coefficients is nonsingular then it can be proved, using a standard type of change of variables which converts the equation to an Itô equation, as we now show.

Lemma 4.4. *Let $1 < \gamma < 2$, let U be an open subset of $\mathbb{R}^n$, let y be a solution of the Stratonovich equation $dy_i = f^i_j \circ dW^j$ in the sense of Definition 3.1 on $\tau_1 \le t \le \tau$, where $f \in C^\gamma$ and suppose $y([\tau_1, \tau]) \subseteq U$. Suppose $\psi : U \to \mathbb{R}^d$ is $C^{1+\gamma}$ and each component of ψ satisfies $\sigma^{kh}(x)\partial_{kh}\psi^i(x) + \rho^k \partial_k \psi^i(x) = 0$ where*

$$\sigma_{kh}(y) = f^k_j(y) f^h_j(y) \quad \text{and} \quad \rho^i(y) = f^k_j(y)\partial_k f^i_j(y) \tag{26}$$

Suppose also that $\tilde{f}$ is C^γ on $\mathbb{R}^d$ and $\tilde{f}^i_j(\psi(y)) = \partial_h \psi^i(y) f^h_j(y)$ for $y \in U$.

Then $x(t) = \psi(y(t))$ is a solution of the Itô equation $dx^i = \tilde{f}^i_j dW^j$ on $[\tau_1, \tau]$ in the sense of Definition 3.1.

Proof. Fix β with $\frac{1}{3} < \beta < \frac{1}{2}$. By assumption

$$y^i(t) - y^i(s) = f^i_j(y(s))W^j(s,t) + g^i_{rj}(y(s))A^{rj}_S(s,t) + R^i(s,t)$$

where $|R^i(s,t)| \le \theta(\tilde{\omega}_{st})$ (where θ and $\tilde{\omega}$ are as in (10)). Now $x(t) = \psi(x(t))$ and expanding $\psi(y)$ about $y = y(s)$ gives

$$\begin{aligned} x^i(t) =& \psi^i(y(s)) + \partial_h \psi^i(y(s)) f^h_j(y(s)) W^j(s,t) + \frac{1}{2}\partial_{hk}\psi^i(y(s)) f^h_j f^k_r W^j(s,t) W^r(s,t) \\ &+ \partial_h \psi^i g^h_{rj} A^{rj}_S(s,t) + O\left(\theta(\tilde{\omega}_{st}) + (t-s)^{3\beta}\right) \end{aligned}$$

Now a calculation shows that

$$\frac{1}{2}\partial_{hk}\psi^i(y(s)) f^h_j f^k_r W^j(s,t) W^r(s,t) + \partial_h \psi^i G^h_{rj} A^{rj}_S(s,t) = \tilde{g}^i_{rj}(x) A^{rj}_I(s,t)$$

where $\tilde{g}^i_{rj}(x) = \tilde{f}^h_r(x)\partial_h \tilde{f}^i_j(x)$. Hence

$$x^i(t) = x^i(s) + \tilde{f}^i_j(x(s))W^j(s,t) + \tilde{g}^i_{rj} A^{rj}(s,t) + O\left(\theta(\tilde{\omega}_{st}) + (t-s)^{3\beta}\right)$$

and the result follows. □

Lemma 4.5. *Let $0 < \alpha < 1$ and consider the PDE*

$$\sigma^{kh}(y)\partial_{kh}\psi(y) + \rho^k \partial_k \psi(y) = 0 \tag{27}$$

where σ is a matrix and ρ a vector of C^α functions on a neighbourhood of the origin in $\mathbb{R}^n$, such that $\sigma(0)$ is positive definite. Then for any $\eta > 0$, we can find a solution ψ in $C^{2+\alpha}$ on a neighbourhood of the origin, such that $|D\psi(0) - e_1| < \eta$, where e_1 is the vector $(1, 0, \cdots, 0)$.

Proof. We use the change of variable $y = \epsilon x$ to obtain the equation $\sigma_{kh}(\epsilon x)\partial_{kh}\phi(x) + \epsilon\rho^k(\epsilon x)\partial_k\phi(x) = 0$. By the Schauder theory (see [3]), for small $\epsilon \geq 0$ this equation has a unique $C^{2+\alpha}$ solution ϕ_ϵ in the unit ball satisfying $\phi_\epsilon(y) = y^1$ on the boundary $\{|y| = 1\}$, and ϕ_ϵ depends continuously on ϵ. Also $\phi_0(y) = y^1$ so $D\phi_0(0) = e_1$. Hence for small enough $\epsilon > 0$ we have $|D\phi_\epsilon(0) - e_1| < \eta$ and then we can take $\psi(y) = \epsilon\phi_\epsilon(y/\epsilon)$. □

Proposition 4.6. *Suppose $f \in C^\gamma$ and $1 < p < \gamma$. Suppose also that the matrix $(f^i_j(y))$ has rank n for every $y \in \mathbb{R}^n$. Then, with probability 1, for any choice of y_0 the Stratonovich equation $dy_i = f^i_j \circ dW^j$ with $y(0) = y_0$ has either a solution in the sense of Definition 3.1 (with $\omega(t) = t$) for all $t \geq 0$ or, for some $T > 0$, a solution on $0 \leq t < T$ with $y(t) \to \infty$ as $t \to T$. Moreover the solution is unique in the sense that if $\tilde{y}$ is any solution on $0 \leq t < \tau$ in the sense of Definition 3.1 then $\tilde{y} = y$ for $0 \leq t < \tau$.*

Proof. Let $y_1 \in \mathbb{R}^n$. Then by Lemma 4.5 we can find $C^{1+\gamma}$ functions ψ^i for $i = 1, \cdots, n$ satisfying (27) with (26) and such that $\psi^i(y_1)$ is close to e_i, where $e_1, \cdots, e_n$ is the standard basis of $\mathbb{R}^n$. Then $\psi = (\psi^1, \cdots, \psi^n)$ has non-zero Jacobian at y_1 and so is a diffeomorphism on a neighbourhood U of y_1, so that we can find $\tilde{f} \in C^\gamma_0$ such that $\tilde{f}^i_j(\psi(y)) = \partial_h\psi^i(y)f^h_j(y)$ for $y \in U$.

Hence we can cover $\mathbb{R}^n$ by a sequence of open sets (U_m) such that for each m there is a $C^{1+\gamma}$ mapping $\psi^{(m)} : U \to \mathbb{R}^n$ satisfying (27) with (26)and $f^{(m)} \in C^\gamma_0$ such that $f^{(m)i}_j(\psi^{(m)}(y)) = \partial_h\psi^{(m)i}(y)f^{(m)h}_j(y)$ for $y \in U_m$. To the Itô equation $dy^i = f^{(m)i}_j dW^j$ we associate a solution flow and $Z^{(m)}(x, s, t)$ as before, and then by Lemma 4.2, with probability 1 there is a double sequence (C_{rm}) such that $|Z^{(m)}(s, t, x)| \leq C_{rm}(t-s)^q$ whenever $r, m \in \mathbb{N}$, $0 \leq s < t < r$ and $x \in \psi^{(m)}(U_m)$. We fix a Brownian path W for which this holds, and now prove uniqueness as in the Itô case.

If we have two solutions y and $\tilde{y}$ with the same initial condition which are not identical, then we define τ_1 and y_1 just as in the proof of Proposition 4.3. Then $y_1 \in U_m$ for some m. Let $\tau > \tau_1$ be such that for $\tau_1 \leq t \leq \tau$, $y(t)$ and $\tilde{y}(t)$ are in U_m. Then $x(t) = \psi^{(m)}(y(t))$ and $\tilde{x}(t) = \psi^{(m)}(\tilde{y}(t))$ are, by Lemma 4.4 both solutions of the Itô equation $dx_i = f^{(m)i}_j(x)dW^j$ on $[\tau_1, \tau]$ in the sense of Definition 3.1 and the proof is concluded just as for Proposition 4.3. □

We remark that versions of Propositions 4.3 and 4.6 can be proved in the same way when f is given to be C^γ on an open set V in $\mathbb{R}^n$ and $y_0 \in V$, and in the case of Proposition 4.6 the matrix $(f^i_j(y))$ is assumed to be nonsingular for all $y \in V$. Then we obtain a solution $y(t) \in V$ which is either defined for all $t > 0$, or defined on $[0, T)$ and $y(t)$ leaves V as $t \to T$ in the sense that $y^{-1}([0, t])$ is a compact subset of V for all $0 \leq t < T$.

Next we show that uniqueness can fail for $f \in C^{2-\epsilon}$.

Lemma 4.7. *Let $W(t)$ denote standard Brownian motion in $\mathbb{R}^d$ where $d \geq 5$. Let $0 < \alpha < d - 4$. Then for $\epsilon > 0$,*

$$\mathbb{P}(\mathrm{dist}(W([0, 2]), W([3, \infty))) < \epsilon) < C\epsilon^\alpha$$

where C is a constant depending only on d and α.

Proof. We use $c_1, c_2, \cdots$ for constants which depend only on d and α. First we make the observation that, for any given ball $B(a,r)$, we have $\mathbb{P}(W(t) \in B(a,r)$ for some $t \geq 1) \leq c_1 r^{d-2}$, which can be verified by a straightforward calculation.

Let $\gamma = (d-2-\alpha)^{-1} < \frac{1}{2}$ and fix a positive integer N. We write $t_j = \frac{j}{N}$. Then, for $j = 0, 1, \cdots, 2N-1$ we have $\mathbb{P}(W([t_j, t_{j+1}]) \nsubseteq B(W(t_j), N^{-\gamma})) \leq c_2 e^{-\frac{1}{2}N^{1-2\gamma}}$ and so

$$\mathbb{P}(\text{for all } j = 0, 1, \cdots, 2N-1 \text{ we have } W([t_j, t_{j+1}]) \subseteq B(W(t_j), N^{-\gamma})) \geq 1 - 2c_2 N e^{-\frac{1}{2}N^{1-2\gamma}}$$

Also, by the above observation (applied to W starting at time 2), we have for each $j = 0, 1, \cdots, 2N-1$ that

$$\mathbb{P}(W([3,\infty)) \text{ meets } B(W(t_j), 2N^{-\gamma})) \leq c_3 N^{-(d-2)\gamma}$$

and so

$$\mathbb{P}(W([3,\infty)) \text{ avoids } B(W(t_j), 2N^{-\gamma}) \text{ for } j = 0, 1, \cdots, 2N-1) \geq 1 - 2c_3 N^{1-(d-2)\gamma}$$

Putting these facts together we obtain

$$\mathbb{P}(\operatorname{dist}(W([0,2]), W([3,\infty))) < 2N^{-\gamma}) < c_3 N^{1-(d-2)\gamma} + 2Nc_2 e^{-\frac{1}{2}N^{1-2\gamma}} < c_4 N^{1-(d-2)\gamma}$$

and the lemma follows on choosing N so that $\epsilon \approx N^{-\gamma}$. □

Theorem 4.8. *Let $\epsilon > 0$ and let $W(t)$ be standard Brownian motion on $\mathbb{R}^6$. Then, with probability 1, there exists a compactly supported $C^{2-\epsilon}$ function f on $\mathbb{R}^6$, such that the system*

$$dy^i = dW^i,\ i = 1, \cdots, 5, \qquad dy^6 = f(y)dW^6$$

has infinitely many solutions, in the sense of Definition 3.1, satisfying the initial condition $y(0) = 0$.

Proof. We use $b_1, b_2, \cdots$ to denote positive constants which can depend on ϵ but on nothing else. We write $\eta = \epsilon/3$ and introduce the notation $I_n = [(n+1)^{-\eta}, n^{-\eta}]$, $\tilde{I}_n = [(n+\frac{4}{3})^{-\eta}, (n-\frac{1}{3})^{-\eta}]$. We also write $\rho_k = k^{-4}2^{-k(1+\eta)/2}$ for $k = 1, 2, \cdots$. We write $W^*(t) = (W^1(t), \cdots, W^5(t))$.

For a given path $W^*(t)$, we define Ω_k to be the set of odd integers n with $2^k \leq n < 2^{k+1}$ and $\operatorname{dist}(W^*(I_n), W^*([0,1]\backslash\tilde{I}_n)) \geq \rho_k$. Using Lemma 4, and the fact that $|I_n| \sim \eta n^{-1-\eta}$, we see that for a given odd n with $2^k \leq n < 2^{k+1}$, we have $\mathbb{P}(n \notin \Omega_k) \leq b_1 k^{-2}$.

For $n \in \Omega_k$ we find f_n in $C^2(\mathbb{R}^5)$ such that $f_n(x) = 1$ if $x \in W^*(I_n)$ and $f_(x) = 0$ if $\operatorname{dist}(x, W^*(I_n)) \geq \rho_k$, and such that $\|f_n\|_{C^\alpha} \leq b_2\rho_k^\alpha$ for $0 < \alpha \leq 2$. Note that then $f_n(W^*(t)) = 0$ for $t \in [0,1]\backslash\tilde{I}_n$. Now let $\alpha_n = \int f_n(W^*(t))dW^6(t)$; then (for a fixed path W^*), α_n is normally distributed with mean 0 and variance $\int f_n(W^*(t))^2 dt \geq |I_n|$, so $\mathbb{P}(|\alpha_n| \leq k^{-2}2^{-k(1+\eta)/2}) \leq b_3 k^{-2}$.

Now we define V_k to be the set of odd integers n with $2^k \leq n < 2^{k+1}$ such that either $n \notin \Omega_k$, or $n \in \Omega_k$ and $|\alpha_n| \leq k^{-2}2^{-k(1+\eta)/2}$. Then for each odd n with $2^k \leq n < 2^{k+1}$ we have $\mathbb{P}(n \in V_k) \leq b_4 k^{-2}$. So if X_k is the cardinality of V_k then $\mathbb{E}X_k \leq b_4 k^{-2}2^{k-1}$ and so $\mathbb{P}(X_k \geq 2^{k-2}) \leq 2b_4 k^{-2}$. Hence almost surely there is k_0 such that for $k > k_0$ we have

$X_k \leq 2^{k-2}$, which implies $\sum_{n\in\Omega_k} |\alpha_n| \geq \frac{1}{4}k^{-2}2^{k(1-\eta)/2}$. Now let σ_n =sign(α_n); then we have, for $k > k_0$, that $\sum_{n\in\Omega_k} \sigma_n\alpha_n \geq \frac{1}{4}k^{-2}2^{k(1-\eta)/2}$.

We also have that for $n \in \Omega_k$, $\mathbb{P}(\max_t \left|\int_t^1 f_n(W^*(s))dW^6(s)\right| > 1) \leq b_5 e^{-b_6 n}$, and so with probability 1 we can (redefining k_0 if necessary) suppose that $\left|\int_t^1 f_n(W^*(s))dW^6(s)\right| \leq 1$ for $n \in \Omega_k$, when $k > k_0$.

Let ϕ be a smooth function of one variable, vanishing outside $[-2, 2]$, such that $\phi(x) = x$ for $|x| \leq 1$. Then define f_k on $\mathbb{R}^6$ by $f_k(x) = \rho_k^{2-\epsilon}\phi(x^6/\rho_k)\sum_{n\in\Omega_k} \sigma_n f_n(x_1, \cdots, x_5)$, and set $f(x) = \sum f_k(x)$. Next, let $\psi(t) = \sum_k \sum_{n\in\Omega_k} \psi_n^{\sigma_n}(t)$ where

$$\psi_n^\sigma(t) = -\rho_k^{1-\epsilon}\sigma \int_t^1 f_n(W^*(s))dW^6 + \frac{1}{2}\rho^{2(1-\epsilon)} \int_t^1 f_n(W^*(s))^2 ds$$

for $\sigma = \pm 1$.

Now for $t < 2^{-(k+1)\eta}$, where $k > k_0$, we have

$$\rho_k^{1-\epsilon} \sum_{n\in\Omega_k} \int_t^1 f_n(W^*(s))dW^6(s) = \rho_k^{1-\epsilon} \sum_{n\in\Omega_k} |\alpha_n| \geq \frac{1}{4}\rho_k^{1-\epsilon}k^{-2}2^{k(1-\eta)/2} \geq k^{-6}2^{k(\epsilon+\eta\epsilon-2\eta)/2} \geq 2^{k\eta/2}$$

for k large enough. Note also that for all n, $\int_t^1 f_n(W^*)dW^6$ is either α_n or 0, unless $t \in \tilde{I}_n$, which for a given t can occur for at most 2 values of n. Taking into account the fact that $\left|\int_t^1 f_n(W^*(s))dW^6(s)\right| \leq 1$, and noting that the second term in the expression for $\psi(t)$ is bounded by 1 in absolute value, it follows that $\psi(t) \leq -bt^{-1/2}$ for t small.

Now let $y(t) = A\exp\psi(t)$, where A is a constant. Then

$$dy(t) = \sum_k \rho_k^{1-\epsilon} \sum_{n\in\Omega_k} \sigma_n f_n(W^*(t))y(t)dW^6(t) \tag{28}$$

and if $|A|$ is small enough, $|y(t)| < \rho_k$ whenever any $f_n(W^*(t))$ is non-zero, so (28) can be written as $dy(t) = f(W(t))dW^6(t)$.

It follows that the system in the statement of the theorem has solution $y^i = W^i$, $i = 1, \cdots, 5$, $y^6(t) = A\exp\psi(t)$ for any sufficiently small A, provided we can verify that (28) holds in the sense of (10). This can be done easily as follows: for $2^k \leq n < 2^{k+1}$ and $\sigma = \pm 1$ let $y_n^\sigma(t) = \exp\psi_n^\sigma(t)$ and then

$$dy_n^\sigma = \rho_k^{1-\epsilon}\sigma f_n(W^*(t))y_n^\sigma(t)dW^6(t)$$

and then from Hölder bounds for W we can deduce that

$$y_n^\sigma(t) - y_n^\sigma(s) - \rho_k^{1-\epsilon}\sigma f_n(W^*(s))y_n^\sigma(s)\{W^6(t) - W^6(s)\} - \rho_k^{1-\epsilon}\sigma\sum_{j=1}^5 \partial_j f_n(W^*(s))A_I^{j6}(s,t) \tag{29}$$

is bounded in absolute value by $C_k(t-s)^\gamma$ where C_k is dominated by a suitable power of 2^k. Now note that on $\tilde{I}_n$, y is $y_n^{\sigma_n}$ multiplied by a positive constant which is $\leq A\exp(-b2^{k\eta/2})$.

This rapid exponential decay as $t \to 0$ means that the required bound for y follows easily from the above bound for (29). $\square$

Next we apply Theorem 3.6, with the improvement described in Remark 5, to show that with probability 1 uniqueness holds for every f in C^2. The proof requires a variant of the Law of the Iterated Logarithm. To state this we introduce the following notation: given $\tau \geq 0$, let

$$M(\tau) = \max \frac{\sum_i |W^i(t) - W^i(s)|^2 + \sum_{r,j} |A^{rj}(s,t)|}{(t-s)\log\log(t-s)}$$

where the max is over all s, t with $0 \leq s \leq \tau \leq t \leq T$ and $t - s \leq \frac{1}{10}$. Then we have

Lemma 4.9. *For any $T > 0$ there are constants c_1 and c_2 such that $\mathbb{P}(M(\tau) \geq K) \leq c_1 e^{-c_2 K}$ for any $\tau \in (0,T)$ and $K > 0$.*

Proof. For any interval $I = (s,t)$ we write $X_I = \sum_i |x^i(t) - x^i(s)|^2 + \sum_{r,j} |A^{rj}(s,t)|$. Then if I has length 2^{-k} we have $\mathbb{P}(X_I \geq \mu 2^{-k}) \leq C_1 e^{-C_2\mu}$ for all $\mu > 0$. Let $\lambda > 1$. Then the probability that, for some $k > 1$ and some dyadic $I \subseteq [0,T]$ with length 2^{-k}, we have $X_I > \lambda 2^{-k/2}(|\tau - s| + 2^{-k})^{1/2}$, does not exceed

$$C_1 \sum_{k=2}^{\infty} \sum_I \exp(-C_2\lambda \log k(1 + 2^k|\tau - s|)^{1/2} \leq C_3 \sum_{k=1}^{\infty} k^{-C_2\lambda} \leq C_4 2^{-C_2\lambda}$$

where $\sum_I$ denotes a sum over all dyadic intervals of length 2^{-K} in $[0,T]$. Hence, with probability at least $1 - C_4 2^{-C_2\lambda}$ we have

$$X_I \leq \lambda 2^{-k/2}(|\tau - s| + 2^{-k})^{1/2} \tag{30}$$

for all $k \geq 2$ and dyadic intervals $I \subseteq [0,T]$ with length 2^{-k}. Now if $I = (s,t)$ is any subinterval of $[0,T]$ containing τ, then we can express I as the union of non-overlapping dyadic intervals, such that not more then 2 of them can have the same length. Then when (30) holds we obtain $X_I \leq C_5\lambda(t-s)\log\log\frac{1}{t-s}$. Then $M(\tau) \leq C_5\lambda$, and this holds with probability at least $1 - C_4 2^{-C_2\lambda}$, which gives the result. $\square$

Theorem 4.10. *If $W^i(t)$ are independent Brownian motions, then with probability 1, for all $f \in C^2$ the equation (1), with $x^i = W^i$, has a unique solution in the sense of Definition 3.1.*

Proof. We work on a fixed interval $[0,T]$. By Remark 5, it suffices to show that, with probability 1, there is an increasing function $\omega(t)$ on $[0,T]$ such that $|W^j(t) - W^j(s)| \leq \omega(s,t)^{1/2}\left(\log\log\frac{1}{\omega(s,t)}\right)^{1/2}$ and $|A^{rj}(s,t)| \leq \omega(s,t)\log\log\frac{1}{\omega(s,t)}$ for all sufficiently small intervals $[s,t]$.

To do this, we apply Lemma 4.9 to assert that $\mathbb{E}M(\tau)^2 \leq C_3$ for each $\tau \in [0,T]$. Then $\mathbb{E}\int_0^T M(\tau)^2 < \infty$ so with probability 1, $\int_0^T M(\tau)^2 d\tau < \infty$. When this integral is finite we can define $\omega(t) = \int_0^t M(\tau)d\tau$ and note that by Cauchy-Schwartz $\omega(s,t) \leq C(t-s)^{1/2}$. Then $|W^i(t) - W^i(s)|^2 \leq \omega(s,t)\log\log\frac{1}{t-s} \leq C'\omega(s,t)\log\log\frac{1}{\omega(s,t)}$ with a similar bound for $A^{rj}(s,t)$, which completes the proof. $\square$

We conclude this section with an example showing that continuous differentiability of f is not sufficient for (even local) existence. The construction, which is similar to Theorem 4.8 is based on the following lemmas.

Lemma 4.11. *Let $W(t)$ be standard Brownian motion in $\mathbb{R}^d$ where $d \geq 5$, let $\gamma > \frac{1}{2} + \frac{1}{d-4}$ and suppose α satisfies $(\gamma - \frac{1}{2})^{-1} < \alpha < d - 4$. Let $\eta > 0$. Then with probability at least $1 - C\eta^\alpha$ we have that*

$$|W(s) - W(t) \geq \eta |s - t|^\gamma$$

for all $s, t \in [0, 1]$.

Proof. For integers $r, k \geq 0$ let E_{rk} denote the event $\operatorname{dist}(W(k2^{-r}, (k+1)2^{-r}), W((k+2)2^{-r}, \infty)) \leq \eta 2^{-r\gamma}$. Then by Lemma 4.7, $\mathbb{P}(E_{rk}) \leq C_1 \eta^\alpha 2^{-r(\gamma - \frac{1}{2})\alpha}$ and so, writing $\delta = (\gamma - \frac{1}{2})\alpha - 1$, we have $\mathbb{P}(\cup_{k=0}^{2^r - 1} E_{rk}) \leq C_1 \eta^\alpha 2^{-r\delta}$. The result follows by summing over r. □

Lemma 4.12. *Let $M > 0$ and let $W(t)$ be standard Brownian motion on $\mathbb{R}^8$. Then with probability 1 we can find a compactly supported smooth function f on $\mathbb{R}^7$ such that* $\sup |f| \leq 1$, $\sup |Df| \leq 1$ *and* $\int_0^1 f(W^*(t)) dW_8(t) > M$, *where* $W^* = (W_1, \cdots, W_7)$.

Proof. Fix γ with $\frac{5}{6} < \gamma < 1$ and then choose α so that $(\gamma - \frac{1}{2})^{-1} < \alpha < 3$. Note that then $\alpha > 1$, so we can fix β with $0 < \beta < 1$ and $\alpha\beta > 1$.

Let k be a positive integer. For $n = 1, 2, \cdots, 2^k$ let $I_{kn} = [(n-1)2^{-k}, n2^{-k}]$ and let $\tilde{I}_{kn} = ((n-2)2^{-k}, (n+1)2^{-k})$. Given a path W^*, let Ω_k be the set of odd integers n with $0 < n < 2^k$ and $\operatorname{dist}(W^*(I_{kn}), W^*([0,1] \backslash \tilde{I}_{kn})) \geq \rho_k$, where $\rho_k = k^{-\beta} 2^{-k/2}$, and such that also

$$|W^*(s) - W^*(t)| \geq k^{-\beta} 2^{(\gamma - \frac{1}{2})k} |s - t|^\gamma$$

for all $s, t \in \tilde{I}_{kn}$. Let N_k be the cardinality of Ω_k. By Lemmas 4.7 and 4.11, with $d = 7$ and scaling of t, we see that for any odd n we have $\mathbb{P}(n \notin \Omega_k) \leq C_1 k^{-\alpha\beta}$ so $\mathbb{E}(2^{k-1} - N_k) \leq C_1 2^{k-1} k^{-\alpha\beta}$ and hence $\mathbb{P}(N_k \leq 2^{k-2}) \leq 2C_1 k^{-\alpha\beta}$. It follows that, with probability 1, there exists k_0 such that $N_k > 2^{k-2}$ for all $k \geq k_0$.

Still considering a fixed path W_*, with $k \geq k_0$, we find for each $n \in \Omega_k$ a function g_{kn} on [0,1] such that $0 \leq g_{kn} \leq \rho_k$ everywhere, $g_{kn} = 0$ outside I_{kn}, $|g_{kn}(s) - g_{kn}(t)| \leq 2\rho_k |s - t|$ for all s, t, and $\int g_{kn}^2 = \frac{1}{3} \rho_k^2 2^{-k}$. Let $F = W^*([0,1])$ and define f_{kn} on F by $f_{kn}(W^*(t)) = g_{kn}(t)$ and note that from the definition of Ω_k we have

$$|f_{kn}(x) - f_{kn}(y)| \leq C_2 \rho_k \min \left(1, \left\{\frac{|x - y|}{\rho_k}\right\}^{1/\gamma}\right) \tag{31}$$

for all $x, y \in F$.

Now let $\alpha_{kn} = \int f_{kn}(W^*(t)) dW_8(t) = \int g_{kn}(t) dW_8(t)$. Conditional on W^*, for fixed k the α_{kn} are independent normally distributed random variables with mean 0, and $\operatorname{Var}(\alpha_{kn}) = \frac{1}{3} 2^{-k} \rho_k^2$. Now let $X_k = \sum_{n \in \Omega_k} |\alpha_{kn}|$. Then, using $\rho_k = 2^{-k/2} k^{-\beta}$, we obtain $\mathbb{E}X_k = \sqrt{2/3\pi} N 2^{-k} k^{-\beta}$ and $\operatorname{Var}(X_k) = \frac{1}{3} N_k 2^{-2k} k^{-2\beta}$. Then by Chebychev's theorem $\mathbb{P}(X_k \leq N_k 2^{-k-1} k^{-\beta}) \leq C_3 N_k^{-1}$. Since $N_k \geq 2^{k-2}$ we deduce $\mathbb{P}(X_k \leq \frac{1}{8} k^{-\beta}) \leq C_4 2^{-k}$. It follows that with probability 1 we have $\sum_{k=k_0}^\infty X_k = \infty$, so we can find k_1 so that $\sum_{k=k_0}^{k_1} X_k > M$.

We now need to extend f_{kn} to the whole of $\mathbb{R}^7$ and smooth it. To this end, we use Whitney's extension theorem (see Section VI.2 of [10]) which gives a bounded linear mapping T from the space of Lipschitz functions on F to the Lipschitz functions on $\mathbb{R}^7$. We also let $\phi \in C_0^\infty(\mathbb{R}^7)$ with $\int \phi = 1$, set $\phi_\epsilon(x) = \epsilon^{-7}\phi(x/\epsilon)$, and let $f_{kn}^\epsilon = \phi_\epsilon * Tf_{kn}$ for $\epsilon > 0$. Let $\alpha_{kn}^\epsilon = \int f_{kn}^\epsilon(W^*(t))dW_8(t)$. Then with probability 1, $\alpha_{kn}^\epsilon \to \alpha_{kn}$ as $\epsilon \to 0$. So if ϵ is chosen small enough, we have $\sum_{k=k_0}^{k_1}\sum_n |\alpha_{kn}^\epsilon| > M$. We fix such an ϵ and let σ_{kn} =sign(α_{kn}^ϵ). Now if $h = \sum_{k=k_0}^{k_1}\sum_n \sigma_{kn} f_{kn}$ then (31) implies a Lipschitz bound $|h(x) - h(y)| \le C_5|x-y|$ for $x, y \in F$. Now let $f = \sum_{k=k_0}^{k_1}\sum_n \sigma_{kn} f_{kn}^\epsilon = \phi_\epsilon * Th$. Then f is smooth and $|Df| \le C_5$ everywhere. Moreover $\int f(W^*(t))dW_8(t) = \sum_{k=k_0}^{k_1} |\alpha_{kn}^\epsilon| > M$, completing the proof. □

Theorem 4.13. *Let $W(t)$ be standard Brownian motion on $\mathbb{R}^8$. Then with probability 1 there exists a compactly supported continuously differentiable function f on $\mathbb{R}^7$, which is C^∞ on $\mathbb{R}^7\backslash\{0\}$, such that the system*

$$dy_i = dW_i,\ i = 1, \cdots, 7, \qquad dy_8 = f(y_1, \cdots, y_7)dW_8$$

has no solution satisfying $y(0) = 0$. To be more precise, there is no continuous $y(t)$ on [0,1], such that $y(0) = 0$ and the above equation is satisfied locally on $(0, 1)$ in the sense of Definition 3.1, this notion being well-defined since, with probability 1, W^* avoids the origin for $t > 0$, and f is smooth away from 0.

Proof. For k even and nonnegative let $I_k = [2^{-k-1}, 2^{-k}]$. With probability 1 the intervals $W^*(I_k)$ are disjoint so we can find a sequence of smooth functions ψ_k with disjoint compact supports ($k = 0, 2, 4, \cdots$) such that $\psi_k = 1$ on $W^*(I_k)$. By Lemma 4.12 we can find smooth f_k such that $\|f_k\|_{C^1}\|\psi_k\|_{C^1} \le k^{-2}$ and $\int_{I_k} f_k(W^*(t))dt > 1$. Let $f = \sum f_k\psi_k$; then f is C^1 and $\int_{I_k} f_k(W^*(t))dt > 1$ for even k. For this f, any solution to the system must satisfy $W_8(2^{-k}) - W_8(2^{-k-1}) > 1$ for all even k, and so cannot be continuous at 0. □

One may expect that the dimensions of the spaces in Theorems 4.8 and 4.13 could be considerably reduced. The point of the high-dimensional Brownian paths is to give good separation between segments of the path, which avoids technical problems in the proofs. Constructions in lower dimensions would probably be more complicated.

We remark that rough path theory can be used to interpret anticipating stochastic differential equations of the form $dy^i = f_j^i(y)dW^j$ where f_j^i is random in the sense that it depends on the path W, without any adaptedness condition, provided f has, with probability 1, the required smoothness w.r.t. y for the theory to apply. Theorem 4.8 and the results following it can be interpreted in this light. Thus when f is almost surely C^2 as a function of y, Theorem 4.10 asserts the existence of a unique solution, with this interpretation. The proofs of Theorems 4.8 and 4.13, in which f is constructed given the path W, can easily be modified so that f depends measurably on W, and give counterexamples in this setting.

Other interpretations of anticipating SDEs can be found for example in [9]. See [1] for a recent study of the relation of the rough path approach to such other approaches.

5. Other examples

The examples below indicate that the smoothness requirements on f in the results of sections 2 and 3 are sharp in respect of the inequalities relating γ and p.

Example 1. Nonuniqueness of solutions for $f \in C^\gamma$ when $1 < \gamma < p < 2$.

Suppose $1 < \gamma < p < 2$. Let β and ρ be large positive numbers with $\gamma < \frac{\rho}{\beta} < \frac{\rho+1}{\beta} < p$, and let $\alpha = p^{-1}$. Let $x^1(t) = t^\beta \cos(t^{-\rho})$, $x^2(t) = t^\beta(2 + \sin(t^{-\rho}))$. Then $x^i \in C^\alpha$ since $\alpha < \beta/(\rho+1)$. Next, we can find a C^γ function f such that $f(y^1, y^2) = (y^2)^\gamma$ if $|y^1| > y^2 > 0$ and it is 0 if $y^1 = 0$. Then the system

$$dy^1 = f(y^1, y^2)dx^1, \qquad dy^2 = dx^2, \qquad y^i(0) = 0$$

has two solutions in C^α for small $t \geq 0$:
$y^2 = x_2$, $y^1 = 0$ and $y^2 = x^2$, $y^1 = \int (x^2)^\gamma dx^1$.

To verify the second solution, one needs to check that $y^1 \geq \text{const } t^{\beta(\gamma+1)-\rho} \geq 3t^\beta \geq x^2$ for t small.

Example 2. Nonuniqueness of solutions for $f \in C^\gamma$ when $2 < \gamma < p < 3$.

When $2 < \gamma < p < 3$ we can use the same construction as in example 1, again with $\gamma < \frac{\rho}{\beta} < \frac{\rho+1}{\beta} < p$. Again we get the same solutions as above, provided we interpret the differential equation naively (everything being smooth for $t > 0$). However this does not fit in with the theory in section 3, because it requires $A^{ij}(s,t) = \int_s^t \{x^i(u) - x^i(s)\}dx^j(u)$ (interpreting the integrals naively), which does not satisfy the $p/2$-variation requirement.

One can get round this problem by defining $A^{ij}(s,t) = -x^i(s)\{x^j(t) - x^j(s)\}$. One can check that this satisfies the consistency condition and the variation requirement, and that then both choices of y^1, y^2 are solutions, in the sense of Definition 3.1 of the modified system

$$dy^1 = (1-\rho)f(y^1, y^2)dx^1, \qquad dy^2 = dx^2, \qquad y^i(0) = 0$$

Example 3. Nonexistence of solutions for $f \in C^{p-1}$ when $1 < p < 2$.

Let $1 < p < 2$ and let $\alpha = 1/p$. For $k = 1, 2, \cdots$ let n_k be the smallest integer $\geq 2^{k-1}/(k\pi)$ and let $t_k = \pi n_k 2^{1-k}$; then $0 < t_k \leq \pi$ and $t_k \sim 1/k$ for k large.

For $t \in [0, \pi]$ let $x^1(t) = \sum 2^{-\alpha k} \sin(2^k t)$ where the sum is over those integers $k \geq 1$ with $t_k \geq t$. Then $x^1 \in C^\alpha$, and is locally Lipschitz on $(0, \pi]$. Also define $z(t) = \sum_{k=1}^\infty 2^{-(1-\alpha)k} \cos(2^k)$. Then $z \in C^{1-\alpha}$. Now, using Lemma 5.1 below, we can find x^2 and x^3 in C^α such that $|(x^2(s), x^3(s)) - (x^2(t), x^3(t))| \geq$const$|s-t|^\alpha$. Then, using Whitney's extension theorem, we can write $z(t) = f(x^2(t), x^3(t)$ where $f \in C^{p-1}$.

Now consider the system

$$dy^1 = f(y^2, y^3)dx^1, \;\; dy^2 = dx^2, \;\; dy^3 = dx^3; \quad y^1(0) = 0, \;\; y^2(0) = x^2(0), \;\; y^3(0) = x^3(0)$$

Suppose we have a solution (in the sense of Definition 2.1) of this system on an interval $[0, \tau]$, where $0 < \tau < \pi$. Then we must have $y^2 = x^2$, $y^3 = x^3$, and, since x^1 is locally Lipschitz for $t > 0$, the equation can be interpreted naively for $t > 0$, and we have, for any $0 < s < \tau$, that

$$\begin{aligned} y^1(\tau) - y^1(s) &= \int_s^\tau z dx^1 = \sum_k{}' \sum_{l=1}^\infty 2^{(1-\alpha)(k-l)} \int_s^{\tau_k} \cos(2^k t)\cos(2^l t) dt \\ &= \frac{1}{2}\sum_k{}' \sum_{l=1}^\infty 2^{(1-\alpha)(k-l)} \frac{\sin(2^k+2^l)\tau_k - \sin(2^k+2^l)s}{2^k+2^l} + \frac{1}{2}\sum_k{}'(\tau_k - s) \\ &\quad + \frac{1}{2}\sum_k{}' \sum_{l\neq k} 2^{(1-\alpha)(k-l)} \frac{\sin(2^k-2^l)\tau_k - \sin(2^k-2^l)s}{2^k-2^l} \\ &= \frac{1}{2}\log\frac{1}{s} + O(1) \end{aligned}$$

as $s \to 0$. Here $\tau_k = \min(\tau, t_k)$ and $\sum_k'$ denotes a sum over those k for which $t_k > s$. But then $y^1(s) \to \infty$ as $s \to 0$, so (2) is not satisfied at 0.

Hence no solution exists on any interval $[0, \tau]$.

The above proof used the following (probably known) lemma:

Lemma 5.1. *Suppose $\frac{1}{2} < \alpha < 1$. Then we can find positive constants c_1 and c_2, and a function u on [0,1] taking values in $\mathbb{R}^2$, such that*

$$c_1|s-t|^\alpha \le |u(s) - u(t)| \le c_2|s-t|^\alpha$$

for all $s, t \in [0,1]$.

Proof. We shall use the following terminology: given a lattice of squares of side ϵ, a *chain of squares* of side ϵ is a sequence $Q_1, \cdots, Q_n$ of squares in the lattice, such that Q_i and Q_{i+1} have one side in common, Q_i and Q_j are disjoint if $|i-j| > 2$ and have at most a corner in common if $|i-j| = 2$.

Now, since $\frac{1}{2} < \alpha < 1$, it is not hard to construct bounded sequences of integers k_r and m_r, such that $k_r \ge 2$, m_r is odd, $n_r \le m_r \le k_r^2$ where $n_r = 2k_r + 1$, and such that the sequence $\epsilon_r/\delta_r^\alpha$ is bounded above and away from 0, where $\epsilon_r = (n_1 n_2 \cdots n_r)^{-1}$ and $\delta_r = (m_1 m_2 \cdots m_r)^{-1}$.

Next, we construct inductively a sequence $C_0, C_1, C_2, \cdots$ where C_r is a chain of squares of side ϵ_r. We start by letting C_0 be a single square of side 1. Next, supposing C_r constructed as a chain of squares of side ϵ_r, we divide each square Q of C_r into a $n_{r+1} \times n_{r+1}$ grid of squares of side ϵ_{r+1}. Two of the sides of Q abut other squares of C_r and we now construct a chain of squares of side ϵ_{r+1} consisting of squares of this grid, joining the middle edge squares of these two sides, containing no other edge squares, and consisting of m_{r+1} squares.

The sequence C_r converges to a curve C, which can be parametrised by $t \to u(t)$, $t \in [0,1]$, in such a way that $u(t)$ spends time δ_r in each square of C_r. To see that $u(t)$ satisfies the required inequality, suppose $s, t \in [0,1]$ and suppose $\delta_r < |s-t| \le \delta_{r-1}$. Then $u(s)$ and $u(t)$ belong to the same or adjoining squares of C_{r-1}, so $u(s) - u(t)| \le 3\epsilon_{r-1}$. On the other hand $u(s)$ and $u(t)$ are not in the same square of C_r, and not in adjoining squares of C_{r+1}, so $|u(s) - u(t)| \ge \epsilon_{r+1}$, which completes the proof. □

Example 4. Nonexistence of solutions for $f \in C^{p-1}$ when $2 < p < 3$.

The construction in Example 3 works for $2 < p < 3$, with the following modifications. We use the same definitions for x^1 and z .We need a modified Lemma 5.1, proved in a similar way, which assumes $\frac{1}{3} < \alpha < 1$ and gives $u = (x^2, x^3, x^4)$ taking values in $\mathbb{R}^3$. Using this and, for example, the version of Whitney's extension theorem in Theorem 4 of Section VI.2 of [10], we get $z(t) = f(x^2(t), x^3(t), x^4(t))$ where $f \in C^{p-1}$ and now $Df(x^2(t), x^3(t), x^4(t)) = 0$. We then consider the same system as in example 3 (with x^4, y^4 added in obvious fashion). Then, because $Df(y^2, y^3, y^4)$ is always 0, whatever choice is made for A^{rj}, the term involving A^{rj} in (10) always vanishes, and any solution in the sense of Definition 3.1 will be a naive solution for $t > 0$. Then the same argument as before shows that no solution exists.

6. Global existence and explosions.

When $x(t)$ is defined on $[0, \infty)$ and f is globally defined, Theorems 2.3-3.2 show that, under suitable conditions, equation (1) has either a solution for all positive t or a solution such that $|y(t)|$ goes to ∞ at some finite time (an explosion). In this section we investigate what conditions will ensure that no explosion occurs, so that a solution exists for all time.

For equations of the form (1) where $x(t)$ has (locally) bounded variation, it is not hard to show that if $D(R)$ is a positive increasing function for $R \geq 1$ with $\int_1^\infty D(R)^{-1} dR = \infty$ and if f is continuous on $\mathbb{R}^n$ and satisfies $|f(y)| \leq D(|y|)$ for all y then no solution can explode in finite time. The following theorem gives an analogous result for the case when x has finite p-variation for $p > 1$. In this case we require control of the growth of Hölder continuity bounds of f as well as $|f|$ itself.

We suppose that either (i) $1 < p < \gamma < 2$ or (ii) $2 \leq p < \gamma < 3$ and let $\beta = \gamma - 1$.

Theorem 6.1. *(a) Suppose $D(R)$ and $A(R)$ are positive increasing functions on $1 \leq R < \infty$ with $D(R) \leq R^\beta A(R)$, such that $|f(y)| \leq D(R)$ and that, in case (i) $|f(y') - f(y)| \leq A(R)|y' - y|^\beta$, while in case (ii) $|Df(y') - Df(y)| \leq A(R)|y' - y|^{\beta - 1}$ for $|y, y'| \leq R$. Suppose $x(t)$ has finite p-variation on each bounded interval, and in case (ii) $A(s,t)$ satisfies assumption 1 on each bounded interval. Then, provided*

$$\int_1^\infty \left\{A(R)^{1-p} D(R)^{p-1-\beta p}\right\}^{1/\beta} dR = \infty$$

no solution of (1) can explode in finite time.

(b) Conversely, suppose $D(R)$ and $A(R)$ are positive increasing functions on $1 \leq R < \infty$ with $D(R) \leq R^\beta A(R)$, and suppose

$$\int_1^\infty \left\{A(R)^{1-p} D(R)^{p-1-\beta p}\right\}^{1/\beta} dR < \infty$$

Then we construct f, $x(t)$, and in case (ii) $A(s,t)$, with the same conditions as in (a), such that (1) has a solution which explodes in finite time.

Remark. The condition $D(R) \leq R^\beta A(R)$ is natural, since the second condition on f in part (a) implies the existence of a constant C such that $|f(y)| \leq C + A(R)R^\beta$ for $|y| \leq R$.

Proof. (a) This is essentially a case of keeping track of the bounds in the arguments of Sections 2 and 3. We start with case (i).

Let $A_k = A(2^k)$ and $D_k = D(2^k)$. Then $\sum 2^k \left\{A_k^{1-p} D_k^{p-1-\beta p}\right\}^{1/\beta} = \infty$. Let k_0 be the smallest nonnegative integer such that $2^k > |y(0)|$. Then for $k = k_0, k_0+1, \cdots$ let t_k be the first time that $|y(t)| = 2^k$ (if for a given k no such time exists, then there can be no explosion).

Then for $t_k \le t \le t_{k+1}$ we have $|y| \le 2^{k+1}$, in which region $|f(y)| \le D_{k+1}$ and $|f(y) - f(y')| \le A_{k+1}|y-y'|^\beta$.

Now we apply the estimates of Lemma 2.4 (which, in view of Remark 1, apply to any solution) on the interval $[t_k, t_{k+1}]$, and note that in the proof of Lemma 2.4 we can take $B_1 = D_{k+1}$ and $B_2 = 2A_{k+1}D_{k+1}^\beta$. We can then take $L = 4A_{k+1}D_{k+1}^\beta(1-2^{1-\gamma/p})^{-1}$ and $\delta = (D_{k+1}^{1-\beta}A_{k+1}^{-1}(1-2^{1-\gamma/p})/4)^{p/\beta}$. Then on any time interval with $\Delta\omega \le \delta$ we have $\Delta y \le c_1(D/A)^{1/\beta}$ where $c_1 = 2(\frac{1}{4}(1-2^{1-\gamma/p}))^{1/\beta} < \frac{1}{4}$. Then, since $|y(t_{k+1}) - y(t_k)| \ge 2^k$, the number of intervals of length δ that fit into $[\omega(t_k), \omega(t_{k+1})]$ is at least the integer part of $c_1^{-1}2^k(A_{k+1}/D_{k+1})^{1/\beta}$ which, in view of the fact that $(D_{k+1}/A_{k+1})^{1/\beta} \le 2^{k+1}$, is $\ge c_1^{-1}2^{k-1}(A_{k+1}/D_{k+1})^{1/\beta}$.

Hence

$$\omega(t_{k+1}) - \omega(t_k) \ge 2^{k-1}c^{-1}(A_{k+1}/D_{k+1})^{1/\beta}\delta = \text{const } 2^k\{A_{k+1}^{1-p}D_{k+1}^{p-1-\beta p}\}^{1/\beta}$$

so $\sum(\omega(t_{k+1}) - \omega(t_k)) = \infty$, so $\omega(t_k) \to \infty$ as $k \to \infty$, which means there is no explosion.

The arguments for case (ii) is similar, using now the estimates of Lemma 3.4. First note that n this case we have for $t_k \le t \le t_{k+1}$ that $|Df(y) - Df(y')| \le A_{k+1}|y-y'|^{\beta-1}$ which together with $|f| \le D_{k+1}$ gives by interpolation (using the fact that $D_{k+1} \le 2^{(k+1)\beta}A_{k+1}$) that $|Df(y)| \le c(A_{k+1}D_{k+1}^{\beta-1})^{1/\beta}$.

Now we apply the estimates of Lemma 3.4 on $[t_k, t_{k+1}]$ and note that we can take $B_1 = A_{k+1}$, $B_2 = c_2D_{k+1}(A_{k+1}D_{k+1})^{1/\beta}$ and $B_3 = D_{k+1}$ provided $\delta < (D_{k+1}^{1-\beta}A_{k+1}^{-1})^{-1}$ and then bounding each term in (12) we find we can take $B_4 = c_3A_{k+1}D_{k+1}^\beta$. So apart from constants we get the same bounds for δ and L as in case (i) and the proof concludes in the same way.

(b) We construct a system of the form (1) with $d = 2$ and $n = 1$.

Let $\rho' = (\beta p + 1 - p)/\beta$, $\rho'' = (p-1)/\beta$, $\rho = \min(\rho', \rho'')$ and for $y \ge 1$ let $F(y) = A(y)^{-\rho''}D(y)^{-\rho'}$, so that the hypothesis gives $\int_1^\infty F(y)dy < \infty$.

We need to 'smooth' the functions A and D. Choose $r > \rho^{-1}$ and define $\tilde{D}(y) = \inf_{u\ge1} u^r D(y/u)$, $\tilde{A}(y) = \inf_{u\ge1} u^r A(y/u)$. Then $\tilde{D}(y) \le D(y)$ and $\tilde{A}(y) \le A(y)$. We let $\tilde{F}(y) = \tilde{A}(y)^{-\rho''}\tilde{D}(y)^{-\rho'}$. Then $\tilde{F}(y) \le \sup_{u\ge1} u^{-r\rho}F(y/u)$. Now if we extend F to $[0,\infty)$ by setting $F(y) = F(1)$ for $0 \le y < 1$ then we have

$$u^{-r\rho}F(y/u) \le r\rho F(y/u)\int_u^\infty v^{-r\rho-1}dv \le r\rho\int_u^\infty F(y/v)v^{-r\rho-1}dv$$

and so $\tilde{F}(y) \le r\rho\int_1^\infty F(y/v)v^{-r\rho-1}dv$ for $y \ge 1$.

Hence

$$\int_1^\infty \tilde{F}(y)dy \le r\rho \int_1^\infty \int_1^\infty F(y/v)v^{-r\rho-1}F(y/v)dvdy \le r\rho \int_1^\infty v^{-r\rho}dv \int_0^\infty F(y)dy < \infty$$

Next we fix a smooth non-negative function ϕ supported on the interval [1,2] such that $\int \phi = 1$, and define $D^*(y) = 2^{-r}\int \tilde{D}(yu)\phi(u)du$ for $y \ge 1$, and A^* similarly. Then $2^{-r}\tilde{D}(y) \le D^*(y) \le \tilde{D}(y)$ with a similar inequality for A^*. Then we set $F^*(y) = A^*(y)^{-\rho''}D^*(y)^{-\rho'}$ and we have $\int_1^\infty F^*(y)dy < \infty$.

For $y \ge 1$ we write $\lambda(y) = \int_1^y (A*/D^*)^{1/\beta}$, $\alpha(y) = (D^*(y)^{1-\beta}A^*(y)^{-1})^{1/\beta}$ and

$$f(y) = D^*(y)(-\sin\lambda(y), \cos\lambda(y)) \in \mathbb{R}^2$$

Let $t_* = \int_1^\infty F^*$ and define $y(t)$ on $[0, t_*)$ by $t = \int_1^y F^*$ and note that $y(t) \to \infty$ as $t \to t_*$. Then let $x(t) = \alpha(y(t))(\cos\lambda(y(t)), \sin\lambda(y(t))) \in \mathbb{R}^2$ for $0 \le t < t_*$, and let $x(t) = 0$ for $t \ge t_*$. Then the equation $dy(t) = f(y).dx(t)$ is satisfied, as a classical ODE on $[0, t_*)$. If we can show that x has finite p-variation and f satisfies the required β-Hölder bound then in case (i) y will satisfy (1) in the sense of Definition 2.1 and we will have the required example. In case (ii) we define $A^{ij}(s,t) = -x^i(s)\{x^j(t) - x^j(s)\}$; one can then check that the term involving A^{rj} in (10) vanishes, so that (10) will hold on any compact subinterval of $[0, t_*)$, and again we have the required example provided A^{ij} satisfies the $\frac{p}{2}$-variation condition.

As preparation for proving the required bounds we note that the assumption $D(y) \le y^\beta A(y)$ implies $D^*(y) \le y^\beta A^*(y)$ and so $\lambda'(y) \ge y^{-1}$. Hence if $y_1 < y$ and $\lambda(y) - \lambda(y_1) \le 1$ then it follows that $y \le ey_1$, and hence $D^*(y) \le e^r D^*(y_1)$ with a similar inequality for A^*, so the relative variation of each of the functions $D^*, A^*, \lambda', \alpha$ is bounded by a constant on $[y_1, y]$. Also we have

$$D^*(y) - D^*(y_1) \le C_1 D^*(y_1)(y - y_1)/y_1 \le C_2 D^*(y_1)(\lambda(y) - \lambda(y_1))$$

with similar bounds for A^* and α.

We now consider $x(t)$, and show in fact that it satisfies a $\frac{1}{p}$-Hölder condition. Let $t_1 < t_2 < t_*$, and we write y_1 for $y(t_1)$ etc. First we suppose that $\lambda_2 - \lambda_1 \le 1$. Then the discussion in the preceding paragraph shows that $|\alpha_2 - \alpha_1| \le C_3\alpha_1(\lambda_2 - \lambda_1)$ and so

$$|x_1 - x_2| \le |\alpha_2 - \alpha_1| + \alpha_1(\lambda_2 - \lambda_1) \le C_4\alpha_1(\lambda_2 - \lambda_1)$$

Also $\lambda_2 - \lambda_1 \le C - 5\alpha_1^{-p}(t_2 - t_1)$ so $\alpha_1 \le C_6(\frac{t_2 - t_1}{\lambda_2 - \lambda_1})^{1/p}$ and so

$$|x_2 - x_1| \le C_7(t_2 - t_1)^{1/p}(\lambda_2 - \lambda_1)^{1-1/p} \le (t_2 - t_1)^{1/p}$$

proving the Hölder estimate in this case. In the case $\lambda_2 - \lambda_1 > 1$ we have $|x_1|, |x_2| \le C(t_2 - t_1)^{1/p}$ giving the estimate in this case also, and the case $t_2 \ge t_*$ follows similarly. In case (ii) the $\frac{p}{2}$-variation condition for A^{ij} follows easily.

The treatment of $f(y)$ is similar. We consider case (i) first. When $\lambda_2 - \lambda_1 \le 1$ we have $|f_2 - f_1| \le C_8 D_1^*(\lambda_2 - \lambda_1)$ and $\lambda_2 - \lambda_1 \le C_9(A_1^*/D_1^*)^{1/\beta}(y_2 - y_1)$ so $D_1^* \le C_{10}A_1^*(\frac{y_2 - y_1}{\lambda_2 - \lambda_1})^\beta$ so

$$|f_2 - f_1| \le C_{11}A_1^*(y_2 - y_1)^\beta \le C_{11}A_1(y_2 - y_1)^\beta$$

The case $\lambda_2 - \lambda_1 > 1$ is treated in the same way as before.

For case (ii), first suppose $\lambda_2 - \lambda_1 \le 1$. We have the derivative bounds $\lambda' = (A^*/D^*)^{1/\beta}$, $|\lambda''| \le C_{12} y^{-1}(A^*/D^*)^{1/\beta}$, $(D^*)' \le C_{12} y^{-1} D^*$, $|(D^*)''| \le C_{12} y^{-2} D^*$ from which we deduce, remembering that $D^*(y) \le y^\beta A^*(y)$, that $|f''(y)| \le C_{13} D^*(A^*/D^*)^{2/\beta}$. Hence we have

$$\begin{aligned}|f_2' - f_1'| &\le C_{14} D_1^*(A_1^*/D_1^*)^{2/\beta}(y_2 - y_1) = C_{14} D_1^*(A_1^*/D_1^*)^{2/\beta}(y_2-y_1)^{2-\beta}(y_2-y_1)^{\beta-1} \\ &\le C_{15} D_1^*(A_1^*/D_1^*)^{2/\beta}(A_1^*/D_1^*)^{1-2/\beta}(y_2-y_1)^{\beta-1} \\ &= C_{15} A_1^*(y_2-y_1)^{\beta-1} \le C_{15} A_1 (y_2-y_1)^\beta\end{aligned}$$

as required. The case $\lambda_2 - \lambda_1 > 1$ is treated as before, using the bound $|f'| \le C_{13} D^*(A^*/D^*)^{1/\beta}$. □

7. Convergence of Euler approximations

In the situation of Section 2 ($1 \le p < \gamma \le 2$), Theorem 2.3 establishes the convergence of Euler approximations (3) to the solution as the mesh size of the partition tends to 0. In fact the proof gives a bound for the rate of convergence: from Remark 1 (and the fact that $f \in C^\gamma$ implies $f \in C^1$) we see that the solution satisfies (1) with $\theta(\delta) = C\delta^{2/p}$; then the last paragraph of the proof of Theorem 2.3 gives

$$|y_K - y(t)| \le \text{const} \sum_{k=0}^{K-1} \omega_{k,k+1}^{2/p},$$

where $\{y_k\}$ is given by (3) for a partition such that $t_K = t$.

In the situation of Section 3 ($2 \le p < \gamma \le 3$) similar reasoning leads to a bound

$$|y_K - y(t)| \le \text{const} \sum_{k=0}^{K-1} \omega_{k,k+1}^{3/p}$$

where now $\{y_k\}$ is given by the scheme (11).

Neither of the above results covers the known fact that Euler approximations of the form (3), containing no A^{rj} term, converge almost surely to solutions of Itô equations driven by Brownian motion. In this section we obtain a convergence result in the setting of Section 3, but using the Euler approximation (3) rather than (11); for this to work we need to impose an additional condition on the driving path, which can be thought of as a 'pathwise' version of the 'independent increments' property of Brownian motion.

One form of the convergence result for Itô equations states that, if $x(t)$ is a d-dimensional Brownian motion and f satisfies a global Lipschitz condition, and $T > 0$ is fixed, then with probability 1, for any $\epsilon > 0$ there is a constant C such that if $0 < t \le T$ then

$$|y(t) - y_K| \le C t^{1-\epsilon} K^{-\frac{1}{2}+\epsilon} \tag{32}$$

where $y(t)$ is the solution of (1), interpreted as an Itô equation, and $\{y_k\}$ is given by the Euler scheme (2) with $t_k = kt/K$. There is also a convergence result for non-uniform step sizes, provided the mesh points are stopping times. But convergence can fail if no restriction is imposed on the partition, as shown in [2].

For simplicity we use uniform step sizes, and then it is convenient to assume a Hölder condition of order $\alpha = \frac{1}{p}$ on the driving path rather than a p-variation condition.

We suppose $\frac{1}{3} < \alpha < \frac{1}{2}$ and $1-\alpha < \beta < 2\alpha$. We assume that the driving path $x \in C^\alpha[0,T]$ and that there is a constant B such that $A^{ij}(s,t)$ satisfies

$$\left|\sum_{l=k}^{m-1} A^{ij}(lh,(l+1)h)\right| \le B(m-k)^\beta h^{2\alpha} \tag{33}$$

whenever $0 < k < m$ are integers and $h > 0$ such that $mh \le T$. Under these hypotheses we have:

Theorem 7.1. *Suppose $f \in C^\gamma$ where $\gamma > \alpha^{-1}$. Let $0 < t \le T$, let K be a positive integer, and let z_k be defined by the Euler recurrence relation*

$$z^i_{k+1} = z^i_k + f^i_j(z_k)(x^j(t_{k+1}) - x^j(t_k))$$

where $t_k = kt/K$ and $z_0 = y_0$.

Let also y be the solution of (1) in the sense of Definition 3.1.

Then $|z_k - y(t)| \le Ct^{2\alpha}K^{\beta-2\alpha}$, where C is a constant independent of K and t.

Proof. For $0 \le l < l \le K$ we define $T_{kl}(z)$ to be y_l where $\{y_m\}$ satisfies (11) with $y_k = z$. From the estimates in Section 3 we have

$$|T_{kl}(z) - T_{kl}(z') - (z-z')| \le C_1\{(l-k)h\}^\alpha |z-z'| \tag{34}$$

where $h = t/K$. We also write $R^{ij}_k = \sum_{l=0}^{k-1} A^{ij}(kh,(k+1)h))$ and $g^i_r(y) = f^h_r(y)\partial_h f^i_j(y)$. Then we define (suppressing indices for notational simplicity)

$$u_{kl} = z_l - T_{kl}(z_k) + g(z_k)(R_l - R_k)$$

for $0 < k < l \le K$.

Now if $0 \le k < l < m \le K$ then

$$\begin{aligned} z_m &= T_{lm}(z_l) - g(z_l)(R_m - R_l) + u_{lm} \\ &= T_{lm}(T_{kl}(z_k) - g(z_k)(R_l - R_k) + u_{kl}) - g(z_l)(R_m - R_l) + u_{lm} \\ &= T_{km}(z_k) + v_{klm} - g(z_k)(R_l - R_k) - g(z_l)(R_m - R_l) + u_{kl} + u_{lm} \end{aligned}$$

where

$$v_{klm} = T_{lm}(T_{kl}(z_k) - g(z_k)(R_l - R_k) + u_{kl}) - T_{lm}(T_{kl}(z_k)) + g(z_k)(R_l - R_k) - u_{kl}$$

and we have used the fact that $T_{lm}(T_{kl}(z)) = T_{km}(z)$. Hence

$$u_{km} = v_{klm} + w_{klm} + u_{kl} + u_{lm} \tag{35}$$

where $w_{klm} = (g(z_k) - g(z_l))(R_m - R_l)$. By (33) we have the bound $|R_m - R_l| \le B(m-l)^\beta h^{2\alpha}$ and

$$|z_k - z_l| = |u_{kl} + (T_{kl}(z_l) - z_k) - g(z_k)(R_l - R_k)| \le C_2\{((l-k)h)^\alpha + |u_{kl}|\}$$

so

$$|w_{klm}| \le C_3\{((l-k)h)^\alpha\} + |u_{kl}|\}(m-l)^\beta h^{2\alpha}$$

And from (34) we obtain $|v_{klm}| \le C_4\{(m-l)h\}^\alpha\{|u_{kl}| + (l-k)^\beta h^{2\alpha}\}$. Putting these bounds into (35) gives

$$|u_{km}| \le |u_{kl}|\{1 + C_5((m-k)h)^\alpha\} + |u_{lm}| + C_5(m-k)^{\alpha+\beta}h^{3\alpha}$$

Also $u_{k,k+1} = 0$. It then follows by an inductive argument similar to that used in the proof of Lemma 3.4 that $|u_{km}| \le C(m-k)^{\alpha+\beta}h^{3\alpha}$ for $0 \le k < m \le K$. We apply this to u_{0K} and use the fact that, by (9) and Remark 3, $|T_{0K}(z_0) - y(t)| \le CKh^{2\alpha}$ together with the bound $|R_K - R_0| \le BK^\beta h^{2\alpha}$ to get the required bound for $z_K - y(t)$. □

The condition (33) holds for Brownian motion (with the Itô interpretation of A^{rj}) for all $\alpha < \frac{1}{2}$ and $\beta > \frac{1}{2}$ with probability 1, and so the bound (32) for the error of the Euler approximation follows from Theorem 6.1. Indeed Theorem 6.1 implies that, for almost all Brownian paths $x(t)$, for any $f \in C^\gamma$ where $\gamma > 2$ and $T > 0$, there is a constant C such that the bound (32) holds for uniform-step Euler approximations to the solution of (1) on $[0, T]$. The construction in Theorem 4.8 can be modified to show that this last statement can fail for $f \in C^\gamma$ if $\gamma < 2$. Indeed, for almost all 6-dimensional Brownian paths one can construct f in C^γ for all $\gamma < 2$ such that the Euler approximations to (1) fail to converge.

We also note that Theorem 7.1 implies that the solution can be obtained from the path $x(t)$ alone, since the $A(s,t)$ do not appear in the approximation. This indicates that when (33) holds the $A(s,t)$ are determined by the path $x(t)$. And indeed it is not hard to deduce from (33) that $A^{ij}(s,t)$ is the limit as $N \to \infty$ of $\sum_{k=0}^N x^i(t_k)\{x^j(t_{k+1}) - x^j(t_k)\}$ where $t_k = s + k(t-s)/N$. Then (33) is effectively a condition on the path $x(t)$.

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