DEEP- AND SHALLOW-WATER LIMITS OF STATISTICAL EQUILIBRIA FOR THE INTERMEDIATE LONG WAVE EQUATION

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Abstract. XXX

Contents

Date: November 16, 2023.

2020 *Mathematics Subject Classification.* 35Q35, 60F15, 60H30.

Key words and phrases. intermediate long wave equation; Benjamin-Ono equation; Korteweg-de Vries equation; Gibbs measures.

1. Introduction

1.1. **The intermediate long wave equation.** We consider the intermediate long wave equation (ILW) on the one-dimensional torus $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$:

$$\begin{cases} \partial_t u - \mathcal{G}_\delta \partial_x^2 u = \partial_x(u^2), \\ u|_{t=0} = u_0, \end{cases} \qquad (t,x) \in \mathbb{R} \times \mathbb{T}. \tag{1.1}$$

This equation was derived in [20, 28] to model the propagation of an internal wave at the interface of a stratified fluid of depth $\delta > 0$, where u denotes the amplitude of the interface. The phase speed is determined by the Fourier multiplier operator $\mathcal{G}_\delta$ defined by

$$\widehat{\mathcal{G}_\delta f}(n) = -i\Big(\coth(\delta n) - \frac{1}{\delta n}\Big)\widehat{f}(n), \quad n \in \mathbb{Z}^*,$$

where $\mathbb{Z}^* = \mathbb{Z}\setminus\{0\}$, with the convention that $\coth(\delta n) - \frac{1}{\delta n} = 0$ for $n = 0$. The ILW equation (1.1) has garnered much attention due to its physical relevance, playing a crucial rule in the study of gravitational waves in stratified fluids, wave propagation in atmospheric sciences and oceanography, and quantum field theory [13, 38, 47, 27, 52, 48, 49, 37, 35, 40, 5].

From an analysis perspective, (1.1) displays a rich structure, such as the existence of soliton solutions, an inverse scattering transform, a Lax pair structure, an infinite number of conservation laws, and a strong connection with the Korteweg-de Vries (KdV) and the Benjamin-Ono (BO) equations. However, the study of ILW (1.1) remains mostly open when compared to KdV and BO. For an overview on ILW, see the recent book [24] and the survey [51].

Our main goal in this work is to further the rigorous understanding of the convergence of ILW to BO and KdV from a *statistical viewpoint*. In particular, we construct an infinite family of measures which are invariant under the flow of ILW (1.1) and converge to corresponding invariant measures for BO ($\delta \to \infty$) and KdV ($\delta \to 0$), establishing the first probabilistic convergence result with uniqueness.

1.2. **BO and KdV as limits of ILW.** The ILW equation can be seen as an intermediate model between BO and KdV. In the deep-water limit (as $\delta \to \infty$), since the symbol of the operator $\mathcal{G}_\delta$ converges to that of the Hilbert transform $\mathcal{H}$ (see Lemma 2.2), the ILW equation (1.1) formally converges to BO:

$$\partial_t u - \mathcal{H}\partial_x^2 u = \partial_x(u^2). \tag{1.2}$$

In the shallow-water regime (as $\delta \to 0$), to avoid a dispersionless limiting equation, we consider the scaling transformation introduced in [1]:

$$v(t,x) = \tfrac{1}{\delta} u(\tfrac{1}{\delta} t, x), \tag{1.3}$$

which leads to the following scaled ILW (sILW):

$$\partial_t v - \widetilde{\mathcal{G}}_\delta \partial_x^2 v = \partial_x(v^2), \tag{1.4}$$

where $\widetilde{\mathcal{G}}_\delta := \frac{1}{\delta}\mathcal{G}_\delta$. In particular, u is a solution to ILW (1.1) if and only if v is a solution to sILW (1.4). Similarly to the deep-water regime, as $\delta \to 0$, due to the convergence of the symbol $\widetilde{\mathcal{G}}_\delta$ (see Lemma 2.3), we see that sILW (1.1) formally converges to KdV:

$$\partial_t v + \partial_x^3 v = \partial_x(v^2). \tag{1.5}$$

There has been significant interest in rigorously establishing these questions of convergence [20, 28, 50, 2, 30, 26, 31, 49, 34, 33, 12]. We emphasize the recent works [34, 12] where the regularity assumptions on the solutions were significantly lowered, in the periodic and Euclidean settings. These are deterministic results, where the convergence is established for each fixed initial data. In our work, we instead consider this convergence from a probabilistic viewpoint, as in [33].

In the present paper, we are concerned with the construction, invariance, and convergence of an infinite family of measures associated with ILW (1.1) (and sILW (1.4)), from which we infer on the limiting behavior of solutions as a statistical ensemble. Analogously to BO (1.2) and KdV (1.5), ILW (1.1) is completely integrable and it exhibits an infinite number of conserved quantities [50, 26, 16, 30, 25, 31, 39, 1, 11]. Alongside the mean $E_{\delta,-1}(u) = \int u\,dx$, (1.1) has a conserved quantity $E_{\delta,\frac{k}{2}}(u)$ at the level of the L^2-based Sobolev norm $\|u\|_{\dot{H}^{\frac{k}{2}}}$ for each $k \in \mathbb{N} \cup \{0\}$:

$$E_{\delta,0}(u) = \frac{1}{2}\|u\|_{L^2}^2, \qquad E_{\delta,\frac{k}{2}}(u) = \frac{1}{2}\sum_{\substack{\ell=0 \\ \ell \text{ even}}}^{k} a_\ell \|\mathcal{G}_\delta^{\frac{k-\ell}{2}} u\|_{\dot{H}^{\frac{k}{2}}}^2 + R_{\delta,\frac{k}{2}}(u), \qquad k \geq 1, \tag{1.6}$$

for positive constants a_ℓ and a remainder $R_{\delta,\frac{k}{2}}(u)$ which contains terms which are cubic and higher in u. The first conserved quantities for (1.1) with $k = 1, 2, 3$ are as follows:

$$E_{\delta,\frac{1}{2}}(u) = \tfrac{1}{2}\|\mathcal{G}_\delta^{\frac{1}{2}} u\|_{\dot{H}^{\frac{1}{2}}}^2 + \tfrac{1}{3}\int u^3\,dx, \tag{1.7}$$

$$E_{\delta,1}(u) = \tfrac{1}{8}\|u\|_{\dot{H}^1}^2 + \tfrac{3}{8}\|\mathcal{G}_\delta u\|_{\dot{H}^1}^2 + \int \big[\tfrac{1}{4}u^4 + \tfrac{3}{4}u^2\mathcal{G}_\delta\partial_x u + \tfrac{1}{4\delta}u^3\big]\,dx, \tag{1.8}$$

$$E_{\delta,\frac{3}{2}}(u) = \ldots \tag{1.9}$$

See Subsection 3.1 for further details on the structure of these conserved quantities and Subsection 4.1 for the structure of the corresponding conserved quantities $\widetilde{E}_{\delta,\frac{k}{2}}(v)$ for sILW (1.4).

In [33], the convergence of ILW (1.1) to BO (1.2) and of sILW (1.4) to KdV (1.5) was shown with respect to the Gibbs measure, i.e., a weighted Gaussian measure associated with $E_{\delta,\frac{1}{2}}(u)$ via the compactness argument introduced in []. A natural question is if this convergence holds if we sample initial data from measures associated with the higher order conservation laws in 1.6. Since these measures are supported on smoother spaces as k increases (see Lemma ??), this is a physically relevant question.

Our strategy is three-fold: (1) for fixed $0 < \delta < \infty$, we construct weighted Gaussian measures associated with each conservation law $E_{\delta,\frac{k}{2}}(u)$ (and $\widetilde{E}_{\delta,\frac{k}{2}}(v)$ for (1.4)); (2) prove the convergence of these measures to the corresponding BO measures as $\delta \to \infty$ (and to the KdV ones as $\delta \to 0$, resp.); (3) establish invariance of the measures and dynamical convergence.

1.3. **Construction and convergence of weighted Gaussian measures.** The Hamiltonian structure of ILW (1.1), associated with the conserved quantity $E_{\delta,\frac{1}{2}}(u)$ in (1.7), naturally leads to the question of the existence of invariant measures for this system. In particular, for $k \in \mathbb{N}$, we are interested in constructing invariant measures $\rho_{\delta,\frac{k}{2}}$ associated with each conserved quantity $E_{\delta,\frac{k}{2}}(u)$ and establishing their convergence as we vary the depth parameter δ.

The study of invariant measures for infinite-dimensional Hamiltonian systems was initiated by Lebowitz-Rose-Speer [32] and Bourgain [7, 8], has regained popularity in the past 15 years []. We emphasize the known results for invariant measures for ILW, BO, and KdV associated with polynomial conservation laws. In [58], Zhidkov constructed an infinite family of invariant measures for KdV associated with its higher order conserved quantities, while the question for BO was addressed by an extensive program due to Deng, Tzvetkov, and Visciglia [53, 54, 55, 56, 14, 15]. For ILW, in [33], Li-Oh-Zheng constructed the Gibbs measure $\rho_{\delta,\frac{1}{2}}$ and established its convergence in the deep- and shallow-water limits. Here, we make the first step towards completing this program by constructing, establishing invariance and convergence of the measures $\rho_{\delta,\frac{k}{2}}$ for $k \geq 3$ for ILW. See Remark ?? for further details on $k = 1, 2$.

We first focus on the deep-water regime. For $0 < \delta \leq \infty$ and $k \geq 2$, we construct the measure $\rho_{\delta,\frac{k}{2}}$ associated with the k-th conserved quantity $E_{\delta,\frac{k}{2}}(u)$ on (1.6), formally given by

$$\begin{aligned} \rho_{\delta,\frac{k}{2}}(du) \text{``}=\text{''} & Z_{\delta,\frac{k}{2}}^{-1} \exp\big(-E_{\delta,\frac{k}{2}}(u)\big)\, du \\ &= Z_{\delta,\frac{k}{2}}^{-1} \exp\big(-R_{\delta,\frac{k}{2}}(u)\big) d\mu_{\delta,\frac{k}{2}}(u). \end{aligned} \tag{1.10}$$

In particular, we construct $\rho_{\delta,\frac{k}{2}}$ as a weighted Gaussian measure with base Gaussian $\mu_{\delta,\frac{k}{2}}$:

$$\mu_{\delta,\frac{k}{2}}(du) = Z_{\delta,\frac{k}{2}}^{-1} \exp\Bigg(-\sum_{\substack{\ell=0 \\ \ell \text{ even}}}^{k} a_\ell \|\mathcal{G}_\delta^{\frac{k-\ell}{2}} u\|_{\dot{H}^{\frac{k}{2}}}^2\Bigg)\, du. \tag{1.11}$$

Here, $\mu_{\delta,\frac{k}{2}}$ can be understood as the induced probability measure under the map

$$\omega \in \Omega \mapsto X_{\delta,\frac{k}{2}}(x;\omega) = \frac{1}{\sqrt{2\pi}} \sum_{n \in \mathbb{Z}^*} \frac{g_n(\omega)}{(T_{\delta,\frac{k}{2}}(n))^{\frac{1}{2}}} e^{inx}, \tag{1.12}$$

where

$$T_{\delta,\frac{k}{2}}(n) := \sum_{\substack{\ell=0 \\ \ell \text{ even}}}^{k} a_\ell |n|^\ell |K_\delta(n)|^{k-\ell}, \tag{1.13}$$

$K_\delta(n) = in\widehat{\mathcal{G}_\delta}(n)$ and $\{g_n\}_{n\in\mathbb{Z}^*}$ is a sequence of independent standard complex-valued Gaussian random variables satisfying $g_{-n} = \overline{g_n}$. One can easily show that $\mu_{\delta,\frac{k}{2}}$ (and consequently $\rho_{\delta,\frac{k}{2}}$) is supported on $H^{(k-1)/2-\varepsilon}(\mathbb{T})$ for all $\varepsilon > 0$, but $\mu_{\delta,\frac{k}{2}}\big(H^{(k-1)/2}(\mathbb{T})\big) = 0$.

Due to the involved structure of the remainder $R_{\delta,\frac{k}{2}}(u)$ in (1.10) (see also (3.2)) and the fact that it is not sign-definite, we do not expect the weight $\exp(-R_{\delta,\frac{k}{2}}(u))$ to be integrable with respect to $\mu_{\delta,\frac{k}{2}}$. This difficulty was also present for KdV and BO in [58, 54]; see also [32, 7, 8] for the analogous issue in the context of Schrödinger equations. There, they propose to renormalized the measure $\rho_{\delta,\frac{k}{2}}$ in (1.10) by restricting to invariant sets where the lower order energies $E_{\delta,\frac{j}{2}}(u)$, for $j = 0,\dots,k-1$, are bounded. This was sufficient for KdV in [58], but a more involved renormalization was needed for BO in [54] due to $E_{\infty,(k-1)/2}(u)$ being infinite in the support of $\mu_{\infty,\frac{k}{2}}$.

Although the same phenomenon is true for ILW (1.1), in our construction, we need only introduce an L^2-cutoff:

$$\rho_{\delta,\frac{k}{2}}(du) = Z^{-1}_{\delta,\frac{k}{2}}\eta_K(\|u\|^2_{L^2})\exp(-R_{\delta,\frac{k}{2}}(u))\,d\mu_{\delta,\frac{k}{2}}(u), \tag{1.14}$$

where $\eta_K(\cdot) = \eta(\cdot/K)$ and η a smooth cutoff with $\eta \equiv 1$ on $[-1,1]$ and supported on $[-2,2]$. More precisely, we first consider the following normalized truncated measures

$$\rho_{\delta,\frac{k}{2},N,K}(du) = Z^{-1}_{\delta,\frac{k}{2},N}\ F_{\delta,\frac{k}{2},N,K}(u)\ d\mu_{\delta,\frac{k}{2}}(du), \tag{1.15}$$

where

$$F_{\delta,\frac{k}{2},N,K}(u) = \eta_K\big(\|\mathbf{P}_N u\|^2_{L^2}\big)\exp\big(-R_{\delta,\frac{k}{2}}(\mathbf{P}_N u)\big), \tag{1.16}$$

and construct $\rho_{\delta,\frac{k}{2}}$ in (1.14) as the limit of the truncated measures $\rho_{\delta,\frac{k}{2},N,K}$ as $N\to\infty$. This construction extends to BO (1.2) when $\delta = \infty$, where the base Gaussian measure associated with (1.2) is given by

$$\mu_{\infty,\frac{k}{2}}(du) = Z^{-1}_{\infty,\frac{k}{2}}\exp\big(-\|u\|^2_{\dot H^{\frac{k}{2}}}\big)\,du, \tag{1.17}$$

and it can be understood as the induced probability measure under the map (1.12), where we extend the definition of (1.13) to $T_{\infty,\frac{k}{2}}(n) := |n|^k$.

We can now state our first main result regarding the construction and convergence of the measures $\rho_{\delta,\frac{k}{2}}$ in the deep-water limit ($\delta\to\infty$).

Theorem 1.1 (Construction and convergence in the deep-water regime)**.** *Let $k\in\mathbb{N}$ with $k\geq 2$. Then, the following statements hold:*
(i) *Let $0<\delta\leq\infty$ and $K>0$. Then, for any $1\leq p<\infty$, we have that*

$$\lim_{N\to\infty} F_{\delta,\frac{k}{2},N,K}(u) = F_{\delta,\frac{k}{2},K}(u) \quad \text{in } L^p(d\mu_{\delta,\frac{k}{2}}), \tag{1.18}$$

where the convergence holds uniformly in $2\leq\delta\leq\infty$. Consequently, the truncated measure $\rho_{\delta,\frac{k}{2},N,K}$ in (1.15) converges to the limiting measure $\rho_{\delta,\frac{k}{2},K}$ given by

$$\rho_{\delta,\frac{k}{2},K}(du) = Z^{-1}_{\delta,\frac{k}{2}}F_{\delta,\frac{k}{2},K}(u)\,d\mu_{\delta,\frac{k}{2}}(u).$$

In particular, $\rho_{\delta,\frac{k}{2},N,K}$ converges to $\rho_{\delta,\frac{k}{2},K}$ in total variation uniformly in δ.

(ii) *The limiting measure* $\rho_{\delta,\frac{k}{2},K}$ *and the base Gaussian measure with smooth* L^2*-cutoff* $\eta_K(\|u\|_{L^2}^2)d\mu_{\delta,\frac{k}{2}}(u)$ *are equivalent, and*

$$\bigcup_{K>0} \operatorname{supp}(\rho_{\delta,\frac{k}{2},K}) = \mu_{\delta,\frac{k}{2}}.$$

(iii) *For* $2 \le \delta < \infty$*, the measures* $\rho_{\delta,\frac{k}{2},K}$ *for* (1.1) *and* $\rho_{\infty,\frac{k}{2},K}$ *for* (1.2) *are equivalent and, as* $\delta \to \infty$*, the measure* $\rho_{\delta,\frac{k}{2},K}$ *converges to* $\rho_{\infty,\frac{k}{2},K}$ *in total variation.*

We now turn our attention to the shallow-water regime and sILW (1.4). The first difficulty in this setting comes from identifying the correct structure for the conservation laws. In light of the scaling (1.3), $E_{\delta,\frac{k}{2}}(\delta v)$ are conserved quantities for sILW (1.4), however, for large k, it is not clear if these are suitable to study the limit as $\delta \to 0$, as they may have terms with negative powers of δ. Consequently, we must instead introduce conservation laws $\widetilde{E}_{\delta,\frac{k}{2}}(v)$ which are do not necessarily agree with $E_{\delta,\frac{k}{2}}(\delta v)$, and thus require an altogether new description of their structure; see Remark 1.4 for a further discussion on this. Here, we find $\widetilde{E}_{\delta,\frac{k}{2}}(v)$ for $k \in \mathbb{N}$ satisfying

$$\begin{aligned} \widetilde{E}_{\delta,\frac{2k-1}{2}}(v) &= \frac{1}{2} \sum_{\substack{\ell=1 \\ \text{odd}}}^{2k-1} a_\ell \delta^{\ell-1} \|\widetilde{\mathcal{G}}_\delta^{\frac{\ell}{2}} v\|_{\dot{H}^{\frac{2k-1}{2}}}^2 + \widetilde{R}_{\delta,\frac{2k-1}{2}}(v), \\ \widetilde{E}_{\delta,\frac{2k}{2}}(v) &= \frac{1}{2} \sum_{\substack{\ell=0 \\ \text{even}}}^{2k} a_\ell \delta^{\ell} \|\widetilde{\mathcal{G}}_\delta^{\frac{\ell}{2}} v\|_{\dot{H}^{\frac{2k}{2}}}^2 + \widetilde{R}_{\delta,\frac{2k}{2}}(v), \end{aligned} \tag{1.19}$$

for some positive constants a_ℓ and remainders $\widetilde{R}_{\delta,\frac{k}{2}}(v)$ with cubic and higher order terms in v; see Subsection 4.1 for further detail on their structure. For $k = 1, 2, 3$, these conservation laws are as follows:

$$\begin{aligned} \widetilde{E}_{\delta,\frac{1}{2}}(v) &= \tfrac{1}{2} \|\widetilde{\mathcal{G}}_\delta^{\frac{1}{2}} v\|_{\dot{H}^{\frac{1}{2}}}^2 + \tfrac{1}{3} \int v^3 \, dx, \\ \widetilde{E}_{\delta,1}(v) &= \dots, \\ \widetilde{E}_{\delta,\frac{3}{2}}(v) &= \dots. \end{aligned} \tag{1.20}$$

Similarly to the deep-water setting, we are interested in constructing the renormalized weighted Gaussian measures $\widetilde{\rho}_{\delta,\frac{k}{2},K}$

$$\widetilde{\rho}_{\delta,\frac{k}{2},K}(dv) = \widetilde{Z}_{\delta,\frac{k}{2}}^{-1} \eta_K(\|v\|_{L^2}^2) \exp\big(-\widetilde{R}_{\delta,\frac{k}{2}}(v)\big) d\widetilde{\mu}_{\delta,\frac{k}{2}}(v),$$

where the Gaussian measures $\widetilde{\mu}_{\delta,\frac{k}{2}}$ formally given by

$$\widetilde{\mu}_{\delta,\frac{k}{2}}(dv) = \widetilde{Z}_{\delta,\frac{k}{2}}^{-1} \exp\Bigg(-\sum_{\substack{\ell=0 \\ \ell\equiv k \bmod 2}}^{k} a_\ell \delta^{\ell-\mathbf{1}_{\ell\equiv k \bmod 2}} \|\widetilde{\mathcal{G}}_\delta^{\frac{\ell}{2}} v\|_{\dot{H}^{\frac{k}{2}}}^2\Bigg) dv, \tag{1.21}$$

can be understood as the induced probability measure under the map

$$\omega \in \Omega \mapsto \widetilde{X}_{\delta,\frac{k}{2}}(\omega; x) = \frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}^*} \frac{g_n(\omega)}{(\widetilde{T}_{\delta,\frac{k}{2}}(n))^{\frac{1}{2}}} e^{inx},$$

where $\widetilde{T}_{\delta,\frac{k}{2}}$ is defined in (4.2).

The construction of the measures $\widetilde{\rho}_{\delta,\frac{k}{2}}$ is analogous to the deep-water regime, albeit more algebraically involved due to the structure of the conservation laws $\widetilde{E}_{\delta,\frac{k}{2}}$. However, there are stark distinctions between the two regimes regarding the convergence. Before discussing these, we introduce relevant notation related to KdV (1.5). For $k \in \mathbb{N}$, let $\widetilde{E}_{0,k}(v)$ denote the k-th conservation law for KdV, which can be written as

$$\widetilde{E}_{0,k}(v) = \tfrac{1}{2}\|v\|^2_{\dot{H}^k} + \widetilde{R}_{0,k}(v),$$

where $\widetilde{R}_{0,k}(v)$ contains terms which are cubic or higher order in v. We also introduce the Gaussian measure $\widetilde{\mu}_{0,k}$ as the induced probability measure under the map

$$\omega \in \Omega \mapsto \widetilde{X}_{0,k}(x;\omega) = \frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}^*} \frac{g_n(\omega)}{|n|^k} e^{inx},$$

and the weighted renormalized measures $\widetilde{\rho}_{0,k,K}$

$$\widetilde{\rho}_{0,k,K}(dv) = \widetilde{Z}^{-1}_{0,k,K} \eta_K(\|v\|^2_{L^2}) \exp\big(-\widetilde{R}_{0,k}(v)\big) d\widetilde{\mu}_{0,k}(v).$$

Firstly, we observe a 2-to-1 collapse of the measures $\widetilde{\rho}_{\delta,\frac{k}{2}}$ to sILW (1.4) as we take the limit $\delta \to 0$. In particular, for $k \in \mathbb{N}$ we observe that $\widetilde{\rho}_{\delta,\frac{2k-1}{2}}$ and $\widetilde{\rho}_{\delta,\frac{2k}{2}}$ converge to the same KdV measure $\widetilde{\rho}_{0,k}$! Secondly, as observed in [33] for $\widetilde{\rho}_{\delta,\frac{1}{2}}$, this convergence holds only weakly, as opposed to the convergence in total variation in the deep-water regime. Moreover, the measures $\widetilde{\rho}_{\delta,\frac{2k-1}{2}}$, $\widetilde{\rho}_{\delta,\frac{2k}{2}}$, and $\widetilde{\rho}_{0,k}$ are mutually singular. These results are summarized in the following theorem.

Theorem 1.2 (Construction and convergence in the shallow-water regime)**.** *Let $k \in \mathbb{N}$ with $k \geq 2$. Then, the following statements hold:*
(i) *Let $0 \leq \delta < \infty$ and $K > 0$. Then, for any $1 \leq p < \infty$, we have that*

$$\lim_{N\to\infty} \widetilde{F}_{\delta,\frac{k}{2},N,K}(u) = \widetilde{F}_{\delta,\frac{k}{2},K}(u) \quad \text{in } L^p(d\widetilde{\mu}_{\delta,\frac{k}{2}}), \tag{1.22}$$

*where the convergence holds uniformly in $0 \leq \delta \leq 1$. Consequently, the truncated measure $\widetilde{\rho}_{\delta,\frac{k}{2},N,K}$ in (**??**) converges in total variation to the limiting measure $\widetilde{\rho}_{\delta,\frac{k}{2},K}$ given by*

$$\widetilde{\rho}_{\delta,\frac{k}{2},K}(du) = Z^{-1}_{\delta,k} \widetilde{F}_{\delta,\frac{k}{2},K}(u)\, d\widetilde{\mu}_{\delta,\frac{k}{2}}(u).$$

In particular, this convergence is uniform in δ. Moreover, the limiting measure $\widetilde{\rho}_{\delta,\frac{k}{2},K}$ and the base Gaussian measure with smooth L^2-cutoff $\eta_K(\|u\|^2_{L^2})d\widetilde{\mu}_{\delta,\frac{k}{2}}(u)$ are equivalent.
(ii) *For $0 < \delta < \infty$, the measures $\widetilde{\rho}_{\delta,\frac{2k-1}{2},K}$, $\widetilde{\rho}_{\delta,\frac{2k}{2},K}$ for (1.4) and $\widetilde{\rho}_{0,k,K}$ for (1.5) are singular and, as $\delta \to 0$, the measures $\widetilde{\rho}_{\delta,\frac{2k-1}{2},K}$ and $\widetilde{\rho}_{\delta,\frac{2k}{2},K}$ converge weakly to $\widetilde{\rho}_{0,k,K}$.*

Remark 1.3. (i) The difficulty in constructing the measures $\rho_{\delta,\frac{k}{2}}$ and the convergence as the depth parameter δ varies is not due to the rough support of the measures (since for $k \geq 2$ this is smooth(er)). Instead, we must tackle two issues: (1) the δ-dependence of both the base Gaussian measure $\mu_{\delta,\frac{k}{2}}$ in (1.11) and the truncated density $F_{\delta,\frac{k}{2},N,K}$ in (1.16) which require uniform in N and δ bounds to first construct the measure $\rho_{\delta,\frac{k}{2}}$ for a given δ and then prove its convergence as $\delta \to \infty$. This difficulty was already observed in [33] for the Gibbs measure case $k = 1$; (2) the increased complexity of the conserved quantities

as k becomes larger, for which we need a good description in order to control the density $F_{\delta,\frac{k}{2},N,K}$ and choose the correct base Gaussian $\mu_{\delta,\frac{k}{2}}$.
(ii) To establish the uniform in N and δ bounds needed to prove Theorems 1.1-1.2, we use a variational approach. For $\delta = \infty$ and $\delta = 0$, we provide an alternative construction to that in [53] and [58] for the BO and KdV measures $\rho_{\infty,\frac{k}{2}}, \widetilde{\rho}_{0,k}$, respectively. Our construction only requires an L^2-cutoff instead of cutoffs involving the conserved quantities of order $0 \le \ell \le k-1$. This alternative construction also simplifies the invariance argument in Theorem 1.5.

Remark 1.4. Due to the scaling in (1.3), one possible choice for the shallow-water conservation laws would be

$$\widetilde{E}_{\delta,\frac{k}{2}}(v) := \delta^{-2}E_{\delta,\frac{k}{2}}(\delta v), \tag{1.23}$$

which are conserved under the sILW dynamics for each fixed $0 < \delta < \infty$. Although this definition leads to a suitable choice of conservation laws for the explicit quantities in (1.6) with $k = 1, 2, 3$, it is not clear from our definition of $E_{\delta,\frac{k}{2}}(u)$ for arbitrarily large values of k that (1.23) defines a suitable conservation law to consider the limit as $\delta \to 0$. In Proposition A.6 in Appendix A, we see that $E_{\delta,\frac{k}{2}}(u)$ are defined as suitable linear combinations of conserved quantities whose densities χ_j (see (A.5)) which satisfy a given recurrence relation:

$$E_{\delta,\frac{k}{2}}(u) \sim \int \chi_{k+2}\,dx - \sum_{j=1}^{k+1} \frac{1}{\delta^{k+2-j}} c_j \int \chi_j\,dx. \tag{1.24}$$

This definition guarantees that the quadratic in u terms of $E_{\delta,\frac{k}{2}}(u)$ are of the form in (1.6), where all coefficients are positive, and thus suitable to define the Gaussian measure $\mu_{\delta,\frac{k}{2}}$ in (1.11). It is not clear from the definition of the terms χ_j, if the transformation $u \mapsto \delta v$ if all the terms appearing in (1.24) will have only positive powers of δ. To avoid this issue, we instead define $\widetilde{E}_{\delta,\frac{k}{2}}$ as linear combinations of analogues of $\int \chi_j\,dx$ which are already adapted to the scaling dynamics (1.4), with no need for a posteriori rescaling. See Appendix B and (B.17) for precise definitions. We note that this approach is not without difficulties, as we are motivated by the construction in [16, 30] where half of the conservation laws derived have trivial limits as $\delta \to 0$. Fortunately, this derivation hides all the relevant information for our analysis and through careful algebraic manipulations, for each $k \in \mathbb{N}$, we can recover conservation laws $\widetilde{E}_{\delta,\frac{k}{2}}(v)$ as in (1.19) and show that they all have a non-trivial limit to a corresponding KdV conservation law. Moreover, we observe the 2-to-1 collapse of these conservation laws in the limit $\delta \to 0$ responsible for the analogous phenomenon described above for the corresponding measures $\widetilde{\rho}_{\delta,\frac{k}{2},K}$:

$$\lim_{\delta\to 0} \widetilde{E}_{\delta,\frac{2k-1}{2}}(v) = \lim_{\delta\to 0} \widetilde{E}_{\delta,\frac{2k}{2}}(v) = \widetilde{E}_{0,k}(v).$$

We believe this is a novel observation in the study of the convergence of ILW in the shallow-water regime. See Proposition B.7 for a precise statement.

1.4. **Dynamical problem and invariance.** We now focus on the dynamics of (1.1) and (1.4). Since the approach is analogous in the deep- and shallow-water regimes, we restrict our discussion to the former.

Our goal is to establish the invariance of the measures $\rho_{\delta,\frac{k}{2}}$ for $k \geq 3$ under the flow of ILW (1.1). To address this question, we require global-in-time dynamics for ILW (1.1) in the support of the measures $\rho_{\delta,\frac{k}{2}}$. In [1], Abdelouhab-Bona-Felland-Saut established global well-posedness in $H^s(\mathbb{T})$ for $s > \frac{3}{2}$ via the energy method, without exploiting the dispersive nature of the equation. The best known result is due to Molinet-Vento [41] where they proved global well-posedness of (1.1) in $H^{\frac{1}{2}}(\mathbb{T})$ through a refinement of the energy method. Consequently, we know that (1.1) is globally well-posed in the support of $\mu_{\delta,\frac{k}{2}}$ for $k \geq 3$.

In order to establish the invariance of $\mu_{\delta,\frac{k}{2}}$, we will focus on the following truncated ILW dynamics:

$$\begin{cases} \partial_t u_N - \mathcal{G}_\delta \partial_x^2 u_N = \mathbf{P}_N \partial_x (\mathbf{P}_N u_N)^2, \\ u_N(0) = u_0. \end{cases} \tag{1.25}$$

The arguments in [41] also apply to the truncated dynamics (1.25), from which we can show that for a fixed $0 < \delta \leq \infty$, $k \geq 3$, and $\frac{1}{2} < s < \sigma < \frac{k-1}{2}$, then for all $R > 0$ there exists $\bar{t} = \bar{t}(R) > 0$ such that

$$\lim_{N\to\infty} \sup_{\substack{t\in(-\bar{t},\bar{t}) \\ A\subset B^\sigma(R)}} \|\Phi_t^N(u_0) - \Phi_t(u_0)\|_{H^s} = 0, \tag{1.26}$$

where Φ_t^N, Φ_t denote the unique global data-to-solution maps of (1.25) and (1.1), respectively. This approximation property between the truncated and full solutions to ILW is essential to establishing invariance. See Proposition 6.3 for further details.

In [54, 55, 56], it was observed that for BO (1.2), the conserved quantities $E_{\infty,\frac{k}{2}}(u)$ for the original equation are no longer conserved for the analogous truncated dynamics, for $k \geq 2$. This problem persists for the ILW dynamics (1.1). In fact, we have that

$$\frac{d}{dt} E_{\delta,\frac{k}{2}}(\mathbf{P}_N \Phi_t^N(u_0)) \neq 0, \qquad k \geq 2.$$

To bypass this difficulty, we proceed as in [54, 55, 56] and reduce the problem to time $t = 0$ and establish an almost almost-sure conservation of the energies for the truncated dynamics with Gaussian initial data:

$$\lim_{N\to\infty} \left\| \frac{d}{dt} E_{\delta,\frac{k}{2}}(\mathbf{P}_N u) \right\|_{L^q(\mu_{\delta,\frac{k}{2}})} = 0, \tag{1.27}$$

for all $1 \leq q < \infty$. This result relies heavily on the orthogonality properties of the Gaussian data. Combining (1.27) with a global-in-time improvement of the good approximation of the full flow by the truncated flow in (1.26), we establish the invariance of the measures $\rho_{\delta,\frac{k}{2}}$ for $k \geq 3$.

Theorem 1.5 (Invariance)**.** *Let $k \geq 3$, $2 \leq \delta \leq \infty$, and $K > 0$. Then, the measures $\rho_{\delta,\frac{k}{2},K}$ are invariant under the unique global-in-time flow Φ_t of ILW (1.1) defined on $H^{\frac{k-1}{2}-}(\mathbb{T})$. Similarly, the measures $\widetilde{\rho}_{\delta,\frac{k}{2},K}$ are invariant under the unique global-in-time flow $\widetilde{\Phi}_t$ of sILW (1.4) defined on $H^{\frac{k-1}{2}-}(\mathbb{T})$.*

We complete the introduction with some additional remarks.

Remark 1.6. (i) Theorem 1.5 establishes the first result on construction and convergence of invariant measures for ILW (1.1) (and sILW (1.4), resp.) with uniqueness. Note that the result in [33] for the Gibbs measures $\rho_{\delta,\frac{1}{2}}, \widetilde{\rho}_{\delta,\frac{1}{2}}$ relies on a compactness argument, therefore uniqueness is not yet known. The measures we are considering are supported at higher regularity (where global well-posedness is known), which allows us to establish a stronger notion of invariance in Theorem 1.5, from which one can infer on the recurrence properties of the flow of (1.1) via Poincaré's recurrence theorem, for example. It would be of interest to improve the result in [33] by extending Theorem 1.5 to $k = 1$, which would require a significant improvement on the well-posedness theory for ILW due to the low regularity support of $\rho_{\delta,\frac{1}{2}}$.

(ii) The almost almost-sure conservation for the truncated energies also holds for $k = 2$. Unfortunately, at this time, there is no known available local well-posedness for ILW (1.1) in a space which contains the support of the measure $\rho_{\delta,1}$ which does not rely on gauge transforms. Note that at the level of L^2-based Sobolev spaces, this support is contained in $H^{\frac{1}{2}-\varepsilon}(\mathbb{T})$ for any $\varepsilon > 0$, which is just missed by the results in [41]. In future work, we hope to extend Theorem 1.5 to the case $k = 2$ by combining the recent well-posedness in [12] and the ideas in [15, 14]. Another alternative is to consider a different scale of spaces, namely adapt the improved energy method of [41] to the Fourier-Lebesgue spaces $\mathcal{F}L^{s,p}$. Since ILW (1.1) is a completely integrable equation which admits a Lax pair, it may also be possible to use the method of commuting flows in [23, 22] as for KdV and BO, to obtain an optimal result in the H^s-scale.

2. Preliminaries

2.1. **Notations and function spaces.** We start by introducing some useful notations. Let $A \lesssim B$ denote an estimate of the form $A \leq CB$ for some constant $C > 0$. We write $A \sim B$ if $A \lesssim B$ and $B \lesssim A$, while $A \ll B$ will denote $A \leq \varepsilon B$, for some small constant $0 < \varepsilon \ll 1$. When relevant, we may write $\lesssim_\delta, \sim_\delta$ to emphasize the dependence of the implicit constant on the parameter δ. The notations $a+$ and $a-$ represent $a + \varepsilon$ and $a - \varepsilon$ for arbitrarily small $\varepsilon > 0$, respectively. We will use the shorthand notation $n_{1\cdots k}$ for the sum $n_1 + \cdots + n_k$.

Throughout this paper, we fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The realization $\omega \in \Omega$ is often omitted in writing. We will use $\mathcal{L}(X)$ to denote the law of the random variable X.

Our conventions for the Fourier transform are as follows. Let $f : \mathbb{T} \to \mathbb{R}$ be a mean zero function and $\mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$. Then, we have the following

$$\widehat{f}(n) = \frac{1}{\sqrt{2\pi}} \int f(x) e^{-inx}\, dx, \qquad f(x) = \frac{1}{\sqrt{2\pi}} \sum_{n \in \mathbb{Z}^*} \widehat{f}(n) e^{inx}.$$

Given $N \in \mathbb{N}$, we denote by $\mathbf{P}_N$ the Dirichlet projection onto spatial frequencies $\{|n| \leq N\}$ defined as follows

$$\mathbf{P}_N f(x) := (D_N * f)(x) = \frac{1}{\sqrt{2\pi}} \sum_{0<|n|\leq N} \widehat{f}(n) e^{inx},$$

where $D_N(x) = \sum_{|n|\leq N} e^{inx}$ is the Dirichlet kernel.

We also introduce some relevant function spaces. Let $s \in \mathbb{R}$ and $1 \le p \le \infty$. We define the L^p-based Sobolev space $W^{s,p}(\mathbb{T})$ through the norm

$$\|f\|_{W^{s,p}} := \|\mathcal{F}_x^{-1}(\langle n\rangle^s \widehat{f}(n))\|_{L^p_x},$$

where $\langle n\rangle = (1+n^2)^{\frac{1}{2}}$. When $p=2$, we use $H^s(\mathbb{T})$ for the L^2-based Sobolev spaces with norm

$$\|f\|_{H^s} = \|\langle n\rangle^s \widehat{f}(n)\|_{\ell^2_n}.$$

We will often use the short-hand notations $L^q_T H^s_x$ and $L^p_\omega H^s_x$ for $L^q\big([-T,T]; H^s(\mathbb{T})\big)$ and $L^p(\Omega; H^s(\mathbb{T}))$, respectively. Lastly, we recall some known results in these function spaces; see [3, 17, 19], for example.

Lemma 2.1. *The following estimates hold:*
(i) (interpolation) *For* $0 < s_1 < s_2$, $r_1, r_2 \in \mathbb{R}$, $\theta \in (0,1)$, *and* $r = \theta r_1 + (1-\theta) r_2$, *we have*

$$\|u\|_{H^{s_1}_x} \lesssim \|u\|_{H^{s_2}_x}^{\frac{s_1}{s_2}} \|u\|_{L^2_x}^{\frac{s_2-s_1}{s_2}}, \tag{2.1}$$

$$\|u\|_{H^r_x} \lesssim \|u\|^{\theta}_{H^{r_1}_x} \|u\|^{1-\theta}_{H^{r_2}_x}. \tag{2.2}$$

(ii) (fractional Leibniz rule) *Let* $0 \le s \le 1$, $1 < p_j, q_j, r < \infty$, $\frac{1}{p_j} + \frac{1}{q_j} = \frac{1}{r}$, $j = 1,2$, *then*

$$\|\langle\nabla\rangle^s (fg)\|_{L^r_x} \lesssim \|\langle\nabla\rangle^s f\|_{L^{p_1}_x} \|g\|_{L^{q_1}_x} + \|f\|_{L^{p_2}_x} \|\langle\nabla\rangle^s g\|_{L^{q_2}_x}. \tag{2.3}$$

2.2. $\mathcal{G}_\delta$ **operator and friends.** In this subsection, we recall important lemmas on the operators $\mathcal{G}_\delta, \mathcal{Q}_\delta = (\mathcal{G}_\delta - \mathcal{H})\partial_x$, and their scaled counterparts $\widetilde{\mathcal{G}}_\delta, \widetilde{\mathcal{Q}}_\delta$ in the shallow-water regime. Recall that for $n \in \mathbb{Z}^*$ we have

$$\widehat{\mathcal{G}_\delta}(n) = -i(\coth(\delta n) - \tfrac{1}{\delta n}), \qquad \widehat{\widetilde{\mathcal{G}}_\delta}(n) = \tfrac{1}{\delta}\widehat{\mathcal{G}_\delta}(n) = -\tfrac{i}{\delta}(\coth(\delta n) - \tfrac{1}{\delta n}),$$

$$\widehat{\mathcal{Q}_\delta}(n) = (n\coth(\delta n) - \tfrac{1}{\delta} - |n|), \qquad \widehat{\widetilde{\mathcal{Q}}_\delta}(n) = \tfrac{1}{\delta}\widehat{\mathcal{Q}_\delta}(n) = \tfrac{1}{\delta}(n\coth(\delta n) - \tfrac{1}{\delta} - |n|),$$

$$\mathrm{K}_\delta(n) = in\widehat{\mathcal{G}_\delta}(n) = n\coth(\delta n) - \tfrac{1}{\delta}, \qquad \mathrm{L}_\delta(n) = \tfrac{1}{\delta}\mathrm{K}_\delta(n) = \tfrac{1}{\delta}(n\coth(\delta n) - \tfrac{1}{\delta}).$$

We recall the following important results on the multipliers $\mathrm{K}_\delta, \mathrm{L}_\delta$. For a proof, see [33], for example.

Lemma 2.2. *For any* $\delta > 0$, *we have*

$$\max\big(0, |n| - \tfrac{1}{\delta}\big) < \mathrm{K}_\delta(n) < |n|, \tag{2.4}$$

for $n \in \mathbb{Z}^*$. *In particular,*

$$\mathrm{K}_\delta(n) \sim_\delta |n|, \quad n \in \mathbb{Z}^*,$$

where the constant can be made independent of $2 \le \delta \le \infty$. *Furthermore, for all* $n \in \mathbb{Z}^*$, $\mathrm{K}_\delta(n)$ *is strictly increasing in* $\delta \ge 1$ *and it converges to* $|n|$ *as* $\delta \to \infty$.

Lemma 2.3. *The scaled multiplier* L_δ *satisfies the following:*
(i) $0 < \mathrm{L}_\delta(n) < \min(\frac{1}{3}n^2, \frac{1}{\delta}|n|)$ *for all* $n \in \mathbb{Z}^*$, $\delta > 0$.
(ii) *For each* $n \in \mathbb{Z}^*$, $\mathrm{L}_\delta(n) \nearrow \frac{1}{3}n^2$ *as* $\delta \to 0$.
(iii) *For all* $\delta_0 > 0$ *and all* $C_0 > 0$, *we have for* $0 < \delta \le \delta_0$

$$\mathrm{L}_\delta(n) \ge \begin{cases} \frac{1}{\delta} C_1 |n|, & \text{if } \delta|n| > C_0, \\ C_2 |n|^2, & \text{if } \delta|n| \le C_0, \end{cases}$$

where $C_1 = 2(\frac{1}{2} - \frac{\arctan(\pi/C_0)}{\pi})$ *and* $C_2 = \sum_{k=1}^{\infty} \frac{2}{k^2\pi^2 + C_0^2}$.

(iv) *We can write* $\mathrm{L}_\delta(n) = \frac{1}{3}n^2 - \frac{1}{3}n^2 h(\delta, n)$ *where*

$$h(\delta, n) = 6\delta^2 \sum_{k=1}^{\infty} \frac{n^2}{k^2\pi^2(k^2\pi^2 + \delta^2 n^2)},$$

from which it follows that $\lim_{n\to\infty} h(\delta, n) = C \neq 0$.

From Lemma 2.2 and properties of the coth function, we can easily conclude the following results on $\mathcal{H} - \mathcal{G}_\delta$, $\mathcal{Q}_\delta$, $\widetilde{\mathcal{Q}}_\delta$, and $\mathcal{G}_\delta$.

Lemma 2.4. *For* $0 < \delta < \infty$ *and* $n \in \mathbb{Z}^*$, *we have that*

$$\big|\mathcal{F}_x(\mathcal{H} - \mathcal{G}_\delta)(n)\big| \le \frac{1}{\delta|n|}, \qquad \big|\mathcal{F}_x(\tfrac{1}{\delta}\mathcal{H} - \widetilde{\mathcal{G}}_\delta)(n)\big| \le \frac{1}{\delta^2|n|} \tag{2.5}$$

$$|\widehat{\mathcal{Q}_\delta}(n)| \le \frac{1}{\delta}, \qquad |\widehat{\widetilde{\mathcal{Q}}_\delta}(n)| \le \frac{1}{\delta^2}. \tag{2.6}$$

Also, for $0 < \delta, \eta < \infty$ *and* $n \in \mathbb{Z}^*$, *we have that*

$$\begin{aligned} &|\widehat{\mathcal{G}_\delta}(n)| \le 1, \\ &|\widehat{\mathcal{G}_\delta}(n)| \le \coth(\eta) - \tfrac{1}{\eta}, \qquad \text{if } \delta|n| \le \eta. \end{aligned} \tag{2.7}$$

Lastly, we recall some important properties of the $\mathcal{G}_\delta$ operator.

Lemma 2.5. *We can write the* $\mathcal{G}_\delta$ *operator as*

$$\mathcal{G}_\delta = \mathcal{T}_\delta - \frac{1}{\delta \partial_x}$$

where $\mathcal{T}_\delta$ *is a Fourier operator with multiplier*

$$\widehat{\mathcal{T}_\delta}(n) = -i\coth(\delta n), \quad n \in \mathbb{Z}^*,$$

and $\widehat{\mathcal{T}_\delta}(0) = 0$. *Moreover, it satisfies the following property for mean zero functions* u, v:

$$\mathcal{T}_\delta[(\mathcal{T}_\delta u)v + u(\mathcal{T}_\delta v)] = (\mathcal{T}_\delta u)(\mathcal{T}_\delta v) - uv.$$

2.3. **Tools from stochastic analysis.** In the following we review some basic facts from stochastic analysis. See, for example, [21, 29, 43] for proofs.

Lemma 2.6 (Wiener chaos estimate)**.** *Let* $\mathbf{g} = \{g_n\}_{n\in\mathbb{Z}}$ *be an independent family of standard complex-valued Gaussian random variables satisfying* $g_{-n} = \overline{g_n}$. *Given* $k \in \mathbb{N}$, *let* $\{Q_j\}_{j\in\mathbb{N}}$ *be a sequence of polynomials in* $\mathbf{g}$ *of degree at most* k. *Then, for any* $2 \le p < \infty$, *we have*

$$\Big\|\sum_{j\in\mathbb{N}} Q_j(\mathbf{g})\Big\|_{L^p(\Omega)} \le (p-1)^{\frac{k}{2}} \Big\|\sum_{j\in\mathbb{N}} Q_j(\mathbf{g})\Big\|_{L^2(\Omega)}.$$

Lemma 2.7 (Kakutani's theorem)**.** *Let* $\{A_n\}_{n\in\mathbb{Z}^*}$ *and* $\{B_n\}_{n\in\mathbb{Z}^*}$ *be two sequences of independent, real-valued, mean-zero Gaussian random variables with* $\mathbb{E}[A_n^2] = a_n > 0$ *and* $\mathbb{E}[B_n^2] = b_n > 0$ *for all* $n \in \mathbb{N}$. *Then, the laws of the sequences* $\{A_n\}_{n\in\mathbb{Z}^*}, \{B_n\}_{n\in\mathbb{Z}^*}$ *are equivalent if and only if*

$$\sum_{n\in\mathbb{N}} \Big(\frac{a_n}{b_n} - 1\Big)^2 < \infty. \tag{2.8}$$

It they are not equivalent, then they are singular.

3. Construction and convergence of measures in the deep-water regime

In this section we prove Theorem 1.1 on construction of the weighted measures $\rho_{\delta,\frac{k}{2}}$ for each fixed $0 < \delta \leq \infty$ and $k \in \mathbb{N}$, and we prove the convergence of $\rho_{\delta,\frac{k}{2}}$ to $\rho_{\infty,\frac{k}{2}}$ in total variation. In Subsections 3.1- 3.3, we establish preliminary results. We first detail the structure of the conservation laws $E_{\delta,\frac{k}{2}}(u)$ of ILW (1.1) in Subsection 3.1. Secondly, in Subsection 3.2, we show the equivalence and convergence of the base Gaussian measures $\mu_{\delta,\frac{k}{2}}$ and $\mu_{\infty,\frac{k}{2}}$ in total variation. Then, in Subsection 3.3 we prove uniform in δ and N bounds on the truncated densities $F_{\delta,\frac{k}{2},N,K}$ via a variational approach. Lastly, in Subsection 3.4 we complete the proof of Theorem 1.1.

3.1. Conservation laws in the deep-water regime.

Conservation laws in the deep-water regime. In this section, we describe the structure of the conserved quantities for ILW (1.1) in the deep-water regime. The derivation of the conserved quantities can be found in [50, 25, 31, 39]. For completeness, we include the derivation and some relevant results on the structure in Appendix A.

Recall that $\mathcal{Q}_\delta = (\mathcal{G}_\delta - \mathcal{H})\partial_x$. Our description of the conservation laws for ILW (1.1) is motivated by seeing ILW as a perturbation of BO and the description of the BO conservation laws in [54]. The following sets are essential to the structure of the conserved quantities. Let $u \in C^\infty(\mathbb{T})$ and define

$$\begin{aligned}
\mathcal{P}_1(u) &:= \big\{\mathcal{H}^{\alpha_1}\mathcal{Q}_\delta^{\beta_1}\partial_x^{\gamma_1}u : \quad \alpha_1 \in \{0,1\}, \beta_1, \gamma_1 \in \mathbb{N}\cup\{0\}\big\},\\
\mathcal{P}_2(u) &:= \big\{\big[\mathcal{H}^{\alpha_1}\mathcal{Q}_\delta^{\beta_1}\partial_x^{\gamma_1}u\big]\big[\mathcal{H}^{\alpha_2}\mathcal{Q}_\delta^{\beta_2}\partial_x^{\gamma_2}u\big] : \quad \alpha_1,\alpha_2 \in \{0,1\}, \beta_1,\beta_2,\gamma_1,\gamma_2 \in \mathbb{N}\cup\{0\}\big\},\\
\mathcal{P}_n(u) &:= \bigg\{\prod_{\ell=1}^{k}\mathcal{H}^{\alpha_\ell}\mathcal{Q}_\delta^{\beta_\ell}p_{j_\ell}(u) : \ \alpha_\ell \in \{0,1\}, \beta_\ell \in \mathbb{N}\cup\{0\},\ j_\ell \in \mathbb{N},\ j_{1\cdots k} = n,\\
&\qquad\qquad k \in \{2,\dots,n\}, p_{j_\ell} \in \mathcal{P}_{j_\ell}(u)\bigg\}.
\end{aligned}$$

Moreover, we define the map $\mathcal{P}_n(u) \ni p_n(u) \mapsto \widetilde{p}_n(u) \in \mathcal{P}_n(u)$ which associates to every $p_n(u) \in \mathcal{P}_n(u)$ the unique essential element $\widetilde{p}_n(u) \in \mathcal{P}_n(u)$ obtained by "dropping" the $\mathcal{Q}_\delta, \mathcal{H}$ operators. Also, we introduce the following quantities associated with $p_n(u)$: let $\widetilde{p}_n(u) = \prod_{i=1}^n \partial_x^{\gamma_i}u$, then

$$\begin{aligned}
|p_n(u)| &:= \sup_{i=1,\dots,n}|\gamma_i|,\\
\|p_n(u)\| &:= \gamma_1 + \ldots + \gamma_n,\\
|||p_n(u)||| &:= \text{ number of } \mathcal{Q}_\delta \text{ terms in } p_n(u).
\end{aligned}$$

We can now write the conserved quantities in a succinct manner. For $k \in \mathbb{N}$, the k-th conserved quantity of ILW (1.1) is given by

$$E_{\delta,\frac{k}{2}}(u) = \frac{1}{2}\sum_{\substack{\ell=0\\ \ell \text{ even}}}^{k} a_\ell \|\mathcal{G}_\delta^{\frac{k-\ell}{2}}u\|^2_{\dot{H}^{\frac{k}{2}}} + R_{\delta,\frac{k}{2}}(u), \tag{3.1}$$

for positive constants a_ℓ with even $\ell \in \{0, \dots, k\}$ which add up to 1, and where the remainder term $R_{\delta, \frac{k}{2}}(u)$ is given by

$$R_{\delta, \frac{k}{2}}(u) = A_{\frac{k}{2}, \frac{k}{2}}(u) + \sum_{\ell=1}^{k-1} \frac{1}{\delta^\ell} A_{\frac{k}{2}, \frac{k-\ell}{2}}(u), \tag{3.2}$$

where $A_{\frac{k}{2}, \frac{\ell}{2}}(u)$ are defined as follows depending on the parity of ℓ:

$$A_{\frac{k}{2}, \frac{2m}{2}}(u) := \sum_{\substack{p(u) \in \mathcal{P}_3(u) \\ \widetilde{p}(u) = u \partial_x^{m-1} u \partial_x^m u \\ |||p(u)||| = 0}} c_{k,m}(p) \int p(u)\, dx + \sum_{\substack{p(u) \in \mathcal{P}_j(u), \\ j = 3, \dots, 2m+2 \\ \|p(u)\| + |||p(u)||| = 2m+2-j \\ |p(u)| \leq m-1}} c_{k,m}(p) \int p(u)\, dx, \tag{3.3}$$

$$\begin{aligned} A_{\frac{k}{2}, \frac{2m+1}{2}}(u) := & \sum_{\substack{p(u) \in \mathcal{P}_3(u) \\ \widetilde{p}(u) = u \partial_x^m u \partial_x^m u \\ |||p(u)||| = 0}} c_{k,m+\frac{1}{2}}(p) \int p(u)\, dx + \sum_{\substack{p(u) \in \mathcal{P}_3(u) \\ \widetilde{p}(u) = \partial_x u \partial_x^{m-1} u \partial_x^m u \\ |||p(u)||| = 0,1}} c_{k,m+\frac{1}{2}}(p) \int p(u)\, dx \\ & + \sum_{\substack{p(u) \in \mathcal{P}_4(u) \\ \widetilde{p}(u) = u^2 \partial_x^{m-1} u \partial_x^m u \\ |||p(u)||| = 0}} c_{k,m+\frac{1}{2}}(p) \int p(u)\, dx + \sum_{\substack{p(u) \in \mathcal{P}_j(u), \\ j = 3, \dots, (2m+1)+2 \\ \|p(u)\| + |||p(u)||| = (2m+1)+2-j \\ |p(u)| \leq m-1}} c_{k,m+\frac{1}{2}}(p) \int p(u)\, dx, \end{aligned} \tag{3.4}$$

for suitable constants $c_{k,m}, c_{k,m+\frac{1}{2}}$. By Lemma A.9 and Lemma A.10, we can rewrite the first contributions in $A_{\frac{k}{2}, \frac{2m}{2}}(u)$ and $A_{\frac{k}{2}, \frac{2m+1}{2}}(u)$ as

$$c_{\frac{k}{2}, m} \int u (\mathcal{H} \partial_x^{m-1} u)(\partial_x^m u)\, dx \quad \text{and} \quad \sum_{(\alpha_1, \alpha_2, \alpha_3) \in C} c_\alpha \int [\mathcal{H}^{\alpha_1} u][\mathcal{H}^{\alpha_2} \partial_x^m u][\mathcal{H}^{\alpha_3} \partial_x^m u]\, dx, \tag{3.5}$$

respectively, with $C = \{(0,0,0), (1,1,0), (0,1,1)\}$, and for some constants $c_{\frac{k}{2}, m}, c_{\alpha_1, \alpha_2}$.

3.2. **Equivalence and convergence of the base Gaussian measures.** Let $k \in \mathbb{N}$ and $0 < \delta \leq \infty$. We focus on the base Gaussian measures $\mu_{\delta, \frac{k}{2}}$ and $\mu_{\infty, \frac{k}{2}}$ defined in (1.11) and (1.17), respectively. Recall that for $n \in \mathbb{Z}^*$, the multiplier $T_{\delta, \frac{k}{2}}(n)$ in (1.13) is given by

$$T_{\delta, \frac{k}{2}}(n) = \sum_{\substack{\ell=0 \\ \text{even}}}^{k} a_\ell |n|^\ell |K_\delta(n)|^{k-\ell}, \qquad T_{\infty, \frac{k}{2}}(n) = |n|^k,$$

for $0 < \delta < \infty$ and constants a_ℓ as in (3.1). Then, $\mu_{\delta, \frac{k}{2}}$ is the probability measure induced by the following random Fourier series in (1.12)

$$X_{\delta, \frac{k}{2}}(x; \omega) = \frac{1}{\sqrt{2\pi}} \sum_{n \in \mathbb{Z}^*} \frac{g_n(\omega)}{\big(T_{\delta, \frac{k}{2}}(n)\big)^{\frac{1}{2}}} e^{inx}, \tag{3.6}$$

where $\{g_n\}_{n \in \mathbb{Z}^*}$ is a sequence of independent standard complex-valued Gaussian random variables conditioned by $g_{-n} = \overline{g_n}$.

Note that from Lemma 2.2, we have that for $0 < \delta \leq \infty$

$$0 \leq T_{\delta, \frac{k}{2}}(n) \sim_\delta |n|^k \quad \text{and} \quad \lim_{\delta \to \infty} T_{\delta, \frac{k}{2}}(n) = |n|^k, \qquad \in \mathbb{Z}^*, \tag{3.7}$$

where the constant is independent of δ for $2 \leq \delta \leq \infty$.

One can easily show that the typical element in the support of the $\mu_{\delta,\frac{k}{2}}$ measure $X_{\delta,\frac{k}{2}}$ is in $H^{\frac{k-1}{2}-\varepsilon}(\mathbb{T}) \setminus H^{\frac{k-1}{2}}(\mathbb{T})$ for all $\varepsilon > 0$, almost surely. Indeed, for $N \in \mathbb{N}$ and $1 \le p < \infty$, with $X_{\delta,\frac{k}{2},N} = \mathbf{P}_N X_{\delta,\frac{k}{2}}$, it follows from the Wiener chaos estimate (Lemma 2.6) and (3.7) that

$$\|X_{\delta,\frac{k}{2},N}\|_{L^p_\omega H_x^{\frac{k-1}{2}-\varepsilon}} \lesssim p^{\frac{1}{2}} \|X_{\delta,\frac{k}{2},N}\|_{L^2_\omega H_x^{\frac{k-1}{2}-\varepsilon}} \lesssim_\delta p^{\frac{1}{2}} \bigg(\sum_{0<|n|\le N} \frac{1}{|n|^{1+2\varepsilon}} \bigg)^{\frac{1}{2}} \sim_\delta p^{\frac{1}{2}},$$

uniformly in $N \in \mathbb{N}$, for $\varepsilon > 0$. From this, we can conclude that $X_{\delta,\frac{k}{2},N}$ converges, in $L^p(\Omega)$ and almost surely, to $X_{\delta,\frac{k}{2}}$ in $H^{\frac{k-1}{2}-\varepsilon}(\mathbb{T})$ for all $\varepsilon > 0$.

The following result establishes the convergence of the random variables $X_{\delta,\frac{k}{2}}$ to $X_{\infty,\frac{k}{2}}$ as well as the convergence of the corresponding Gaussian measures $\mu_{\delta,\frac{k}{2}}$ to $\mu_{\infty,\frac{k}{2}}$, and their equivalence.

Proposition 3.1. *For $k \in \mathbb{N}$, the following results hold:*
(i) *Let $X_{\delta,\frac{k}{2}}$, $X_{\infty,\frac{k}{2}}$ be as in* (3.6)*. Then, given any $\varepsilon > 0$ and $1 \le p < \infty$, $X_{\delta,\frac{k}{2}}$ converges to $X_{\infty,\frac{k}{2}}$ in $L^p(\Omega; H^{\frac{k-1}{2}-\varepsilon}(\mathbb{T}))$ and in $H^{\frac{k-1}{2}-\varepsilon}(\mathbb{T})$ almost surely, as $\delta \to \infty$. In particular, the Gaussian measure $\mu_{\delta,\frac{k}{2}}$ converges weakly to $\mu_{\infty,\frac{k}{2}}$, as $\delta \to \infty$.*
(ii) *For any $0 < \delta < \infty$, the measures $\mu_{\delta,\frac{k}{2}}$ and $\mu_{\infty,\frac{k}{2}}$ are equivalent.*
(iii) *As $\delta \to \infty$, the measure $\mu_{\delta,\frac{k}{2}}$ converges to $\mu_{\infty,\frac{k}{2}}$ in the Kullback-Leibler divergence. In particular, $\mu_{\delta,\frac{k}{2}}$ converges to $\mu_{\infty,\frac{k}{2}}$ in total variation.*

Proof. We first show (i). Let $\varepsilon > 0$ and fix $1 \le p < \infty$. Then, by the Wiener chaos estimate (Lemma 2.6)

$$\begin{aligned} \mathbb{E}\big[\|X_{\delta,\frac{k}{2}} - X_{\infty,\frac{k}{2}}\|^p_{H^{\frac{k-1}{2}-\varepsilon}}\big] &\lesssim_p \mathbb{E}\big[\|X_{\delta,\frac{k}{2}} - X_{\infty,\frac{k}{2}}\|^2_{H^{\frac{k-1}{2}-\varepsilon}}\big] \\ &\sim \bigg(\sum_{n\in\mathbb{Z}^*} \mathbb{E}[|g_n|^2] \langle n \rangle^{k-1-2\varepsilon} \bigg[\frac{1}{(T_{\delta,\frac{k}{2}}(n))^{\frac{1}{2}}} - \frac{1}{|n|^{\frac{k}{2}}} \bigg]^2 \bigg)^{\frac{1}{2}} \\ &\lesssim \bigg(\sum_{n\in\mathbb{Z}^*} \frac{1}{\langle n \rangle^{1+2\varepsilon}} \frac{\big| |n|^k - T_{\delta,\frac{k}{2}}(n) \big|}{T_{\delta,\frac{k}{2}}(n)} \bigg)^{\frac{1}{2}} \end{aligned} \tag{3.8}$$

using the fact that $\sqrt{a} - \sqrt{b} \le \sqrt{a-b}$ for $a \ge b \ge 0$. Looking at the numerator, by (2.4)

$$\begin{aligned} 0 \le |n|^k - T_{\delta,\frac{k}{2}}(n) &= \sum_{\substack{\ell=0 \\ \ell \text{ even}}}^{k} a_\ell |n|^\ell \big[|n|^{k-\ell} - (K_\delta(n))^{k-\ell} \big] \\ &\le \sum_{\substack{\ell=0 \\ \ell \text{ even}}}^{k} a_\ell |n|^\ell \big[|n|^{k-\ell} - (|n| - \tfrac{1}{\delta})^{k-\ell} \big] \\ &= \frac{1}{\delta} \sum_{\substack{\ell=0 \\ \ell \text{ even}}}^{k-1} a_\ell \sum_{j=0}^{k-\ell-1} |n|^{k-1-j} (|n| - \tfrac{1}{\delta})^j, \end{aligned}$$

from which we get that

$$0 \le |n|^k - T_{\delta,\frac{k}{2}}(n) \lesssim \begin{cases} \frac{1}{\delta}|n|^{k-1}, & \text{if } |n| \ge \frac{1}{\delta}, \\ \frac{1}{\delta^k}, & \text{if } |n| \le \frac{1}{\delta}. \end{cases} \tag{3.9}$$

Combining (3.8), (3.7), and (3.9), we get

$$\mathbb{E}\big[\|X_{\delta,\frac{k}{2}} - X_{\infty,\frac{k}{2}}\|^p_{H^{\frac{k-1}{2}-\varepsilon}}\big] \lesssim \bigg(\frac{1}{\delta^k}\sum_{0<|n|\le\frac{1}{\delta}}\frac{1}{\langle n\rangle^{k+1+2\varepsilon}} + \frac{1}{\delta}\sum_{|n|>\frac{1}{\delta}}\frac{1}{\langle n\rangle^{2+2\varepsilon}}\bigg)^{\frac{1}{2}} \lesssim \frac{1}{\delta^{\frac{1}{2}}} \to 0$$

as $\delta \to \infty$, giving the $L^p\big(\Omega; H^{\frac{k-1}{2}-\varepsilon}(\mathbb{T})\big)$ convergence of $X_{\delta,\frac{k}{2}}$. For almost sure convergence, instead of using the fact that $\mathbb{E}[|g_n|^2] = 1$, we use the bound

$$\sup_{n\in\mathbb{Z}^*}\langle n\rangle^{-\varepsilon_0}|g_n(\omega)| \le C_{\varepsilon_0,\omega} < \infty,$$

almost surely, for some random constant $C_{\varepsilon,\omega} > 0$. Proceeding as in the earlier estimate, we have that

$$\begin{aligned}\|X_{\delta,\frac{k}{2}} - X_{\infty,\frac{k}{2}}\|^2_{H^{\frac{k-1}{2}-\varepsilon}} &\le \frac{C_{\varepsilon_0,\omega}}{\delta}\bigg(\frac{1}{\delta^{k-1}}\sum_{0<|n|\le\frac{1}{\delta}}\frac{1}{\langle n\rangle^{k+1+2\varepsilon-\varepsilon_0}} + \sum_{|n|>\frac{1}{\delta}}\frac{1}{\langle n\rangle^{2+2\varepsilon-\varepsilon_0}}\bigg)\\ &\le \frac{\widetilde{C}_{\varepsilon_0,\omega}}{\delta} \to 0\end{aligned}$$

as $\delta \to \infty$ as long as $0 < \varepsilon_0 < 1 + 2\varepsilon$. Since $\mu_{\delta,\frac{k}{2}}, \mu_{\infty,\frac{k}{2}}$ are the laws of $X_{\delta,\frac{k}{2}}, X_{\infty,\frac{k}{2}}$, respectively, the convergence of the random variables implies the weak convergence of the Gaussian measures.

To show (ii), we first rewrite $X_{\delta,\frac{k}{2}}$ as follows

$$X_{\delta,\frac{k}{2}}(x;\omega) = \frac{1}{\sqrt{2\pi}}\sum_{n\in\mathbb{N}}\frac{2}{(T_{\delta,\frac{k}{2}}(n))^{\frac{1}{2}}}\big[\operatorname{Re} g_n\cos(nx) - \operatorname{Im} g_n\sin(nx)\big].$$

The above also holds for $\delta = \infty$. For $n \in \mathbb{N}$, set

$$A_n = \frac{\operatorname{Re} g_n}{(T_{\delta,\frac{k}{2}}(n))^{\frac{1}{2}}}, \quad A_{-n} = -\frac{\operatorname{Im} g_n}{(T_{\delta,\frac{k}{2}}(n))^{\frac{1}{2}}}, \quad B_n = \frac{\operatorname{Re} g_n}{|n|^{\frac{k}{2}}}, \quad B_{-n} = -\frac{\operatorname{Im} g_n}{|n|^{\frac{k}{2}}},$$

with $a_{\pm n} = \mathbb{E}[A^2_{\pm n}] = (T_{\delta,\frac{k}{2}}(n))^{-1}$ and $b_{\pm n} = \mathbb{E}[B^2_{\pm n}] = |n|^{-k}$. Then, using (3.9), we have that

$$\sum_{n\in\mathbb{Z}^*}\bigg(\frac{a_n}{b_n} - 1\bigg)^2 = \sum_{n\in\mathbb{Z}^*}\bigg(\frac{|n|^k}{T_{\delta,\frac{k}{2}}(n)} - 1\bigg)^2 \lesssim \frac{1}{\delta^{2k}}\sum_{0<|n|\le\frac{1}{\delta}}\frac{1}{|n|^{2k}} + \frac{1}{\delta^2}\sum_{|n|>\frac{1}{\delta}}\frac{1}{|n|^2} < \infty.$$

Then, by Kakutani's theorem (Lemma 2.7), we conclude that $\mu_{\delta,\frac{k}{2}}$ and $\mu_{\infty,\frac{k}{2}}$ are equivalent. □

3.3. **Uniform bounds on the density - deep-water regime.** In this subsection, we establish uniform in δ and N bounds on the truncated density $F_{\delta,\frac{k}{2},N,K}$ defined by

$$F_{\delta,\frac{k}{2},N,K}(u) = \eta_K\big(\|\mathbf{P}_N u\|^2_{L^2}\big)\exp\big(-R_{\delta,\frac{k}{2}}(\mathbf{P}_N u)\big), \tag{3.10}$$

where we fix $K > 0$ for the remaining of this section. Our main goal is to prove the following result.

Proposition 3.2. *Let $1 \le p < \infty$, $k \in \mathbb{N}$, and $K > 0$. Then, for any $0 < \delta \le \infty$, we have that*

$$\sup_{N\in\mathbb{N}} \|F_{\delta,\frac{k}{2},N,K}(X_{\delta,\frac{k}{2}})\|_{L^p(\Omega)} = \sup_{N\in\mathbb{N}} \|F_{\delta,\frac{k}{2},N,K}(u)\|_{L^p(d\mu_{\delta,\frac{k}{2}})} \le C_{p,k,\delta,K} < \infty. \tag{3.11}$$

In addition, the following uniform bound holds for $2 \le \delta \le \infty$:

$$\sup_{N\in\mathbb{N}} \sup_{2\le\delta\le\infty} \|F_{\delta,\frac{k}{2},N,K}(X_{\delta,\frac{k}{2}})\|_{L^p(\Omega)} = \sup_{N\in\mathbb{N}} \sup_{2\le\delta\le\infty} \|F_{\delta,\frac{k}{2},N,K}(u)\|_{L^p(d\mu_{\delta,\frac{k}{2}})} \le C_{p,k,K} \tag{3.12}$$

for a finite constant $C_{p,k,K} > 0$.

To establish Proposition 3.2, we use a variational approach, introduced by Barashkov-Gubinelli [4]; see also [44, 19, 45, 9, 46, 33] for other recent applications of this method. In particular, we consider the following truncated density

$$\mathcal{F}_{\delta,\frac{k}{2},N,K}(u) = \exp\Big(- R_{\delta,\frac{k}{2}}(\mathbf{P}_N u) - A\|\mathbf{P}_N u\|_{L^2}^{2\alpha(k)}\Big) \tag{3.13}$$

for $A > 0$ and $\alpha(k) \in \mathbb{N}$ to be chosen later. Noting that

$$\eta_K(x) \le \exp\big(- A|x|^\gamma\big) \exp(AK^\gamma) \tag{3.14}$$

for any $K, A, \gamma > 0$, we have that

$$F_{\delta,\frac{k}{2},N,K}(u) \le C_{A,K} \cdot \mathcal{F}_{\delta,\frac{k}{2},N,K}(u).$$

Hence, Proposition 3.2 follows once we prove the following uniform bounds.

Proposition 3.3. *There exist $A_0, \gamma > 0$ such that*

$$\sup_{N\in\mathbb{N}} \|\mathcal{F}_{\delta,\frac{k}{2},N,K}(X_{\delta,\frac{k}{2}})\|_{L^p(\Omega)} = \sup_{N\in\mathbb{N}} \|\mathcal{F}_{\delta,\frac{k}{2},N,K}(u)\|_{L^p(d\mu_{\delta,\frac{k}{2}})} \le C_{p,\delta,k,K,A_0} \tag{3.15}$$

for any $0 < \delta \le \infty$, $1 \le p < \infty$, $K > 0$, $A \ge A_0$, and a finite constant $C_{p,\delta,k,K,A_0} > 0$. In addition, the following uniform bound holds for $2 \le \delta \le \infty$:

$$\sup_{N\in\mathbb{N}} \sup_{2\le\delta\le\infty} \|\mathcal{F}_{\delta,\frac{k}{2},N,K}(X_{\delta,\frac{k}{2}})\|_{L^p(\Omega)} = \sup_{N\in\mathbb{N}} \sup_{2\le\delta\le\infty} \|\mathcal{F}_{\delta,\frac{k}{2},N,K}(u)\|_{L^p(d\mu_{\delta,\frac{k}{2}})} \le C_{p,k,K,A_0} \tag{3.16}$$

for any $1 \le p < \infty$, $K > 0$, $A \ge A_0$, and a finite constant $C_{p,k,K,A_0} > 0$.

Let us first introduce some notations. Let $W(t)$ be a cylindrical Brownian motion in $L^2_0(\mathbb{T}) = \mathbf{P}_{\neq 0} L^2(\mathbb{T})$ of mean-zero functions on $\mathbb{T}$, where $\mathbf{P}_{\neq 0}$ denotes the projection onto non-zero frequencies. Namely, we have

$$W(t) = \frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}^*} B_n(t) e^{inx}, \tag{3.17}$$

where $\{B_n\}_{n\in\mathbb{Z}^*}$ is a sequence of mutually independent complex-valued[1] Brownian motions such that $\overline{B_n} = B_{-n}$, $n \in \mathbb{Z}^*$. Then, we define a centered Gaussian process $Y_{\delta,\frac{k}{2}}(t)$ by

$$Y_{\delta,\frac{k}{2}}(t) = (\mathcal{T}_{\delta,\frac{k}{2}})^{-\frac{1}{2}} W(t), \tag{3.18}$$

where $(\mathcal{T}_{\delta,\frac{k}{2}})^{-\frac{1}{2}}$ is the Fourier multiplier operator with the multiplier $(T_{\delta,\frac{k}{2}}(n))^{-\frac{1}{2}}$ as in (1.13). In view of (1.12), we have $\mathcal{L}(Y_{\delta,\frac{k}{2}}(1)) = \mu_{\delta,\frac{k}{2}}$. Given $N \in \mathbb{N}$, we set $Y_{\delta,\frac{k}{2},N} = \mathbf{P}_N Y_{\delta,\frac{k}{2}}$. Let $\mathbb{H}_a$ denote the collection of drifts, which are progressively measurable

[1] By convention, we normalize B_n such that $\mathrm{Var}(B_n(t)) = \sqrt{2\pi} t$.

processes belonging to $L^2([0,1];L^2_0(\mathbb{T}))$, $\mathbb{P}$-almost surely. We now state the Boué-Dupuis variational formula [6, 57].

Lemma 3.4. *Given $0<\delta\le\infty$, let $Y_{\delta,\frac{k}{2}}$ be as in (3.18) and fix $N\in\mathbb{N}$. Suppose that $F: C^\infty(\mathbb{T})\to\mathbb{R}$ is measurable such that $\mathbb{E}\big[|F(Y_{\delta,\frac{k}{2}}(1))|^p\big]<\infty$ and $\mathbb{E}\big[|e^{-F(Y_{\delta,\frac{k}{2}}(1))}|^q\big]<\infty$ for some $1<p,q<\infty$ with $\frac{1}{p}+\frac{1}{q}=1$. Then, we have*

$$-\log\mathbb{E}\Big[e^{-F(Y_{\delta,\frac{k}{2}}(1))}\Big]=\inf_{\theta\in\mathbb{H}_a}\mathbb{E}\bigg[F(Y_{\delta,\frac{k}{2}}(1)+I_{\delta,\frac{k}{2}}(\theta)(1))+\frac{1}{2}\int_0^1\|\theta(t)\|^2_{L^2_x}dt\bigg], \tag{3.19}$$

where $I_{\delta,\frac{k}{2}}(\theta)$ is defined by

$$I_{\delta,\frac{k}{2}}(\theta)(t)=\int_0^t(\mathcal{T}_{\delta,\frac{k}{2}})^{-\frac{1}{2}}\theta(t')dt'$$

and the expectation $\mathbb{E}=\mathbb{E}_\mathbb{P}$ is taken with respect to the underlying probability measure $\mathbb{P}$.

We first show a preliminary result on the pathwise regularity of $Y_{\delta,\frac{k}{2},N}(1)$ and $I_{\delta,\frac{k}{2}}(\theta)(1)$.

Lemma 3.5. *Let $0<\delta\le\infty$ and $k\in\mathbb{N}$. For any finite $p\ge1$ and $\sigma<\frac{k-1}{2}$, we have that*

$$\sup_{N\in\mathbb{N}}\mathbb{E}\Big[\|Y_{\delta,\frac{k}{2}N}(1)\|^p_{W^{\sigma,\infty}_x}\Big]<C_{p,\delta}<\infty,$$

where $C_{p,\delta}$ is independent of δ if $2\le\delta\le\infty$. Moreover, for any $\theta\in\mathbb{H}_a$, we have

$$\|I_{\delta,\frac{k}{2}}(\theta)(1)\|^2_{H^{\frac{k}{2}}_x}\le C_\delta\int_0^t\|\theta(t)\|^2_{L^2_x}\,dt$$

where the constant $C_\delta>0$ can be chosen independently of δ if $2\le\delta\le\infty$.

Proof. For the first inequality, fix $\sigma<\frac{k-1}{2}$ and $\varepsilon>0$ sufficiently small such that $\sigma+\varepsilon<\frac{k-1}{2}$. Then, for $r\gg1$ with $r\varepsilon>1$, Sobolev embedding gives that

$$\|Y_{\delta,\frac{k}{2},N}(1)\|_{W^{\sigma,\infty}_x}\lesssim\|\langle\nabla\rangle^{\sigma+\varepsilon}Y_{\delta,\frac{k}{2},N}(1)\|_{L^r_x}.$$

Fix $p\ge r$. Note that the result follows for $1\le p<r$ from the embedding $L^r(\Omega)\subset L^p(\Omega)$. Then, by Minkowski's inequality and the Wiener chaos estimate (Lemma 2.6), we have

$$\begin{aligned}\big\|\|Y_{\delta,\frac{k}{2},N}(1)\|_{W^{\sigma,\infty}_x}\big\|_{L^p(\Omega)}&\le Cp^{\frac{1}{2}}\big\|\|\langle\nabla\rangle^{\sigma+\varepsilon}Y_{\delta,\frac{k}{2},N}(1)\|_{L^2(\Omega)}\big\|_{L^r_x}\\&\le C_p\bigg\|\mathbb{E}\bigg[\sum_{0<|n|,|m|\le N}\langle n\rangle^{\sigma+\varepsilon}\langle m\rangle^{\sigma+\varepsilon}\frac{g_n\overline{g_m}}{(T_{\delta,\frac{k}{2}}(n)T_{\delta,\frac{k}{2}}(m))^{\frac{1}{2}}}e^{i(n-m)x}\bigg]\bigg\|_{L^r_x}\\&\le C_{p,\delta}\sum_{0<|n|\le N}\frac{1}{|n|^{k-2(\sigma+\varepsilon)}}<C_{p,\delta}<\infty\end{aligned}$$

due to (3.7), for some constant $C_{p,\delta}>0$ independent of $N\in\mathbb{N}$. Moreover, for $2\le\delta<\infty$, $C_{p,\delta}$ can be chosen independently of δ. For the second estimate, from Minkowski and Cauchy-Schwarz inequalities, and (3.7), we have that

$$\|I_{\delta,\frac{k}{2}}(\theta)(1)\|^2_{H^{\frac{k}{2}}_x}\le C\bigg(\int_0^1\bigg[\sum_{n\neq0}\frac{|n|^k}{T_{\delta,\frac{k}{2}}(n)}|\widehat{\theta}(t,n)|^2\bigg]^{\frac{1}{2}}dt\bigg)^2\le C_\delta\bigg(\int_0^1\|\theta(t)\|_{L^2_x}dt\bigg)^2$$

$$\leq C_\delta \int_0^1 \|\theta(t)\|_{L^2_x}^2 \, dt,$$

where C_δ is independent of δ for $2 \leq \delta < \infty$. □

Note that (3.19) is equivalent to

$$\log \mathbb{E}\Big[e^{-F(Y_{\delta,\frac{k}{2}}(1))}\Big] = \sup_{\theta \in \mathbb{H}_a} \mathbb{E}\bigg[- F(Y_{\delta,\frac{k}{2}}(1) + I_{\delta,\frac{k}{2}}(\theta)(1)) - \frac{1}{2}\int_0^1 \|\theta(t)\|_{L^2_x}^2 dt\bigg].$$

Then, setting $F(u) = p\big[R_{\delta,\frac{k}{2},K}(\mathbf{P}_N u) + A\|\mathbf{P}_N u\|_{L^2}^{2\alpha(k)}\big]$, Proposition 3.3 follows from Lemma 3.4 once we establish an upper bound on

$$\mathcal{M}_{\delta,\frac{k}{2},N}(v) = \mathbb{E}\bigg[- pR_{\delta,\frac{k}{2}}(Y_{\delta,\frac{k}{2},N}(1) + \mathbf{P}_N v) - pA\|Y_{\delta,\frac{k}{2},N}(1) + \mathbf{P}_N v\|_{L^2}^{2\alpha(k)} - \frac{1}{2}C\|v\|_{H_x^{\frac{k}{2}}}^2\bigg] \tag{3.20}$$

uniformly in $N \in \mathbb{N}$ and $v \in H^{\frac{k}{2}}(\mathbb{T})$. In the following, we will use drop the δ, k dependence on $Y_{\delta,\frac{k}{2},N}$ and use notation $v_N = \mathbf{P}_N v$. From the definition of the remainder $R_{\delta,\frac{k}{2}}$ in (3.2), we can rewrite it as

$$R_{\delta,\frac{k}{2}}(Y_N + v_N) = A_{\frac{k}{2},\frac{k}{2}}(Y_N + v_N) + \sum_{\ell=1}^{k-1} \frac{1}{\delta^{k-\ell}} A_{\frac{k}{2},\frac{\ell}{2}}(Y_N + v_N).$$

The following lemma controls the lower order terms in the remainder.

Lemma 3.6. *Let $k \in \mathbb{N}$ with $k \geq 2$, $1 \leq \ell \leq k-1$, $0 < \delta \leq \infty$, $u_1 \in H^{\frac{k-1}{2}-}(\mathbb{T})$, and $u_2 \in H^{\frac{k}{2}}(\mathbb{T})$. Then, for $0 < \varepsilon < 1$, there exists $C > 0$ independent of δ such that*

$$|A_{\frac{k}{2},\frac{\ell}{2}}(u_1 + u_2)| \leq C\Big(1 + \frac{1}{\delta^{\ell-1}}\Big)\Big[1 + \|u_1\|_{H^{\frac{k-1}{2}-}}^q + \|u_2\|_{L^2}^{2\alpha(k)}\Big] + \varepsilon\|u_2\|_{H^{\frac{k}{2}}}^2,$$

for $1 \ll q < \infty$ large enough and $\alpha(k) \in \mathbb{N}$ sufficiently large.

Proof. Fix $k \in \mathbb{N}$ with $k \geq 2$. Since the dependence on k in the definition of $A_{\frac{k}{2},\frac{\ell}{2}}$ in (3.3)-(3.4) is only seen in the constants, we will write $A_{\frac{\ell}{2}}$ for simplicity. For $\ell = 1$, since $A_{\frac{1}{2}}(u) \sim \int u^3 \, dx$, using Sobolev's inequality, (2.1), and Young's inequality gives

$$\begin{aligned} |A_{\frac{1}{2}}(u_1 + u_2)| &\lesssim \|u_1\|_{H^{\frac{1}{6}}}^3 + \|u_2\|_{H^{\frac{k}{2}}}^{\frac{1}{k}} \|u_2\|_{L^2}^{\frac{3k-1}{k}} \\ &\leq C_\varepsilon\big(\|u_1\|_{H^{\frac{k-1}{2}-}}^3 + \|u_2\|_{L^2}^{\frac{2(3k-1)}{2k-1}}\big) + \varepsilon\|u_2\|_{H^{\frac{k}{2}}}^2 \end{aligned} \tag{3.21}$$

for $C, C_\varepsilon > 0$ and any $0 < \varepsilon \ll 1$. Also, note that $\frac{2(2k-1)}{2k-1} \leq k+1$.

We now estimate $A_{\frac{\ell}{2}}$ when $k \geq 3$ and $2 \leq \ell \leq k-1$. First, consider $\ell = 2m$ for $1 \leq m \leq \frac{k-1}{2}$. For the first terms in (3.3), it suffices to consider $p(u) = u\partial_x^{m-1}u\partial_x^m u$, since the same estimate follows for the remaining terms from the boundedness of the Hilbert transform. Let $\frac{1}{2} < \theta < \frac{3}{4}$, then it follows from Cauchy-Schwarz, (2.3), Sobolev inequality, (2.1), and Young's inequality that for $0 < \varepsilon < 1$ there exists $C_\varepsilon > 0$ such that

$$\bigg|\int (u_1 + u_2)\partial_x^{m-1}(u_1 + u_2)\partial_x^m(u_1 + u_2) \, dx\bigg| \tag{3.22}$$

$$
\begin{aligned}
&= \bigg| \int \partial_x^\theta[(u_1+u_2)\partial_x^{m-1}(u_1+u_2)]\partial_x^{m-\theta}(u_1+u_2)\,dx \bigg| \\
&\lesssim \|u_1+u_2\|_{H^{m-\theta}}^2 \|u_1+u_2\|_{H^{2\theta-\frac12}} \\
&\lesssim \|u_1\|^3_{H^{\frac{k-1}{2}-}} + \|u_1\|^2_{H^{\frac{k-1}{2}-}}\|u_2\|_{H^{\frac k2}} + \|u_2\|_{L^2}^{\frac{2k-4m+1}{k}}\|u_2\|_{H^{\frac k2}}^{\frac{4m-1}{k}} \\
&\quad + \|u_1\|_{H^{\frac{k-1}{2}-}}\big(\|u_2\|_{L^2}^{\frac{2k-4m+4\theta}{k}}\|u_2\|_{H^{\frac k2}}^{\frac{4m-4\theta}{k}} + \|u_2\|_{L^2}^{\frac{2k-4m+4\theta}{k}}\|u_2\|_{H^{\frac k2}}^{\frac{4m-4\theta}{k}}\big) \\
&\le C_\varepsilon\big(1+\|u_1\|^q_{H^{\frac{k-1}{2}-}} + \|u_2\|_{L^2}^{2(k-1)}\big) + \varepsilon\|u_2\|^2_{H^{\frac k2}}
\end{aligned} \tag{3.23}
$$

since $2\theta - \frac12 < 1 \le \frac{k-1}{2}$ and $m \le \frac{k-1}{2}$, for $1 \ll q < \infty$ sufficiently large. For the second contribution in (3.3), using (2.6), Young's convolution inequality and Hölder's inequality, we estimate it by

$$
\begin{aligned}
&\sum_{j=3}^{2m+2}\sum_{i=0}^{2m+2-j}\sum_{\substack{m-1\ge\alpha_1\ge\ldots\ge\alpha_j\ge0,\\ \alpha_{1\ldots j}=i}} \frac{1}{\delta^{2m+2-j-i}}\|u_1+u_2\|_{H^{\alpha_1}}\|u_1+u_2\|_{H^{\alpha_2}}\prod_{l=3}^{j}\|u_1+u_2\|_{H^{\alpha_l+\frac12+}} \\
&\lesssim \Big(1+\frac{1}{\delta^{2m-1}}\Big)\bigg(1+\|u_1\|^{2m+2}_{H^{\frac{k-1}{2}-}} + \|u_1\|^{2m+1}_{H^{\frac{k-1}{2}-}}\|u_2\|_{H^{\frac k2}} + \|u_1\|^{2m}_{H^{\frac{k-1}{2}-}}\|u_2\|^2_{H^{m-1}} \\
&\qquad + \sum_{j=3}^{2m+2}\sum_{\substack{m-1\ge\alpha_1\ge\ldots\ge\alpha_j\ge0,\\ \alpha_{1\ldots j}=2m+2-j}}\Big[\|u_1\|^{j-3}_{H^{m-1}}\|u_2\|_{H^{\alpha_1}}\|u_2\|_{H^{\alpha_2}}\|u_2\|_{H^{\alpha_3+\frac12+}} + \cdots \\
&\qquad\qquad + \|u_2\|_{H^{\alpha_1}}\|u_2\|_{H^{\alpha_2}}\prod_{l=3}^{j}\|u_2\|_{H^{\alpha_l+\frac12+}}\Big]\bigg).
\end{aligned}
$$

For fixed $j \in \{3,\ldots,2m+2\}$, from (2.1) and Young's inequality, we have that for $0<\varepsilon<1$, there exists $C_\varepsilon > 0$ and $1 \le q < \infty$ such that

$$
\|u_1\|^{2m+1}_{H^{\frac{k-1}{2}-}}\|u_2\|_{H^{\frac k2}} + \|u_1\|^{2m}_{H^{\frac{k-1}{2}-}}\|u_2\|^2_{H^{m-1}} \le C_\varepsilon(1+\|u_1\|^q_{H^{\frac{k-1}{2}-}} + \|u_2\|^6_{L^2}) + \varepsilon\|u_2\|^2_{H^{\frac k2}},
$$

while for $3 \le l \le j$ we have

$$
\begin{aligned}
&\|u_1\|^{j-l}_{H^{\frac{k-1}{2}-}}\|u_2\|_{H^{\alpha_1}}\|u_2\|_{H^{\alpha_2}}\prod_{\kappa=3}^{l}\|u_2\|_{H^{\alpha_\kappa+\frac12+}} \\
&\lesssim \|u_1\|^{j-l}_{H^{\frac{k-1}{2}-}}\|u_2\|_{H^{\frac k2}}^{\frac{2\alpha_{1\ldots l}+l-2}{k}}\|u_2\|_{L^2}^{\frac{lk-2\alpha_{1\ldots l}-l+2}{k}} \\
&\le C_\varepsilon\Big(1+\|u_1\|^q_{H^{\frac{k-1}{2}-}} + \|u_2\|_{L^2}^{2\frac{lk-2\alpha_{1\ldots l}-l+2}{2k-2\alpha_{1\ldots l}+l-2}+}\Big) + \varepsilon\|u_2\|^2_{H^{\frac k2}}
\end{aligned}
$$

for some $1 \ll q < \infty$. Replacing this estimate above gives that for $0<\varepsilon<1$ there exists $C_\varepsilon > 0$ and $1 \ll q < \infty$ such that

$$
\cdots \lesssim \Big(1+\frac{1}{\delta^{2m-1}}\Big)\Big[C_\varepsilon\Big(1+\|u_1\|^q_{H^{\frac{k-1}{2}-}} + \|u_2\|_{L^2}^{2(k-1)}\Big) + \varepsilon\|u_2\|^2_{H^{\frac k2}}\Big]. \tag{3.24}
$$

Thus, for even $k \geq 3$ and $2 \leq \ell \leq k-1$ even, for $0 < \varepsilon \ll 1$ sufficiently small, there exists $C_\varepsilon > 0$ such that

$$|A_{\frac{\ell}{2}}(u_1+u_2)| \leq \Big(1+\frac{1}{\delta^{\ell-1}}\Big)C_\varepsilon\Big(1+\|u_1\|^q_{H^{\frac{k-1}{2}-}} + \|u_2\|^{2(k-1)}_{L^2}\Big) + \varepsilon\|u_2\|^2_{H^{\frac{k}{2}}}, \tag{3.25}$$

for some $1 \ll q < \infty$.

Now let $k \geq 3$ and $2 \leq \ell \leq k-1$ odd, i.e., $\ell = 2m+1$ for $1 \leq m \leq \frac{k-2}{2}$. From (3.4), we see that from (2.6) and Young's convolution inequality, we have

$$\begin{aligned}
&|A_{\frac{2m+1}{2}}(u_1+u_2)| \\
&\lesssim \sum_{j=3}^{2m+3}\sum_{i=0}^{2m+3-j} \sum_{\substack{p(u)\in\mathcal{P}_j(u)\\ |||p(u)|||=2m+3-j-i\\ \widetilde{p}(u)=\prod_{\kappa=1}^j \partial_x^{\alpha_\kappa}u\\ m\geq\alpha_{1\dots j}\geq 0}} \Big|\int p(u_1+u_2)\,dx\Big| \\
&\lesssim \Big(1+\frac{1}{\delta^{2m}}\Big)\sum_{j=3}^{2m+3}\sum_{\substack{\alpha_{1\dots j}=2m+3-j\\ m\geq\alpha_1\geq\dots\geq\alpha_j\geq 0}} \|u_1+u_2\|_{H^{\alpha_1}}\|u_1+u_2\|_{H^{\alpha_2}}\prod_{\kappa=3}^{j}\|u_1+u_2\|_{H^{\alpha_\kappa+\frac{1}{2}+}}.
\end{aligned}$$

For $3 \leq l \leq j$, for $0 < \varepsilon < 1$ there exists $C_\varepsilon > 0$ such that

$$\begin{aligned}
&\|u_1\|^{j-l}_{H^m}\|u_2\|_{H^{\alpha_1}}\|u_2\|_{H^{\alpha_2}}\prod_{\kappa=3}^{l}\|u_2\|_{H^{\alpha_\kappa+\frac{1}{2}+}} \\
&\leq C_\varepsilon\Big(1+\|u_1\|^q_{H^{\frac{k-1}{2}-}} + \|u_2\|_{L^2}^{2\frac{(k-2)j-2}{2k-2m-1}}\Big) + \varepsilon\|u_2\|^2_{H^{\frac{k}{2}}}
\end{aligned}$$

for $1 \ll q < \infty$ sufficiently large. Then, we have that for $0 < \varepsilon < 1$ there exists $C_\varepsilon > 0$ such that

$$\begin{aligned}
&|A_{\frac{2m+1}{2}}(u_1+u_2)| \\
&\leq \big(1+\frac{1}{\delta^{2m}}\big)\Big(1+\|u_1\|^q_{H^{\frac{k-1}{2}-}} + \|u_2\|^{2(2k-1)(k-2)}_{L^2}\Big) + \varepsilon\|u_2\|^2_{H^{\frac{k}{2}}}
\end{aligned} \tag{3.26}$$

for $1 \ll q < \infty$ sufficiently large. Thus, for odd $3 \leq \ell \leq k-1$, for $0 < \varepsilon \ll 1$ sufficiently small, there exist $C_\varepsilon > 0$ and $1 \ll q < \infty$ such that

$$|A_{\frac{\ell}{2}}(u_1+u_2)| \leq \Big(1+\frac{1}{\delta^{\ell-1}}\Big)C_\varepsilon\Big[1+\|u_1\|^q_{H^{\frac{k-1}{2}-}} + \|u_2\|^{2(k-2)(2k-1)}_{L^2}\Big] + \varepsilon\|u_2\|^2_{H^{\frac{k}{2}}},. \tag{3.27}$$

The estimate follows from combining (3.21), (3.25), and (3.27). □

The following lemma controls the leading contribution in the remainder.

Lemma 3.7. *Let $k \in \mathbb{N}$, $0 < \delta \leq \infty$, and Y_N, v_N as defined earlier. Then, for $0 < \varepsilon < 1$, there exists $C_{\delta,\varepsilon} > 0$ independent of $N \in \mathbb{N}$ such that*

$$\big|\mathbb{E}\big[A_{\frac{k}{2},\frac{k}{2}}(Y_N+v_N)\big]\big| \leq \mathbb{E}\Big[C_{\delta,\varepsilon}\big(1+\|Y_N\|^q_{H^{\frac{k-1}{2}-}} + \|v_N\|^{2\alpha(k)}_{L^2}\big) + \varepsilon\|v_N\|^2_{H^{\frac{k}{2}}}\Big]$$

for $1 \ll q < \infty$ and $\alpha(k) \in \mathbb{N}$ sufficiently large enough depending on k. Moreover, $C_{\delta,\varepsilon}$ can be chosen independently of $2 \leq \delta \leq \infty$.

Proof. We first consider odd integers $k \geq 1$. Let $k = 2\ell + 1$ for $\ell \geq 0$. We can estimate the last contribution in (3.4) as in (3.26) for $\ell \geq 1$ (since this contribution does not appear in $A_{\frac12,\frac12}$), to get that for $\varepsilon > 0$ there exist $C_\varepsilon > 0$, $1 \ll q < \infty$, and $\alpha(k) \in \mathbb{N}$ such that

$$\sum_{j=3}^{k+2} \sum_{i=0}^{k+2-j} \sum_{\substack{p(u) \in \mathcal{P}_j(u) \\ |||p(u)||| = k+2-j-i \\ \widetilde{p}(u) = \prod_{\kappa=1}^{j} \partial_x^{\alpha_\kappa} u \\ \alpha_{1\dots j} = i \\ \ell - 1 \geq \alpha_1 \geq \dots \geq \alpha_j \geq 0}} \bigg| \int p(Y_N + v_N)\, dx \bigg|$$
$$\leq C\Big(1 + \frac{1}{\delta^{k-1}}\Big)\Big[1 + \|Y_N\|^q_{H^{\frac{k-1}{2}-}} + \|v_N\|^{2\alpha(k)}_{L^2}\Big] + \varepsilon \|v_N\|^2_{H^{\frac{k}{2}}}.$$

For $\ell \geq 0$, to estimate the first term in (3.4), it suffices to estimate $\int p(Y_N + v_N)\, dx$ where $p(u) = u \partial_x^\ell u \partial_x^\ell u$:

$$\mathbb{E}\bigg[\int \Big(Y_N (\partial_x^\ell Y_n)^2 + v_N (\partial_x^\ell Y_N)^2 + (Y_N + v_N)\partial_x^\ell (Y_N + v_N) \partial_x^\ell v_N\Big)\, dx\bigg].$$

The first contribution vanishes by Isserlis' theorem. For the second contribution, using Hölder's and Young's inequality, for $0 < \varepsilon < 1$, there exist $C > 0$ such that

$$\begin{aligned}
\bigg| \int v_N (\partial_x^\ell Y_N)^2\, dx \bigg| &= \bigg| \int \langle \partial_x \rangle^{\ell + \frac12} (v_N) \langle \partial_x \rangle^{-\ell - \frac12} (\partial_x^\ell Y_N)^2 \bigg| \\
&\lesssim \|v_N\|_{H^{\ell+\frac12}} \|\mathbf{P}_{\neq 0} (\partial_x^\ell Y_N)^2\|_{H^{-\ell-\frac12}} \\
&\leq \varepsilon \|v_N\|^2_{H^{\frac{k}{2}}} + C \|\mathbf{P}_{\neq 0}(\partial_x^\ell Y_N)^2\|^2_{H^{-\ell-\frac12}},
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}\Big[\|(\partial_x^\ell Y_N)^2\|^2_{H^{-\ell-\frac12}}\Big] &= \mathbb{E}\bigg[\sum_{n \neq 0} \frac{1}{\langle n \rangle^{2\ell+1}} \bigg| \sum_{\substack{n = n_1 + n_2 \\ 0 < |n_j| \leq N}} \frac{g_{n_1} g_{n_2}}{(T_{\delta, \ell + \frac12}(n_1) T_{\delta, \ell+\frac12}(n_2))^{\frac12}} (n_1 n_2)^\ell \bigg|^2\bigg] \\
&= \sum_{n \neq 0} \frac{1}{\langle n \rangle^{2\ell+1}} \sum_{\substack{n = n_1 + n_2 \\ 0 < |n_j| \leq N}} \frac{|n_1 n_2|^{2\ell}}{T_{\delta,\ell+\frac12}(n_1) T_{\delta, \ell+\frac12}(n_2)} \\
&\sim_\delta \sum_{n, n_1} \frac{1}{\langle n \rangle \langle n_1 \rangle \langle n - n_1 \rangle} < C_\delta,
\end{aligned}$$

by (3.7), for some $C_\delta > 0$ independent of $N \in \mathbb{N}$, which can be chosen independently of $2 \leq \delta \leq \infty$. For the last contribution, using Cauchy's inequality, (2.3), Sobolev inequality, (2.1), and Young's inequality, we get that for $0 < \varepsilon < 1$, there exists $C > 0$ such that

$$\begin{aligned}
&\bigg| \int (Y_N + v_N) \partial_x^\ell v_N \partial_x^\ell (Y_N + v_N)\, dx \bigg| \\
&= \bigg| \int \langle \partial_x \rangle^\theta ((Y_N + v_N) \partial_x^\ell v_N) \langle \partial_x \rangle^{-\theta} \partial_x^\ell (Y_N + v_N)\, dx \bigg| \\
&\lesssim \|(Y_N + v_N)\partial_x^\ell v_N\|_{H^\theta} \|Y_N + v_N\|_{H^{\ell-\theta}} \\
&\lesssim \|Y_N + v_N\|_{H^{\ell-\theta}} \|Y_N + v_N\|_{H^{\frac14}} \|v_N\|_{H^{\ell+\theta+\frac14}} \\
&\lesssim 1 + \|Y_N\|^4_{H^{\ell-}} + \|Y_N\|_{H^{\ell-}} \Big(\|v_N\|_{H^{\frac{k}{2}}}^{\frac{2(\ell+\theta)+1}{k}} \|v_N\|_{L^2}^{\frac{2(k-\ell-\theta)-1}{k}} + \|v_N\|_{H^{\frac{k}{2}}}^{\frac{8\ell+1}{2k}} \|v_N\|_{L^2}^{\frac{4(k-2\ell)-1}{2k}}\Big)
\end{aligned}$$

$$
\begin{aligned}
&+ \|v_N\|_{H^{\frac{k}{2}}}^{\frac{4\ell+1}{k}} \|v_N\|_{L^2}^{\frac{3k-4\ell-1}{k}} \\
&\le C\big(1 + \|Y_N\|^q_{H^{\frac{k-1}{2}-}} + \|v_N\|_{L^2}^{2(k+1)}\big) + \varepsilon\|v_N\|^2_{H^{\frac{k}{2}}}
\end{aligned}
$$

for $\ell = 1$, $0 < \theta < \frac{1}{4}$, and $1 \ll q < \infty$ sufficiently large. For $\ell = 0$, it remains to estimate the cubic contribution $\int v_N^3 \, dx$:

$$
\left| \int v_N^3 \, dx \right| \lesssim \|v_N\|^3_{H^{\frac{1}{6}}} \lesssim \|v_N\|^2_{L^2}\|v_N\|_{H^{\frac{1}{2}}} \le C\|v_N\|^4_{L^2} + \varepsilon\|v_N\|^2_{H^{\frac{1}{2}}}.
$$

For the cubic terms of the form $\int p(Y_N + v_N)\, dx$ where $\widetilde{p}(u) = \partial_x u \cdot \partial_x^{\ell-1} u \cdot \partial_x^{\ell} u$, we need only estimate $\ell \ge 2$, as for $\ell = 1$ this is the same as the previous term. Then, proceeding as before, for $0 < \varepsilon < 1$, there exists $C > 0$ such that

$$
\begin{aligned}
&\left| \int \partial_x(Y_N + v_N)\partial_x^{\ell-1}(Y_N + v_N)\partial_x^{\ell}(Y_N + v_N)\, dx \right| \\
&= \left| \int \langle \partial_x\rangle^{\theta}(\partial_x(Y_N + v_N)\partial_x^{\ell-1}(Y_N + v_N))\langle\partial_x\rangle^{-\theta}\partial_x^{\ell}(Y_N + v_N)\, dx \right| \\
&\lesssim \|\partial_x(Y_N + v_N)\partial_x^{\ell-1}(Y_N + v_N)\|_{H^{\theta}}\|Y_N + v_N\|_{H^{\ell-\theta}} \\
&\lesssim \|Y_N + v_N\|_{H^{\ell-\theta}}\|Y_N + v_N\|_{H^{\frac{1}{2}+2\theta}}\|v_N\|_{H^{\ell-\theta}} \\
&\lesssim 1 + \|Y_N\|^4_{H^{\ell-}} + \|Y_N\|_{H^{\ell-}}\big(\|v_N\|_{H^{\frac{k}{2}}}^{\frac{4(\ell-\theta)}{k}}\|v_N\|_{L^2}^{\frac{2(k-2\ell+2\theta)}{k}} + \|v_N\|_{H^{\frac{k}{2}}}^{\frac{2(\ell+\theta)+1}{k}}\|v_N\|_{L^2}^{\frac{2(k-\ell-\theta)-1}{k}}\big) \\
&\quad + \|v_N\|_{H^{\frac{k}{2}}}^{\frac{4\ell+1}{k}}\|v_N\|_{L^2}^{\frac{3k-4\ell-1}{k}} \\
&\le C\big(1 + \|Y_N\|^q_{H^{\frac{k-1}{2}-}} + \|v_N\|_{L^2}^{2(k+1)}\big) + \varepsilon\|v_N\|^2_{H^{\frac{k}{2}}} \qquad (3.28)
\end{aligned}
$$

for $\frac{1}{4} < \theta < \frac{3}{4}$ and $1 \ll q < \infty$ sufficiently large. The cubic terms with $\widetilde{p}(u) = u \cdot \partial_x^{\ell-1}u \cdot \partial_x^{\ell}u$ and $|||p(u)||| = 1$ are controlled by the term above multiplied by $\frac{1}{\delta}$. It only remains to estimate the quartic terms with $\widetilde{p}(u) = u^2 dx^{\ell-1} u \partial_x^{\ell} u$. Proceeding as before, for $0 < \varepsilon < 1$, there exists $C > 0$ such that

$$
\begin{aligned}
&\left| \int (Y_N + v_N)^2 \partial_x^{\ell-1}(Y_N + v_N)\partial_x^{\ell}(Y_N + v_N)\, dx \right| \qquad (3.29) \\
&= \left| \int \langle\partial_x\rangle^{\theta}[(Y_N + v_N)^2\partial_x^{\ell-1}(Y_N + v_N)]\langle\partial_x\rangle^{-\theta}[\partial_x^{\ell}(Y_N + v_N)]\, dx \right| \\
&\lesssim \|(Y_N + v_N)^2\partial_x^{\ell-1}(Y_N + v_N)\|_{H^{\theta}}\|Y_N + v_N\|_{H^{\ell-\theta}} \\
&\lesssim \|Y_N + v_N\|^2_{H^{\theta}}\|Y_N + v_N\|^2_{H^{\ell-\theta}} \\
&\lesssim \|Y_N\|^4_{H^{\ell-}} + \|Y_N\|^3_{H^{\ell-}}\|v_N\|_{H^{\ell-\theta}} + \|Y_N\|^2_{H^{\ell-}}\|v_N\|_{H^{\frac{k}{2}}}^{\frac{4(\ell-\theta)}{k}}\|v_N\|_{L^2}^{\frac{2(k-2\ell+2\theta)}{k}} \qquad (3.30) \\
&\quad + \|Y_N\|_{H^{\ell-}}\|v_N\|_{H^{\frac{k}{2}}}^{\frac{4\ell-2\theta}{k}}\|v_N\|_{L^2}^{\frac{3k-4\ell+2\theta}{k}} + \|v_N\|_{H^{\frac{k}{2}}}^{\frac{4\ell}{k}}\|v_N\|_{L^2}^{\frac{4(k-\ell)}{k}} \\
&\le C\big(1 + \|Y_N\|^q_{H^{\frac{k-1}{2}-}} + \|v_N\|_{L^2}^{2(k+1)}\big) + \varepsilon\|v_N\|^2_{H^{\frac{k}{2}}}, \qquad (3.31)
\end{aligned}
$$

for $\frac{1}{4} < \theta < \frac{1}{2}$ and $1 \ll q < \infty$ sufficiently large.

Now, we consider even integers $k \ge 2$, namely, $k = 2\ell$ for $\ell \ge 1$. We can estimate the last contribution in (3.3) as in (3.24) for $k \ge 4$, while for $k = 2$, we have that these contributions are of the form $\int p_j(u)\,dx$ where $p_j(u) \in \mathcal{P}_j(u)$, $j = 3, 4$, with $\widetilde{p}_3(u) = u^3$, $|||p_3(u)||| = 1$, and $\widetilde{p}_4(u) = u^4$ with $|||p_4(u)||| = 0$, therefore controlled by

$$\frac{1}{\delta}\|Y_N + v_N\|_{L^3}^3 + \|Y_N + v_N\|_{L^4}^4 \le C_\varepsilon\Big(1 + \frac{1}{\delta}\Big)(1 + \|Y_N\|^q_{H^{\frac{1}{2}-}} + \|v_N\|_{L^2}^6) + \varepsilon\|v_N\|_{H^1}^2,$$

for $0 < \varepsilon \ll 1$, $C_\varepsilon > 0$, and some $1 \ll q < \infty$. Thus, we get that the last contribution in (3.3) for $k = 2\ell \ge 2$ is controlled by

$$\begin{aligned}
&\sum_{j=3}^{k+2}\sum_{i=0}^{k+2-j}\sum_{\substack{p(u)\in\mathcal{P}_j(u),\ j=3,\dots,k+2\\ \|p(u)\|+|||p(u)|||=k+2-j\\ |p(u)|\le \ell-1}} \bigg|\int p(Y_N + v_N)\,dx\bigg| \\
&\le C\Big(1 + \frac{1}{\delta^{k+1}}\Big)\Big(1 + \|Y_N\|^q_{H^{\frac{k-1}{2}-}} + \|v_N\|_{L^2}^{2\alpha(k)}\Big) + \varepsilon\|v_N\|^2_{H^{\frac{k}{2}}}, \qquad (3.32)
\end{aligned}$$

for $0 < \varepsilon < 1$, for some $C > 0$, $1 \ll q < \infty$, and $\alpha(k) \in \mathbb{N}$ sufficiently large. It remains to estimate the leading order cubic terms in (3.3), for which it suffices to estimate the term corresponding to $p(u) = u\partial_x^{\ell-1}u\partial_x^\ell u$:

$$\begin{aligned}
\mathbb{E}\bigg[\int \Big(&Y_N\partial_x^{\ell-1}Y_N\partial_x^\ell Y_N + (Y_N + v_N)\partial_x^{\ell-1}(Y_N + v_N)\partial_x^\ell v_N \\
&+ (Y_N + v_N)\partial_x^{\ell-1}v_N\partial_x^\ell Y_N + v_N\partial_x^{\ell-1}Y_N\partial_x^\ell Y_N\Big)\,dx\bigg].
\end{aligned}$$

The first term vanishes by Isserlis' theorem. For the second term, we use Cauchy-Schwarz inequality, Sobolev inequality, (2.1), and Young's inequality, to show that for $0 < \varepsilon < 1$ there exists $C > 0$ such that

$$\begin{aligned}
&\bigg|\int (Y_N + v_N)\partial_x^{\ell-1}(Y_N + v_N)\partial_x^\ell v_N\,dx\bigg| \\
&\lesssim \|Y_N + v_N\|_{H^{\frac{1}{4}}}\|Y_N + v_N\|_{H^{\ell-\frac{3}{4}}}\|v_N\|_{H^\ell} \\
&\lesssim \|Y_N\|^2_{H^{\ell-\frac{3}{4}}}\|v_N\|_{H^\ell} + \|Y_N\|_{H^{\ell-\frac{3}{4}}}\|v_N\|_{H^\ell}\|v_N\|_{H^{\ell-\frac{3}{4}}} + \|v_N\|_{H^\ell}\|v_N\|_{H^{\ell-\frac{3}{4}}}\|v_N\|_{H^{\frac{1}{4}}} \\
&\lesssim \|Y_N\|^2_{H^{\ell-\frac{3}{4}}}\|v_N\|_{H^\ell} + \|Y_N\|_{H^{\ell-\frac{3}{4}}}\|v_N\|^{\frac{4k-3}{2k}}_{H^{\frac{k}{2}}}\|v_N\|^{\frac{3}{2k}}_{L^2} + \|v_N\|^{\frac{2k-1}{k}}_{H^{\frac{k}{2}}}\|v_N\|^{\frac{k+1}{k}}_{L^2} \\
&\le C(1 + \|Y_N\|^q_{H^{\frac{k-1}{2}-}} + \|v_N\|_{L^2}^{2(k+1)}) + \varepsilon\|v_N\|^2_{H^{\frac{k}{2}}} \qquad (3.33)
\end{aligned}$$

since $\ell = \frac{k}{2}$ and for $1 \ll q < \infty$ large enough. The third contribution can be handled by integration by parts and earlier estimates since

$$\begin{aligned}
\int (Y_N + v_N)\partial_x^{\ell-1}v_N\partial_x^\ell Y_N\,dx = &-\int \partial_x(Y_N + v_N)\partial_x^{\ell-1}v_N\partial_x^{\ell-1}Y_N\,dx \\
&- \int (Y_N + v_N)\partial_x^\ell v_N\partial_x^{\ell-1}Y_N\,dx,
\end{aligned}$$

where the first term can be estimate as in (3.32) and the second term was estimated in (3.33). Thus, it only remains to estimate the last term. Applying Cauchy-Schwarz and Young's inequality, we get that for $0 < \varepsilon < 1$, there exists $C > 0$ such that

$$\bigg|\mathbb{E}\bigg[\int v_N\partial_x^{\ell-1}Y_N\partial_x^\ell Y_N\,dx\bigg]\bigg| = \bigg|\mathbb{E}\bigg[\int (\langle\partial_x\rangle^\ell v_N)\langle\partial_x\rangle^{-\ell}(\partial_x^{\ell-1}Y_N\partial_x^{\ell-1}Y_N)\,dx\bigg]\bigg|$$

$$
\begin{aligned}
&\lesssim \big|\mathbb{E}\big[\|v_N\|_{H^\ell}\|\mathbf{P}_{\neq 0}(\partial_x^{\ell-1}Y_N\partial_x^\ell Y_N)\|_{H^{-\ell}}\big]\big|\\
&\le \big|\mathbb{E}\big[\varepsilon\|v_N\|^2_{H^{\frac{k}{2}}}\big] + \big[C\|\mathbf{P}_{\neq 0}(\partial_x^{\ell-1}Y_N\partial_x^\ell Y_N)\|^2_{H^{-\ell}}\big]\big|.
\end{aligned}
$$

Then, from Isserlis' theorem and (3.7), we have

$$
\begin{aligned}
\mathbb{E}\Big[\|\mathbf{P}_{\neq 0}(\partial_x^{\ell-1}Y_N\partial_x^\ell Y_N)\|^2_{H^{-\ell}}\Big] &= \mathbb{E}\bigg[\sum_{n\neq 0}\frac{1}{\langle n\rangle^{2\ell}}\bigg|\sum_{\substack{n=n_1+n_2\\0<|n_j|\le N}}\frac{(in_1)^{\ell-1}(in_2)^\ell g_{n_1}g_{n_2}}{(T_{\delta,k}(n_1)T_{\delta,k}(n_2))^{\frac12}}\bigg|^2\bigg]\\
&\sim_\delta \sum_{n\neq 0}\frac{1}{\langle n\rangle^{2\ell}}\sum_{\substack{n=n_1+n_2\\0<|n_j|\le N}}\frac{|n_1|^{k-2}|n_2|^k+|n_1n_2|^{k-1}}{|n_1|^k|n_2|^k}\\
&\sim_\delta \sum_{n}\frac{1}{\langle n\rangle^{2\ell}}\sum_{0<|n_1|\le N}\bigg(\frac{1}{|n_1||n-n_1|}+\frac{1}{|n_1|^2}\bigg)<\infty
\end{aligned}
$$

uniformly in $N\in\mathbb{N}$. The implicit constant can also be made independent of $2\le\delta\le\infty$.

Combining all the estimates above, we get the intended result. □

We can now focus on $\mathcal{M}_{\delta,\frac{k}{2},N}$ defined in (3.20). Fix $1\le p<\infty$. Applying Lemma 3.6 and Lemma 3.7, for $0<\varepsilon<1$ there exists $C_\delta>0$ independent of $N\in\mathbb{N}$ (and also independent of $2\le\delta\le\infty$) such that for any $v\in H^{\frac{k}{2}}(\mathbb{T})$, we have

$$
\begin{aligned}
\mathcal{M}_{\delta,\frac{k}{2},N}(v)\le \mathbb{E}\big[&pC_\delta\big(1+\|Y_N\|^q_{H^{\frac{k-1}{2}-}}\big)+(pC_\delta\varepsilon-\tfrac12 C)\|v_N\|^2_{H^{\frac{k}{2}}}+pC_\delta\|v_N\|^{2\alpha(k)}_{L^2}\\
&-A_0\|Y_N+v_N\|^{2\alpha(k)}_{L^2}\big]
\end{aligned}
$$

for $1\ll q<\infty$ and $\alpha(k)\in\mathbb{N}$ sufficiently large. By picking ε so that $pC_\delta\varepsilon<\frac12 C$ and using Lemma 3.5, there exists a constant $C_{\delta,p}>0$ independent of $N\in\mathbb{N}$ (and independent of $2\le\delta\le\infty$) such that

$$
\mathcal{M}_{\delta,\frac{k}{2},N}(v)\le C_{p,\delta}+\mathbb{E}\big[pC_\delta\|v_N\|^{2\alpha(k)}_{L^2}-A_0\|Y_N+v_N\|^{2\alpha(k)}_{L^2}\big].
$$

Note that by using Cauchy-Schwarz and Young's inequality, we get that for $0<\varepsilon<1$, there exists $C>0$ such that

$$
\begin{aligned}
-\|Y_N+v_N\|^{2\alpha(k)}_{L^2} &= -\bigg(\|Y_N\|^2_{L^2}+\|v_N\|^2_{L^2}+2\int Y_Nv_N\,dx\bigg)^{\alpha(k)}\\
&= -\sum_{\substack{\ell_1+\ell_2+\ell_3=\alpha(k)\\ \ell_i\ge 0}}\binom{\alpha(k)}{\ell_1,\ell_2,\ell_3}\|Y_N\|^{2\ell_1}_{L^2}\|v_N\|^{2\ell_2}_{L^2}\bigg(2\int Y_Nv_N\,dx\bigg)^{\ell_3}\\
&\le -\|v_N\|^{2\alpha(k)}_{L^2}+\sum_{\substack{\ell_1+\ell_2+\ell_3=\alpha(k)\\ \ell_1,\ell_2\ge 0,\ell_3\ge 1}}\binom{\alpha(k)}{\ell_1,\ell_2,\ell_3}\|Y_N\|^{2\ell_1}_{L^2}\|v_N\|^{2\ell_2}_{L^2}\bigg|2\int Y_Nv_N\,dx\bigg|^{\ell_3}\\
&\le -\|v_N\|^{2\alpha(k)}_{L^2}+\sum_{\substack{\ell_1+\ell_2+\ell_3=\alpha(k)\\ \ell_1,\ell_2\ge 0,\ell_3\ge 1}}\binom{\alpha(k)}{\ell_1,\ell_2,\ell_3}2^{\ell_3}\|Y_N\|^{2\ell_1+\ell_3}_{L^2}\|v_N\|^{2\ell_2+\ell_3}_{L^2}\\
&\le -(1-\varepsilon)\|v_N\|^{2\alpha(k)}_{L^2}+C\|Y_N\|^q_{L^2},
\end{aligned}
$$

for $1 \ll q < \infty$ sufficiently large, since $2\ell_1 + \ell_3 < 2\alpha(k)$. Therefore, by choosing $A_0 > 0$ large enough such that $pC_\delta - A_0(1-\varepsilon) \le 0$, we get that

$$\sup_{v \in H^{\frac{k}{2}}} \sup_{N \in \mathbb{N}} \mathcal{M}_{\delta, \frac{k}{2}, N}(v) < C_{p,\delta}$$

for some constant $C_{p,\delta} > 0$, which can be chosen independently of $2 \le \delta \le \infty$. This completes the proof of Proposition 3.2.

3.4. **Construction of the measures $\rho_{\delta, \frac{k}{2}}$ for $0 < \delta \le \infty$.** Fix $k \in \mathbb{N}$ with $k \ge 2$, $0 < \delta \le \infty$, and $K > 0$. Define the limiting density $F_{\delta, \frac{k}{2}, K}(u)$ by

$$F_{\delta, \frac{k}{2}, K}(u) = \eta_K\big(\|u\|_{L^2}^2\big) \exp(-R_{\delta, \frac{k}{2}}(u)).$$

Proposition 3.8. *Let $k \in \mathbb{N}$, with $k \ge 2$, and $0 < \delta \le \infty$. Given $1 \le p < \infty$, the sequences $\{\mathbf{P}_N X_{\delta, \frac{k}{2}}\}_{N \in \mathbb{N}}$ and $\{R_{\delta, \frac{k}{2}}(\mathbf{P}_N X_{\delta, \frac{k}{2}})\}_{N \in \mathbb{N}}$ are Cauchy in $L^p(\Omega; H^{\frac{k-1}{2}-}(\mathbb{T}))$ and $L^p(\Omega)$, thus converging to limits denoted by $X_{\delta, \frac{k}{2}}$ and $R_{\delta, \frac{k}{2}}(X_{\delta, \frac{k}{2}})$, respectively. Moreover, given any $1 \le p < \infty$ and $\theta > 0$, we have that*

$$\sup_{N \in \mathbb{N}} \sup_{2 \le \delta \le \infty} \big\| \|\mathbf{P}_N X_{\delta, \frac{k}{2}}\|_{H_x^{\frac{k-1}{2}-\theta}} \big\|_{L^p(\Omega)} < Cp < \infty, \tag{3.34}$$

$$\sup_{2 \le \delta \le \infty} \big\| \|\mathbf{P}_N X_{\delta, \frac{k}{2}} - \mathbf{P}_M X_{\delta, \frac{k}{2}}\|_{H_x^{\frac{k-1}{2}-\theta}} \big\|_{L^p(\Omega)} \le \frac{Cp}{N^\theta} \to 0, \tag{3.35}$$

for any $M \ge N$ tending to ∞. In particular, the rate of convergence is uniform in $2 \le \delta \le \infty$. In addition, for $1 \le p < \infty$, there exists $\theta > 0$ such that

$$\sup_{N \in \mathbb{N} \cup \{\infty\}} \sup_{2 \le \delta \le \infty} \|R_{\delta, \frac{k}{2}}(\mathbf{P}_N X_{\delta, \frac{k}{2}})\|_{L^p(\Omega)} < \infty, \tag{3.36}$$

$$\|R_{\delta, \frac{k}{2}}(\mathbf{P}_M X_{\delta, \frac{k}{2}}) - R_{\delta, \frac{k}{2}}(\mathbf{P}_N X_{\delta, \frac{k}{2}})\|_{L^p(\Omega)} \le \frac{C_{\delta,k} p^{k+1}}{N^\theta}, \tag{3.37}$$

for any $M \ge N \ge 1$. For $2 \le \delta \le \infty$, we can choose the constant $C_{\delta,k}$ independently of δ, and hence the rate of convergence is uniform in $2 \le \delta \le \infty$.

Before proceeding to the proof of Proposition 3.8, we show Theorem 1.1(i).

Proof of Theorem 1.1(i). Theorem 1.1(i) follows from (i) the uniform in $N \in \mathbb{N}$ (and $2 \le \delta < \infty$) $L^p(\Omega)$-bounds on the density $F_{\delta, \frac{k}{2}, N, K}(X_{\delta, \frac{k}{2}})$, (ii) the $L^p(\Omega)$ convergence of the remainder $R_{\delta, \frac{k}{2}}(\mathbf{P}_N X_{\delta, \frac{k}{2}})$ and of $\|\mathbf{P}_N X_{\delta, \frac{k}{2}}\|_{L^2}$ in Proposition 3.8, (iii) and the continuous mapping theorem which allows us to conclude the convergence in probability of the density.

Lastly, we show the convergence of $\rho_{\delta, \frac{k}{2}, N, K}$ to $\rho_{\delta, \frac{k}{2}, K}$ in total variation as $N \to \infty$, uniformly in $2 \le \delta \le \infty$. From the $L^p(\Omega)$-convergence of the density, we have the following for the partition functions $Z_{\delta,N}, Z_\delta$ of $\rho_{\delta, \frac{k}{2}, N, K}, \rho_{\delta, \frac{k}{2}, K}$, respectively:

$$Z_{\delta,N} = \|F_{\delta, \frac{k}{2}, N, K}(X_{\delta, \frac{k}{2}})\|_{L^1(\Omega)} \to \|F_{\delta, \frac{k}{2}, K}(X_{\delta, \frac{k}{2}})\|_{L^1(\Omega)} = Z_\delta \qquad \text{as } N \to \infty,$$

where the convergence is uniform in $2 \le \delta \le \infty$. Let $\varepsilon > 0$ and $\mathcal{B}$ denote the family of Borel sets of $H^{\frac{k-1}{2}-\varepsilon}(\mathbb{T})$. Then,

$$\lim_{N \to \infty} \sup_{A \in \mathcal{B}} \big|\rho_{\delta, \frac{k}{2}, N, K}(A) - \rho_{\delta, \frac{k}{2}, K}(A)\big|$$

$$
\begin{aligned}
&= Z_\delta^{-1} \lim_{N\to\infty} \sup_{A\in\mathcal{B}} \bigg| \frac{Z_\delta}{Z_{\delta,N}} \int_A F_{\delta,\frac{k}{2},N,K}(u)d\mu_{\delta,\frac{k}{2}} - \int_A F_{\delta,\frac{k}{2},K}(u)d\mu_{\delta,\frac{k}{2}} \bigg| \\
&\le Z_\delta^{-1} \big(\sup_{A\in\mathcal{B}} \mu_{\delta,\frac{k}{2}}(A)\big)^{\frac12} \Big[\sup_{N\in\mathbb{N}} \|F_{\delta,\frac{k}{2},N,K}(u)\|_{L^2(d\mu_{\delta,\frac{k}{2}})} \lim_{N\to\infty} \Big| \frac{Z_\delta}{Z_{\delta,N}} - 1 \Big| \\
&\qquad\qquad + \lim_{N\to\infty} \|F_{\delta,\frac{k}{2},N,K}(u) - F_{\delta,\frac{k}{2},K}(u)\|_{L^p(d\mu_{\delta,\frac{k}{2}})} \Big] = 0
\end{aligned}
$$

from the uniform in N bounds on the density in Proposition 3.2, the convergence of partition functions $Z_{\delta,N}$, and the convergence of the densities in $L^p(\Omega)$. ☐

Proof of Proposition 3.8. Fix $k \in \mathbb{N}$ with $k \ge 2$, $2 \le p < \infty$, and integers $M \ge N \ge 1$. For simplicity, let $X_N = \mathbf{P}_N X_{\delta,\frac{k}{2}}$. The estimate in (3.34) follows from that in Lemma 3.5, since X_N and $Y_{\delta,\frac{k}{2},N}(1)$ have the same law. For the difference, for $\theta > 0$, by Minkowski's inequality, the Wiener chaos estimate (Lemma 2.6), and (3.7), we have that

$$
\begin{aligned}
&\|X_N - X_M\|_{L^p_\omega H_x^{\frac{k-1}{2}-\theta}} \\
&\le p \|\langle\partial_x\rangle^{\frac{k-1}{2}-\theta}(X_N - X_M)\|_{L^2_x L^2_\omega} \\
&\sim p \bigg\| \bigg(\mathbb{E} \bigg| \sum_{N<|n|\le M} \langle n\rangle^{\frac{k-1}{2}-\theta} \frac{g_n}{(T_{\delta,\frac{k}{2}}(n))^{\frac12}} e^{inx} \bigg|^2 \bigg)^{\frac12} \bigg\|_{L^2_x} \\
&= p \bigg\| \bigg(\mathbb{E} \bigg[\sum_{N<|n|,|m|\le M} \langle n\rangle^{\frac{k-1}{2}-\theta} \langle m\rangle^{\frac{k-1}{2}-\theta} \frac{g_n \overline{g_m}}{(T_{\delta,\frac{k}{2}}(n) T_{\delta,\frac{k}{2}}(m))^{\frac12}} e^{i(n-m)x} \bigg] \bigg)^{\frac12} \bigg\|_{L^2_x} \\
&= p \bigg(\sum_{N<|n|\le M} \frac{\langle n\rangle^{k-1-2\theta}}{T_{\delta,\frac{k}{2}}(n)} \bigg)^{\frac12} \\
&\sim_\delta p \bigg(\sum_{N<|n|\le M} \frac{1}{|n|^{1+2\theta}} \bigg)^{\frac12} \lesssim_\delta \frac{p}{N^\theta}.
\end{aligned}
$$

Note that the implicit constants can be chosen independently of $2 \le \delta \le \infty$.

We now prove (3.37), which suffices to conclude that $\{R_{\delta,\frac{k}{2}}(X_N)\}_{N\in\mathbb{N}}$ is Cauchy in $L^p(\Omega)$. Let $M \ge N \ge 1$. We will show that for $2 \le p < \infty$, $k \in \mathbb{N}$, and $1 \le \ell \le k$,

$$
\|A_{\frac{k}{2},\frac{\ell}{2}}(X_M) - A_{\frac{k}{2},\frac{\ell}{2}}(X_N)\|_{L^p(\Omega)} \le \frac{C_{\delta,k,\ell}\, p^{\ell+2}}{N^{\frac13-}}, \tag{3.38}
$$

where $C_{\delta,k,\ell}$ can be chosen independently of $2 \le \delta \le \infty$.

We first prove (3.38) for $\ell = 1$. Note that by Hölder's inequality, Sobolev inequality, (3.34), and (3.35),

$$
\begin{aligned}
\|A_{\frac{k}{2},\frac12}(X_M) - A_{\frac{k}{2},\frac12}(X_N)\|_{L^p_\omega} &\sim_{\ell,k} \bigg\| \int (X_M - X_N)(X_M^2 + X_M X_N + X_N^2)\, dx \bigg\|_{L^p_\omega} \\
&\lesssim \|X_M - X_N\|_{L^{3p}_\omega H_x^{\frac16}} \|X_M\|^2_{L^{3p}_\omega H_x^{\frac16}} \\
&\lesssim_{\ell,k,\delta} \frac{p^3}{N^{\frac13-}},
\end{aligned}
$$

since $\frac{k-1}{2} - \frac{1}{6} \leq \frac{1}{3}$. Now consider odd $\ell = 2\ell' + 1$ and $1 \leq \ell' \leq \frac{k-1}{2}$. By (3.4),

$$
\begin{aligned}
A_{\frac{k}{2}, \frac{2\ell'+1}{2}}(X_M) - A_{\frac{k}{2}, \frac{2\ell'+1}{2}}(X_N) &= \sum_{\substack{p(u)\in\mathcal{P}_3(u) \\ \widetilde{p}(u)=\partial_x^{\alpha_1}u\partial_x^{(\ell'-1)+\alpha_2}u\partial_x^{\ell'}u \\ 0\leq\alpha_{12}\leq 1 \\ |||p(u)|||=1-(\alpha_{12})}} \big[p(X_M) - p(X_N)\big]\,dx \\
&\quad + \sum_{\substack{p(u)\in\mathcal{P}_4(u) \\ \widetilde{p}(u)=u^2\partial_x^{\ell'-1}u\partial_x^{\ell'}u \\ |||p(u)|||=0}} \int \big[p(X_M) - p(X_N)\big]\,dx \\
&\quad + \sum_{j=3}^{2\ell'+3}\sum_{i=0}^{2\ell'+3-j} \sum_{\substack{p(u)\in\mathcal{P}_j(u) \\ \|p(u)\|=i \\ |||p(u)|||=2\ell'+3-j-i \\ |p(u)|\leq\ell-1}} \int \big[p(X_M) - p(X_N)\big]\,dx \\
&=: \mathrm{I} + \mathrm{II} + \mathrm{III}.
\end{aligned}
$$

For I, it suffices to estimate the term with $p(u) = \partial_x^{\alpha_1}u\partial_x^{\ell'-1+\alpha_2}u\partial_x^{\ell'}u$, where $\alpha_{12} = 1$. We have that

$$
\begin{aligned}
\|\mathrm{I}\|_{L^2(\Omega)}^2 &\lesssim_\delta \mathbb{E}\Bigg|\sum_{0=n_{123}} (\mathbf{1}_{\substack{0<|n_i|\leq M \\ i=1,2,3}} - \mathbf{1}_{\substack{0<|n_i|\leq N \\ i=1,2,3}}) \frac{g_{n_1}g_{n_2}g_{n_3}(in_1)^{\alpha_1}(in_2)^{\ell'-1+\alpha_2}(in_3)^{\ell'}}{(T_{\delta,\frac{\ell}{2}}(n_1)T_{\delta,\frac{\ell}{2}}(n_2)T_{\delta,\frac{\ell}{2}}(n_3))^{\frac{1}{2}}}\Bigg|^2 \\
&\lesssim_\delta \sum_{0=n_1+n_2+n_3} \mathbf{1}_{B_{N,M}(n_1,n_2,n_3)} \frac{1}{|n_1n_2n_3|^{2\ell'+1}} \big(|n_1|^{2\alpha_1}|n_2|^{2(\ell'-1+\alpha_2)}|n_3|^{2\ell'} \\
&\qquad + |n_1|^{2\alpha_1}|n_2n_3|^{2\ell'-1+\alpha_2} + |n_1n_2|^{\ell'}|n_3|^{2\ell'} + |n_1|^{\ell'}|n_2|^{2\ell'-1+\alpha_2}|n_3|^{\ell'+\alpha_1} \\
&\qquad + |n_1n_3|^{\ell'+\alpha_1}|n_2|^{2(\ell'-1+\alpha_2)}\big)
\end{aligned}
$$

by Isserlis' theorem and (3.7), where

$$
\begin{aligned}
B_{N,M}(n_1,n_2,n_3) = \Big\{&(n_1,n_2,n_3) \in (\mathbb{Z}^*)^3 : 0 < |n_{i_1}| \leq M, \frac{N}{2} < |n_{i_2}|, |n_{i_3}| \leq M, \\
&\text{for some } \{i_1,i_2,i_3\} = \{1,2,3\}\Big\}.
\end{aligned}
$$

Then,

$$
\|\mathrm{I}\|_{L^2(\Omega)} \lesssim_\delta \Bigg(\sum_{\substack{0<|n_{\min}|\leq M \\ \frac{N}{2}<|n_{\mathrm{med}}|\leq M}} \frac{1}{|n_{\min}|^{2\ell'+1}|n_{\mathrm{med}}|^2}\Bigg)^{\frac{1}{2}} \lesssim_\delta \frac{1}{N^{\frac{1}{2}}},
$$

where the implicit constants are independent of $N, M \in \mathbb{N}$ and $2 \leq \delta \leq \infty$. For II, it suffices to control the contribution with $p(u) = u^2\partial_x^{\ell'-1}u\partial_x^{\ell'}u$ and $|||p(u)||| = 0$. Note that writing $p(u_1,\dots,u_4) = u_1u_2\partial_x^{\ell'-1}u_3\partial_x^{\ell'}u_4$, and proceeding as in (3.31), we get that for $0 < \theta \leq \frac{1}{3}$,

$$
\bigg|\int p(u_1,u_2,u_3,u_4)\,dx\bigg| \lesssim \|u_4\|_{H^{\ell-\theta}} \sum_{\substack{\theta_1+\theta_2+\theta_3=\theta \\ \theta_i\in\{0,\theta\}}} \|u_1\|_{H^{\frac{1}{3}+\theta_1}}\|u_2\|_{H^{\frac{1}{3}+\theta_2}}\|u_3\|_{H^{\ell-\frac{2}{3}+\theta_3}}
$$

from which, by Hölder's inequality, (3.34) and (3.35), we have

$$\|\mathrm{II}\|_{L^p(\Omega)} \lesssim_\delta \|X_N - X_M\|_{L^{4p}_\omega H_x^{\frac{\ell-1}{2}-\theta}} \|X_M\|^3_{L^{4p}_\omega H_x^{\frac{\ell-1}{2}-}} \lesssim_\delta p^4 \frac{1}{N^\theta},$$

where the implicit constants are independent of $2 \le \delta \le \infty$. For III, fix $3 \le j \le k+2$, $0 \le i \le \ell + 2 - j$, $\alpha_{1\cdots j} = i$ with $\ell' - 1 \ge \alpha_1 \ge \ldots \ge \alpha_j \ge 0$, and $\widetilde{p}(u) = \prod_{\kappa=1}^j \partial_x^{\alpha_\kappa} u$, then

$$\bigg| \int [p(X_M) - p(X_N)]\, dx \bigg| \lesssim \frac{1}{\delta^{\ell+2-j-i}} \|X_M - X_N\|_{H^{\alpha_1}} \|X_M\|_{H^{\alpha_2}} \prod_{\kappa=3}^{j} \|X_M\|_{H^{\alpha_\kappa + \frac12 +}},$$

from which we get

$$\|\mathrm{III}\|_{L^p(\Omega)} \le C_\delta \sum_{j=3}^{\ell+2} \|X_M - X_N\|_{L^{jp}_\omega H_x^{\ell'-1}} \|X_M\|^{j-1}_{L^{jp}_\omega H_x^{\frac{\ell-1}{2}-}} \le \frac{C_{\delta,\ell} p^{\ell+2}}{N^{1-}}$$

from (3.34) and (3.35), where the constant $C_{\delta,\ell}$ can be chosen independently of $2 \le \delta \le \infty$.

Now, consider even $\ell = 2\ell'$ with $\ell' \ge 1$. Then, from (3.3), we can write

$$\begin{aligned} A_{\frac{k}{2},\frac{\ell}{2}}(X_M) - A_{\frac{k}{2},\frac{\ell}{2}}(X_N) = &\sum_{\substack{p(u) \in \mathcal{P}_3(u) \\ \widetilde{p}(u) = u\partial_x^{\ell'-1} u \partial_x^{\ell'} u \\ |||p(u)||| = 0}} \int [p(X_M) - p(X_N)]\, dx \\ &+ \sum_{j=3}^{k+2} \sum_{i=0}^{k+2-j} \sum_{\substack{p(u) \in \mathcal{P}_j(u) \\ \|p(u)\| = i \\ |||p(u)||| = k+2-j-i \\ |p(u)| \le \ell' - 1}} \int \big[p(X_M) - p(X_N)\big]\, dx =: \mathrm{I} + \mathrm{II}. \end{aligned}$$

To estimate I, it suffices to control the contribution from $p(u) = u\partial_x^{\ell'-1} u \partial_x^{\ell'} u$ and $|||p(u)||| = 0$. Proceeding as in the estimate of I in the odd case, we get that

$$\begin{aligned} \|\mathrm{I}\|^2_{L^2(\Omega)} &\lesssim \sum_{0 = n_1 + n_2 + n_3} \mathbf{1}_{B_{N,M}(n_1,n_2,n_3)} \frac{1}{|n_1 n_2 n_3|^{2\ell'}} \big(|n_2|^{2\ell'-2} |n_3|^{2\ell'} + |n_2 n_3|^{2\ell'-1} \\ &\qquad + |n_1 n_2|^{\ell'-1} |n_3|^{\ell'} (|n_2|^{\ell'} + |n_3|^{\ell'}) + |n_1 n_3|^{\ell'} |n_2|^{\ell'-1} (|n_2|^{\ell'-1} + |n_3|^{\ell'-1}) \big) \\ &\lesssim \sum_{\substack{0 < |n_{\min}| \le M \\ \frac{N}{2} < |n_{\mathrm{med}}| \le M}} \frac{1}{|n_{\min}|^{2\ell'} |n_{\mathrm{med}}|^2} \lesssim \frac{1}{N}, \end{aligned}$$

where the implicit constants are independent of N, M, δ. For II, fix $3 \le j \le \ell + 2$, $0 \le i \le \ell + 2 - j$, $\alpha_{1\cdots j} = i$ with $\ell - 1 \ge \alpha_1 \ge \ldots \ge \alpha_j \ge 0$, and $\widetilde{p}(u) = \prod_{\kappa=1}^j \partial_x^{\alpha_\kappa} u$, then

$$\bigg| \int [p(X_M) - p(X_N)]\, dx \bigg| \lesssim \frac{1}{\delta^{\ell+2-j-i}} \|X_M - X_N\|_{H^{\alpha_1}} \|X_M\|_{H^{\alpha_2}} \prod_{\kappa=3}^{j} \|X_M\|_{H^{\alpha_\kappa + \frac12 +}},$$

from which we get

$$\|\mathrm{III}\|_{L^p(\Omega)} \le C_\delta \sum_{j=3}^{\ell+2} \|X_M - X_N\|_{L^{jp}_\omega H_x^{\ell'-1}} \|X_M\|^{(j-1)}_{L^{jp}_\omega H_x^{\frac{\ell-1}{2}-}} \le \frac{C_{\delta,k} p^{\ell+2}}{N^{1-}}$$

from (3.34) and (3.35), where the constant $C_{\delta,k}$ can be chosen independently of $2 \le \delta \le \infty$.

From (3.38), we have that

$$\|R_{\delta,\frac{k}{2}}(X_M) - R_{\delta,\frac{k}{2}}(X_N)\|_{L^p(\Omega)} \le \sum_{\ell=1}^{k} \frac{1}{\delta^{k-\ell}} \|A_{\frac{k}{2},\frac{\ell}{2}}(X_M) - A_{\frac{k}{2},\frac{\ell}{2}}(X_N)\|_{L^p(\Omega)} \le \frac{C_{\delta,k} p^{k+2}}{N^{\frac{1}{3}-}}$$

from which (3.37) follows. Therefore, the sequence $\{R_{\delta,\frac{k}{2}}(X_N)\}_{N\in\mathbb{N}}$ is Cauchy in $L^p(\Omega)$ and it has a limit in $L^p(\Omega)$. The estimate in (3.36) follows by the same arguments above. □

3.5. **Convergence of $\rho_{\delta,\frac{k}{2}}$ as $\delta \to \infty$.** In this subsection, we present the proof of Theorem 1.1(ii). First, we establish the $L^p(\Omega)$-convergence of the truncated densities.

Lemma 3.9. *Given $N \in \mathbb{N}$, we have that*

$$\lim_{\delta\to\infty} \|F_{\delta,\frac{k}{2},N,K}(X_{\delta,\frac{k}{2}}) - F_{\infty,\frac{k}{2},N,K}(X_{\infty,\frac{k}{2}})\|_{L^p(\Omega)} = 0, \tag{3.39}$$

$$\lim_{\delta\to\infty} \|F_{\delta,\frac{k}{2},N,K}(X_{\infty,\frac{k}{2}}) - F_{\infty,\frac{k}{2},N,K}(X_{\infty,\frac{k}{2}})\|_{L^p(\Omega)} = 0. \tag{3.40}$$

Moreover,

$$\lim_{\delta\to\infty} \|F_{\delta,\frac{k}{2},K}(X_{\delta,\frac{k}{2}}) - F_{\infty,\frac{k}{2},K}(X_{\infty,\frac{k}{2}})\|_{L^p(\Omega)} = 0, \tag{3.41}$$

and the partition function Z_δ of $\rho_{\delta,\frac{k}{2},K}$ converges to the partition function Z_∞ for $\rho_{\infty,\frac{k}{2}}$ of $\rho_{\infty,\frac{k}{2}}$, as $\delta \to \infty$.

Proof. For simplicity, we omit the K, k dependence in the proof and let $X_{\delta,N} = \mathbf{P}_N X_{\delta,\frac{k}{2}}$ for $2 \le \delta \le \infty$. Fix $N \in \mathbb{N}$ and $2 \le \delta \le \infty$. For fixed $x \in \mathbb{T}$ and $\omega \in \Omega$, we have that

$$X_{\delta,N}(x;\omega) = \frac{1}{\sqrt{2\pi}} \sum_{0<|n|\le N} \frac{g_n(\omega)}{T_{\delta,\frac{k}{2}}(n)^{\frac{1}{2}}} e^{inx} \to \frac{1}{\sqrt{2\pi}} \sum_{0<|n|\le N} \frac{g_n(\omega)}{|n|^{\frac{k}{2}}} e^{inx} = X_{\infty,N}(x;\omega)$$

as $\delta \to \infty$ from (3.7).

Now, we show that $R_{\delta,N}(X_\delta(\omega)) \to R_{\infty,N}(X_\infty(\omega))$ as $\delta \to \infty$. First note that from Lemma 3.6 with $u_2 \equiv 0$ and $u_1 = X_{\delta,N}$, we get that

$$\bigg|\sum_{\ell=1}^{k-1} \frac{1}{\delta^{k-\ell}} A_{\frac{k}{2},\frac{\ell}{2}}(X_{\delta,N}(\omega))\bigg| \lesssim \frac{1}{\delta}(1 + \|X_{\delta,N}(\omega)\|^q_{H^{\frac{k-1}{2}-}})$$

for some $1 \ll q < \infty$ and $2 \le \delta \le \infty$. From (3.7)

$$\|X_{\delta,N}(\omega)\|^2_{H^{\frac{k-1}{2}-}} \sim \sum_{0<|n|\le N} \frac{|g_n(\omega)|^2}{|n|^{1+}} \le C_{N,\omega} < \infty, \tag{3.42}$$

$$|X_{\delta,N}(x;\omega)| \sim \sum_{0<|n|\le N} \frac{|g_n(\omega)|}{|n|^{\frac{k}{2}}} \le \widetilde{C}_{N,\omega} < \infty, \tag{3.43}$$

where the constants are independent of δ. Thus, for fixed $2 \le \delta \le \infty$ and $\omega \in \Omega$,

$$\bigg|\sum_{\ell=1}^{k-1} \frac{1}{\delta^{k-\ell}} A_{\frac{k}{2},\frac{\ell}{2}}(X_{\delta,N}(\omega))\bigg| \lesssim \frac{1}{\delta} C_{N,\omega} \to 0 \qquad \text{as } \delta \to \infty. \tag{3.44}$$

To complete the convergence of the remainder, we must show that $A_{\frac{k}{2},\frac{k}{2}}(X_{\delta,N}(\omega)) \to R_\infty(X_{\infty,N})$ as $\delta \to \infty$. Let $j = 3, \ldots, k+2$ and $p_j(u) \in \mathcal{P}_j(u)$ with $\widetilde{p}_j(u) = \prod_{i=1}^{j} \partial_x^{\alpha_i} u$ and $\alpha_{1\cdots j} \leq k+2-j$. If $|||p_j(u)||| \geq 1$, then from (3.42)

$$\begin{aligned}\left|\int_{\mathbb{T}} p_j(X_{\delta,N}(x;\omega))\,dx\right| &\lesssim \frac{1}{\delta}\|X_{\delta,N}(\omega)\|_{H^{\alpha_1+\frac{1}{2}+}}\|X_{\delta,N}(\omega)\|_{H^{\alpha_2+\frac{1}{2}+}}\prod_{i=3}^{j}\|X_{\delta,N}(\omega)\|_{H^{\alpha_i+\frac{1}{2}+}}\\ &\leq \frac{1}{\delta}C_{N,\omega} \to 0\end{aligned}$$

as $\delta \to \infty$. The remaining contributions in $A_{\frac{k}{2}}(X_{\delta,N}(\omega))$ correspond to polynomials $p_j(u)$ as above with $|||p_j(u)||| = 0$, which are the terms that appear in $R_{\infty,\frac{k}{2}}$ and are therefore independent of δ (see Lemma A.7 for details). Since $X_{\delta,N}(x;\omega) \to X_{\infty,N}(x;\omega)$ for all $x \in \mathbb{T}$ and $\omega \in \Omega$, and $X_{\delta,N}(x,\omega)$ is uniformly bounded in x, δ per (3.43), by dominated convergence theorem we obtain that for each polynomial $p_j(u) \in \mathcal{P}_j(u)$ with $|||p_j(u)||| = 0$,

$$\int p_j(X_{\delta,N}(x;\omega))\,dx \to \int p_j(X_{\infty,N}(x;\omega))\,dx \qquad \text{as } \delta \to \infty.$$

Combining these results, we conclude that

$$R_{\delta,N}(X_\delta(\omega)) \to R_{\infty,N}(X_\infty(\omega)) \quad \text{as } \delta \to \infty$$

for each fixed $\omega \in \Omega$. Since, $\eta_K(\|X_{\delta,N}(\omega)\|_{L^2}^2) \to \eta_K(\|X_{\infty,N}(\omega)\|_{L^2}^2)$ as $\delta \to \infty$ for all $\omega \in \Omega$, we conclude that $F_{\delta,N}(X_\delta(\omega)) \to F_{\infty,N}(X_\infty(\omega))$ as $\delta \to \infty$ and for fixed $\omega \in \Omega$. The convergence in $L^p(\Omega)$ in (3.39) follows from the pointwise in ω convergence, the uniform in N bounds in Proposition 3.2, and dominated convergence theorem. The same argument above shows that $R_{\delta,N}(X_\infty(\omega)) \to R_{\infty,N}(X_\infty(\omega))$ as $\delta \to \infty$, for all $\omega \in \Omega$, and both $F_{\delta,N}(X_\infty)$ and $F_{\infty,N}(X_\infty)$ share the same cutoff function. Thus, the convergence in (3.40) follows again by the ω-pointwise convergence and the dominated convergence theorem.

To show (3.41), note that

$$\begin{aligned}&\|F_\delta(X_\delta) - F_\infty(X_\infty)\|_{L^p(\Omega)}\\ &\leq \|F_\delta(X_\delta) - F_{\delta,N}(X_\delta)\|_{L^p(\Omega)} + \|F_{\delta,N}(X_\delta) - F_{\infty,N}(X_\infty)\|_{L^p(\Omega)}\\ &\qquad + \|F_{\infty,N}(X_\infty) - F_\infty(X_\infty)\|_{L^p(\Omega)}\\ &\leq 2\sup_{2\leq\delta\leq\infty}\|F_\delta(X_\delta) - F_{\delta,N}(X_\delta)\|_{L^p(\Omega)} + \|F_{\delta,N}(X_\delta) - F_{\infty,N}(X_\infty)\|_{L^p(\Omega)}.\end{aligned}$$

Taking a limit as $\delta \to \infty$ followed by a limit as $N \to \infty$, from (3.39) and (1.18), we get the intended convergence in (3.41). □

Lastly, we establish a uniform in $2 \leq \delta \leq \infty$ and $\omega \in \Omega$ estimate on $F_{\delta,\frac{k}{2},K,N}(X_{\infty,\frac{k}{2}}(\omega))$.

Lemma 3.10. *For fixed $N \in \mathbb{N}$, there exists $C_{k,K,N} > 0$ independent of $2 \leq \delta \leq \infty$ and $\omega \in \Omega$, such that*

$$\sup_{\substack{2\leq\delta\leq\infty\\ \omega\in\Omega}}\left|F_{\frac{k}{2},\delta,N,K}(X_{\infty,\frac{k}{2}}(\omega))\right| \leq C_{k,K,N} < \infty. \tag{3.45}$$

Proof. Fix $N \in \mathbb{N}$, and let $\omega \in \Omega$ and $2 \leq \delta \leq \infty$. For simplicity, let $F_{\delta,N} = F_{\delta,\frac{k}{2},N,K}$, $R_\delta = R_{\delta,\frac{k}{2}}$, and $X_{\infty,N} = \mathbf{P}_N X_{\infty,\frac{k}{2}}$. Recall that

$$F_{\delta,N}(X_\infty(\omega)) = \eta_K(\|X_{\infty,N}(\omega)\|_{L^2}^2)\exp(-R_\delta(X_{\infty,N}(\omega))).$$

Due to the cutoff, this quantity is 0 for all ω such that $\|X_{\infty,N}(\omega)\|_{L^2}^2 \geq 2K$. Therefore, it suffices to consider $\omega \in \Omega$ such that $\|X_{\infty,N}(\omega)\|_{L^2}^2 \leq 2K$. This means that

$$0 \leq \frac{1}{2\pi} \sum_{0<|n|\leq N} \frac{|g_n(\omega)|^2}{|n|^k} \leq 2K. \tag{3.46}$$

For $R_\delta(X_{\infty,N}(\omega))$, from the structure in (3.2), applying Young's convolution inequality, (2.2), and (3.46), we get

$$\begin{aligned}
|R_\delta(X_{\infty,N}(\omega))| &\leq \sum_{\ell=0}^{k-1} \frac{1}{\delta^\ell} \big|A_{\frac{k}{2},\frac{k-\ell}{2}}(X_{\infty,N}(\omega))\big| \\
&\lesssim \sum_{\ell=0}^{k-1} \frac{1}{\delta^\ell} \sum_{j=3}^{k-\ell+2} \sum_{\kappa=0}^{k-\ell+2-j} \sum_{\substack{p(u)\in\mathcal{P}_j(u),\\ \|p(u)\|=k-\ell+2-j-\kappa\\ |||p(u)|||=\kappa}} \int |p(X_{\infty,N}(\omega))|\,dx \\
&\lesssim \sum_{\ell=0}^{k-1} \frac{1}{\delta^\ell} \sum_{j=3}^{k-\ell+2} \sum_{\kappa=0}^{k-\ell+2-j} \sum_{\substack{p(u)\in\mathcal{P}_j(u),\\ \|p(u)\|=k-\ell+2-j-\kappa\\ |||p(u)|||=\kappa}} N^{(k-\ell+2-j-\kappa)+j/2+}\frac{1}{\delta^\kappa}\|X_{\infty,N}(\omega)\|_{L^2}^j \\
&\leq C_k N^{k+1}\big(1+\tfrac{1}{\delta^{k-1}}\big)\big(1+\|X_{\infty,N}(\omega)\|_{L^2}^{k+2}\big) \\
&\leq C_k N^{k+1}\big(1+\tfrac{1}{\delta^{k-1}}\big)\big(1+(2K)^{\frac{k+2}{2}}\big) = C_{k,K,N} < \infty,
\end{aligned}$$

where $C_{k,K,N} > 0$ is independent of $2 \leq \delta \leq \infty$ and $\omega \in \Omega$. Therefore,

$$|F_{\delta,N}(X_\infty(\omega))| \leq \exp(C_{k,K,N}) < \infty,$$

as intended. □

Proof of Theorem 1.1(ii). Fix $k \in \mathbb{N}$ with $k \geq 2$ and $K > 0$. For simplicity, we omit the dependence on k, K in the proof, using the notation $\rho_\delta = \rho_{\delta,\frac{k}{2}}$, $\mu_\delta = \mu_{\delta,\frac{k}{2}}$, $F_\delta = F_{\delta,\frac{k}{2}}$, for $0 < \delta \leq \infty$.

For $0 < \delta \leq \infty$, it follows from the construction of the measure ρ_δ that it is equivalent to the base Gaussian measure $\mu_{\delta,}$. Moreover, Proposition 3.1(ii) says that the measures μ_δ are equivalent for all $0 < \delta \leq \infty$. Consequently, ρ_δ is equivalent to ρ_∞.

Now, it remains to show the convergence $\rho_\delta \to \rho_\infty$ in total variation as $\delta \to \infty$. From (3.41), we have convergence of the partition functions $Z_\delta = \|F_\delta(X_{\delta,\frac{k}{2}})\|_{L^1(\Omega)} \to \|F_\infty(X_{\infty,\frac{k}{2}})\|_{L^1(\Omega)} = Z_\infty$ as $\delta \to \infty$. For any $N \in \mathbb{N}$, by triangle inequality, we have that

$$\begin{aligned}
d_{\mathrm{TV}}(\rho_\delta, \rho_\infty) &\leq d_{\mathrm{TV}}(\rho_\delta, \rho_{\delta,N}) + d_{\mathrm{TV}}(\rho_{\delta,N}, \rho_{\infty,N}) + d_{\mathrm{TV}}(\rho_{\infty,N}, \rho_\infty) \\
&\leq 2 \sup_{2\leq\delta\leq\infty} d_{\mathrm{TV}}(\rho_\delta, \rho_{\delta,N}) + d_{\mathrm{TV}}(\rho_{\delta,N}, \rho_{\infty,N}).
\end{aligned}$$

From Theorem 1.1(i), we know that $\rho_{\delta,N} \to \rho_\delta$ as $N \to \infty$ in total variation uniformly in δ, thus the first contribution above converges to zero. Therefore, taking a limit as $\delta \to \infty$ followed by a limit as $N \to \infty$, we get that

$$\lim_{\delta\to\infty} d_{\mathrm{TV}}(\rho_\delta, \rho_\infty) \leq \lim_{N\to\infty} \lim_{\delta\to\infty} d_{\mathrm{TV}}(\rho_{\delta,N}, \rho_{\infty,N}). \tag{3.47}$$

For the distance on the right hand-side, we have

$$\begin{aligned}\mathrm{d}_{\mathrm{TV}}(\rho_{\delta,N},\rho_{\infty,N}) &= \sup_{A\in\mathcal{B}}|\rho_{\delta,N}(A)-\rho_{\infty,N}(A)|\\ &= \sup_{A\in\mathcal{B}}\bigg|Z_{\delta,N}^{-1}\int \mathbf{1}_A(u)F_{\delta,N}(u)\,d\mu_\delta(u) - Z_{\infty,N}^{-1}\int \mathbf{1}_A(u)F_{\infty,N}(u)\,d\mu_\infty(u)\bigg|\\ &\le Z_{\infty,N}^{-1}\big|Z_{\infty,N}-Z_{\delta,N}\big|\sup_{A\in\mathcal{B}}\rho_\delta(A)\\ &\quad + Z_{\infty,N}^{-1}\sup_{A\in\mathcal{B}}\bigg|\int \mathbf{1}_A(u)F_{\delta,N}(u)\,d\mu_\delta(u) - \int \mathbf{1}_A(u)F_{\infty,N}(u)\,d\mu_\infty(u)\bigg|.\end{aligned} \tag{3.48}$$

For the first contribution in (3.48), note that from (3.39) and the fact that ρ_δ is a probability measure, we get that

$$Z_{\infty,N}^{-1}\lim_{\delta\to\infty}\big|Z_{\infty,N}-Z_{\delta,N}\big|\sup_{A\in\mathcal{B}}\rho_\delta(A)=0.$$

Focusing on the latter contribution in (3.48), we obtain

$$\begin{aligned}&\bigg|\int \mathbf{1}_A(u)F_{\delta,N}(u)\,d\mu_\delta(u) - \int \mathbf{1}_A(u)F_{\infty,N}(u)\,d\mu_\infty(u)\bigg|\\ &\le \bigg|\int \mathbf{1}_A(u)\big[F_{\delta,N}(u)-F_{\infty,N}(u)\big]\,d\mu_\infty(u)\bigg| + \bigg|\int \mathbf{1}_A(u)\Big[\frac{d\mu_\delta}{d\mu_\infty}(u)-1\Big]F_{\delta,N}(u)\,d\mu_\infty(u)\bigg|\\ &=: \mathrm{I}+\mathrm{II}.\end{aligned}$$

Using (3.40), we have that for all $N\in\mathbb{N}$

$$\lim_{\delta\to\infty} Z_{\infty,N}^{-1}\sup_{A\in\mathcal{B}}\mathrm{I}\le \sup_{A\in\mathcal{B}}\mu_\infty(A)^{\frac12}\lim_{\delta\to\infty}\|F_{\delta,N}(X_\infty)-F_{\infty,N}(X_\infty)\|_{L^2(\Omega)}=0.$$

From Scheffé's lemma, we have that

$$\mathrm{d}_{\mathrm{TV}}(\mu_\delta,\mu_\infty)=\frac12\int_{H^{\frac{k-1}{2}-\varepsilon}}\Big|\frac{d\mu_\delta}{d\mu_\infty}(u)-1\Big|\,d\mu_\infty(u).$$

Since from Proposition 3.1, we know that $\mathrm{d}_{\mathrm{TV}}(\mu_\delta,\mu_\infty)\to 0$ as $\delta\to\infty$, we get

$$\begin{aligned}\lim_{\delta\to\infty}\sup_{A\in\mathcal{B}}\mathrm{II} &\le \Big(\sup_{2\le\delta\le\infty}\sup_{\omega\in\Omega}|F_{\delta,N}(X_\infty(\omega))|\Big)\lim_{\delta\to\infty}\int\Big|\frac{d\mu_\delta}{d\mu_\infty}(u)-1\Big|\,d\mu_\infty(u)\\ &\le 2C_{k,K,N}\lim_{\delta\to\infty}\mathrm{d}_{\mathrm{TV}}(\mu_\delta,\mu_\infty)=0\end{aligned}$$

using the uniform in δ,ω estimate in (3.45). Therefore, for each $N\in\mathbb{N}$, $\lim_{\delta\to\infty}\mathrm{d}_{\mathrm{TV}}(\rho_{\delta,N},\rho_{\infty,N})=0$, and the intended limit follows from (3.47). □

4. Construction and convergence of measures in the shallow-water regime

In this section we prove Theorem 1.2 on construction of the weighted measures $\widetilde{\rho}_{\delta,\frac{k}{2},K}$ for each fixed $0<\delta\le\infty$ and $k\ge 2$, and we prove the weak convergence of $\widetilde{\rho}_{\delta,\frac{k}{2}-\frac12,K}$ and $\widetilde{\rho}_{\delta,\frac{k}{2},K}$ to $\widetilde{\rho}_{0,k,K}$. We first detail the structure of the conservation laws $\widetilde{E}_{\delta,\frac{k}{2}}(v)$ for sILW (1.4) in Subsection 4.1. In Subsection 4.2, we establish the singularity of the Gaussian measures $\widetilde{\mu}_{\delta,k-\frac12}$, $\widetilde{\mu}_{\delta,k}$, and $\widetilde{\mu}_{0,k}$, and the weak convergence of the former to the latter as $\delta\to 0$. In

Subsection 4.3, we prove uniform in δ and N bounds on the truncated densities $\widetilde{F}_{\delta,\frac{k}{2},N,K}$, and we complete the proof of Theorem 1.2 in Subsections 4.4-4.5.

4.1. **Conservation laws in the shallow-water regime.** In this subsection, we describe the structure of the conserved quantities for sILW (1.4), focusing on the shallow-water regime. Derivation of the conservation laws can be found in [50, 25, 16, 30]. However, the format of these was not suitable for our construction. For completeness, we include the derivation and some relevant results on the structure in Appendix B.

We first introduce some notations. Consider the following sets

$$\begin{aligned}
\widetilde{\mathcal{P}}_1(u) &:= \big\{\widetilde{\mathcal{G}}_\delta^{\alpha}\partial_x^{\beta}u : \quad \alpha,\beta\in\mathbb{N}\cup\{0\}\big\},\\
\widetilde{\mathcal{P}}_2(u) &:= \big\{\big[\widetilde{\mathcal{G}}_\delta^{\alpha_1}\partial_x^{\beta_1}u\big]\big[\widetilde{\mathcal{G}}_\delta^{\alpha_2}\partial_x^{\beta_2}u\big] : \quad \alpha_1,\alpha_2,\beta_1,\beta_2\in\mathbb{N}\cup\{0\}\big\},\\
\widetilde{\mathcal{P}}_n(u) &:= \bigg\{\prod_{\ell=1}^{k}\widetilde{\mathcal{G}}_\delta^{\alpha_\ell}\partial_x^{\beta_\ell}p_{j_\ell}(u) : \ \alpha_\ell,\beta_\ell\in\mathbb{N}\cup\{0\},\ j_\ell\in\mathbb{N},\ j_{1\cdots k}=n,\\
&\qquad\qquad k\in\{2,\dots,n\}, p_{j_\ell}(u)\in\widetilde{\mathcal{P}}_{j_\ell}(u)\bigg\}.
\end{aligned}$$

We also define the map $\widetilde{\mathcal{P}}_n(u)\ni p(u)\mapsto\widetilde{p}(u)\in\widetilde{\mathcal{P}}_n(u)$ which associates to every $p(u)\in\widetilde{\mathcal{P}}_n(u)$ the unique essential element $\widetilde{p}(u)\in\widetilde{\mathcal{P}}_n(u)$ obtained by "dropping" the $\widetilde{\mathcal{G}}_\delta$ operators. Also, we defined the following quantities associated with $p(u)$: let $\widetilde{p}(u)=\prod_{i=1}^n\partial_x^{\beta_i}u$, then

$$\begin{aligned}
|p(u)| &:= \sup_{i=1,\dots,n}|\beta_i|,\\
\|p(u)\| &:= \beta_1+\ldots+\beta_n,\\
|||p(u)||| &:= \text{ number of } \widetilde{\mathcal{G}}_\delta \text{ terms in } p(u).
\end{aligned}$$

Using these sets, we can now describe our conserved quantities $\widetilde{E}_{\delta,\frac{k}{2}}(u)$. For $k\in\mathbb{N}$, we have that

$$\begin{aligned}
\widetilde{E}_{\delta,\frac{2k-1}{2}}(u) &= \frac{3}{4k}\sum_{\substack{m=1\\ \text{odd}}}^{2k-1}\binom{2k}{m}\delta^{m-1}\|\widetilde{\mathcal{G}}_\delta^{\frac{m}{2}}u\|^2_{\dot{H}^{k-\frac12}}+\widetilde{R}_{\delta,\frac{2k-1}{2}}(u),\\
\widetilde{E}_{\delta,\frac{2k}{2}}(u) &= \frac{1}{2}\sum_{\substack{m=0\\ \text{even}}}^{2k}\binom{2k+1}{m}\delta^{m}\|\widetilde{\mathcal{G}}_\delta^{\frac{m}{2}}u\|^2_{\dot{H}^{k}}+\widetilde{R}_{\delta,\frac{2k}{2}}(u),
\end{aligned}$$

where the remainders $\widetilde{R}_{\delta,\frac{k}{2}}(u)$ denote the remainder terms, which include contributions which are cubic or higher in u. These can be written as follows

$$\widetilde{R}_{\delta,\frac{2k-1}{2}}(u)=\sum_{\ell=0}^{2k-3}\delta^\ell\bigg[\sum_{\substack{p(u)\in\widetilde{\mathcal{P}}_3(u)\\ \ell+1\le\|p(u)\|\le 2k-3\\ |p(u)|\le k-1\\ |||p(u)|||=\ell+1}}c(p)\int p(u)\,dx+\sum_{\substack{p(u)\in\widetilde{\mathcal{P}}_4(u)\\ \ell+1\le\|p(u)\|\le 2k-5\\ |p(u)|\le k-2\\ |||p(u)|||=\ell+1}}c(p)\int p(u)\,dx$$

$$
\begin{aligned}
&+\sum_{j=5}^{2k-\ell}\sum_{\substack{p(u)\in\widetilde{\mathcal{P}}_j(u)\\ \ell+1\le\|p(u)\|\le 2k+1-j\\ |p(u)|\le k-1\\ |||p(u)|||=\ell+1}} c(p)\int p(u)\,dx\Bigg]\\
&+\sum_{\ell=0}^{2k-2}\delta^\ell\sum_{j=3}^{2k+1}\sum_{\substack{p(u)\in\widetilde{\mathcal{P}}_j(u)\\ \|p(u)\|\le 2k+1-j\\ |p(u)|\le k-1\\ |||p(u)|||\le\ell}} c(p)\int p(u)\,dx\\
&=:\widetilde{R}^{[<]}_{\delta,\frac{2k-1}{2}}(u)+\widetilde{R}^{[\ge]}_{\delta,\frac{2k-1}{2}}(u),
\end{aligned}
$$

$$
\widetilde{R}_{\delta,\frac{2k}{2}}(u)=\sum_{\ell=0}^{2k-1}\delta^\ell\sum_{j=3}^{2k+2}\sum_{\substack{p(u)\in\widetilde{\mathcal{P}}_j(u)\\ \|p(u)\|\le 2k+2-j\\ |p(u)|\le k\\ |||p(u)|||\le\ell}} c(p)\int p(u)\,dx. \tag{4.1}
$$

4.2. **Singularity and convergence of the base Gaussian measures.** Let $k\in\mathbb{N}$ and $0<\delta<\infty$. We focus on the base Gaussian measures $\widetilde{\mu}_{\delta,\frac{k}{2}}$ in (1.21). For $n\in\mathbb{Z}^*$, let $\widetilde{T}_{\delta,\frac{k}{2}}(n)$ denote the following multipliers

$$
\begin{aligned}
\widetilde{T}_{\delta,\frac{2k-1}{2}}(n)&=\frac{3}{2k}\sum_{\substack{m=1\\ \text{odd}}}^{2k-1}\binom{2k}{m}\delta^{m-1}\mathrm{L}_\delta(n)^m|n|^{2k-1-m},\\
\widetilde{T}_{\delta,\frac{2k}{2}}(n)&=\sum_{\substack{m=0\\ \text{even}}}^{2k}\binom{2k+1}{m}\delta^{m}\mathrm{L}_\delta(n)^m|n|^{2k-m},\\
\widetilde{T}_{0,k}(n)&=|n|^{2k},
\end{aligned}
\tag{4.2}
$$

and let $\widetilde{\mu}_{\delta,\frac{k}{2}}$ denote the induced probability measure induced under the map

$$
\omega\in\Omega\mapsto\widetilde{X}_{\delta,\frac{k}{2}}(x;\omega)=\frac{1}{\sqrt{2\pi}}\sum_{n\in\mathbb{Z}^*}\frac{g_n(\omega)}{(\widetilde{T}_{\delta,\frac{k}{2}}(n))^{\frac12}}e^{inx}, \tag{4.3}
$$

where $\{g_n\}_{n\in\mathbb{Z}^*}$ is a sequence of independent standard complex-valued Gaussian random variables satisfying $g_{-n}=\overline{g_n}$. We also extend the definition (4.3) to $\delta=0$ then $k=2m$ for $m\in\mathbb{N}$, and use $\widetilde{\mu}_{0,m}$ to denote the associated measure.

We have the following results on convergence of the Gaussian measures $\widetilde{\mu}_{\delta,\frac{2k-1}{2}},\widetilde{\mu}_{\delta,\frac{2k}{2}}$ associated with sILW to $\widetilde{\mu}_{0,k}$.

Lemma 4.1. *Let $k\in\mathbb{N}$. Then, the following results hold:*
(i) *Given any $\varepsilon>0$ and $1\le p<\infty$, $\widetilde{X}_{\delta,k-\frac12}$ converges to $\widetilde{X}_{0,k}$ in $L^p(\Omega;H^{k-1-\varepsilon}(\mathbb{T}))$ and almost surely in $H^{k-1-\varepsilon}(\mathbb{T})$, as $\delta\to0$, while $\widetilde{X}_{\delta,k}$ converges to $\widetilde{X}_{0,k}$ in $L^p(\Omega;H^{k-\frac12-\varepsilon}(\mathbb{T}))$ and almost surely in $H^{k-\frac12-\varepsilon}(\mathbb{T})$, as $\delta\to0$. Moreover, the Gaussian measures $\widetilde{\mu}_{\delta,k-\frac12},\widetilde{\mu}_{\delta,k}$ converge weakly to the Gaussian measure $\widetilde{\mu}_{0,k}$, as $\delta\to0$.*

(ii) *Given* $\varepsilon > 0$ *and* $0 < \delta < \eta < \infty$*, the Gaussian measures* $\widetilde{\mu}_{\delta,k}$*,* $\widetilde{\mu}_{\eta,k}$*, and* $\widetilde{\mu}_{0,k}$ *are singular.*

Before proving Lemma 4.1, we need some auxiliary results on the multipliers $\widetilde{T}_{\delta,\frac{k}{2}}$.

Lemma 4.2. *Let* $k \in \mathbb{N}$, $0 < \delta < \infty$*, and* $n \in \mathbb{Z}^*$ *fixed. Then, we have that*

$$\lim_{\delta\to 0} \widetilde{T}_{\delta,k-\frac{1}{2}}(n) = \lim_{\delta\to 0} \widetilde{T}_{\delta,k}(n) = \widetilde{T}_{0,k}(n) = |n|^{2k}. \tag{4.4}$$

Proof. From Lemma 2.3(i)-(ii), we have that

$$\begin{aligned}
\big|\widetilde{T}_{\delta,k-\frac{1}{2}}(n) - |n|^{2k}\big| &= \Bigg|3\mathrm{L}_\delta(n)|n|^{2k-2} - |n|^{2k} + \sum_{\substack{m=3\\ \text{odd}}}^{2k-1} \binom{2k}{m} \delta^{m-1} \mathrm{L}_\delta(n)^m |n|^{2k-1-m}\Bigg| \\
&\le |n|^{2k-2}|3\mathrm{L}_\delta(n) - |n|^2| + \sum_{\substack{m=3\\ \text{odd}}}^{2k-1} \binom{2k}{m} \delta^{m-1} \Big(\frac{1}{\delta}|n|\Big)^{m-2} \big(|n|^2\big)^2 |n|^{2k-1-m} \\
&\lesssim |n|^{2k-2}\big|3\mathrm{L}_\delta(n) - |n|^2\big| + \delta|n|^{2k+1} \to 0 \quad \text{ as } \delta\to 0, \\
\big|\widetilde{T}_{\delta,2k+1}(n) - |n|^{2k}\big| &= \sum_{\substack{m=2\\ \text{even}}}^{2k} \binom{2k+1}{m} \delta^m \mathrm{L}_\delta(n)^m |n|^{2k-m} \\
&\le \sum_{\substack{m=2\\ \text{even}}}^{2k} \binom{2k+1}{m} \delta^m \Big(\frac{1}{\delta}|n|\Big)^{m-1} \big(|n|^2\big)|n|^{2k-m} \\
&\lesssim \delta|n|^{2k+1} \to 0 \qquad \text{as } \delta \to 0.
\end{aligned}$$

as intended. ☐

We can now prove our results on the convergence of the Gaussian measures $\widetilde{\mu}_{\delta,\frac{k}{2}}$.

Proof of Lemma 4.1. We start by proving (i). Let $\varepsilon > 0$. By the Wiener chaos estimate, we have that for $m = 2k-1$ or $m = 2k$,

$$\begin{aligned}
\|\widetilde{X}_{\delta,\frac{m}{2}} - \widetilde{X}_{0,k}\|_{L^p_\omega H^{\frac{m-1}{2}-\varepsilon}_x} &\lesssim_p \|\langle\nabla\rangle^{m-\frac{1}{2}-\varepsilon}(\widetilde{X}_{\delta,\frac{m}{2}} - \widetilde{X}_{0,k})\|_{L^2_x L^2_\omega} \\
&\sim \bigg(\sum_{n\in\mathbb{Z}^*} \langle n\rangle^{2(m-\frac{1}{2}-\varepsilon)} \bigg[\frac{1}{(\widetilde{T}_{\delta,\frac{m}{2}}(n))^{\frac{1}{2}}} - \frac{1}{|n|^k}\bigg]^2\bigg)^{\frac{1}{2}}.
\end{aligned} \tag{4.5}$$

First, from the definition of $\widetilde{T}_{\delta,\frac{m}{2}}$ (4.2) and Lemma 2.3(iii), we have that

$$\widetilde{T}_{\delta,k}(n) \gtrsim |n|^{2k}, \qquad \widetilde{T}_{\delta,k-\frac{1}{2}}(n) \gtrsim \begin{cases} \frac{1}{\delta}|n|^{2k-1}, & \text{if } \delta|n| \gtrsim 1, \\ |n|^{2k}, & \text{if } 0 < \delta|n| \ll 1. \end{cases}$$

Consequently,

$$\begin{aligned}
\|\widetilde{X}_{\delta,k-\frac{1}{2}} - \widetilde{X}_{0,k}\|^2_{L^p_\omega H^{k-1-\varepsilon}_x} &\lesssim \sum_{0<\delta|n|\ll 1} \frac{\langle n\rangle^{2(k-1-\varepsilon)}}{\langle n\rangle^{2k}} + \sum_{\delta|n|\gtrsim 1} \langle n\rangle^{2(k-1-\varepsilon)} \frac{\delta\langle n\rangle + 1}{\langle n\rangle^{2k}} \\
&\lesssim \sum_{n\in\mathbb{Z}^*} \frac{1}{\langle n\rangle^{1+2\varepsilon}} < \infty,
\end{aligned}$$

$$\|\widetilde{X}_{\delta,k}-\widetilde{X}_{0,k}\|^2_{L^p_\omega H_x^{k-\frac{1}{2}-\varepsilon}} \lesssim \sum_{n\in\mathbb{Z}^*}\frac{\langle n\rangle^{2(k-\frac{1}{2}-\varepsilon)}}{\langle n\rangle^{2k}} \lesssim \sum_{n\in\mathbb{Z}^*}\frac{1}{\langle n\rangle^{1+2\varepsilon}} < \infty.$$

From (4.4), we have that $\widetilde{T}_{\delta,\frac{m}{2}}(n)^{\frac{1}{2}} \to |n|^k$ for all $n\in\mathbb{Z}^*$, for $m=2k-1$ and $m=2k$, which means that the summand on the RHS of (4.5) converges to 0 as $\delta\to 0$. The intended convergence then follows from the dominated convergence theorem. The almost sure convergence follows from a similar calculation, replacing the fact that $\mathbb{E}[|g_n|^2]=1$ with $\sup_n\langle n\rangle^{-\varepsilon}|g_n(\omega)| < C_{\varepsilon,\omega} < \infty$.

Fix $0<\delta<\infty$. We first show the singularity of the measures $\widetilde{\mu}_{\delta,k-\frac{1}{2}}, \widetilde{\mu}_{\delta,k}$ and $\widetilde{\mu}_{0,k}$. We can write

$$\begin{aligned}
\widetilde{X}_{\delta,\frac{m}{2}}(x;\omega) &= \sum_{n\in\mathbb{N}}\frac{1}{(\widetilde{T}_{\delta,\frac{m}{2}}(n))^{\frac{1}{2}}}\sqrt{\frac{2}{\pi}}[\operatorname{Re} g_n(\omega)\cos(nx)-\operatorname{Im} g_n(\omega)\sin(nx)],\\
\widetilde{X}_{0,k}(x;\omega) &= \sum_{n\in\mathbb{N}}\frac{1}{|n|^k}\sqrt{\frac{2}{\pi}}[\operatorname{Re} g_n\cos(nx)-\operatorname{Im} g_n\sin(nx)],
\end{aligned}$$

for $m\in\{2k-1,2k\}$. Then, for $n\in\mathbb{N}$, define

$$\begin{aligned}
A_{\delta,\frac{m}{2},n} &:= \sqrt{\frac{2}{\pi}}\frac{\operatorname{Re} g_n}{(\widetilde{T}_{\delta,\frac{m}{2}}(n))^{\frac{1}{2}}}, & A_{\delta,\frac{m}{2},-n} &:= -\sqrt{\frac{2}{\pi}}\frac{\operatorname{Im} g_n}{(\widetilde{T}_{\delta,\frac{m}{2}}(n))^{\frac{1}{2}}},\\
B_n &:= \sqrt{\frac{2}{\pi}}\frac{\operatorname{Re} g_n}{|n|^k}, & B_{-n} &:= -\sqrt{\frac{2}{\pi}}\frac{\operatorname{Im} g_n}{|n|^k},\\
a_{\delta,\frac{m}{2},\pm n} &:= \mathbb{E}[A^2_{\delta,\frac{m}{2},\pm n}] = \frac{1}{\pi\widetilde{T}_{\delta,\frac{m}{2}}(n)}, & b_{\pm n} &:= \mathbb{E}[B^2_{\pm n}] = \frac{1}{\pi|n|^{2k}}.
\end{aligned}$$

Then, by Kakutani's theorem, it suffices to show that

$$\sum_{n\in\mathbb{N}}\left(\frac{a_{\delta,\frac{m}{2},n}}{b_n}-1\right)^2 = \sum_{n\in\mathbb{N}}\left(\frac{\widetilde{T}_{\delta,\frac{m}{2}}(n)-|n|^{2k}}{|n|^{2k}}\right)^2 = +\infty. \tag{4.6}$$

For $m=2k-1$, from Lemma 2.3, we have that for $\delta|n|\gtrsim 1$, $\mathrm{L}_\delta(n)\sim\frac{|n|}{\delta}$, which gives

$$\begin{aligned}
\frac{\widetilde{T}_{\delta,k-\frac{1}{2}}(n)-|n|^{2k}}{|n|^{2k}} &= \frac{1}{|n|^{2k}}\Bigg(-|n|^{2k}h(\delta,n)+\frac{3}{2k}\sum_{\substack{m=3\\ \text{odd}}}^{2k-1}\binom{2k}{m}\delta^{m-1}\mathrm{L}_\delta(n)^m|n|^{2k-1-m}\Bigg)\\
&\sim -h(\delta,n)+\frac{3}{2k}\sum_{\substack{m=3\\ \text{odd}}}^{2k-1}\binom{2k}{m}\delta^{m-1}\frac{|n|^m}{\delta^m|n|^{m+1}}\\
&\sim -h(\delta,n)+\frac{1}{\delta|n|},\\
\frac{\widetilde{T}_{\delta,k}(n)-|n|^{2k}}{|n|^{2k}} &= \sum_{\substack{m=2\\ \text{even}}}^{2k}\binom{2k+1}{m}\delta^m\frac{\mathrm{L}_\delta(n)^m}{|n|^m}\gtrsim 1.
\end{aligned}$$

Since from Lemma 2.3(iv), $\lim_{n\to\infty} h(\delta,n) = C \neq 0$, we conclude that $\frac{\widetilde{T}_{\delta,\frac{m}{2}}(n)-|n|^{2k}}{|n|^{2k}} \not\to 0$. Therefore, $(\frac{a_{\frac{m}{2},n}}{b_n}-1)^2$ is not summable and (4.6) holds, from which we conclude that $\widetilde{\mu}_{\delta,k-\frac{1}{2}}$ and $\widetilde{\mu}_{0,k}$, and $\widetilde{\mu}_{\delta,k}$ and $\widetilde{\mu}_{0,k}$ are singular.

It only remains to show that for $0<\eta<\delta<\infty$ the measures $\widetilde{\mu}_{\delta,\frac{k}{2}}$ and $\widetilde{\mu}_{\eta,\frac{k}{2}}$ are singular, which follows from Kakutani's theorem once we show that

$$\sum_{n\in\mathbb{N}}\bigg(\frac{a_{\delta,\frac{m}{2},n}}{a_{\eta,\frac{m}{2},n}}-1\bigg)^2 = \sum_{n\in\mathbb{N}}\bigg(\frac{\widetilde{T}_{\eta,\frac{m}{2}}(n)-\widetilde{T}_{\delta,\frac{m}{2}}(n)}{\widetilde{T}_{\delta,\frac{m}{2}}(n)}\bigg)^2 = +\infty. \tag{4.7}$$

Note that

$$\begin{aligned}
&\widetilde{T}_{\delta,k-\frac{1}{2}}(n)-\widetilde{T}_{\eta,k-\frac{1}{2}}(n)\\
&= \frac{3}{2k}\sum_{\substack{m=1\\ \text{odd}}}^{2k-1}\binom{2k}{m}|n|^{2k-1-m}(\delta^{m-1}\mathrm{L}_\delta(n)^m-\eta^{m-1}\,\mathrm{L}_\eta(n)^m)\\
&= \frac{3}{2k}\sum_{\substack{m=3\\ \text{odd}}}^{2k-1}\binom{2k}{m}|n|^{2k-1-m}\mathrm{L}_\delta(n)^m(\delta^{m-1}-\eta^{m-1})\\
&\quad + \frac{3}{2k}\sum_{\substack{m=3\\ \text{odd}}}^{2k-1}\binom{2k}{m}|n|^{2k-1-m}\eta^{m-1}[\mathrm{L}_\delta(n)-\mathrm{L}_\eta(n)]\sum_{j=0}^{m-1}\mathrm{L}_\delta(n)^{m-1-j}\,\mathrm{L}_\eta(n)^j\\
&\widetilde{T}_{\delta,k}(n)-\widetilde{T}_{\eta,k}(n)\\
&= \sum_{\substack{m=2\\ \text{even}}}^{2k}\binom{2k+1}{m}|n|^{2k-m}\mathrm{L}_\delta(n)^m(\delta^m-\eta^m)\\
&\quad + \sum_{\substack{m=2\\ \text{even}}}^{2k}\binom{2k+1}{m}|n|^{2k-m}\eta^m(\mathrm{L}_\delta(n)-\mathrm{L}_\eta(n))\sum_{j=0}^{m-1}\mathrm{L}_\delta(n)^{m-1-j}\,\mathrm{L}_\eta(n)^j.
\end{aligned}$$

Note that since $0<\eta<\delta<\infty$, for $\delta|n|\geq\eta|n|\gtrsim 1$, from Lemma 2.3 we get

$$\begin{aligned}
&|\widetilde{T}_{\delta,k-\frac{1}{2}}(n)-\widetilde{T}_{\eta,k-\frac{1}{2}}(n)|\\
&\gtrsim_{\delta,\eta} \sum_{\substack{m=3\\ \text{odd}}}^{2k-1}|n|^{2k-1-m}|\mathrm{L}_\delta(n)-\mathrm{L}_\eta(n)||n|^{m-1} - \sum_{\substack{m=3\\ \text{odd}}}^{2k-1}(\delta^{m-1}-\eta^{m-1})|n|^{2k-1-m}|n|^m\\
&\gtrsim_{\delta,\eta} |n|^{2k}|h(\delta,n)-h(\eta,n)| - |n|^{2k-1}\\
&\sim_{\delta,\eta} |n|^{2k}\sum_{\ell\in\mathbb{N}}\frac{n^2(\delta^2-\eta^2)}{(\ell^2\pi^2+\delta^2n^2)(\ell^2\pi^2+\eta^2n^2)} - |n|^{2k-1}\\
&\gtrsim_{\delta,\eta} |n|^{2k},\\
&|\widetilde{T}_{\delta,k}(n)-\widetilde{T}_{\eta,k}(n)|\\
&\gtrsim_{\delta,\eta} |n|^{2k+1}|h(\delta,n)-h(\eta,n)| - |n|^{2k}\\
&\gtrsim_{\delta,\eta} |n|^{2k+1}.
\end{aligned}$$

Consequently, we get that for $\delta|n| > \eta|n| \gtrsim 1$

$$\left|\frac{\widetilde{T}_{\delta,\frac{m}{2}}(n) - \widetilde{T}_{\eta,\frac{m}{2}}(n)}{\widetilde{T}_{\delta,\frac{m}{2}}(n)}\right| \gtrsim_{\delta,\eta} \frac{|n|^{m+1} - |n|^m}{|n|^m} \gtrsim |n|$$

from which (4.7) follows. □

4.3. **Uniform bounds on the density - shallow-water regime.** In this section, we establish the shallow-water counterpart of the results in Subsection 3.3

Given $0 < \delta < \infty$, $K > 0$, and $N \in \mathbb{N}$, we recall the definition of the truncated densities

$$\widetilde{F}_{\delta,\frac{k}{2},N,K}(u) = \eta_K(\|\mathbf{P}_N u\|_{L^2}^2) \exp(-\widetilde{R}_{\delta,\frac{k}{2}}(\mathbf{P}_N u)), \tag{4.8}$$

where $\eta : \mathbb{R} \to [0,1]$ denotes a smooth cutoff function with $\eta(x) = 1$ for $x \in [-1,1]$ and $\operatorname{supp}\chi \subset [-2,2]$, $\eta_K(x) = \eta(x/K)$, and $\widetilde{R}_{\delta,\frac{k}{2}}(u)$ denotes the remainder in (4.1). Our main goal is to prove the following result.

Proposition 4.3. *Let $1 \le p < \infty$, $k \in \mathbb{N}$, and $K > 0$. Then, for any $0 < \delta < \infty$, we have that*

$$\sup_{N\in\mathbb{N}} \|\widetilde{F}_{\delta,\frac{k}{2},N,K}(\widetilde{X}_{\delta,\frac{k}{2}})\|_{L^p(\Omega)} = \sup_{N\in\mathbb{N}} \|\widetilde{F}_{\delta,\frac{k}{2},N,K}(u)\|_{L^p(d\widetilde{\mu}_{\delta,\frac{k}{2}})} \le C_{p,k,\delta,K}. \tag{4.9}$$

Moreover, for $\delta_0 > 0$ and $0 \le \delta \le \delta_0$, the constant $C_{p,k,\delta,K}$ above can be chosen independently of δ.

As in the deep-water regime, we reduce the problem to estimating the following truncated density:

$$\widetilde{\mathcal{F}}_{\delta,\frac{k}{2},N,K}(u) = \exp\Big(- \widetilde{R}_{\delta,\frac{k}{2}}(\mathbf{P}_N u) - A\|\mathbf{P}_N u\|_{L^2}^{2\alpha(k)}\Big), \tag{4.10}$$

for some $A \gg 1$ and $\alpha(k) \in \mathbb{N}$. In particular, it suffices to prove the following proposition.

Proposition 4.4. *Let $1 \le p < \infty$, $0 \le \delta < \infty$, $k \in \mathbb{N}$, and $K > 0$. Then, there exist $sA_0 > 0$ such that for $A \ge A_0$*

$$\sup_{N\in\mathbb{N}} \|\widetilde{\mathcal{F}}_{\delta,\frac{k}{2},N,K}(\widetilde{X}_{\delta,\frac{k}{2}})\|_{L^p(\Omega)} = \sup_{N\in\mathbb{N}} \|\widetilde{\mathcal{F}}_{\delta,\frac{k}{2},N,K}(u)\|_{L^p(d\widetilde{\mu}_{\delta,\frac{k}{2}})} \le C_{p,\delta,k,K,A_0} < \infty. \tag{4.11}$$

In addition, for $\delta_0 > 0$, the following uniform bound holds for $0 \le \delta \le \delta_0$:

$$\begin{aligned} \sup_{N\in\mathbb{N}} \sup_{0\le\delta\le\delta_0} \|\widetilde{\mathcal{F}}_{\delta,\frac{k}{2},N,K}(\widetilde{X}_{\delta,\frac{k}{2}})\|_{L^p(\Omega)} &= \sup_{N\in\mathbb{N}} \sup_{0\le\delta\le\delta_0} \|\widetilde{\mathcal{F}}_{\delta,\frac{k}{2},N,K}(u)\|_{L^p(d\widetilde{\mu}_{\delta,\frac{k}{2}})} \\ &\le C_{p,k,K,A_0} < \infty. \end{aligned} \tag{4.12}$$

Recalling $W(t)$ in (3.17), we define a centered Gaussian process $\widetilde{Y}_{\delta,\frac{k}{2}}(t)$ by

$$\widetilde{Y}_{\delta,\frac{k}{2}}(t) = (\widetilde{\mathcal{T}}_{\delta,\frac{k}{2}})^{-\frac{1}{2}} W(t), \tag{4.13}$$

where $(\widetilde{\mathcal{T}}_{\delta,\frac{k}{2}})^{-\frac{1}{2}}$ is the Fourier multiplier operator with the multiplier $(\widetilde{T}_{\delta,\frac{k}{2}}(n))^{-\frac{1}{2}}$ as in (4.2). In view of (4.3), we have $\mathcal{L}(\widetilde{Y}_{\delta,\frac{k}{2}}(1)) = \widetilde{\mu}_{\delta,\frac{k}{2}}$. Given $N \in \mathbb{N}$, we set $\widetilde{Y}_{\delta,\frac{k}{2},N} = \mathbf{P}_N \widetilde{Y}_{\delta,\frac{k}{2}}$. We can now state the analogue of the Boué-Dupuis variational formula in this setting.

Lemma 4.5. *Given $0 \le \delta < \infty$, let $\widetilde{Y}_{\delta,\frac{k}{2}}$ be as in (4.13) and fix $N \in \mathbb{N}$. Suppose that $F : C^\infty(\mathbb{T}) \to \mathbb{R}$ is measurable such that $\mathbb{E}\big[|F(\widetilde{Y}_{\delta,\frac{k}{2}}(1))|^p\big] < \infty$ and $\mathbb{E}\big[|e^{-F(\widetilde{Y}_{\delta,\frac{k}{2}}(1))}|^q\big] < \infty$ for some $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then, we have*

$$-\log \mathbb{E}\Big[e^{-F(\widetilde{Y}_{\delta,\frac{k}{2}}(1))}\Big] = \inf_{\theta \in \mathbb{H}_a} \mathbb{E}\Big[F(\widetilde{Y}_{\delta,\frac{k}{2}}(1) + \widetilde{I}_{\delta,\frac{k}{2}}(\theta)(1)) + \frac{1}{2}\int_0^1 \|\theta(t)\|_{L^2_x}^2 dt\Big], \tag{4.14}$$

where $\widetilde{I}_{\delta,\frac{k}{2}}(\theta)$ is defined by

$$\widetilde{I}_{\delta,\frac{k}{2}}(\theta)(t) = \int_0^t (\widetilde{\mathcal{T}}_{\delta,\frac{k}{2}})^{-\frac{1}{2}} \theta(t')dt'.$$

We first show a preliminary result on the pathwise regularity of $\widetilde{Y}_{\delta,\frac{k}{2},N}(1)$ and $\widetilde{I}_{\delta,\frac{k}{2}}(\theta)(1)$.

Lemma 4.6. *Let $\delta_0 > 0$ and $k \in \mathbb{N}$. For any $1 \le p < \infty$, $0 \le \delta \le \delta_0$, and $\sigma < \frac{k-1}{2}$, we have that*

$$\sup_{N \in \mathbb{N}} \mathbb{E}\Big[\|\widetilde{Y}_{\delta,\frac{k}{2},N}(1)\|_{W^{\sigma,\infty}_x}^p\Big] < C_{p,\delta_0} < \infty,$$

where C_{p,δ_0} is independent of δ. Moreover, for any $\theta \in \mathbb{H}_a$, we have

$$\|\widetilde{I}_{\delta,\frac{k}{2}}(\theta)(1)\|_{H^{\frac{k}{2}}_x}^2 \le C_{\delta_0} \int_0^t \|\theta(t)\|_{L^2_x}^2 \, dt$$

where the constant $C_{\delta_0} > 0$ can be chosen independently of $0 \le \delta \le \delta_0$.

Proof. For the first inequality, fix $\sigma < \frac{k-1}{2}$ and $\varepsilon > 0$ sufficiently small such that $2(\sigma+\varepsilon) < k-1$. Then, for $r \gg 1$ with $r\varepsilon > 1$, Sobolev embedding gives that

$$\|\widetilde{Y}_{\delta,\frac{k}{2},N}(1)\|_{W^{\sigma,\infty}_x} \lesssim \|\langle\nabla\rangle^{\sigma+\varepsilon}\widetilde{Y}_{\delta,\frac{k}{2},N}(1)\|_{L^r_x}.$$

Fix $p \ge r$. Note that the result follows for $1 \le p < r$ from the embedding $L^r(\Omega) \subset L^p(\Omega)$. Then, by Minkowski's inequality and the Wiener chaos estimate (Lemma 2.6), we have

$$\begin{aligned}
\big\|\|\widetilde{Y}_{\delta,\frac{k}{2},N}(1)\|_{W^{\sigma,\infty}_x}\big\|_{L^p(\Omega)} &\le Cp^{\frac{1}{2}}\big\|\|\langle\nabla\rangle^{\sigma+\varepsilon}\widetilde{Y}_{\delta,\frac{k}{2},N}(1)\|_{L^2(\Omega)}\big\|_{L^r_x}\\
&\le C_p\Bigg\|\mathbb{E}\Bigg[\sum_{0<|n|,|m|\le N}\langle n\rangle^{\sigma+\varepsilon}\langle m\rangle^{\sigma+\varepsilon}\frac{g_n\overline{g_m}}{(\widetilde{\mathcal{T}}_{\delta,\frac{k}{2}}(n)\widetilde{\mathcal{T}}_{\delta,\frac{k}{2}}(m))^{\frac{1}{2}}}e^{i(n-m)x}\Bigg]^{\frac{1}{2}}\Bigg\|_{L^r_x}\\
&\le C_p\Bigg(\sum_{0<|n|\le N}\frac{\langle n\rangle^{2\sigma+2\varepsilon}}{\widetilde{\mathcal{T}}_{\delta,\frac{k}{2}}(n)}\Bigg)^{\frac{1}{2}}.
\end{aligned}$$

From (4.2) and Lemma 2.3, we have that for $n \in \mathbb{Z}^*$, $0 < \delta \le \delta_0$, and $m \in \mathbb{N}$,

$$\begin{aligned}
\widetilde{\mathcal{T}}_{\delta,\frac{2m-1}{2}}(n) \ge 3\mathrm{L}_\delta(n)|n|^{2m-2} &\ge \begin{cases}\frac{1}{\delta}|n|^{2m-1}, & \delta|n| \gg 1,\\ |n|^{2m}, & \delta|n| \lesssim 1\end{cases}\\
&\gtrsim_{\delta_0} |n|^{2m-1}
\end{aligned}$$

while $\widetilde{\mathcal{T}}_{\delta,\frac{2m}{2}}(n) \ge |n|^{2m}$. Therefore, for $n \in \mathbb{Z}^*$ and $0 < \delta \le \delta_0$, we have that

$$\widetilde{\mathcal{T}}_{\delta,\frac{k}{2}}(n) \gtrsim_{\delta_0} |n|^k, \tag{4.15}$$

from which we conclude that

$$\big\| \|\widetilde{Y}_{\delta,\frac{k}{2},N}(1)\|_{W_x^{\sigma,\infty}} \big\|_{L^p(\Omega)} \lesssim_{p,\delta_0} \bigg(\sum_{0<|n|\le N} \frac{\langle n\rangle^{2\sigma+2\varepsilon}}{\langle n\rangle^k} \bigg)^{\frac{1}{2}} < C_{p,\delta_0} < \infty.$$

For the second estimate, from Minkowski and Cauchy-Schwarz inequalities, and (4.15), we have that

$$\begin{aligned} \|\widetilde{I}_{\delta,\frac{k}{2}}(\theta)(1)\|^2_{H_x^{\frac{k}{2}}} &\le C\bigg(\int_0^1 \bigg[\sum_{n\neq 0} \frac{|n|^k}{\widetilde{T}_{\delta,\frac{k}{2}}(n)} |\widehat{\theta}(t,n)|^2 \bigg]^{\frac{1}{2}} dt \bigg)^2 \\ &\le C_{\delta_0} \bigg(\int_0^1 \|\theta(t)\|_{L^2_x} dt \bigg)^2 \\ &\le C_{\delta_0} \int_0^1 \|\theta(t)\|^2_{L^2_x} \, dt, \end{aligned}$$

for some constant $C_{\delta_0} > 0$ which only depends on δ_0. □

Note that (4.14) is equivalent to

$$\begin{aligned} \log \mathbb{E}\Big[e^{-F(\widetilde{Y}_{\delta,\frac{k}{2},N}(1))} \Big] &= -\inf_{\theta\in\mathbb{H}_a} \mathbb{E}\bigg[F(\widetilde{Y}_{\delta,\frac{k}{2},N}(1) + \widetilde{I}_{\delta,\frac{k}{2},N}(\theta)(1)) + \frac{1}{2}\int_0^1 \|\theta(t)\|^2_{L^2_x} dt \bigg] \\ &= \sup_{\theta\in\mathbb{H}_a} \mathbb{E}\bigg[- F(\widetilde{Y}_{\delta,\frac{k}{2},N}(1) + \widetilde{I}_{\delta,\frac{k}{2},N}(\theta)(1)) - \frac{1}{2}\int_0^1 \|\theta(t)\|^2_{L^2_x} dt \bigg]. \end{aligned}$$

Fix $L > 0$. Then, setting $F_L = p\big[\max(\widetilde{F}_{\delta,\frac{k}{2},N,K}, -L) + A\|\mathbf{P}_N \cdot \|^{2\alpha(k)}_{L^2}\big]$, we easily see that

$$\begin{aligned} &\mathbb{E}[|e^{-F_L(\widetilde{Y}_{\delta,\frac{k}{2}}(1))}|^q] \le \mathbb{E}[e^{-q\max(\widetilde{F}_{\delta,\frac{k}{2},N,K},-L)}] = \mathbb{E}[e^{q\min(-\widetilde{F}_{\delta,\frac{k}{2},N,K},L)}] \le e^{qL} < \infty, \\ &\mathbb{E}\bigg[F_L(\widetilde{Y}_{\delta,\frac{k}{2}}(1) + \widetilde{I}_{\delta,\frac{k}{2}}(\theta)(1)) + \frac{1}{2}\int_0^1 \|\theta(t)\|^2_{L^2_x} \, dt \bigg] \\ &= \mathbb{E}\bigg[- F_L(\widetilde{Y}_{\delta,\frac{k}{2}}(1) + \widetilde{I}_{\delta,\frac{k}{2}}(\theta)(1)) - \frac{1}{2}\int_0^1 \|\theta(t)\|^2_{L^2_x} \, dt \bigg] \\ &= \mathbb{E}\bigg[- \max(\widetilde{F}_{\delta,\frac{k}{2}}(\widetilde{Y}_{\delta,k}(1) + \widetilde{I}_{\delta,\frac{k}{2}}(\theta)(1)), -L) - \frac{1}{2}\int_0^1 \|\theta(t)\|^2_{L^2_x} \, dt \bigg] \\ &= \mathbb{E}\bigg[\min(-\widetilde{F}_{\delta,\frac{k}{2}}(\widetilde{Y}_{\delta,\frac{k}{2}}(1) + \widetilde{I}_{\delta,\frac{k}{2}}(\theta)(1)), L) - \frac{1}{2}\int_0^1 \|\theta(t)\|^2_{L^2_x} \, dt \bigg] \\ &\le \mathbb{E}\bigg[- \widetilde{F}_{\delta,\frac{k}{2}}(\widetilde{Y}_{\delta,\frac{k}{2}}(1) + \widetilde{I}_{\delta,\frac{k}{2}}(\theta)(1)) - \frac{1}{2}\int_0^1 \|\theta(t)\|^2_{L^2_x} \, dt \bigg]. \end{aligned}$$

Therefore, Proposition 4.4 follows from Lemma 4.5 once we establish an upper bound on

$$\widetilde{\mathcal{M}}_{\delta,\frac{k}{2},N}(v) = \mathbb{E}\bigg[- p\widetilde{R}_{\delta,\frac{k}{2}}(\mathbf{P}_N\widetilde{Y}_{\delta,\frac{k}{2}}(1) + \mathbf{P}_N v) - pA\|\mathbf{P}_N\widetilde{Y}_{\delta,\frac{k}{2}}(1) + \mathbf{P}_N v\|^{2\alpha(k)}_{L^2} - \frac{1}{2}C\|v\|^2_{H^{\frac{k}{2}}} \bigg] \tag{4.16}$$

uniformly in $N \in \mathbb{N}$ and $v \in H^{\frac{k}{2}}(\mathbb{T})$, and take a limit as $L \to \infty$. In the following, we will use $\widetilde{Y}_{\frac{k}{2},N} = \mathbf{P}_N\widetilde{Y}_{\delta,\frac{k}{2}}(1)$, which has the same law as $\mathbf{P}_N\widetilde{X}_{\delta,\frac{k}{2}}$, and $v_N = \mathbf{P}_N v$, for a fixed $v \in H^{\frac{k}{2}}(\mathbb{T})$.

Lemma 4.7. *Let $k \in \mathbb{N}$, $\delta_0 > 0$, $0 < \delta \le \delta_0$, and $N \in \mathbb{N}$. Then, for $0 < \varepsilon < 1$, there exist $C_{\delta_0,\varepsilon} > 0$ independent of N and δ, and $\alpha(k) \in \mathbb{N}$ such that*

$$|\mathbb{E}[\widetilde{R}_{\delta,k}(\widetilde{Y}_{k,N} + v_N)]| \le \mathbb{E}[C_{\delta_0,\varepsilon}(1 + \|\widetilde{Y}_{k,N}\|^q_{H^{k-\frac{1}{2}-}} + \|v_N\|^{2\alpha(k)}_{L^2}) + \varepsilon\|v_N\|^2_{H^k}],$$

for $1 \ll q < \infty$ sufficiently large.

Proof. For $0 \le \ell \le 2k-1$ in (4.1), we see that each term to be estimated has number of $\widetilde{\mathcal{G}}_\delta$ operators at most ℓ. Thus, from (2.7), since the Fourier multiplier $|\widehat{\delta\widetilde{\mathcal{G}}_\delta}(n)| = |\widehat{\mathcal{G}_\delta}(n)| \le 1$ is bounded, it suffices to estimate the contributions $\int p(u)\,dx$ where $|||p(u)||| = 0$. Moreover, any additional positive powers of δ can be controlled by δ_0. Then, it suffices to consider two cases: (i) $p(u) = u\partial_x^{k-1}u\partial_x^k u$, and (ii) $j = 3, \dots, 2k+2$, $p(u) \in \widetilde{\mathcal{P}}_j(u)$ with $\|p(u)\| \le 2k+2-j$ and $|p(u)| \le k-1$.

<u>**Case (i):** $p(u) = u\partial_x^{k-1}u\partial_x^k u$</u>
Using multilinearity of p, we have that

$$\begin{aligned} p(\widetilde{Y}_{k,N} + v_N, \widetilde{Y}_{k,N} + v_N, \widetilde{Y}_{k,N} + v_N) &= p(\widetilde{Y}_{k,N}, \widetilde{Y}_{k,N}, \widetilde{Y}_{k,N}) \\ &\quad + p(\widetilde{Y}_{k,N} + v_N, \widetilde{Y}_{k,N} + v_N, v_N) + p(v_N, \widetilde{Y}_{k,N}, \widetilde{Y}_{k,N}) + p(\widetilde{Y}_{k,N} + v_N, v_N, \widetilde{Y}_{k,N}). \end{aligned}$$

By Isserlis' theorem, the first contribution $\mathbb{E}[p(\widetilde{Y}_{k,N}, \widetilde{Y}_{k,N}, \widetilde{Y}_{k,N})] = 0$. For the second contribution, Young's convolution inequality, Hölder's inequality, (2.1), and Young's inequality, give

$$\bigg|\int p(u_1, u_1, u_2)\,dx\bigg| \lesssim \|\widehat{u}_1(n)\|_{\ell_n^{\frac{4}{3}}} \|\langle n\rangle^{k-1}\widehat{u}_1(n)\|_{\ell_n^{\frac{4}{3}}} \|u_2\|_{H^k} \lesssim \|u_1\|_{H^{\frac{1}{4}+}} \|u_1\|_{H^{k-\frac{3}{4}+}} \|u_2\|_{H^k},$$

$$\begin{aligned} &\bigg|\int p(\widetilde{Y}_{k,N} + v_N, \widetilde{Y}_{k,N} + v_N, v_N)\,dx\bigg| \\ &\lesssim \|v_N\|_{H^k} \|\widetilde{Y}_{k,N} + v_N\|_{H^{k-\frac{3}{4}+}} \|\widetilde{Y}_{k,N} + v_N\|_{H^{\frac{1}{4}+}} \\ &\lesssim \|v_N\|_{H^k} (\|\widetilde{Y}_{k,N}\|^2_{H^{k-\frac{1}{2}-}} + \|v_N\|_{H^{k-\frac{3}{4}+}} \|\widetilde{Y}_{k,N}\|_{H^{k-\frac{1}{2}-}} + \|v_N\|_{H^{k-\frac{3}{4}+}} \|v_N\|_{H^{\frac{1}{4}+}}) \\ &\lesssim \|v_N\|_{H^k} \|\widetilde{Y}_{k,N}\|^2_{H^{k-\frac{1}{2}-}} + \|v_N\|^{2-\frac{3}{4k}+}_{H^k} \|v_N\|^{\frac{3}{4k}-}_{L^2} \|\widetilde{Y}_{k,N}\|_{H^{k-\frac{1}{2}-}} + \|v_N\|^{2-\frac{1}{2k}}_{H^k} \|v_N\|^{1+\frac{1}{2k}}_{L^2} \\ &\lesssim \varepsilon\|v_N\|^2_{H^k} + C_\varepsilon\big(\|\widetilde{Y}_{k,N}\|^q_{H^{k-\frac{1}{2}-}} + \|v\|^{2\alpha(k)}_{L^2}\big) \end{aligned}$$

for $1 \ll q < \infty$ and $\alpha(k) \in \mathbb{N}$ large enough. For the third contribution, by Cauchy-Schwarz inequality, we get

$$\begin{aligned} &\bigg|\int p(v_N, \widetilde{Y}_{k,N}, \widetilde{Y}_{k,N})\,dx\bigg| \\ &= \bigg|\sum_{0<|n_1|\le N} (in_1)^k \widehat{v}_N(n_1) \sum_{-n_1 = n_{23}} \frac{(in_2)^{k-1}(in_3)^k}{(in_1)^k} \widehat{\widetilde{Y}}_{k,N}(n_2)\widehat{\widetilde{Y}}_{k,N}(n_3)\bigg| \\ &\lesssim \|v_N\|_{H^k} \bigg(\sum_{0<|n|\le N} \bigg|\sum_{n=n_{23}} \frac{(in_2)^{k-1}(in_3)^k}{(in_1)^k} \widehat{\widetilde{Y}}_{k,N}(n_2)\widehat{\widetilde{Y}}_{k,N}(n_3)\bigg|^2\bigg)^{\frac{1}{2}} \\ &\lesssim \varepsilon\|v_N\|^2_{H^k} + C_\varepsilon \sum_{0<|n|\le N} \bigg|\sum_{n=n_{23}} \frac{(in_2)^{k-1}(in_3)^k}{(in)^k} \widehat{\widetilde{Y}}_{k,N}(n_2)\widehat{\widetilde{Y}}_{k,N}(n_3)\bigg|^2. \end{aligned}$$

Moreover, by Isserlis' theorem, since $n \neq 0$, and (4.15), we have

$$\begin{aligned}
&\mathbb{E}\bigg[\sum_{0<|n|\le N}\bigg|\sum_{n=n_{23}}\frac{(in_2)^{k-1}(in_3)^k}{(in)^k}\widehat{\widetilde{Y}}_{k,N}(n_2)\widehat{\widetilde{Y}}_{k,N}(n_3)\bigg|^2\bigg]\\
&\le \sum_{0<|n|\le N}\frac{1}{\langle n\rangle^{2k}}\sum_{\substack{n=n_{23}\\ n=m_{23}\\ 0<|n_j|,|m_j|\le N}}\frac{|n_2m_2|^{k-1}|n_3m_3|^k}{(\widetilde{T}_{\delta,k}(n_2)\widetilde{T}_{\delta,k}(n_3)\widetilde{T}_{\delta,k}(m_2)\widetilde{T}_{\delta,k}(m_3))^{\frac12}}|\mathbb{E}[g_{n_2}g_{n_3}\overline{g_{m_2}g_{m_3}}]|\\
&\lesssim \sum_{0<|n|\le N}\frac{1}{\langle n\rangle^{2k}}\sum_{\substack{n=n_{23}\\ 0<|n_j|\le N}}\frac{|n_2|^{2k-2}|n_3|^{2k}+|n_2n_3|^{2k-1}}{|n_2n_3|^{2k}}\\
&\lesssim \sum_{0<|n|\le N}\frac{1}{\langle n\rangle^{2k}}\sum_{\substack{n=n_{23}\\ 0<|n_j|\le N}}\bigg(\frac{1}{|n_2|^2}+\frac{1}{|n_2n_3|}\bigg)\le C<\infty
\end{aligned}$$

for some constant $C>0$ independent of N and δ. For the last contribution, by integration by parts we get

$$\begin{aligned}
&p(\widetilde{Y}_{k,N}+v_N, v_N, \widetilde{Y}_{k,N})\\
&=\int(\widetilde{Y}_{k,N}+v_N)(\partial_x^{k-1}v_N)(\partial_x^k\widetilde{Y}_{k,N})\,dx\\
&=-\int(\widetilde{Y}_{k,N}+v_N)(\partial_x^{k}v_N)(\partial_x^{k-1}\widetilde{Y}_{k,N})\,dx-\int\partial_x(\widetilde{Y}_{k,N}+v_N)(\partial_x^{k-1}v_N)(\partial_x^{k-1}\widetilde{Y}_{k,N})\,dx
\end{aligned}$$

where we see that the first term is of the form $p(\widetilde{Y}_{k,N}+v_N, \widetilde{Y}_{k,N}, v_N)$ while the second for $k=1$ is of the form $p(\widetilde{Y}_{k,N}, v_N, \widetilde{Y}_{k,N}+v_N)$, both estimated above. For the second contribution when $k\ge 2$, each factor has at most $k-1$ derivatives, so this contribution is handled in Case (ii).

Case (ii): $p(u)$ with $|p(u)|\le k-1$

Fix $3\le j\le 2k+2$ and consider the terms in (4.1) arising from $p(u)\in\widetilde{\mathcal{P}}_j(u)$ of the form $p(u)=\prod_{\ell=1}^j\partial_x^{\alpha_j}u$ where $0\le\alpha_j\le\dots\le\alpha_1\le k-1$ and $\alpha_{1\cdots j}\le 2k+2-j$. Using Young's convolution inequality and Hölder's inequality, we have

$$\bigg|\int p(u)\,dx\bigg|\le\sum_{0=n_{1\cdots j}}\prod_{\ell=1}^j\langle n_\ell\rangle^{\alpha_\ell}|\widehat{u}(n_\ell)|\lesssim\prod_{\ell=1}^j\|\langle n\rangle^{\alpha_\ell}\widehat{u}(n_\ell)\|_{\ell_n^{\frac{j}{j-1}}}\lesssim\prod_{\ell=1}^j\|u\|_{H^{\alpha_\ell+\frac12-\frac1j+}}. \tag{4.17}$$

Then, by (2.1) and Young's inequality, we have that

$$\begin{aligned}
&\bigg|\int p(\widetilde{Y}_{k,N}+v_N)\,dx\bigg|\\
&\lesssim\prod_{\ell=1}^j\|\widetilde{Y}_{k,N}+v_N\|_{H^{\alpha_\ell+\frac12-\frac1j+}}\\
&\lesssim\sum_{\ell=0}^j\|\widetilde{Y}_{k,N}\|^\ell_{H^{k-\frac12-}}\|v_N\|_{H^k}^{\frac1k(\alpha_{1\cdots(j-\ell)}+(j-\ell)(\frac12-\frac1j))+}\|v_N\|_{L^2}^{\frac1k((j-\ell)(k-\frac12+\frac1j)-\alpha_{1\cdots(j-\ell)})-}\\
&\le\varepsilon\|v_N\|^2_{H^k}+C_\varepsilon(1+\|\widetilde{Y}_N\|^q_{H^{k-\frac12-}}+\|v_N\|_{L^2}^{2\alpha(k)}),
\end{aligned}$$

for $1 \ll q < \infty$ and $\alpha(k) \in \mathbb{N}$ sufficiently large. $\square$

Remark 4.8. Note that the all the terms in Case (ii) are estimated in a deterministic manner, and the estimate extends to $\int p(u_1 + u_2)\,dx$ for any $u_1 \in H^{k-\frac{1}{2}-}$ and $u_2 \in H^k$. The random nature of $\widetilde{Y}_{k,N}$ only comes into play in Case (i), more specifically for the terms of the form

$$\delta^{\alpha_{123}} \int \widetilde{\mathcal{G}}_\delta^{\alpha_1}(v_N + \widetilde{Y}_{k,N})(\widetilde{\mathcal{G}}_\delta^{\alpha_2}\partial_x^{k-1}\widetilde{Y}_{k,N})(\widetilde{\mathcal{G}}_\delta^{\alpha_3}\partial_x^{k}\widetilde{Y}_{k,N})\,dx.$$

Lemma 4.9. *Let $k \in \mathbb{N}$, $\delta_0 > 0$, $0 < \delta \le \delta_0$, $0 < \delta \le \delta_0$, and $N \in \mathbb{N}$. Then, for $0 < \varepsilon < 1$, there exists $C_{\delta_0,\varepsilon} > 0$ independent of N and δ, and $\alpha(k) \in \mathbb{N}$ such that*

$$|\mathbb{E}[\widetilde{R}_{\delta,k-\frac{1}{2}}(\widetilde{Y}_{k-\frac{1}{2},N} + v_N)]| \le \mathbb{E}[C_{\delta_0,\varepsilon}(1 + \|\widetilde{Y}_{k-\frac{1}{2},N}\|^q_{H^{k-1-}} + \|v_N\|_{L^2}^{2\alpha(k)}) + \varepsilon\|v_N\|^2_{H^{k-\frac{1}{2}}}],$$

for $1 \ll q < \infty$ sufficiently large.

Proof. We first focus on estimating $\widetilde{R}^{[<]}_{\delta,k-\frac{1}{2}}(u)$ in (4.1), which only appears when $k \ge 2$. For $\ell = 0, \dots, 2k-3$, since all contributions have one extra operator $\widetilde{\mathcal{G}}_\delta$ compared to the number of powers of δ, we use the boundedness of the multiplier of $\delta\widetilde{\mathcal{G}}_\delta$ in (2.7) for ℓ of these operators. The last $\widetilde{\mathcal{G}}_\delta$ operator leads to a loss of derivative, since Lemma 2.3 shows that $|\widehat{\widetilde{\mathcal{G}}_\delta}(n)| \le |n|$ for all $n \in \mathbb{Z}^*$ uniformly in δ. Consequently, it suffices to consider terms coming from $p_1(u) = u\partial_x^{k-1}u\partial_x^{k-1}u$ or from $p_2(u) = \prod_{\ell=1}^{j}\partial_x^{\alpha_\ell}u$ with $j = 3, \dots, 2k$, $0 \le \alpha_j \le \cdots \le \alpha_1 \le k-1$, $(\alpha_1, \alpha_2) \ne (k-1, k-1)$, and $\alpha_{1\cdots j} \le 2k+2-j$. In the above, note the increase in the number of derivatives compared to the polynomials appearing in $\widetilde{R}^{[<]}_{\delta,k-\frac{1}{2}}(u)$, which are a consequence of the derivative loss coming from the extra $\widetilde{\mathcal{G}}_\delta$ operator.

Case (i): $p_1(u) = u\partial_x^{k-1}u\partial_x^{k-1}u$

Using multilinearity, we can write $p_1(u_1, u_2, u_3) = u_1\partial_x^{k-1}u_2\partial_x^{k-1}u_3$ and

$$\begin{aligned} p_1(\widetilde{Y}_{k-\frac{1}{2},N} + v_N) = p_1(\widetilde{Y}_{k-\frac{1}{2},N}, \widetilde{Y}_{k-\frac{1}{2},N}, \widetilde{Y}_{k-\frac{1}{2},N}) + 2p_1(\widetilde{Y}_{k-\frac{1}{2},N} + v_N, \widetilde{Y}_{k-\frac{1}{2},N} + v_N, v_N) \\ + p_1(v_N, \widetilde{Y}_{k-\frac{1}{2},N}, \widetilde{Y}_{k-\frac{1}{2},N}). \end{aligned}$$

From Isserlis' theorem, $\mathbb{E}[p_1(\widetilde{Y}_{k-\frac{1}{2},N}, \widetilde{Y}_{k-\frac{1}{2},N}, \widetilde{Y}_{k-\frac{1}{2},N})] = 0$. For the second contribution, from Young's convolution inequality, Hölder's inequality, (2.1), and Young's inequality, we have that

$$\begin{aligned} &\bigg|\int p_1(u_1, u_2, u_3)\,dx\bigg| \\ &\lesssim \sum_{\substack{0=n_{123}\\ n_\ell\ne 0}} |n_2|^{k-1-\theta}|n_3|^{k-1}|n_{13}|^\theta|\widehat{u}_1(n_1)\widehat{u}_2(n_2)\widehat{u}_3(n_3)| \lesssim \|u_1\|_{H^{\frac{1}{2}+\theta+}}\|u_2\|_{H^{k-1-\theta}}\|u_3\|_{H^{k-1+\theta}}, \end{aligned}$$

$$\begin{aligned} &\bigg|\int p_1(\widetilde{Y}_{k-\frac{1}{2},N} + v_N, \widetilde{Y}_{k-\frac{1}{2},N} + v_N, v_N)\,dx\bigg| \\ &\lesssim \|\widetilde{Y}_{k-\frac{1}{2},N} + v_N\|_{H^{\frac{1}{2}+}}\|\widetilde{Y}_{k-\frac{1}{2},N} + v_N\|_{H^{k-1-}}\|v_N\|_{H^{k-1+}} \\ &\lesssim \|\widetilde{Y}_{k-\frac{1}{2},N}\|^2_{H^{k-1-}}\|v_N\|_{H^{k-\frac{1}{2}}} + (\|\widetilde{Y}_{k-\frac{1}{2},N}\|_{H^{k-1-}} + \|v_N\|_{H^{\frac{1}{2}+}})\|v_N\|^2_{H^{k-1+}} \\ &\lesssim \|\widetilde{Y}_{k-\frac{1}{2},N}\|^2_{H^{k-1-}}\|v_N\|_{H^{k-\frac{1}{2}}} + \|\widetilde{Y}_{k-\frac{1}{2},N}\|_{H^{k-1-}}\|v_N\|_{H^{k-\frac{1}{2}}}^{2-\frac{2}{2k-1}+}\|v_N\|_{L^2}^{\frac{2}{2k-1}-} + \|v_N\|_{H^{k-\frac{1}{2}}}^{2-\frac{1}{2k-1}+}\|v_N\|_{L^2}^{\frac{1}{2k-1}-} \end{aligned}$$

$$\le C_\varepsilon(\|\widetilde{Y}_{k-\frac{1}{2},N}\|_{H^{k-1-}}^q + \|v_N\|_{L^2}^{2\alpha(k)}) + \varepsilon\|v_N\|_{H^{k-\frac{1}{2}}}^2,$$

for $0 < \theta \ll 1$, for some $C_\varepsilon > 0$ independent of N, δ and some $\alpha(k) \in \mathbb{N}$. For the third contribution, by Cauchy-Schwarz inequality, Young's inequality, Isserlis' theorem, and (4.15), we have

$$\begin{aligned}
&\bigg|\int p_1(v_N, \widetilde{Y}_{k-\frac{1}{2},N}, \widetilde{Y}_{k-\frac{1}{2},N})\,dx\bigg| \\
&= \bigg|\sum_{\substack{0=n_{123}\\ 0<|n_\ell|\le N}} \frac{(-n_2n_3)^{k-1}}{|n_1|^{k-\frac{1}{2}}}|n_1|^{k-\frac{1}{2}}\widehat{v}_N(n_1)\widehat{\widetilde{Y}}_{k-\frac{1}{2}}(n_2)\widehat{\widetilde{Y}}_{k-\frac{1}{2}}(n_3)\bigg| \\
&\le \|v_N\|_{H^{k-\frac{1}{2}}}\bigg(\sum_{0<|n|\le N}\frac{1}{|n|^{2k-1}}\bigg|\sum_{n=n_{23}}(-n_2n_3)^{k-1}\widehat{\widetilde{Y}}_{k-\frac{1}{2}}(n_2)\widehat{\widetilde{Y}}_{k-\frac{1}{2}}(n_3)\bigg|^2\bigg)^{\frac{1}{2}} \\
&\le \varepsilon\|v_N\|_{H^{k-\frac{1}{2}}}^2 + C_\varepsilon\sum_{0<|n|\le N}\frac{1}{|n|^{2k-1}}\bigg|\sum_{n=n_{23}}(-n_2n_3)^{k-1}\widehat{\widetilde{Y}}_{k-\frac{1}{2}}(n_2)\widehat{\widetilde{Y}}_{k-\frac{1}{2}}(n_3)\bigg|^2,
\end{aligned}$$

$$\begin{aligned}
&\mathbb{E}\bigg[\sum_{0<|n|\le N}\frac{1}{|n|^{2k-1}}\bigg|\sum_{n=n_{23}}(-n_2n_3)^{k-1}\widehat{\widetilde{Y}}_{k-\frac{1}{2}}(n_2)\widehat{\widetilde{Y}}_{k-\frac{1}{2}}(n_3)\bigg|^2\bigg] \\
&= \sum_{0<|n|\le N}\frac{1}{|n|^{2k-1}}\sum_{\substack{n=n_{23}=m_{23}\\ 0<|n_\ell|,|m_\ell|\le N}}\frac{|n_2n_3m_2m_3|^{k-1}}{(\widetilde{T}_{\delta,\frac{2k-1}{2}}(n_2)\widetilde{T}_{\delta,\frac{2k-1}{2}}(n_3)\widetilde{T}_{\delta,\frac{2k-1}{2}}(m_2)\widetilde{T}_{\delta,\frac{2k-1}{2}}(m_2))^{\frac{1}{2}}}|\mathbb{E}[g_{n_2}g_{n_3}\overline{g_{m_2}g_{m_3}}]| \\
&\lesssim \sum_{0<|n|\le N}\frac{1}{|n|^{2k-1}}\sum_{\substack{n=n_{23}\\ 0<|n_\ell|\le N}}\frac{|n_2n_2|^{2k-2}}{|n_2n_3|^{2k-1}} \\
&\lesssim_{\delta_0} \sum_{0<|n|\le N}\frac{1}{|n|^{2k-1}}\sum_{\substack{n=n_{23}\\ 0<|n|\le N}}\frac{1}{|n_2n_3|} < C_{\delta_0} < \infty
\end{aligned}$$

for some constant $C_{\delta_0} > 0$ only depending on δ_0, recalling that $k \ge 2$.

Case (ii): remaining contributions in $\widetilde{R}^{[>]}_{\delta,k-\frac{1}{2}}(u)$

Let $k \in \mathbb{N}$ and $3 \le j \le 2k+1$. Consider $p(u) \in \widetilde{\mathcal{P}}_j(u)$ with

$$p(u) = \prod_{\ell=1}^{j}\partial_x^{\alpha_\ell}u, \quad 0 \le \alpha_j \le \cdots \le \alpha_1 \le k-1, \quad (\alpha_1,\alpha_2) \ne (k-1,k-1),$$

$$\alpha_{1\cdots j} \le \begin{cases} 2k-2, & j=3, \\ 2k-4, & j=4, \\ 2k+2-j, & j \ge 5. \end{cases}$$

From Young's convolution inequality and Hölder's inequality, for $0 < \theta \ll 1$ such that $\alpha_1 - \theta < k-1$, we have

$$\bigg|\int p(u)\,dx\bigg| \lesssim \|u\|_{H^{\alpha_1-\theta}}\prod_{\ell=2}^{j}\|\langle n\rangle^{\alpha_\ell+\theta}\widehat{u}(n)\|_{\ell_n^{\frac{2j-2}{2j-3}}} \lesssim \|u\|_{H^{\alpha_1-\theta}}\prod_{\ell=2}^{j}\|u\|_{H^{\alpha_\ell+\frac{1}{2}-\frac{1}{2(j-1)}+\theta}}. \tag{4.18}$$

Then, using (2.1), and Young's inequality, we have we obtain

$$\begin{aligned}
&\bigg| \int p(\widetilde{Y}_{k-\frac{1}{2},N} + v_N)\, dx \bigg| \\
&\lesssim \|\widetilde{Y}_{k-\frac{1}{2},N}\|^j_{H^{k-1-}} + \prod_{\kappa=1}^{j} \|\widetilde{Y}_{k-\frac{1}{2},N}\|^{j-\kappa}_{H^{k-1-}} \|v_N\|_{H^{k-1}} \prod_{\ell=2}^{\kappa} \|v_N\|_{H^{\alpha_\ell + \frac{1}{2} - \frac{1}{2(j-1)}+}} \\
&\lesssim \|\widetilde{Y}_{k-\frac{1}{2},N}\|^j_{H^{k-1-}} + \prod_{\kappa=1}^{j} \|\widetilde{Y}_{k-\frac{1}{2},N}\|^{j-\kappa}_{H^{k-1-}} \|v_N\|^{2\frac{\alpha_{1\cdots\kappa} + (\kappa-1)(\frac{1}{2} - \frac{1}{2(j-1)})}{2k-1}}_{H^{k-\frac{1}{2}}} \|v_N\|^{2\frac{2k+1+\alpha_{1\cdots\kappa} + (\kappa-1)(2k - \frac{j}{2(j-1)})}{2k-1}}_{L^2} \\
&\le C_\varepsilon(\|\widetilde{Y}_{k-\frac{1}{2},N}\|^q_{H^{k-1-}} + \|v_N\|^{2\alpha(k)}_{L^2}) + \varepsilon \|v_N\|^2_{H^{k-\frac{1}{2}}},
\end{aligned}$$

for $1 \le q < \infty$ and $\alpha(k) \in \mathbb{N}$ sufficiently large, since

$$\begin{aligned}
2\frac{\alpha_{1\cdots\kappa} + (\kappa-1)(\frac{1}{2} - \frac{1}{2(j-1)})}{2k-1} &\le 2\frac{2k-2+2\frac{1}{4}}{2k-1} = \frac{4k-3}{2k-1} < 2, && \text{for } j = 3, \\
2\frac{\alpha_{1\cdots\kappa} + (\kappa-1)(\frac{1}{2} - \frac{1}{2(j-1)})}{2k-1} &\le 2\frac{2k-4+1}{2k-1} = 2\frac{2k-3}{2k-1} < 2, && \text{for } j = 4, \\
2\frac{\alpha_{1\cdots\kappa} + (\kappa-1)(\frac{1}{2} - \frac{1}{2(j-1)})}{2k-1} &\le 2\frac{2k+2-j+\frac{j-1}{2} - \frac{1}{2}}{2k-1} \\
&\le \frac{4k+2-j}{2k-1} \le \frac{4k-3}{2k-1} < 2, && \text{for } j \ge 5.
\end{aligned}$$

This completes the estimate for $\widetilde{R}^{[<]}_{\delta,k-\frac{1}{2}}(u)$. It remains to estimate the terms in $\widetilde{R}^{[\ge]}_{\delta,k-\frac{1}{2}}(u)$. Note that all the contributions here have at least as many powers of δ as $\widetilde{\mathcal{G}}_\delta$ operators, thus we can use the boundedness of the Fourier multiplier of $\delta\widetilde{\mathcal{G}}_\delta$ in (2.7) and focus on contributions with no $\widetilde{\mathcal{G}}_\delta$ operators. All such contributions were estimated in Case (ii) above, which completes our proof. $\square$

Remark 4.10. As in Lemma 4.7, in Lemma 4.9 most terms in $\widetilde{R}_{\delta,k-\frac{1}{2}}(u)$ can be handled deterministically. The problematic contributions which require an orthogonality argument are of the form

$$\delta^{\alpha_{123}-1} \int (\widetilde{\mathcal{G}}^{\alpha_1}_\delta u_1)(\widetilde{\mathcal{G}}^{\alpha_2}_\delta \partial_x^{k-1} u_2)(\widetilde{\mathcal{G}}^{\alpha_3}_\delta \partial_x^{k-1} u_3)\, dx,$$

in particular, when $(u_1, u_2, u_3) = (v_N, \widetilde{Y}_{k-\frac{1}{2},N}, \widetilde{Y}_{k-\frac{1}{2},N})$ or $(u_1, u_2, u_3) = (\widetilde{Y}_{k-\frac{1}{2},N}, \widetilde{Y}_{k-\frac{1}{2},N}, \widetilde{Y}_{k-\frac{1}{2},N})$.

We can now complete the proof of Proposition 4.4 by showing the upper bound on $\mathcal{M}_{\delta,\frac{k}{2},N}$ defined in (3.20). Fix $1 \le p < \infty$, $N \in \mathbb{N}$, and $\delta_0 > 0$. Then, for $0 < \delta \le \delta_0$ and $0 < \varepsilon < 1$, Lemma 4.7 and Lemma 4.9 guarantee that there exists $C_{\varepsilon,\delta_0} > 0$ independent of N and δ, $1 \ll q < \infty$, and $\alpha(k) \in \mathbb{N}$ such that for any $v \in H^{\frac{k}{2}}(\mathbb{T})$, we have

$$\begin{aligned}
\mathcal{M}_{\delta,\frac{k}{2},N}(v) \le \mathbb{E}\big[&pC_{\varepsilon,\delta_0}(1 + \|\widetilde{Y}_{\frac{k}{2},N}\|^q_{H^{\frac{k-1}{2}-}}) + (p\varepsilon - \tfrac{1}{2}C)\|v_N\|^2_{H^{\frac{k}{2}}} + pC_{\varepsilon,\delta_0}\|v_N\|^{2\alpha(k)}_{L^2} \\
&- A_0\|\widetilde{Y}_{\frac{k}{2},N} + v_N\|^{2\alpha(k)}_{L^2}\big].
\end{aligned}$$

By picking $0 < \varepsilon < 1$ so that $p\varepsilon < \frac{1}{2}C$ and using Lemma 4.6, there exists a constant $C_{\delta_0,p} > 0$ independent of $N \in \mathbb{N}$ and δ such that

$$\mathcal{M}_{\delta,\frac{k}{2},N}(v) \le \mathbb{E}\big[C_{p,\delta_0} + pC_{\varepsilon,\delta_0}\|v_N\|_{L^2}^{2\alpha(k)} - A_0\|\widetilde{Y}_{\frac{k}{2},N} + v_N\|_{L^2}^{2\alpha(k)}\big].$$

Note that by using Cauchy-Schwarz and Young's inequality, we get that for $0 < \eta < 1$, there exists $C > 0$ such that

$$\begin{aligned}
&- \|\widetilde{Y}_{\frac{k}{2},N} + v_N\|_{L^2}^{2\alpha(k)} \\
&= -\bigg(\|\widetilde{Y}_{\frac{k}{2},N}\|_{L^2}^2 + \|v_N\|_{L^2}^2 + 2\int \widetilde{Y}_{\frac{k}{2},N} v_N \, dx\bigg)^{\alpha(k)} \\
&= - \sum_{\substack{\ell_1+\ell_2+\ell_3=\alpha(k)\\ \ell_i \ge 0}} \binom{\alpha(k)}{\ell_1,\ell_2,\ell_3} \|\widetilde{Y}_{\frac{k}{2},N}\|_{L^2}^{2\ell_1} \|v_N\|_{L^2}^{2\ell_2} \bigg(2\int \widetilde{Y}_{\frac{k}{2},N} v_N \, dx\bigg)^{\ell_3} \\
&\le -\|v_N\|_{L^2}^{2\alpha(k)} + \sum_{\substack{\ell_1+\ell_2+\ell_3=\alpha(k)\\ \ell_1,\ell_2\ge 0, \ell_3 \ge 1}} \binom{\alpha(k)}{\ell_1,\ell_2,\ell_3} \|\widetilde{Y}_{\frac{k}{2},N}\|_{L^2}^{2\ell_1} \|v_N\|_{L^2}^{2\ell_2} \bigg|2\int \widetilde{Y}_{\frac{k}{2},N} v_N \, dx\bigg|^{\ell_3} \\
&\le -\|v_N\|_{L^2}^{2\alpha(k)} + \sum_{\substack{\ell_1+\ell_2+\ell_3=\alpha(k)\\ \ell_1,\ell_2\ge 0, \ell_3 \ge 1}} \binom{\alpha(k)}{\ell_1,\ell_2,\ell_3} 2^{\ell_3} \|\widetilde{Y}_{\frac{k}{2},N}\|_{L^2}^{2\ell_1+\ell_3} \|v_N\|_{L^2}^{2\ell_2+\ell_3} \\
&\le -(1-\eta)\|v_N\|_{L^2}^{2\alpha(k)} + C\|\widetilde{Y}_{\frac{k}{2},N}\|_{L^2}^{q},
\end{aligned}$$

for $1 \ll q < \infty$ sufficiently large, since $2\ell_1 + \ell_3 = \ell_2 - \ell_1 + \alpha(k) \le \ell_2 + \alpha(k) \le 2\alpha(k)$. Therefore, by choosing $A_0 > 0$ large enough such that $pC_{\varepsilon,\delta_0} - A_0(1-\eta) \le 0$, we get that

$$\sup_{v \in H^{\frac{k}{2}}} \sup_{N\in\mathbb{N}} \mathcal{M}_{\delta,\frac{k}{2},N}(v) < C_{p,\delta_0} < \infty$$

for some constant $C_{p,\delta_0} > 0$, which can be chosen independently of $0 < \delta \le \delta_0$.

4.4. **Construction of the measures $\widetilde{\rho}_{\delta,\frac{k}{2}}$ for $0 \le \delta < \infty$.** Fix $k \in \mathbb{N}$, $\delta_0 > 0$, $0 < \delta \le \delta_0$, and $K > 0$. Define the limiting density $\widetilde{F}_{\delta,\frac{k}{2},K}(u)$ by

$$\widetilde{F}_{\delta,\frac{k}{2},K}(u) = \eta_K\big(\|u\|_{L^2}^2\big)\exp(-\widetilde{R}_{\delta,\frac{k}{2}}(u)),$$

where $\widetilde{R}_{\delta,\frac{k}{2}}$ denotes the remainder in (4.1).

Proposition 4.11. *Let $k \in \mathbb{N}$, $\delta_0 > 0$, and $0 < \delta \le \delta_0$. Given $1 \le p < \infty$, the sequences $\{\mathbf{P}_N\widetilde{X}_{\delta,\frac{k}{2}}\}_{N\in\mathbb{N}}$ and $\{\widetilde{R}_{\delta,\frac{k}{2}}(\mathbf{P}_N\widetilde{X}_{\delta,\frac{k}{2}})\}_{N\in\mathbb{N}}$ are Cauchy in $L^p(\Omega; H^{\frac{k-1}{2}-}(\mathbb{T}))$ and $L^p(\Omega)$, thus converging to limits denoted by $\widetilde{X}_{\delta,\frac{k}{2}}$ and $\widetilde{R}_{\delta,\frac{k}{2}}(\widetilde{X}_{\delta,\frac{k}{2}})$, respectively. Moreover, given any $1 \le p < \infty$ and $\theta > 0$, we have that*

$$\sup_{N\in\mathbb{N}} \sup_{0<\delta\le\delta_0} \big\|\|\mathbf{P}_N\widetilde{X}_{\delta,\frac{k}{2}}\|_{H_x^{\frac{k-1}{2}-}}\big\|_{L^p(\Omega)} < Cp < \infty, \tag{4.19}$$

$$\sup_{0<\delta\le\delta_0} \big\|\|\mathbf{P}_N\widetilde{X}_{\delta,\frac{k}{2}} - \mathbf{P}_M\widetilde{X}_{\delta,\frac{k}{2}}\|_{H_x^{\frac{k-1}{2}-\theta}}\big\|_{L^p(\Omega)} \le \frac{Cp}{N^\theta} \to 0, \tag{4.20}$$

for any $M \geq N$ tending to ∞. In particular, the rate of convergence is uniform in $0 < \delta \leq \delta_0$. In addition, for $1 \leq p < \infty$, there exists $\theta > 0$ such that

$$\sup_{N\in\mathbb{N}} \sup_{0<\delta\leq\delta_0} \|\widetilde{R}_{\delta,\frac{k}{2}}(\mathbf{P}_N\widetilde{X}_{\delta,\frac{k}{2}})\|_{L^p(\Omega)} < \infty, \tag{4.21}$$

$$\|\widetilde{R}_{\delta,\frac{k}{2}}(\mathbf{P}_M\widetilde{X}_{\delta,\frac{k}{2}}) - \widetilde{R}_{\delta,\frac{k}{2}}(\mathbf{P}_N\widetilde{X}_{\delta,\frac{k}{2}})\|_{L^p(\Omega)} \leq \frac{C_{k,\delta_0}p^{k+1}}{N^\theta}, \tag{4.22}$$

for any $M \geq N \geq 1$ where C_{k,δ_0} independently of δ. Hence, the rate of convergence of $\widetilde{R}_{\delta,\frac{k}{2}}(\mathbf{P}_N u)$ to $\widetilde{R}_{\delta,\frac{k}{2}}(u)$ is uniform in $0 < \delta \leq \delta_0$.

Once we have Proposition 4.11, then the proof of Theorem 1.2(i) follows from the same approach as in Theorem 1.1(i), where instead we use the convergence of the random variables $\mathbf{P}_N\widetilde{X}_{\delta,\frac{k}{2}}$ and $\widetilde{R}_{\delta,\frac{k}{2}}(\mathbf{P}_N\widetilde{X}_{\delta,\frac{k}{2}})$ in Proposition 4.11 and the uniform bounds on the density in Proposition 4.3. Therefore, we omit the details, and proceed to the proof of Proposition 4.11.

Proof of Proposition 4.11. For simplicity, let $X_N = \mathbf{P}_N\widetilde{X}_{\delta,\frac{k}{2}}$. The estimate in (4.19) follows from that in Lemma 4.6, since X_N and $Y_{\delta,\frac{k}{2},N}(1)$ have the same law. For the difference, we proceed as in the proof of Proposition 3.8, using (4.15), and omit details here.

We now prove (4.22), which suffices to conclude that $\{\widetilde{R}_{\delta,\frac{k}{2}}(\mathbf{P}_N X_N)\}_{N\in\mathbb{N}}$ is Cauchy in $L^p(\Omega)$. Let $M \geq N \geq 1$. The estimate in (4.21) follows by analogous arguments.

Case 1: $\widetilde{R}_{\delta,k}$ All the cubic in u terms in $\widetilde{R}_{\delta,k}(u)$ (4.1) are of the form

$$\int p_1(u)\,dx = \delta^{\alpha_{123}+\gamma}\int [(\delta\widetilde{\mathcal{G}}_\delta)^{\alpha_1}\partial_x^{\beta_1}u][(\delta\widetilde{\mathcal{G}}_\delta)^{\alpha_2}\partial_x^{\beta_2}u][(\delta\widetilde{\mathcal{G}}_\delta)^{\alpha_3}\partial_x^{\beta_3}u]\,dx,$$

where $\gamma, \alpha_\ell, \beta_\ell \geq 0$, $k \geq \beta_1 \geq \beta_2 \geq \beta_3$, and $\beta_{123} \leq 2k-1$. Using the Wiener chaos estimate, Lemma 2.7, Isserlis' theorem, and (4.15), we have that

$$\begin{aligned}
&\|p_1(X_N) - p_1(X_M)\|_{L^p(\Omega)}^2 \\
&\leq p^3\|p_1(X_N) - p_1(X_M)\|_{L^2(\Omega)}^2 \\
&\lesssim_{\delta_0} p^3\mathbb{E}\Bigg[\bigg|\sum_{0=n_{123}}(\mathbf{1}_{0<|n_i|<M} - \mathbf{1}_{0<|n_i|<N})(in_1)^{\beta_1}(in_2)^{\beta_2}(in_3)^{\beta_3}\prod_{\ell=1}^3\frac{\delta\widehat{\widetilde{\mathcal{G}}_\delta}(n_\ell)g_{n_\ell}}{T_{\delta,k}(n_\ell)^{\frac12}}\bigg|^2\Bigg] \\
&\lesssim_{\delta_0} p^3\sum_{0=n_{123}}\frac{\mathbf{1}_{B_{N,M}(n_1,n_2,n_3)}}{|n_1n_2n_3|^{2k}}\Big[|n_1|^{2\beta_1}|n_2|^{2\beta_2}|n_3|^{2\beta_3} + |n_1|^{2\beta_1}|n_2n_3|^{\beta_{23}} \\
&\qquad + |n_1n_2|^{\beta_{12}}|n_3|^{2\beta_3} + |n_1|^{\beta_{12}}|n_2|^{\beta_{23}}|n_3|^{\beta_{13}} + |n_1n_3|^{\beta_{13}}|n_2|^{2\beta_2} + |n_1|^{\beta_{13}}|n_2|^{\beta_{12}}|n_3|^{\beta_{23}}\Big]
\end{aligned}$$

since the restriction $|n_\ell|, |m_\ell| > 0$ does not allow for any pairings $n_i+n_j = 0$ or $m_i+m_j = 0$, and $B_{N,M}$ is defined as follows

$$\begin{aligned}
B_{N,M}(n_1,n_2,n_3) = \Big\{(n_1,n_2,n_3) \in (\mathbb{Z}^*)^3 : 0 < |n_{i_1}| \leq M, \frac{N}{2} < |n_{i_2}| \leq M, N < |n_{i_3}| \leq M, \\
\text{for some } \{i_1,i_2,i_3\} = \{1,2,3\}\Big\}.
\end{aligned}$$

Since $\beta_{123} \le 2k-1$ and $2\beta_i \le 2k$ for $i \in \{1,2,3\}$, we have the following bound

$$\|p_1(X_N) - p_1(X_M)\|^2_{L^p(\Omega)} \lesssim_{\delta_0} p^3 \sum_{\substack{0<|n_{\min}|\le M \\ \frac{N}{2}<|n_{\text{med}}|\le M}} \frac{1}{|n_{\min}|^{2k}|n_{\text{med}}|^2} \lesssim p^3 \frac{1}{N}.$$

The remaining terms in (4.1) have $j \in \{4,\dots,2k+2\}$ copies of u and can be written as (omitting the $\delta\widetilde{\mathcal{G}}_\delta$ operators)

$$p_2(u) = \delta^\gamma \int \prod_{\ell=1}^{j} \partial_x^{\beta_\ell} u \, dx$$

for $\gamma, \beta_\ell \ge 0$, $\beta_{1\cdots j} \le 2k+2-j \le 2k-2$, and $\beta_\ell \le k-1$, possibly after doing integration by parts. Since $\delta\widetilde{\mathcal{G}}_\delta$ has a bounded Fourier multiplier due to (2.7), we omit them for simplicity. Proceeding as in (4.17), since $\beta_\ell + \frac{1}{2} - \frac{1}{j} \le k - \frac{1}{2} - \frac{1}{2k+2}$, and using Lemma 4.6, we get

$$\begin{aligned}\|p_2(X_N) - p_2(X_M)\|_{L^p(\Omega)} &\lesssim_{\delta_0} \Big\| \|X_N - X_M\|_{H^{k-\frac{1}{2}-\frac{1}{2k+2}+}} \|X_M\|^{j-1}_{H^{k-\frac{1}{2}-}} \Big\|_{L^p(\Omega)} \\ &\lesssim_{\delta_0} \frac{1}{N^{\frac{1}{2k+2}-}} \|X_M\|^j_{L^{pj}(\Omega) H_x^{k-\frac{1}{2}-}} \\ &\lesssim_{\delta_0,k} p^{k+1} \frac{1}{N^{\frac{1}{2k+2}-}}.\end{aligned}$$

Combining the estimates above, we get that

$$\|\widetilde{R}_{\delta,k}(X_M) - \widetilde{R}_{\delta,k}(X_N)\|_{L^p(\Omega)} \le C_{\delta_0,k} p^{k+1} \frac{1}{N^{\frac{1}{2k+2}-}} \tag{4.23}$$

from which we conclude that the sequence $\{\widetilde{R}_{\delta,2k+1}(X_N)\}_{N\in\mathbb{N}}$ is Cauchy in $L^p(\Omega)$ and therefore has a limit in $L^p(\Omega)$.

Case 2: $\widetilde{R}_{\delta,k-\frac{1}{2}}$ The cubic in u terms in $\widetilde{R}^{[<]}_{\delta,k-\frac{1}{2}}(u)$ in (4.1) are of the form

$$p_3(u) = \delta^{\alpha_{123}-1} \int [(\delta\widetilde{\mathcal{G}}_\delta)^{\alpha_1} \partial_x^{\beta_1} u][(\delta\widetilde{\mathcal{G}}_\delta)^{\alpha_2} \partial_x^{\beta_2} u][(\delta\widetilde{\mathcal{G}}_\delta)^{\alpha_3} \partial_x^{\beta_3} u] \, dx,$$

where $\alpha_\ell, \beta_\ell \ge 0$, $k-1 \ge \beta_1 \ge \beta_2 \ge \beta_3$, and $\beta_{123} \le 2k-3$. Using Lemma 2.7 and Lemma 2.3, we have that $|\widehat{\delta\widetilde{\mathcal{G}}_\delta}(n)| \le 1$ and $|\widehat{\widetilde{\mathcal{G}}_\delta}(n)| \le |n|$, so it suffices to ignore all the $\delta, \widetilde{\mathcal{G}}_\delta$ terms and impose $\beta_{123} \le 2k-2$.

Proceeding as before, using the Wiener chaos estimate (Lemma 2.6), Isserlis' theorem, and (4.15), we obtain

$$\begin{aligned}&\Big\| \int [p_3(X_N) - p_3(X_M)] \, dx \Big\|^2_{L^p(\Omega)} \\ &\lesssim p^3 \mathbb{E}\Bigg[\Bigg| \sum_{0=n_{123}} (\mathbf{1}_{0<|n_i|<M} - \mathbf{1}_{0<|n_i|<N})(in_1)^{\beta_1}(in_2)^{\beta_2}(in_3)^{\beta_3} \prod_{\ell=1}^{3} \frac{\widehat{\delta\widetilde{\mathcal{G}}_\delta}(n_\ell) g_{n_\ell}}{T_{\delta,k-\frac{1}{2}}(n_\ell)^{\frac{1}{2}}} \Bigg|^2\Bigg] \\ &\lesssim_{\delta_0} p^3 \sum_{0=n_{123}} \mathbf{1}_{B_{N,M}(n_1,n_2,n_3)} \frac{1}{|n_1 n_2 n_3|^{2k-1}} \Big[|n_1|^{2\beta_1}|n_2|^{2\beta_2}|n_3|^{2\beta_3} + |n_1|^{2\beta_1}|n_2 n_3|^{\beta_{23}} \\ &\qquad + |n_1 n_2|^{\beta_{12}}|n_3|^{2\beta_3} + |n_1|^{\beta_{12}}|n_2|^{\beta_{23}}|n_3|^{\beta_{13}} + |n_1 n_3|^{\beta_{13}}|n_2|^{2\beta_2} + |n_1|^{\beta_{13}}|n_2|^{\beta_{12}}|n_3|^{\beta_{23}} \Big]\end{aligned}$$

$$
\begin{aligned}
&\lesssim_{\delta_0} p^3 \sum_{0=n_{123}} \mathbf{1}_{B_{N,M}}(n_1,n_2,n_3)\frac{1}{|n_1n_2n_3|}\\
&\lesssim_{\delta_0} p^3 \sum_{\substack{0<|n_{\min}|\le M\\ \frac{N}{2}<|n_{\text{med}}|\le M}} \frac{1}{|n_{\min}|^{1+\theta}|n_{\text{med}}|^{2-\theta}}\\
&\lesssim_{\delta_0} p^3 \frac{1}{N^{1-\theta}}.
\end{aligned}
$$

Note that the same estimate above holds for the cubic terms in $\widetilde{R}^{[\geq]}_{\delta,k-\frac{1}{2}}(u)$.

To estimate the remaining contributions in $\widetilde{R}_{\delta,k-\frac{1}{2}}(u)$, it suffices to control the terms

$$
\begin{aligned}
p_4(u) = \delta^\gamma \int \prod_{\ell=1}^{j} \partial_x^{\alpha_\ell} u\, dx,\\
j = 4,\dots,2k+1, \qquad \gamma\ge 0, \qquad 0\le \alpha_\ell \le k-1, \qquad \alpha_{1\cdots j}\le 2k+2-j,
\end{aligned}
$$

as the same estimate holds for all other terms, using Lemma 2.7 and Lemma 2.3 to handle the $\widetilde{\mathcal{G}}_\delta$ operators. Then, for fixed $j=4,\dots,2k+1$, proceeding as in (4.18) and using Lemma 4.6, we have that

$$
\begin{aligned}
\|p_4(X_N)-p_4(X_M)\|_{L^p(\Omega)} &\lesssim_{\delta_0} \big\|\|X_N-X_M\|_{H^{k-1-2\theta}}\|X_M\|^{j-1}_{H^{k-1-}}\big\|_{L^p(\Omega)} && (4.24)\\
&\lesssim_{\delta_0} \frac{1}{N^\theta}\|\|X_M\|_{H^{k-1-}_x}\|^j_{L^{pj}(\Omega)} && (4.25)\\
&\lesssim_{\delta_0,k} p^{k+\frac{1}{2}}\frac{1}{N^\theta} && (4.26)
\end{aligned}
$$

for $0<\theta\ll 1$ sufficiently small.

Combining this estimate with the earlier one for the cubic contributions, we get that

$$
\|\widetilde{R}_{\delta,k-\frac{1}{2}}(X_M)-\widetilde{R}_{\delta,k-\frac{1}{2}}(X_N)\|_{L^p(\Omega)} \le C_{\delta_0,k}p^{k+\frac{1}{2}}\frac{1}{N^\theta} \tag{4.27}
$$

for some $0<\theta<1$, from which we conclude that the sequence $\{\widetilde{R}_{\delta,k-\frac{1}{2}}(X_N)\}_{N\in\mathbb{N}}$ is Cauchy in $L^p(\Omega)$ and therefore has a limit in $L^p(\Omega)$. □

Remark 4.12. All of the results on $\widetilde{X}_{\delta,k}$, $\widetilde{R}_{\delta,k}$, $F_{\delta,k,K,N}$, and $\widetilde{\rho}_{\delta,k,K}$ can be extended to the KdV setting $\delta=0$. In Proposition B.7, we established that for $k\in\mathbb{N}$, the k-th conserved quantity for KdV satisfies

$$
\widetilde{E}_{0,k}(u) = \widetilde{E}^{[0]}_{\delta,k}(u), \tag{4.28}
$$

where the latter, defined in (B.18), collects the terms in $\widetilde{E}_{\delta,k}(u)$ which have no $\widetilde{\mathcal{G}}_\delta$ operators and no powers of δ. Consequently, we can write

$$
\widetilde{E}_{0,k}(u) = \frac{1}{2}\|u\|^2_{\dot{H}^k} + \widetilde{R}_{0,k}(u)
$$

and, from (4.28), we have that all the terms in $\widetilde{R}_{0,k}(u)$ appear in $\widetilde{R}_{\delta,k}(u)$. Consequently, all the results established in Subsections 4.3 and 4.4 for even-indexed quantities extends to $\delta=0$. We briefly mention the relevant results below.

Given $k\in\mathbb{N}$, $K>0$, and $N\in\mathbb{N}$, we recall the definition of the truncated densities

$$
\widetilde{F}_{0,k,N,K}(u) = \eta_K(\|\mathbf{P}_N u\|^2_{L^2})\exp(-\widetilde{R}_{0,k}(\mathbf{P}_N u)), \tag{4.29}
$$

which are uniformly bounded in N (proceeding as in Subsection 4.3): for any $1 \le p < \infty$,

$$\sup_{N\in\mathbb{N}} \|\widetilde{F}_{0,k,N,K}(\widetilde{X}_{0,k})\|_{L^p(\Omega)} = \sup_{N\in\mathbb{N}} \|\widetilde{F}_{0,k,N,K}(u)\|_{L^p(d\widetilde{\mu}_{0,k})} \le C_{p,k,K} < \infty. \tag{4.30}$$

The variables $\{\mathbf{P}_N \widetilde{X}_{0,k}\}_{N\in\mathbb{N}}$ and $\{\widetilde{R}_{0,k}(\mathbf{P}_N \widetilde{X}_{0,k})\}_{N\in\mathbb{N}}$ are Cauchy in $L^p(\Omega; H^{k-\frac{1}{2}-}(\mathbb{T}))$ and $L^p(\Omega)$, respectively, for any $1 \le p < \infty$, and they converge to limits $\widetilde{X}_{0,k}$ and $\widetilde{R}_{0,k}(\widetilde{X}_{0,k})$. Consequently, we can rigorously construct the weighted Gaussian measures $\widetilde{\rho}_{0,k,K}$ for KdV:
(i) For all $1 \le p < \infty$, we have $\lim_{N\to\infty} \widetilde{F}_{0,k,N,K}(u) = \widetilde{F}_{0,k,K}(u)$ in $L^p(d\widetilde{\mu}_{0,k})$, where

$$\widetilde{F}_{0,k,K}(u) = \eta_K(\|u\|_{L^2}^2)\exp(-\widetilde{R}_{0,k})(u).$$

(ii) The truncated measure $\widetilde{\rho}_{0,k,N,K}$ converges in total variation to $\widetilde{\rho}_{0,k,K}$ given by

$$\widetilde{\rho}_{0,k,K}(du) = Z_{0,k}^{-1}\widetilde{F}_{0,K}(u)d\widetilde{\mu}_{0,k}(u).$$

Moreover, the limiting measure above is equivalent to the base Gaussian measure $\eta_K(\|u\|_{L^2}^2)d\widetilde{\mu}_{0,k}(u)$.

4.5. **Convergence of $\widetilde{\rho}_{\delta,\frac{k}{2}}$ as $\delta \to 0$.** In this section, we show Theorem 1.2(ii). The main ingredient to establish the weak convergence of the measures $\widetilde{\rho}_{\delta,k-\frac{1}{2},K}, \widetilde{\rho}_{\delta,k,K}$ to $\widetilde{\rho}_{0,k,K}$ is the following $L^p(\Omega)$-convergence of the truncated densities.

Lemma 4.13. *Let $k, N \in \mathbb{N}$. Then, for all $1 \le p < \infty$, we have*

$$\lim_{\delta\to 0} \|\widetilde{F}_{\delta,k-\frac{1}{2},K,N}(\widetilde{X}_{\delta,k-\frac{1}{2}}) - \widetilde{F}_{0,k,K,N}(\widetilde{X}_{0,k})\|_{L^p(\Omega)} = 0, \tag{4.31}$$

$$\lim_{\delta\to 0} \|\widetilde{F}_{\delta,k,K,N}(\widetilde{X}_{\delta,k}) - \widetilde{F}_{0,k,K,N}(\widetilde{X}_{0,k})\|_{L^p(\Omega)} = 0. \tag{4.32}$$

Assuming Lemma 4.13, we can now proceed to the proof the theorem.

Proof of Theorem 1.2(ii). Fix $K, \delta_0 > 0$ and $k \in \mathbb{N}$. By construction, the measures $\widetilde{\rho}_{\delta,\frac{k}{2},K}$ are equivalent for all $0 < \delta \le \delta_0$ and they are also equivalent to the base Gaussian measure with cutoff $\widetilde{\mu}_{\delta,\frac{k}{2},K}$. The same is true for $\widetilde{\rho}_{0,k,K}$ and the base Gaussian $\widetilde{\mu}_{0,k,K}$ (see Remark 4.12). From Lemma 4.1, we have that $\widetilde{\mu}_{\delta,k-\frac{1}{2},K}, \widetilde{\mu}_{0,k,K}$ and $\widetilde{\mu}_{\delta,k,K}, \widetilde{\mu}_{0,k,K}$ are singular. Consequently, we conclude that $\widetilde{\rho}_{\delta,k-\frac{1}{2},K}, \widetilde{\rho}_{\mathrm{KdV},2k,K}$ and $\widetilde{\rho}_{\delta,k,K}, \widetilde{\rho}_{\mathrm{KdV},2k,K}$ are also singular for all $0 < \delta \le \delta_0$. It remains to show the weak convergence of $\widetilde{\rho}_{\delta,k-\frac{1}{2},K}$ and $\widetilde{\rho}_{\delta,k,K}$ to $\widetilde{\rho}_{0,k,K}$ as $\delta \to 0$.

In the following, for simplicity, we omit the K dependence and only show the convergence for $\widetilde{\rho}_{\delta,k,K}$, as the same ideas apply to $\widetilde{\rho}_{\delta,k-\frac{1}{2},K}$. Let $\varepsilon > 0$ and A be any Borel subset of $H^{k-\frac{1}{2}-\varepsilon}(\mathbb{T})$ with $\widetilde{\rho}_{0,k}(\partial A) = 0$, where ∂A denotes the boundary of the set A. By the portmanteau lemma, the weak convergence of $\widetilde{\rho}_{\delta,k}$ to $\widetilde{\rho}_{0,k}$ follows once we show that

$$\lim_{\delta\to 0}\big[\widetilde{\rho}_{0,k}(A) - \widetilde{\rho}_{\delta,k}(A)\big] = 0. \tag{4.33}$$

By triangle inequality, we have that

$$\begin{aligned} &\big|\widetilde{\rho}_{0,k}(A) - \widetilde{\rho}_{\delta,k}(A)\big| \\ &\le \big|\widetilde{\rho}_{0,k}(A) - \widetilde{\rho}_{0,k,N}(A)\big| + \big|\widetilde{\rho}_{\delta,k}(A) - \widetilde{\rho}_{\delta,k,N}(A)\big| + \big|\widetilde{\rho}_{0,k}(A) - \widetilde{\rho}_{\delta,k,N}(A)\big| \\ &\le \big|\widetilde{\rho}_{0,k}(A) - \widetilde{\rho}_{0,k,N}(A)\big| + \sup_{0<\delta\le\delta_0}\big|\widetilde{\rho}_{\delta,k}(A) - \widetilde{\rho}_{\delta,k,N}(A)\big| + \big|\widetilde{\rho}_{0,k,N}(A) - \widetilde{\rho}_{\delta,k,N}(A)\big|. \end{aligned} \tag{4.34}$$

From Remark 4.12, we know that $\widetilde{\rho}_{0,k,N}$ converges in total variation to $\widetilde{\rho}_{0,k}$, thus

$$\lim_{N\to\infty}\big|\widetilde{\rho}_{0,k}(A)-\widetilde{\rho}_{0,k,N}(A)\big|=0.$$

Moreover, Theorem 1.2(i) guarantees that

$$\lim_{N\to\infty}\sup_{0<\delta\le\delta_0}\big|\widetilde{\rho}_{\delta,k}(A)-\widetilde{\rho}_{\delta,k,N}(A)\big|=0.$$

Therefore, if we show that for $N\gg 1$,

$$\lim_{\delta\to 0}\big|\widetilde{\rho}_{0,k,N}(A)-\widetilde{\rho}_{\delta,k,N}(A)\big|=0, \tag{4.35}$$

then (4.33) follows from taking a limit as $\delta\to 0$ followed by a limit as $N\to\infty$ in (4.34) and the three convergence results above.

We first note that

$$\begin{aligned}
&\big|\widetilde{\rho}_{0,k,N}(A)-\widetilde{\rho}_{\delta,k,N}(A)\big|\\
&=\Big|Z_{0,k,N}^{-1}\mathbb{E}\big[\widetilde{F}_{0,k,N}(\widetilde{X}_{0,k})\mathbf{1}_A(\widetilde{X}_{0,k})\big]-Z_{\delta,k,N}^{-1}\mathbb{E}\big[\widetilde{F}_{\delta,k,N}(\widetilde{X}_{\delta,k})\mathbf{1}_A(\widetilde{X}_{\delta,k})\big]\Big|\\
&\le |Z_{0,k,N}^{-1}-Z_{\delta,k,N}^{-1}|\mathbb{E}[\widetilde{F}_{0,k,N}(\widetilde{X}_{0,k})\mathbf{1}_A(\widetilde{X}_{0,k})]\\
&\quad+Z_{\delta,k,N}^{-1}\big|\mathbb{E}\big[\widetilde{F}_{0,k,N}(\widetilde{X}_{0,k})\mathbf{1}_A(\widetilde{X}_{0,k})\big]-\mathbb{E}\big[\widetilde{F}_{\delta,k,N}(\widetilde{X}_{\delta,k})\mathbf{1}_A(\widetilde{X}_{\delta,k})\big]\big|
\end{aligned}$$

where $Z_{0,k,N}=\mathbb{E}[\widetilde{F}_{0,k,N}(\widetilde{X}_{0,k})]$ and $Z_{\delta,k,N}=\mathbb{E}[\widetilde{F}_{\delta,k,N}(\widetilde{X}_{\delta,k})]$. The first contribution converges to 0 by (4.32) and boundedness of the density $\widetilde{F}_{0,k,N}$ in (4.30). Since $Z_{\delta,k,N}$ is uniformly bounded in δ, N by (4.9), it remains to show that

$$\lim_{\delta\to 0}\big|\mathbb{E}\big[\widetilde{F}_{0,k,N}(\widetilde{X}_{0,k})\mathbf{1}_A(\widetilde{X}_{0,k})\big]-\mathbb{E}\big[\widetilde{F}_{\delta,k,N}(\widetilde{X}_{\delta,k})\mathbf{1}_A(\widetilde{X}_{\delta,k})\big]\big|=0. \tag{4.36}$$

Again, by triangle inequality,

$$\begin{aligned}
&\big|\mathbb{E}\big[\widetilde{F}_{0,k,N}(\widetilde{X}_{0,k})\mathbf{1}_A(\widetilde{X}_{0,k})\big]-\mathbb{E}\big[\widetilde{F}_{\delta,k,N}(\widetilde{X}_{\delta,k})\mathbf{1}_A(\widetilde{X}_{\delta,k})\big]\big|\\
&\qquad\le\big\|\widetilde{F}_{0,k,N}(\widetilde{X}_{0,k})-\widetilde{F}_{\delta,k,N}(\widetilde{X}_{\delta,k})\big\|_{L^1(\Omega)}+\mathbb{E}\big[\widetilde{F}_{0,k,N}(\widetilde{X}_{0,k})|\mathbf{1}_A(\widetilde{X}_{0,k})-\mathbf{1}_A(\widetilde{X}_{\delta,k})|\big],
\end{aligned}$$

where the first term converges to 0 as $\delta\to 0$ by (4.32). For the second term, from Cauchy-Schwarz inequality, we get

$$\mathbb{E}\big[\widetilde{F}_{0,k,N}(\widetilde{X}_{0,k})|\mathbf{1}_A(\widetilde{X}_{0,k})-\mathbf{1}_A(\widetilde{X}_{\delta,k})|\big]\lesssim\|\widetilde{F}_{0,k,N}(\widetilde{X}_{0,k})\|_{L^2(\Omega)}\mathbb{E}\big[|\mathbf{1}_A(\widetilde{X}_{0,k})-\mathbf{1}_A(\widetilde{X}_{\delta,k})|\big],$$

where the first factor on the RHS is uniformly bounded in N from (4.30), thus it suffices to show that

$$\lim_{\delta\to 0}\mathbb{E}\big[|\mathbf{1}_A(\widetilde{X}_{0,k})-\mathbf{1}_A(\widetilde{X}_{\delta,k})|\big]=0. \tag{4.37}$$

Since $\widetilde{\mu}_{0,k}(\partial A)=0$, we have

$$\begin{aligned}
\mathbb{E}\big[|\mathbf{1}_A(\widetilde{X}_{0,k})-\mathbf{1}_A(\widetilde{X}_{\delta,k})|\big]&=\mathbb{E}\big[\mathbf{1}_{\text{int}A}(\widetilde{X}_{0,k})|\mathbf{1}_A(\widetilde{X}_{0,k})-\mathbf{1}_A(\widetilde{X}_{\delta,k})|\big]\\
&\quad+\mathbb{E}\big[\mathbf{1}_{\text{int}A^c}(\widetilde{X}_{0,k})|\mathbf{1}_A(\widetilde{X}_{0,k})-\mathbf{1}_A(\widetilde{X}_{\delta,k})|\big].
\end{aligned}$$

Note that $\text{int}A, \text{int}A^c$ are open sets, the quantities inside the expected values above are always bounded by 1, and $\lim_{\delta\to 0}\widetilde{X}_{\delta,k}=\widetilde{X}_{0,k}$ a.s. from Lemma 4.1, then $\mathbf{1}_A(\widetilde{X}_{\delta,k})-\mathbf{1}_A(\widetilde{X}_{0,k})\to 0$ a.s. as $\delta\to 0$. Then, by the dominated convergence theorem, (4.37) follows, completing the proof of weak convergence. □

It only remains to prove the convergence in δ for the truncated densities in Lemma 4.13.

Proof of Lemma 4.13. For simplicity, we omit the K dependence in the proof. Fix $N \in \mathbb{N}$ and let $\widetilde{X}_{\delta,\frac{k}{2},N} = \mathbf{P}_N \widetilde{X}_{\delta,\frac{k}{2}}$ for $0 \le \delta < \infty$. For fixed $x \in \mathbb{T}$ and $\omega \in \Omega$, from (4.4), we get

$$\lim_{\delta\to 0} \widetilde{X}_{\delta,k-\frac{1}{2},N}(x;\omega) = \lim_{\delta\to 0} \widetilde{X}_{\delta,k,N}(x;\omega) = \widetilde{X}_{0,k}(x;\omega).$$

Also, from (4.15), we have for $\omega \in \Omega$ fixed that for $0 < \delta \le \delta_0$

$$\|\widetilde{X}_{\delta,k-\frac{1}{2},N}(x;\omega)\|^2_{H^k}, \|\widetilde{X}_{\delta,k,N}(x;\omega)\|^2_{H^k} \lesssim_{\delta_0} \sum_{0<|n|\le N} |g_n(\omega)|^2 \le C_{\omega,\delta_0,N} < \infty, \tag{4.38}$$

where $C_{\omega,\delta_0,N}$ depends on ω, δ_0, N. For fixed $\omega \in \Omega$, from (4.38) and the proof of Proposition B.7, we have the following convergence

$$\lim_{\delta\to 0} \widetilde{R}_{\delta,k-\frac{1}{2},N}(\widetilde{X}_{\delta,k-\frac{1}{2}}(\omega)) = \lim_{\delta\to 0} \widetilde{R}_{\delta,k,N}(\widetilde{X}_{\delta,k}(\omega)) = \widetilde{R}_{0,k,N}(\widetilde{X}_{0,k}(\omega)). \tag{4.39}$$

Fixing $\omega \in \Omega$ and using (4.4), we have that for $m = 2k-1$ and $m = 2k$

$$\big|\|\widetilde{X}_{\delta,\frac{m}{2},N}(\omega)\|^2_{L^2} - \|\widetilde{X}_{0,k,N}(\omega)\|^2_{L^2}\big| = \frac{1}{2\pi} \sum_{0<|n|\le N} |g_n(\omega)|^2 \bigg| \frac{1}{\widetilde{T}_{\delta,\frac{m}{2}}(n)} - \frac{1}{|n|^{2k}} \bigg| \to 0,$$

as $\delta \to 0$, we conclude that $\lim_{\delta\to 0} \|\widetilde{X}_{\delta,,\frac{m}{2},N}(\omega)\|_{L^2} \to \|\widetilde{X}_{0,k,N}(\omega)\|_{L^2}$. Moreover,

$$\lim_{\delta\to 0} \eta_K(\|\widetilde{X}_{\delta,k-\frac{1}{2},N}(\omega)\|^2_{L^2}) = \lim_{\delta\to 0} \eta_K(\|\widetilde{X}_{\delta,k,N}(\omega)\|^2_{L^2}) = \eta_K(\|\widetilde{X}_{0,k,N}(\omega)\|^2_{L^2}), \tag{4.40}$$

by continuity of the cutoff function η_K. Lastly, combining (4.39) and (4.40), we conclude that for all $\omega \in \Omega$,

$$\lim_{\delta\to 0} \widetilde{F}_{\delta,k-\frac{1}{2},N}(\widetilde{X}_{\delta,k-\frac{1}{2}}(\omega)) = \lim_{\delta\to 0} \widetilde{F}_{\delta,k,N}(\widetilde{X}_{\delta,k}(\omega)) = \widetilde{F}_{0,k,N}(\widetilde{X}_{0,k}(\omega)).$$

From the pointwise in $\omega \in \Omega$ convergence above, the uniform in $0 < \delta \le \delta_0$ bounds in (4.9) and (4.30), and the dominated convergence theorem, we get the intended $L^p(\Omega)$ convergence of the truncated densities (4.31) and (4.32). $\square$

5. Almost almost-sure conservation for truncated dynamics

In this section we prove the main ingredient needed for the proof of invariance of the measures $\rho_{\delta,\frac{k}{2}}$ and $\widetilde{\rho}_{\delta,\frac{k}{2}}$ for $k \ge 3$. To this end, we consider suitably truncated dynamics which approximate (1.1) and (1.4) for which it is easier to establish invariance. However, as in the analysis for Benjamin-Ono in [54, 55, 56], these dynamics no longer have $E_{\delta,\frac{k}{2}}$ and $\widetilde{E}_{\delta,\frac{k}{2}}$ as conserved quantities for $k \in \mathbb{N}$. We discuss this difficulty below.

In the deep-water regime, we fix $2 \le \delta \le \infty$ for simplicity and let $N \in \mathbb{N}$. Consider the following truncated ILW equation:

$$\begin{cases} \partial_t u_N - \mathcal{G}_\delta \partial_x^2 u_N = 2\mathbf{P}_N\big((\mathbf{P}_N u_N)\partial_x(\mathbf{P}_N u_N)\big), \\ u_N(0) = u_0. \end{cases} \tag{5.1}$$

For $u_0 \in L^2(\mathbb{T})$, one can easily see that (5.1) has a unique global solution $u_N \in C(\mathbb{R}; L^2(\mathbb{T}))$. Recalling that $\mathbf{P}_{>N}$ denotes the projection onto Fourier modes $\{|n| > N\}$, we can write $u_N = \mathbf{P}_N u_N + \mathbf{P}_{>N} u_N$. Then, one can see that the dynamics of (5.1) decouple, where the high frequency piece $\mathbf{P}_{>N} u_N$ solves the linear ILW equation with initial data $P_{>N} u_0$, while the low frequency part $\mathbf{P}_N u_N$ solves a finite-dimensional system of ODEs which preserves the L^2-norm and the Hamiltonian $E_{\delta,\frac{1}{2}}$:

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int (\mathbf{P}_N u_N(t))^2\,dx &= \int \big[2(\mathbf{P}_N u_N)\mathbf{P}_N\big(\mathbf{P}_N u_N \partial_x \mathbf{P}_N u_N\big) + (\mathbf{P}_N u_N)\mathcal{G}_\delta \partial_x^2 u_N\big]\,dx \\
&= \int \big[\tfrac{2}{3}\partial_x(\mathbf{P}_N u_N)^3 + (\mathbf{P}_N u_N)\mathcal{G}_\delta\partial_x^2(\mathbf{P}_N u_N)\big]\,dx = 0, \\
\frac{d}{dt}E_{\delta,\frac{1}{2}}(\mathbf{P}_N u_N) &= \int (\partial_t \mathbf{P}_N u_N)[\mathcal{G}_\delta\partial_x \mathbf{P}_N u_N + (\mathbf{P}_N u_N)^2]\,dx \\
&= \int \big[(\mathcal{G}_\delta\partial_x\mathbf{P}_N u_N)(\mathcal{G}_\delta\partial_x^2\mathbf{P}_N u_N) \\
&\qquad + (\mathbf{P}_N u_N)^2\mathcal{G}_\delta\partial_x^2\mathbf{P}_N u_N + \partial_x(\mathbf{P}_N u_N)^2\mathcal{G}_\delta\partial_x\mathbf{P}_N u_N \\
&\qquad + \mathbf{P}_N(\mathbf{P}_N u_N)^2\mathbf{P}_N\partial_x(\mathbf{P}_N u_N)^2\big]\,dx = 0.
\end{aligned}$$

Unfortunately, the higher order conserved quantities $E_{\delta,\frac{k}{2}}(\mathbf{P}_N u_N)$ for $k \geq 2$ are no longer conserved. This is analogous to the phenomenon observed for the BO and KdV equations in [54, 55, 56, **?**]. As a replacement for conservation, the main ingredient to show invariance of $\rho_{\delta,\frac{k}{2}}$ is the following almost almost-sure conservation for the truncated dynamics.

Proposition 5.1. *Let $2 \leq \delta \leq \infty$, $k \geq 2$, $N \in \mathbb{N}$, and let Φ_t^N denote the data-to-solution map of the truncated ILW* (5.1)*. Then, for all $1 \leq q < \infty$, we have that*

$$\lim_{N\to\infty} \Big\| \frac{d}{dt}E_{\delta,\frac{k}{2}}(\mathbf{P}_N\Phi_t^N(u_0))\Big|_{t=0}\Big\|_{L^q(\mu_{\delta,\frac{k}{2}})} = 0.$$

We postpone the proof of Proposition 5.1 to Subsection 5.1.

A similar phenomenon is observed for the scaled ILW equation (1.4) and its corresponding truncated dynamics:

$$\begin{cases} \partial_t \widetilde{u}_N - \widetilde{\mathcal{G}}_\delta\partial_x^2\widetilde{u}_N = 2\mathbf{P}_N\big((\mathbf{P}_N\widetilde{u}_N)\partial_x(\mathbf{P}_N\widetilde{u}_N)\big), \\ \widetilde{u}_N(0) = u_0. \end{cases} \tag{5.2}$$

As before, (5.2) has a global-in-time solution $\widetilde{u}_N$ and its dynamics decouple, with the low frequency part $\mathbf{P}_N\widetilde{u}_N$ satisfying conservation of the L^2-norm and $\widetilde{E}_{\delta,\frac{1}{2}}$, but $\widetilde{E}_{\delta,\frac{k}{2}}$ is not conserved for $k \geq 2$. As in the deep-water regime, we instead prove the following result.

Proposition 5.2. *Let $\delta_0 > 0$, $0 < \delta \leq \delta_0$, $k \geq 2$, $N \in \mathbb{N}$, and $\widetilde{\Phi}_t^N$ denote the data-to-solution map of the truncated sILW* (5.2)*. Then, for all $1 \leq q < \infty$, we have*

$$\lim_{N\to\infty} \Big\| \frac{d}{dt}\widetilde{E}_{\delta,\frac{k}{2}}(\mathbf{P}_N\widetilde{\Phi}_t^N(u_0))\Big|_{t=0}\Big\|_{L^q(d\widetilde{\mu}_{\delta,\frac{k}{2}})} = 0. \tag{5.3}$$

We postpone the proof of Proposition 5.2 to Subsection **??**

Before proceeding, we recall some relevant results and notations. Following [54], we introduce some useful notations. Consider any polynomial $p(u) \in \mathcal{P}_j(u)$ for some integer $j \geq 2$, with $\mathcal{P}_j(u)$ as defined in Section 3. Recall that $p(u)$ has an associated fundamental polynomial $\widetilde{p}(u)$ of the form

$$\widetilde{p}(u) = \prod_{\ell=1}^{j} \partial_x^{\gamma_\ell} u$$

for suitable $\gamma_\ell \in \mathbb{N} \cup \{0\}$. For $N \in \mathbb{N}$ and $\ell \in \{1, \dots, j\}$, we now define the new polynomials $p^*_{\ell,N}(u)$ as follows

$$p^*_{\ell,N}(u) = p(u)|_{\partial_x^{\gamma_\ell} u = \partial_x^{\gamma_\ell}(\mathbf{P}_{>N}(u\partial_x u))}, \tag{5.4}$$

i.e., the polynomial obtained from $p(u)$ by replacing the $\partial_x^{\gamma_\ell} u$ term by $\partial_x^{\gamma_\ell}(\mathbf{P}_{>N}(u\partial_x u))$. Moreover, we define $p^*_N(u)$ as

$$p^*_N(u) = \sum_{\ell=1}^{j} p^*_{\ell,N}(u). \tag{5.5}$$

Note that the above transformation increases the degree of the original polynomial in u by 1 as well as its total number of derivatives. This definition will become useful in writing $\frac{d}{dt} E_{\delta, \frac{k}{2}}(\mathbf{P}_N u)$. The same definition is extended to polynomials $p(u) \in \widetilde{\mathcal{P}}_j(u)$ used to describe the shallow-water conserved quantities.

We also recall the following estimate on truncated sums; see Lemma 3.2 in [56] for the proof.

Lemma 5.3. *Let $j \geq 2$. Then, the following estimate holds as $N \to \infty$*

$$\sum_{\substack{|n_1+\cdots+n_j|>N \\ 0<|n_1|,\dots,|n_j|\leq N}} \frac{1}{|n_1||n_2\cdots n_j|^2} = O\Big(\frac{\ln N}{N}\Big). \tag{5.6}$$

Lastly, we will require the following property of Gaussian random variables. For all $1 \leq q < \infty$ and $\ell \in \mathbb{N}$, it follows from the Wiener chaos (Lemma 2.6) estimate and Isserlis's theorem that there exists $C = C(k, q) > 0$ such that

$$\sup_{n_1,\dots,n_\ell \in \mathbb{Z}^*} \|g_{n_1} \cdots g_{n_\ell}\|_{L^q(\Omega)} \leq C_{k,q} < \infty. \tag{5.7}$$

5.1. **Proof of Proposition 5.1.** Fix $k \geq 2$, $N \in \mathbb{N}$, and let u_N denote the solution to (5.1) with data $u_0 \in L^2(\mathbb{T})$. Then, the low frequency part $\mathbf{P}_N u_N$ solves the following equation

$$\partial_t \mathbf{P}_N u_N - \mathcal{G}_\delta \partial_x^2 \mathbf{P}_N u_N = (\mathbf{P}_N u_N)\partial_x(\mathbf{P}_N u_N) - \mathbf{P}_{>N}\big((\mathbf{P}_N u_N)\partial_x(\mathbf{P}_N u_N)\big). \tag{5.8}$$

Consequently, when calculating the time derivative of $E_{\frac{k}{2}}(\mathbf{P}_N u_N)$, we can replace the $\partial_t(\mathbf{P}_N u_N)$ terms with

$$\Big[\mathcal{G}_\delta \partial_x^2 \mathbf{P}_N u_N + (\mathbf{P}_N u_N)\partial_x(\mathbf{P}_N u_N)\Big] - \mathbf{P}_{>N}\big((\mathbf{P}_N u_N)\partial_x(\mathbf{P}_N u_N)\big). \tag{5.9}$$

Since the terms obtained by replacing the first contribution above are those we would expect from a solution to the full ILW equation (1.1), these terms collectively vanish. Therefore, the potentially nonzero contributions arise from the $-\mathbf{P}_{>N}\big((\mathbf{P}_N u)\partial_x(\mathbf{P}_N u)\big)$ substitutions. Moreover, note that all the leftover terms are cubic or higher order in u_N, since the quadratic terms vanish. In more detail, note that the high frequency remainder obtained by replacing $\partial_t \mathbf{P}_N u_N$ with the second contribution in (5.9) is

$$\sum_{\substack{\ell=0\\ \ell \text{ even}}}^{k} a_\ell \int \mathbf{P}_N(\partial_x^{\frac{\ell}{2}} u_N) \cdot \mathbf{P}_{>N}\big[\partial_x^{\frac{\ell}{2}}(\mathcal{G}_\delta \partial_x)^{k-\ell}(\mathbf{P}_N u_N \partial_x \mathbf{P}_N u_N)\big]\, dx = 0$$

as all the integrals vanish due to the orthogonality of the frequency supports of the two terms being integrated.

Recalling the structure of the conserved quantities in (3.1) and (3.2) and following the observation above, we get

$$\frac{d}{dt} E_{\delta,\frac{k}{2}}(\mathbf{P}_N u_N) = \sum_{\ell=0}^{k-1} \frac{1}{\delta^\ell} A^*_{\frac{k}{2},\frac{k-\ell}{2},N}(\mathbf{P}_N u_N(t)),$$

where $A^*_{\frac{k}{2},\frac{k}{2},N}(\cdot)$ is defined by replacing each polynomial $p(\cdot)$ in (3.3) and (3.4) by $p^*_N(\cdot)$. In the following, we detail how to establish that for $2 \le \delta \le \infty$, $k \ge 2$, and $1 \le q < \infty$,

$$\lim_{N\to\infty} \Big\| A^*_{\frac{k}{2},\frac{k}{2},N}(\mathbf{P}_N u) \Big\|_{L^q(d\mu_{\delta,\frac{k}{2}})} = 0, \tag{5.10}$$

$$\lim_{N\to\infty} \Big\| A^*_{\frac{k}{2},\frac{1}{2},N}(\mathbf{P}_N u) \Big\|_{L^q(d\mu_{\delta,\frac{k}{2}})} = 0. \tag{5.11}$$

The convergence of the contributions $A^*_{\frac{k}{2},\frac{k-\ell}{2},N}(\mathbf{P}_N \cdot)$ for $1 \le \ell \le k-2$ follow the same approach, thus we omit the details.

Before proving (5.10) and (5.11), we recall a useful result in [56] on the leading order cubic terms in $A_{\frac{k}{2},\frac{2m+1}{2}}$. See Lemma 7.1 in [56] for a proof.

Lemma 5.4. [56, Lemma 7.1] *Let $m \in \mathbb{N}$, $N \in \mathbb{N}$, and $u_N = \mathbf{P}_N u$. Then, for $\alpha = 0, 1$*

$$\begin{aligned}
&\int [u_N][\mathcal{H}^\alpha \partial_x^m u_N] \mathbf{P}_{>N} \mathcal{H}^\alpha \partial_x^m [u_N \partial_x u_N]\, dx \\
&\qquad\qquad = \sum_{j=1}^{m} c_j \int [u_N][\mathcal{H}^\alpha \partial_x^m u_N] \mathbf{P}_{>N} \mathcal{H}^\alpha [\partial_x^j u_N \cdot \partial_x^{m+1-j} u_N]\, dx, \\
&\int \mathbf{P}_{>N}[\mathcal{H} u_N \cdot \mathcal{H}\partial_x^m u_N] \mathbf{P}_{>N} \partial_x^m [u_N \partial_x u_N]\, dx \\
&\qquad\qquad + \int \mathbf{P}_{>N}[\mathcal{H} u_N \cdot \partial_x^m u_N] \mathbf{P}_{>N} \partial_x^m \mathcal{H}[u_N \partial_x u_N]\, dx = 0.
\end{aligned}$$

Proof of (5.11). Since the only contribution in $A_{\frac{k}{2},\frac{1}{2}}(u)$ arises from the polynomial $p(u) = u^3$, using Wiener chaos estimate (Lemma 2.6), it suffices to estimate

$$\| A^*_{\frac{k}{2},\frac{1}{2},N}(\mathbf{P}_N u) \|_{L^2(d\mu_{\delta,\frac{k}{2}})} \sim \bigg\| \int (\mathbf{P}_N u)^2 \mathbf{P}_{>N}(\mathbf{P}_N u \partial_x \mathbf{P}_N u)\, dx \bigg\|_{L^2(d\mu_{\delta,\frac{k}{2}})}.$$

From (3.7), we get

$$\cdots \lesssim \sum_{\substack{n_{1\cdots4}=0\\ 0<|n_j|\le N\\ |n_{34}|>N}} \sum_{\substack{m_{1\cdots4}=0\\ 0<|m_j|\le N\\ |m_{34}|>N}} \frac{|n_4 m_4|}{|n_1 m_1 \cdots n_4 m_4|^{\frac{k}{2}}} |\mathbb{E}[g_{n_1}\cdots g_{n_4} \overline{g_{m_1}\cdots g_{m_4}}]|. \tag{5.12}$$

From Isserlis's theorem, the only possibly nonzero contributions arise from having $n_{i_1} + n_{j_1} = m_{i_2} + m_{j_2} = 0$ for pairs $i_1, i_2, j_1, j_2 \in \{1, \dots, 4\}$ or $\{n_1, \dots, n_4\} = \{m_1, \dots, m_4\}$. Since $|n_{34}| = |n_{12}| > N$, we can only have $n_{i_1} + n_{j_1} = 0$ for $i_1 \in \{1, 2\}, j_1 \in \{3, 4\}$. The same is true for the other family of frequencies. Then, by symmetry and using (5.6), we have

$$
\begin{aligned}
\cdots &\lesssim \sum_{\substack{0<|n_1|,|n_2|\le N\\ 0<|m_1|,|m_2|\le N\\ |n_{12}|,|m_{12}|>N}} \frac{|n_1 m_1|}{|n_1 n_2 m_1 m_2|^k} + \sum_{\substack{0=n_{1\dots4}\\ 0<|n_\ell|\le N\\ |n_{12}|>N}} \frac{|n_4|(|n_1| + |n_3| + |n_4|)}{|n_1 \cdots n_4|^k} \\
&\lesssim \sum_{\substack{0<|n_1|,|n_2|\le N\\ 0<|m_1|,|m_2|\le N\\ |n_{12}|,|m_{12}|>N}} \frac{1}{|n_1 m_1||n_2 m_2|^2} + \sum_{\substack{0<|n_1|,|n_2|,|n_3|\le N\\ |n_{12}|>N}} \left[\frac{1}{|n_1 n_2|^2|n_3|} + \frac{1}{|n_1||n_2 n_3|^2}\right] \\
&\lesssim \mathcal{O}(N^{-1}\log N),
\end{aligned}
$$

for all $k \geq 2$. □

To prove (5.10), we first focus on the leading order cubic terms appearing in (3.5). If k is even, say $k = 2m$ for some $m \in \mathbb{N}$, then the leading order cubic terms are of the form $\int p(u)\, dx$ where

$$
p(u) = u(\mathcal{H}\partial_x^{m-1}u)(\partial_x^m u).
$$

While if k is odd, say $k = 2m+1$ for $m \in \mathbb{N}$ then the relevant contributions are of the form $\int p_j(u)\, dx$, $j = 1, 2, 3$, where

$$
\begin{aligned}
p_1(u) &= \int u[\partial_x^m u]^2\, dx, \\
p_2(u) &= \int u[\mathcal{H}\partial_x^m u]^2\, dx, \\
p_3(u) &= \int [\mathcal{H}u][\mathcal{H}\partial_x^m u][\partial_x^m u]\, dx.
\end{aligned}
\tag{5.13}
$$

Lemma 5.5 establishes the decay of the contributions in $A^*_{\frac{2m}{2},\frac{2m}{2},N}$ and $A^*_{\frac{2m+1}{2},\frac{2m+1}{2},N}$ in (5.10) arising from cubic in u terms in the remainder. The remaining terms are handled in Lemma 5.6.

Lemma 5.5. *Let $m \in \mathbb{N}$, $p(u) = u(\mathcal{H}\partial_x^{m-1}u)(\partial_x^m u)$ and $p_j(u)$, $j = 1, 2, 3$ as given in (5.13). Then, for $1 \leq q < \infty$, we have that for $j = 1, 2, 3$*

$$
\lim_{N\to\infty} \left(\left\| \int p_N^*(\mathbf{P}_N u)\, dx \right\|_{L^q(d\mu_{\delta,m})} + \left\| \int p_{j,N}^*(\mathbf{P}_N u)\, dx \right\|_{L^q(d\mu_{\delta,m+\frac{1}{2}})} \right) = 0.
$$

Proof. Let $m \in \mathbb{N}$ and $u_N = \mathbf{P}_N u$.

We first prove the result for $p(u)$. For $m = 1$, $p(u) = u^2 \mathcal{H}\partial_x u$ and we see that the intended contribution vanishes, using the fact that $\mathcal{H}u = -i(u^+ - u^-)$ where $u^\pm$ is the projection of u onto positive/negative frequencies,

$$
\int p_N^*(u_N)\, dx
$$

$$
\begin{aligned}
&= 2\int \mathbf{P}_{>N}[u_N\partial_x u_N]\mathbf{P}_{>N}[u_N\mathcal{H}\partial_x u_N]\,dx + \int \mathbf{P}_{>N}[\mathcal{H}\partial_x(u\partial_x u)]\mathbf{P}_{>N}[u^2]\,dx\\
&= -2i\int \mathbf{P}_{>N}[u_N\partial_x u_N]\mathbf{P}_{>N}[u_N^+\partial_x u_N^+ - u_N^-\partial_x u_N^-]\,dx - 2\int \mathbf{P}_{>N}\mathcal{H}[u\partial_x u]\mathbf{P}_{>N}[u\partial_x u]\,dx\\
&= -2i\int \mathbf{P}_{>N}[u_N^-\partial_x u_N^-]\mathbf{P}_{>N}[u_N^+\partial_x u_N^+]\,dx + 2i\int \mathbf{P}_{>N}[u_N^+\partial_x u_N^+]\mathbf{P}_{>N}[u_N^-\partial_x u_N^-]\,dx\\
&= 0
\end{aligned}
$$

where we used the fact that $\mathbf{P}_{>N}(\mathbf{P}_N f^+\cdot\mathbf{P}_N g^-) = 0$.

Now let $m\geq 2$. From the definition (5.4), we see that

$$
\begin{aligned}
\int p^*_{1,N}(u)\,dx &= \int \big[\mathbf{P}_{>N}(u\partial_x u)\big]\big[\mathcal{H}\partial_x^{m-1}u\big]\big[\partial_x^m u\big]\,dx + \int u\big[\mathbf{P}_{>N}\partial_x^{m-1}\mathcal{H}(u\partial_x u)\big]\big[\partial_x^m u\big]\,dx\\
&\quad + \int u\big[\mathcal{H}\partial_x^{m-1}u\big]\big[\mathbf{P}_{>N}\partial_x^m(u\partial_x u)\big]\,dx\\
&= \int [\mathcal{H}\partial_x^{m-1}u\partial_x^m u]\mathbf{P}_{>N}[u\partial_x u]\,dx\\
&\quad + \sum_{j=1}^{m-1}\int [c_{1,j}\mathcal{H}(u\partial_x^m u) + c_{2,j}\partial_x u\mathcal{H}\partial_x^{m-1}u + c_{3,j}u\mathcal{H}\partial_x^m u]\mathbf{P}_{>N}[\partial_x^j u\partial_x^{m-j}u]\,dx\\
&\quad + \int [c_{1,0}\mathcal{H}(u\partial_x^m u) + c_{2,0}\partial_x u\mathcal{H}\partial_x^m u + c_{3,0}u\mathcal{H}\partial_x^m u]\mathbf{P}_{>N}[u\partial_x^m u]\,dx\\
&=: \mathrm{I}_N(u) + \mathrm{II}_N(u) + \mathrm{III}_N(u),
\end{aligned}
$$

for some constants $c_{i,j}\in\mathbb{R}$. For the first contribution, using Minkowski's inequality, (5.7), (3.7), and (5.6), we get that

$$
\begin{aligned}
\|\mathrm{I}_N(\mathbf{P}_N u)\|_{L^q(d\mu_{\delta,m})} &\lesssim \sum_{\substack{0<|n_j|\leq N\\ n_{1\cdots 4}=0\\ |n_{12}|>N}} \frac{|n_2||n_3|^{m-1}|n_4|^m}{\prod_{j=1}^4|n_j|^m}\|g_{n_1}(\omega)g_{n_2}(\omega)g_{n_3}(\omega)g_{n_4}(\omega)\|_{L^q(\Omega)}\\
&\lesssim \sum_{\substack{0<|n_j|\leq N\\ n_{1\cdots 4}=0\\ |n_{12}|>N}} \frac{1}{|n_1|^m|n_2|^{m-1}|n_3|}\\
&\leq \sum_{0<|n_3|\leq N}\frac{1}{|n_3|}\sum_{\substack{0<|n_1|,|n_2|\leq N\\ |n_{12}|>N}}\frac{1}{|n_1|^2|n_2|} = \mathcal{O}\Big(\frac{\log^2 N}{N}\Big).
\end{aligned}
$$

For the second contribution, by Minkowski's inequality, (5.7), and (3.7), we get that

$$
\begin{aligned}
\|\mathrm{II}_N(\mathbf{P}_N u)\|_{L^q(d\mu_{\delta,m})} &\lesssim \sum_{j=1}^{m-1}\sum_{\substack{0<|n_j|\leq N\\ n_{1\cdots 4}=0\\ |n_{12}|>N}} \frac{|n_2|^{m-1}(|n_1|+|n_2|)|n_3|^j|n_4|^{m-j}}{|n_1\cdots n_4|^m}\\
&\lesssim \sum_{j=1}^{m-1}\sum_{\substack{0<|n_j|\leq N\\ n_{1\cdots 4}=0\\ |n_{12}|>N}} \frac{1}{\min(|n_1|,|n_2|)^m|n_3|^{m-j}|n_4|^j}.
\end{aligned}
$$

For simplicity, assume that $|n_3| \leq |n_4|$. If $|n_1| \sim |n_2|$, using the fact that $|n_{34}| > N$ implies that $|n_4| \geq \frac{N}{2}$, then

$$\cdots \lesssim N^{-1} \sum_{0<|n_1|,|n_2|,|n_3|\leq N} \frac{1}{|n_1 n_2 n_3|} = \mathcal{O}(N^{-1}\log^3 N).$$

If instead $\max(|n_1|,|n_2|) \gg \min(|n_1|,|n_2|)$, then from $n_{1\cdots 4} = 0$ we have that $\max(|n_1|,|n_2|) \sim |n_{12}| \lesssim |n_4|$ and by using (5.6) we get

$$\cdots \lesssim \sum_{\substack{0<|n_1|,|n_2|,|n_3|\leq N \\ |n_{12}|>N}} \frac{1}{\min(|n_1|,|n_2|)^2 \max(|n_1|,|n_2|)|n_3|} = \mathcal{O}(N^{-1}\log^2 N).$$

For $\mathrm{III}_N(\mathbf{P}_N u)$, note that the first contribution vanishes since $\mathcal{H}$ is anti-self-adjoint, and the third one vanishes since:

$$\begin{aligned} &\int \mathbf{P}_{>N}[u_N \mathcal{H}\partial_x^m u_N]\mathbf{P}_{>N}[u_N\partial_x^m u_N] \\ &= -i\int \mathbf{P}_{>N}[u_N^+\partial_x^m u_N^+ - u_N^-\partial_x^m u_N^-]\mathbf{P}_{>N}[u_N\partial_x^m u_N]\,dx \\ &= -i\int \mathbf{P}_{>N}[u_N^+\partial_x^m u_N^+]\mathbf{P}_{>N}[u_N^-\partial_x^m u_N^-]\,dx + i\int \mathbf{P}_{>N}[u_N^-\partial_x^m u_N^-]\mathbf{P}_{>N}[u_N^+\partial_x^m u_N^+]\,dx = 0. \end{aligned} \tag{5.14}$$

Lastly, the second contribution can be handled as the terms in II_N with $j = 1$. This completes the proof of the first estimate on p_N^*.

In the remaining of the proof, we focus on the $p_{j,N}^*$ contributions from (5.13), for $j = 1, 2, 3$. For $m = 1$, we have $p_j(u) \in \{u[\partial_x u]^2, u[\mathcal{H}\partial_x u]^2, [\mathcal{H}u][\mathcal{H}\partial_x u][\partial_x u]\}$. Moreover, by doing integration by parts, we can write

$$\int p_3(u)\,dx = \frac{1}{2}\int [p_1(u) - p_2(u)]\,dx,$$

so it suffices to estimate the contributions coming from p_1, p_2. Note that

$$\begin{aligned} \int p_{1,N}^*(u) &= \int \mathbf{P}_{>N}[u\partial_x u]\mathbf{P}_{>N}[\partial_x u]^2\,dx, \\ \int p_{2,N}^*(u) &= \int \big(\mathbf{P}_{>N}[u\partial_x u]\mathbf{P}_{>N}[\mathcal{H}\partial_x u]^2\,dx, \end{aligned}$$

since

$$\begin{aligned} &\int \mathbf{P}_{>N}[u\mathcal{H}\partial_x u]\mathbf{P}_{>N}\mathcal{H}\partial_x[u\partial_x u]\,dx \\ &= \int \mathbf{P}_{>N}[u^+\partial_x u^+ - u^-\partial_x u^-]\mathbf{P}_{>N}\partial_x[u^+\partial_x u^+ - u^-\partial_x u^-]\,dx \\ &= \int \big(\mathbf{P}_{>N}[u^+\partial_x u^+]\mathbf{P}_{>N}\partial_x[u^-\partial_x u^-] + \mathbf{P}_{>N}[u^-\partial_x u^-]\mathbf{P}_{>N}\partial_x[u^+\partial_x u^+]\big)dx = 0. \end{aligned}$$

We use an orthogonality argument to handle these two contributions. By the Wiener chaos estimate (Lemma 2.6), it suffices to estimate the L^2-norm, for which we have

$$\Big\| \int \mathbf{P}_{>N}[u_N\mathcal{H}^\alpha\partial_x u_N]\mathbf{P}_{>N}\mathcal{H}^\alpha\partial_x[u_N\partial_x u_N]\,dx\Big\|^2_{L^2(d\mu_{\delta,\frac{3}{2}})}$$

$$
\begin{aligned}
&\lesssim \sum_{\substack{n_{1\cdots4}=0\\ m_{1\cdots4}=0\\ 0<|n_\ell|,|m_\ell|\le N\\ |n_{12}|,|m_{12}|>N}} \frac{|n_2n_3n_4m_2m_3m_4|}{|n_1m_1\cdots n_4m_4|^{\frac32}}|\mathbb{E}[g_{n_1}\cdots g_{n_4}\overline{g_{m_1}\cdots g_{m_4}}]|\\
&\lesssim \bigg(\sum_{\substack{0<|n_1|,|n_2|\le N\\ |n_{12}|>N}} \frac{|n_1||n_2|^2}{|n_1n_2|^3}\bigg)^2 + \sum_{\substack{n_{1\cdots4}=0\\ 0<|n_\ell|\le N\\ |n_{12}|>N}} \frac{|n_2n_3n_4|^2+|n_1n_3||n_2n_4|^2}{|n_1\cdots n_4|^3}\\
&\lesssim \bigg(\sum_{\substack{0<|n_1|,|n_2|\le N\\ |n_{12}|>N}} \frac{1}{|n_1|^2|n_2|}\bigg)^2 + \sum_{\substack{n_{1\cdots4}=0\\ 0<|n_\ell|\le N\\ |n_{12}|>N}} \bigg(\frac{1}{|n_1|^3|n_2n_3n_4|}+\frac{1}{|n_1n_3|^2|n_2n_4|}\bigg)\\
&\lesssim \mathcal{O}(N^{-2}\log^2 N),
\end{aligned}
$$

where $\alpha = 0, 1$, using (3.7), Isserlis' theorem, symmetry, and (5.6).

For $m \ge 2$, by Lemma 5.4, we have

$$
\begin{aligned}
\int p^*_{1,N}(u_N)\,dx &= \int \mathbf{P}_{>N}[u_N\partial_x u_N]\mathbf{P}_{>N}[\partial_x^m u_N]^2\,dx\\
&\quad+\sum_{j=1}^m c_{1,j}\int \mathbf{P}_{>N}[u_N\partial_x^m u_N]\mathbf{P}_{>N}[\partial_x^j u_N\partial_x^{m+1-j}u_N]\,dx\\
&=: \mathrm{I}_{1,N}(u_N)+\mathrm{II}_{1,N}(u_N),\\
\int p^*_{2,N}(u_N)\,dx &= \int \mathbf{P}_{>N}[u_N\partial_x u_N]\mathbf{P}_{>N}[\mathcal{H}\partial_x^m u_N]^2\,dx\\
&\quad+\sum_{j=1}^m c_{2,j}\int \mathbf{P}_{>N}[u_N\mathcal{H}\partial_x^m u_N]\mathbf{P}_{>N}\mathcal{H}[\partial_x^j u_N\partial_x^{m+1-j}u_N]\,dx\\
&=: \mathrm{I}_{2,N}(u_N)+\mathrm{II}_{2,N}(u_N),\\
\int p^*_{3,N}(u_N)\,dx &= \int \mathbf{P}_{>N}\mathcal{H}[u_N\partial_x u_N]\mathbf{P}_{>N}[\partial_x^m u_N\mathcal{H}\partial_x^m u_N]\,dx =: \mathrm{I}_{3,N}(u_N),
\end{aligned}
$$

for some constants $c_{1,j}, c_{2,j} \in \mathbb{R}$. For $\mathrm{I}_{1,N}$, using Minkowski's inequality, (5.7), (3.7), and (5.6), we get that

$$
\begin{aligned}
\|\mathrm{I}_{1,N}(\mathbf{P}_N u)\|_{L^q(d\mu_{\delta,m+\frac12})} &\lesssim \sum_{\substack{n_{1\cdots4}=0\\ 0<|n_\ell|\le N\\ |n_{12}|>N}} \frac{|n_2||n_3n_4|^m}{|n_1\cdots n_4|^{m+\frac12}} \lesssim \sum_{\substack{n_{1\cdots4}=0\\ 0<|n_\ell|\le N\\ |n_{12}|>N}} \frac{1}{|n_1|^2|n_2|\min(|n_3|,|n_4|)}\\
&\lesssim \mathcal{O}(N^{-1}\log^2 N).
\end{aligned}
$$

Note that the same approach works for $\mathrm{I}_{2,N}$. For $\mathrm{II}_{1,N}$, following the same ideas, we have

$$
\|\mathrm{II}_{1,N}\|_{L^q(d\mu_{\delta,m+\frac12})} \lesssim \sum_{j=1}^m \sum_{\substack{n_{1\cdots4}=0\\ 0<|n_\ell|\le N\\ |n_{12}|>N}} \frac{|n_2|^m|n_3|^j|n_4|^{m+1-j}}{|n_1\cdots n_4|^{m+\frac12}}
$$

$$\lesssim \sum_{j=1}^{m} \sum_{\substack{n_{1\cdots 4}=0 \\ 0<|n_\ell|\leq N \\ |n_{12}|>N}} \frac{1}{|n_1|^{m+\frac{1}{2}}|n_2|^{\frac{1}{2}}|n_3|^{m+\frac{1}{2}-j}|n_4|^{j-\frac{1}{2}}}$$

$$\lesssim \sum_{\substack{n_{1\cdots 4}=0 \\ 0<|n_\ell|\leq N \\ |n_{12}|>N}} \frac{1}{|n_1|^2|n_2|\min(|n_3|,|n_4|)^{\frac{3}{2}}} = \mathcal{O}(N^{-1}\log N)$$

since $|n_2| = |n_{134}|$ and $m+\frac{1}{2}-j, j-\frac{1}{2} \geq \frac{1}{2}$. The same argument applies to $\mathrm{II}_{2,N}$. Lastly, for $\mathrm{I}_{3,N}$,

$$\begin{aligned} \| \mathrm{I}_{3,N}\|_{L^q(d\mu_{\delta,m+\frac{1}{2}})}\| &\lesssim \sum_{\substack{n_{1\cdots 4}=0 \\ 0<|n_\ell|\leq N \\ |n_{12}|>N}} \frac{|n_2||n_3n_4|^m}{|n_1\cdots n_4|^{m+\frac{1}{2}}} \lesssim \sum_{\substack{n_{1\cdots 4}=0 \\ 0<|n_\ell|\leq N \\ |n_{12}|>N}} \frac{1}{|n_1|^2|n_2|\min(|n_3|,|n_4|)} \\ &= \mathcal{O}(N^{-1}\log^2 N). \end{aligned}$$

This completes the proof. ∎

Lemma 5.6. *Let $m\in\mathbb{N}$, $j\in\{3,\dots,2m+2\}$, $\ell\in\{3,\dots,2m+3\}$, with $p_1(u)\in\mathcal{P}_j(u)$, $p_2(u)\in\mathcal{P}_\ell(u)$ satisfying*

$$\begin{aligned} &\|p_1(u)\| + |||p_1(u)||| = 2m+2-j, && |p_j(u)| \leq m-1, \\ &\|p_2(u)\| + |||p_2(u)||| = 2m+3-\ell, && |p_2(u)| \leq m, && \widetilde{p}_3(u) \neq u\partial_x^m u\partial_x^m u. \end{aligned}$$

Then, we get that

$$\lim_{N\to\infty}\Big[\big\|\textstyle\int p_{1,N}^*(\mathbf{P}_N u)\,dx\big\|_{L^q(d\mu_{\delta,m})} + \big\|\textstyle\int p_{2,N}^*(\mathbf{P}_N u)\,dx\big\|_{L^q(d\mu_{\delta,m+\frac{1}{2}})}\Big] = 0.$$

Proof. For $m=1$, we have that $p_1(u)\in\{u^2\mathcal{Q}_\delta u, u^4\}$. If $p_1(u)=u^4$, then by Wiener chaos estimate, it suffices to control the $L^2(d\mu_{\delta,1})$-norm, and by using (3.7), we have

$$\bigg\|\int p_{1,N}^*(\mathbf{P}_N u)\,dx\bigg\|_{L^2(d\mu_{\delta,1})}^2 \lesssim \sum_{\substack{0<|n_j|,|m_j|\leq N \\ n_{1\cdots 5}=0 \\ m_{1\cdots 5}=0 \\ |n_{45}|,|m_{45}|>N}} \frac{|n_5m_5|}{|n_1m_1\cdots n_5m_5|}|\mathbb{E}[g_{n_1}\cdots g_{n_5}\overline{g_{m_1}\cdots g_{m_5}}]|.$$

Since $|n_{45}|, |m_{45}| > N$, we cannot have that $n_{45}=0$ or $m_{45}=0$. Also, since $|n_{123} > N, |m_{123} > N$, and all frequencies are nonzero, we cannot have $n_{ij}=0$ or $m_{ij}=0$ for distinct $i,j\in\{1,2,3\}$. So we can only have $n_{ij}=0$ or $m_{ij}=0$ if $i\in\{1,2,3\}, j\in\{4,5\}$. Moreover, there can only be one such pairing among the n's and m's, since all frequencies are nonzero. Then, we have by (5.6)

$$\cdots \lesssim \sum_{\substack{0<|n_\ell|,|m_1|\leq N \\ \ell=1,2,3 \\ |n_{123}|>N}} \frac{1}{|n_1m_1||n_2n_3|^2} + \sum_{\substack{n_{1\cdots 5}=0 \\ 0<|n_\ell|\leq N \\ |n_{123}|>N}} \frac{1}{|n_1n_4||n_2n_3|^2} = \mathcal{O}(N^{-1}\log N).$$

Following the same method for $p_1(u) = u^2\mathcal{Q}_\delta u$ with (2.6), we get that

$$\bigg\|\int p_{1,N}^*(\mathbf{P}_N u)\,dx\bigg\|_{L^2(d\mu_{\delta,1})}^2 \sim \bigg\|\int [2u_N\mathcal{Q}_\delta u_N + \mathcal{Q}_\delta(u_N^2)]\mathbf{P}_{>N}[u_N\partial_x u_N]\,dx\bigg\|_{L^2(d\mu_{\delta,1})}^2$$

$$\lesssim_\delta \sum_{\substack{0<|n_j|,|m_j|\le N\\ n_{1\cdots4}=0\\ m_{1\cdots4}=0\\ |n_{34}|,|m_{34}|>N}} \frac{|n_4 m_4|}{|n_1\cdots n_4 m_1\cdots m_4|}|\mathbb{E}[g_{n_1}\cdots g_{n_4}\overline{g_{m_1}\cdots g_{m_4}}]|.$$

In applying Isserlis' theorem, we note that since $n_{12}, n_{34}, m_{12}, m_{34}\neq 0$, the only possible pairings within families of frequencies are of the form $n_{i_1j_1} = m_{i_2j_2} = 0$ when $i_1, i_2\in \{1,2\}$, $j_1, j_2 \in \{3,4\}$. We can also have $n_{i_k} = m_{j_k}$ for $k = 1,\dots,4$, where $\{i_1,\dots,i_4\} = \{j_1,\dots,j_4\} = \{1,\dots,4\}$. Then, we have

$$\cdots \lesssim \sum_{\substack{0<|n_\ell|,|m_\ell|\le N,\\ \ell=1,2\\ |n_{12}|,|m_{12}|>N}} \bigg[\frac{1}{|n_1m_1|^2|n_2m_2|} + \frac{1}{|n_1m_2||m_1n_2|^2}\bigg] + \sum_{\substack{n_{1\dots4}=0\\ 0<|n_\ell|\le N\\ |n_{12}|>N}} \frac{1}{|n_1n_3||n_2|^2} = \mathcal{O}\Big(\frac{\log^2 N}{N}\Big).$$

For $m\geq 2$, fix $j\in\{3,\dots,2m+2\}$ and $p_1(u)\in\mathcal{P}_j(u)$ satisfying the assumptions. Then, we have

$$\widetilde{p}_1(u) = \prod_{k=1}^{j}\partial_x^{\alpha_j}u$$

with $\alpha_{1\cdots j}\leq 2m+2-j$ and $0\leq \alpha_k\leq m-1$. Thus, $p^*_{1,N}(u)$ has terms of the form (ignoring operators $\mathcal{H}$ and $\mathcal{Q}_\delta$)

$$p^*_{1,N}(u) = \sum_{i=1}^{j}\mathbf{P}_{>N}\bigg[\prod_{\substack{k=1\\k\neq i}}^{j-1}\partial_x^{\alpha_k}u\bigg]\mathbf{P}_{>N}\partial_x^{\alpha_i}[u\partial_x u].$$

Due to symmetry we can set $i=j$ above, and using (5.7), (3.7), Lemma 2.6 and the fact that $\delta\geq 2$, we have that

$$\begin{aligned}\bigg\|\int p^*_{1,N}(u)\,dx\bigg\|_{L^q(d\mu_m)} &\lesssim \sum_{\substack{n_{1\dots(j+1)}=0\\ 0<|n_\ell|\le N\\ |n_{1\cdots(j-1)}|>N}} \frac{|n_{j(j+1)}|^{\alpha_j}|n_{j+1}|\prod_{k=1}^{j-1}|n_k|^{\alpha_k}}{|n_1\cdots n_{j+1}|^m}\\ &\lesssim \sum_{\substack{n_{1\dots(j+1)}=0\\ 0<|n_\ell|\le N\\ |n_{1\cdots(j-1)}|>N}} \frac{|n_{j(j+1)}|^{\alpha_j}}{|n_j|^m|n_{j+1}|^{m-1}|n_1\cdots n_{j-1}|}.\end{aligned}$$

If $\alpha_1\leq m-2$, then using (5.6), we have

$$\cdots\lesssim \sum_{\substack{n_{1\dots(j+1)}=0\\ 0<|n_\ell|\le N\\ |n_{j(j+1)}|>N}} \frac{1}{|n_j|^2|n_{j+1}||n_1\cdots n_{j-1}|} = \mathcal{O}(N^{-1}\log^{j-1}N).$$

If $\alpha_1 = m-1$, then $|n_{j(j+1)}| = |n_{1\cdots(j-1)}| \lesssim \max(|n_1|,\dots,|n_{j-1}|)$, which we let to be $|n_1|$ for simplicity, to obtain

$$\cdots \lesssim \sum_{\substack{n_{1\dots(j+1)}=0\\ 0<|n_\ell|\le N\\ |n_{1\cdots(j-1)}|>N}} \frac{\max(|n_j|,|n_{j+1}|)^{m-2}|n_1|}{\min(|n_j|,|n_{j+1}|)^m \min(|n_j|,|n_{j+1}|)^{m-1}|n_1\cdots n_{j-1}|} = \mathcal{O}(N^{-1}\log^{j-1} N).$$

We now consider the contributions coming from $p_2(u)$. For $m \ge 2$ and $p_2(u)$ as in the statement, we can write $p^*_{2,N}(u)$ (ignoring the $\mathcal{H}$ and $\mathcal{Q}_\delta$ operators) as

$$p^*_{2,N}(u) = \sum_{i=1}^{j} \mathbf{P}_{>N}\Big[\prod_{\substack{k=1\\k\neq i}}^{j-1}\partial_x^{\alpha_k}u\Big]\mathbf{P}_{>N}\partial_x^{\alpha_i}[u\partial_x u].$$

with $3 \le \ell \le 2m+3$, $\alpha_{1\cdots j} \le 2m+3-\ell$, and if $\max_{k=1,\dots,j}\alpha_k = m$, then the remaining $\alpha_k \le m-1$.

By symmetry, we abuse notation and only consider the contribution with $i=1$ and call it $\mathrm{I}_N(u)$. Using (5.7), (3.7), and Lemma (2.6), we have

$$\begin{aligned}\|\mathrm{I}_N(\mathbf{P}_N u)\|_{L^q(d\mu_{\delta,m+\frac12})} &\lesssim_\delta \sum_{\substack{0<|n_\ell|\le N\\ n_{1\cdots(j+1)}=0\\ |n_{12}|>N}} \frac{|n_{12}|^{\alpha_1}|n_2|\prod_{\ell=3}^{j+1}|n_\ell|^{\alpha_\ell}}{|n_{1\cdots(j+1)}|^{m+\frac12}}\\ &\lesssim \sum_{\substack{0<|n_\ell|\le N\\ n_{1\cdots(j+1)}=0\\ |n_1+n_2|>N}} \frac{|n_{12}|^{\alpha_1}}{\min_{\ell=1,2}|n_\ell|^{m+\frac12}\max_{\ell=1,2}|n_\ell|^{m-\frac12}\prod_{\ell=3}^{j+1}|n_\ell|^{m+\frac12-\alpha_\ell}}.\end{aligned}$$

For simplicity, assume that $|n_3| \ge \cdots \ge |n_{j+1}|$ and $|n_1| \ge |n_2|$. If $\alpha_1 \le m-1$, then

$$\begin{aligned}\cdots &\lesssim \sum_{\substack{0<|n_\ell|\le N\\ n_{1\cdots(j+1)}=0\\ |n_1+n_2|>N}} \frac{|n_1|^{m-\frac32}|n_3|^{\frac12}}{|n_2|^{m+\frac12}|n_1|^{m-\frac12}\prod_{\ell=3}^{j+1}|n_\ell|^{m+\frac12-\alpha_\ell}}\\ &\lesssim \sum_{\substack{0<|n_\ell|\le N\\ n_{1\cdots(j+1)}=0\\ |n_1+n_2|>N}} \frac{1}{|n_1||n_2|^2|n_3|^{m-\alpha_3}\prod_{\ell=4}^{j+1}|n_\ell|^{m+\frac12-\alpha_\ell}} = \mathcal{O}(N^{-1}\log^j N)\end{aligned}$$

by using (5.6), and noting that there is at most one $\alpha_\ell = m$, which allows us to do the sum in the remaining frequencies. If $\alpha_1 = m$, then by (5.6)

$$\cdots \lesssim \sum_{\substack{0<|n_\ell|\le N\\ n_{1\cdots(j+1)}=0\\ |n_1+n_2|>N}} \frac{|n_1|^{m-\frac32}|n_3|^{\frac32}}{|n_1|^{m-\frac12}|n_2|^{m+\frac12}|n_3\cdots n_{j+1}|^{\frac32}} = \mathcal{O}(N^{-1}\log N),$$

since for $\ell = 3,\dots,j+1$, $\alpha_\ell \le m-1$ and thus $m+\frac12-\alpha_\ell \ge \frac32$.

It only remains to consider $m = 1$. Let $j = 3, 4, 5$ and $p(u) \in \mathcal{P}_j(u)$ with $\|p(u)\| = 0$. Then, proceeding as before, we have that

$$\bigg\| \int p_N^*(u)\,dx \bigg\|_{L^q(d\mu_{\delta, m+\frac12})} \lesssim \sum_{\substack{n_{1\cdots(j+1)}=0\\ 0<|n_\ell|\le N\\ |n_{12}|>N}} \frac{|n_1|}{|n_1\cdots n_{j+1}|^{\frac32}}$$

$$\lesssim N^{-\frac12} \sum_{\substack{n_{1\cdots(j+1)}=0\\ 0<|n_\ell|\le N\\ |n_{12}|>N}} \frac{1}{\min(|n_1|,|n_2|)^{\frac32}|n_3\cdots n_{j+1}|^{\frac32}} \lesssim N^{-\frac12},$$

since $\max(|n_1|,|n_2|) \ge N/2$. It remains to consider the contributions with 3 and 4 factors and exactly one derivative. Let $p_j(u) \in \mathcal{P}_j(u)$ for $j = 3, 4$, and $\widetilde{p}_3(u) = u^2\partial_x u$, $\widetilde{p}_4(u) = u^3\partial_x u$. By the Wiener chaos estimate, it suffices to estimate the $L^2(d\mu_{m+\frac12})$-norm of $\int p_{j,N}^*(u)\,dx$, so we have

$$\bigg\| \int p_{j,N}^*(u)\,dx \bigg\|^2_{L^2(d\mu_{m+\frac12})} \lesssim \sum_{\substack{n_{1\cdots(j+1)}=0\\ m_{1\cdots(j+1)}=0\\ 0<|n_\ell|,|m_\ell|\le N\\ |n_{12}|,|m_{12}|>N}} \frac{|n_1 m_1 n_{j+1} m_{j+1}|}{|n_1 m_1\cdots n_{j+1}m_{j+1}|^{\frac32}} |\mathbb{E}[g_{n_1}\cdots g_{n_{j+1}}\overline{g_{m_1}\cdots g_{m_{j+1}}}]|.$$

Next we want to apply Isserlis' theorem and consider the possible pairings $n_{ik} = 0$ and $n_i = m_k$. If $j = 3$, then we can only have pairings $n_{i_1} + n_{i_2} = n_{i_3} + n_{i_4} = 0$ where $\{i_1, i_3\} = \{1, 3\}, \{i_2, i_4\} = \{2, 4\}$ and similar for m_ℓ frequencies, or $n_{i_k} = m_{i_k}$ for $k = 1, \dots, 4$ and $\{i_1, \dots, i_4\} = \{j_1, \dots, j_4\} = \{1, \dots, 4\}$. Then, by (5.6), we have

$$\bigg\| \int p_{3,N}^*(u)\,dx \bigg\|^2_{L^2(d\mu_{m+\frac12})} \lesssim \bigg(\sum_{\substack{0<|n_1|,|n_2|\le N\\ |n_{12}|>N}} \frac{1}{|n_1||n_2|^2} \bigg)^2 + \sum_{\substack{n_{1\cdots4}=0\\ 0<|n_\ell|\le N\\ |n_{12}|>N}} \frac{1}{|n_1||n_2|^2|n_3|^2}$$

$$= \mathcal{O}(N^{-1}\log N).$$

If $j = 4$, then we can have (1) $n_{i_1 i_2} = 0, m_{j_1 j_2} = 0$ for $i_1, j_1 \in \{1, 2\}$, $i_2, j_2 \in \{3, 4, 5\}$, and $n_{i_k} = m_{j_k}$ for $k = 3, 4, 5$ where $\{i_3, \dots, i_5\} = \{1, \dots, 5\} \setminus \{i_1, i_2\}$ and $\{j_3, \dots, j_5\} = \{1, \dots, 5\} \setminus \{j_1, j_2\}$; or (2) $n_{i_k} = m_{j_k}$, $k = 1, \dots, 5$, $\{i_1, \dots, i_5\} = \{j_1, \dots, j_5\} = \{1, \dots, 5\}$. Consequently, we get by (5.6)

$$\bigg\| \int p_{4,N}^*(u)\,dx \bigg\|^2_{L^2(d\mu_{m+\frac12})} \lesssim \sum_{\substack{n_{1\cdots5}=0\\ m_{1\cdots5}=0\\ 0<|n_\ell|,|m_\ell|\le N\\ |n_{12}|,|m_{12}|>N}} \frac{|\mathbb{E}[g_{n_1}\cdots g_{n_5}\overline{g_{m_1}\cdots g_{m_5}}]|}{|n_1 m_1 n_3 m_3\cdots n_5 m_5|^{\frac12}|n_2 m_2|^2}$$

$$\lesssim \sum_{\substack{0<|n_\ell|,|m_1|\le N\\ \ell=1,2,3\\ |n_{12}|>N}} \frac{1}{|n_1 m_1 n_3 n_4||n_2|^2} + \sum_{\substack{n_{1\cdots5}=0\\ 0<|n_\ell|\le N\\ |n_{12}|>N}} \frac{1}{|n_1 n_3 n_4 n_5||n_2|^2}$$

$$= \mathcal{O}(N^{-1}\log^4 N),$$

which completes the proof. □

5.2. **Proof of Proposition 5.2.** This follows essentially as in the deep-water regime. Below we describe how to adapt the proof to the shallow-water setting. Fix $k \in \mathbb{N}$ with $k \geq 2$, $N \in \mathbb{N}$, and let $\widetilde{u}_N$ denote the solution to (5.2) with data $u_0 \in L^2(\mathbb{T})$. Then, as before, from (4.1), we have that

$$\frac{d}{dt}\widetilde{E}_{\delta,\frac{k}{2}}(\mathbf{P}_N\widetilde{u}_N) = \widetilde{R}^*_{\delta,\frac{k}{2},N}(\mathbf{P}_N\widetilde{u}_N)$$

where $\widetilde{R}^*_{\delta,\frac{k}{2},N}(\cdot)$ denotes $\widetilde{R}_{\delta,\frac{k}{2}}(\cdot)$ as defined in (4.1) with every polynomial $p(\cdot)$ replaced by $p^*_N(\cdot)$ as defined in (5.5). Therefore, the proof of Proposition 5.2 reduces to showing that for $\delta_0 > 0$, $0 < \delta \leq \delta_0$, and $1 \leq q < \infty$, we have

$$\lim_{N\to\infty} \Big\| \widetilde{R}^*_{\delta,\frac{k}{2},N}(\mathbf{P}_N\widetilde{u}) \Big\|_{L^q(d\widetilde{\mu}_{\delta,\frac{k}{2}})} = 0. \tag{5.15}$$

Since δ is fixed and we do not require uniform in δ bounds here, we can see from (4.1), that all the contributions in $\widetilde{R}_{\delta,\frac{k}{2}}(u)$ are of the form

$$\int p_j(u)\,dx, \qquad p_j(u) \in \widetilde{\mathcal{P}_j}(u), \qquad j = 3,\dots,k+2,$$
$$\|p_j(u)\| \leq k+2-j, \qquad |p_j(u)| \leq \frac{k}{2}.$$

Note that the difficulty in the $\widetilde{R}^{[<]}_{\delta,\frac{2k-1}{2}}(u)$ contributions in controlling the $\widetilde{\mathcal{G}}_\delta$ operator due to insufficient powers of δ, which lead to a derivative loss to get the uniform in δ bounds in (4.3), is no longer an issue here.

By using the lower bound (4.15) and (2.7), proceeding as in the proof of Lemma 5.6 gives

$$\lim_{N\to\infty} \bigg[\bigg\| \int p^*_{1,N}(\mathbf{P}_N u)\,dx \bigg\|_{L^q(d\widetilde{\mu}_{\delta,m})} + \bigg\| \int p^*_{2,N}(\mathbf{P}_N u)\,dx \bigg\|_{L^q(d\widetilde{\mu}_{\delta,m+\frac{1}{2}})} \bigg] = 0 \tag{5.16}$$

where $m \in \mathbb{N}$

$$j = 3,\dots,2m+2, \quad p_1(u) \in \widetilde{\mathcal{P}}_j(u), \quad \|p_1(u)\| \leq 2m+2-j, \quad |p_1(u)| \leq m-1$$
$$\ell = 3,\dots,2m+3, \quad p_2(u) \in \widetilde{\mathcal{P}}_\ell(u), \quad \|p_2(u)\| \leq 2m+3-j, \quad |p_2(u)| \leq m$$

and $\widetilde{p}_\ell(u) \neq u\partial_x^m u \partial_x^m u$. Consequently, (5.15) follows once we show this decay for the terms coming from

$$\widetilde{p_3}(u) = u\partial_x^{m-1}u\partial_x^m u, \qquad \text{for } k = 2m,$$
$$\widetilde{p_4}(u) = u\partial_x^m u\partial_x^m u, \qquad \text{for } k = 2m+1.$$

Following the approach in the deep-water regime, if $p_3(u), p_4(u)$ above have $\widetilde{\mathcal{G}}_\delta$ operators, these can always be associated with ∂_x and replace by

$$\frac{1}{\delta}\mathcal{H}\partial_x + \frac{1}{\delta}(\mathcal{G}_\delta - \mathcal{H})\partial_x =: \frac{1}{\delta}\mathcal{H}\partial_x + \widetilde{\mathcal{Q}_\delta}$$

where by (2.6), $|\widehat{\widetilde{\mathcal{Q}_\delta}}(n)| \leq \frac{1}{\delta^2}$ for all $n \in \mathbb{Z}^*$. Consequently, we can further split the contributions coming from $p_3(u), p_4(u)$ into those with $\frac{1}{\delta}\mathcal{H}\partial_x$ operators and the same fundamental polynomials $\widetilde{p}_3(u), \widetilde{p}_4(u)$, and the contributions with $\widetilde{\mathcal{Q}_\delta}$ which have less derivatives and can therefore be has (5.16). Consequently, the only terms left to consider are

$$p_3(u) = [\mathcal{H}^{\alpha_1} u][\mathcal{H}^{\alpha_2} \partial_x^{m-1} u][\mathcal{H}^{\alpha_3} \partial_x^m u], \qquad p_4(u) = [\mathcal{H}^{\alpha_1} u][\mathcal{H}^{\alpha_2} \partial_x^{m} u][\mathcal{H}^{\alpha_3} \partial_x^m u],$$

for $\alpha_1, \alpha_2, \alpha_3 \in \{0, 1\}$. By Lemma A.9 and Lemma A.10, it suffices to consider the terms in (3.5) as in the deep-water setting, which were already estimated in Lemma 5.5. The only difference in the proof comes is in using (4.15) instead of (3.7).

6. Dynamical problem

In this section we study qualitative properties of solutions to ILW (1.1) and sILW (1.4). Since the argument is analogous for both regimes, we include only the proof in the deep-water regime. For the remaining of this section, we fix $0 < \delta < \infty$ and omit dependence on δ when clear from context. Recall the truncated dynamics in (5.1) and its unique global-in-time solutions by $u_N(t, \cdot) = \Phi_t^N(u_0)$, for $u_0 \in L^2(\mathbb{T})$.

The following lemma collects relevant results on well-posedness and deterministic bounds on the solutions to ILW. For proofs see Theorem 1.9 and Corollary 1.10 in [41], as well as Proposition 3.1 in [34].

Lemma 6.1 ([41, 34])**.** (i) *The ILW equation* (1.1) *is globally well-posed in $H^s(\mathbb{T})$ for $s \geq \frac{1}{2}$.*
(ii) *Let $s > \frac{1}{2}$. For $R > 0$ and $u_0 \in B^s(R)$, there exists $T = T(R) > 0$ such that*

$$\|\Phi_t(u_0)\|_{L^\infty([0,T];H^s)} \lesssim \|u_0\|_{H^s}, \quad \|\Phi_t^N(u_0)\|_{L^\infty([0,T];H^s)} \lesssim \|u_0\|_{H^s}. \tag{6.1}$$

Remark 6.2. Although not explicitly stated in Remark 1.1.(2) in [41], Hypothesis 1 needed for the global well-posedness result also holds when $L_2 = \mathcal{G}_\delta \partial_x^2$, not only for $L_2 = \mathcal{T}_\delta \partial_x^2$. Thus, the result applies to our version of the ILW equation (1.1) and also sILW (1.4). Moreover, although Lemma 6.1(ii) is not explicitly stated for the truncated dynamics, the same approach applies.

6.1. Approximation by the truncated flow.

The main goal of this subsection is to establish the following approximation result between the ILW and truncated ILW flows, Φ_t and Φ_t^N, respectively. A key ingredient of the proof follows from adapting the difference estimate [41, Proposition 3.5] to allow for the difference between the truncated and full flows of ILW.

Proposition 6.3. *Let $k \geq 3$, $\varepsilon > 0$, $\frac{1}{2} < s < \sigma$, and $R > 0$. Then, there exists $T = T(R) > 0$ such that for every $R_0 > 0$ there exists $N_0(R_0)$ with the property*

$$\Phi_t^N(A) \subset \Phi_t(A) + B^s(R_0), \qquad \forall N > N_0, \ \forall t \in (-T, T), \ \forall A \subset B^\sigma(R),$$

where $B^s(R_0)$ denotes the ball on $H^s(\mathbb{T})$ of radius R_0 centered on the origin.

Before proceeding to the proof, we require some notation and auxiliary results from [41]. For $s, b \in \mathbb{R}$ we define the Fourier restriction space $X^{s,b}(\mathbb{T})$ through the norm

$$\|u\|_{X^{s,b}} := \big\| \langle n \rangle^s \langle \tau - p_\delta(n) \rangle^b \mathcal{F}_{t,x} u(\tau, n) \big\|_{L^2_\tau \ell^2_n},$$

where $p_\delta(n) = n(n\coth(\delta n) - \frac{1}{\delta})$ in the deep-water regime, or $p_\delta(n) = \frac{n}{\delta}(n\coth(\delta n) - \frac{1}{\delta})$ in the shallow-water regime. We also define the time-restricted version of this space $X_T^{s,b}$ defined via the norm

$$\|u\|_{X_T^{s,b}} := \inf\big\{\|\widetilde{u}\|_{X^{s,b}} : \ \widetilde{u}|_{[-T,T]} = u\big\}.$$

Moreover, we introduce the function spaces $M_T^s := L_T^\infty H_x^s \cap X_T^{s-1,1}$, endowed with the natural norm

$$\|u\|_{M_T^s} = \|u\|_{L_T^\infty H^s} + \|u\|_{X_T^{s-1,1}} \, .$$

Lemma 6.4 (Lemma 3.1 [41])**.** *Let $0 < T < 2$, $s > 1/2$ and $u \in L_T^\infty H^s$ be a solution to* (1.1)*. Then, $u \in M_T^s$ and it holds*

$$\|u\|_{M_T^s} \lesssim \|u\|_{L_T^\infty H^s} + \|u\|_{L_T^\infty H^s}\|u\|_{L_T^\infty H^{\frac12 +}} \, .$$

Moreover, for any $u_N \in L_T^\infty H^s$ of solutions to (5.1)*, the same statement holds.*

Proof of Proposition 6.3. Let $u_0 \in B^\sigma(R)$. Then, from Lemma 6.1, there exist global-in-time solutions $u(t) = \Phi_t(u_0)$ and $u_N(t) = \Phi_t^N(u_0)$ to (1.1) and (5.1), respectively, and $T = T(R) > 0$ such that (6.1) holds.

From (2.2), we have that

$$\|u(t) - u_N(t)\|_{H^s} \lesssim \|u(t) - u_N(t)\|_{H^{s-1}}^\theta \|u(t) - u_N(t)\|_{H^\sigma}^{1-\theta}$$

where $\theta = \frac{\sigma - s}{1+\sigma-s} \in (0,1)$. From (6.1), the second factor above is uniformly bounded in N and $u_0 \in B^\sigma(R)$, thus the intended approximation follows once we show that

$$\lim_{N\to\infty}\Big(\sup_{\substack{t\in[-T,T]\\ u_0\in B^\sigma(R)}} \|\Phi_t(u_0) - \Phi_t^N(u_0)\|_{H^{s-1}}\Big) = 0\,. \tag{6.2}$$

We show this by adapting the proof of [41, Proposition 3.5] to the difference between the full and the truncated flows for ILW. We see that $w_N = u - u_N$ solves

$$\begin{cases} \partial_t w_N - \mathcal{G}_\delta\partial_x^2 w_N = \partial_x(u^2 - (\mathbf{P}_N u_N)^2) + \mathbf{P}_{>N}\partial_x(\mathbf{P}_N u_N)^2, \\ w_N|_{t=0} = 0. \end{cases} \tag{6.3}$$

With the notation $y_N = u - \mathbf{P}_N u_N$ and $z_N = u + \mathbf{P}_N u_N$, we can write

$$\partial_t w_N - \mathcal{G}_\delta\partial_x^2 w_N = \partial_x(z_N y_N) + \mathbf{P}_{>N}\partial_x(\mathbf{P}_N u_N)^2. \tag{6.4}$$

Let K be a dyadic number and P_K denote the Littlewood-Paley projector with multiplier ϕ_K satisfying $\operatorname{supp}\phi_K \subset [\frac12 K, 2K]$. Applying the operator P_K to (6.4), taking the H^{s-1} scalar product with $P_K w_N$, and integrating in time, we obtain

$$\begin{aligned} \|P_K w_N\|_{L_T^\infty H_x^{s-1}}^2 \lesssim \sup_{|t|\le T}\langle K\rangle^{2(s-1)}&\Big|\int_0^t\int_{\mathbb{T}} P_K(z_N y_N)P_K\partial_x w_N\,dx\,dt'\Big| \\ &+ \sup_{|t|\le T}\langle K\rangle^{2(s-1)}\Big|\int_0^t\int_{\mathbb{T}} P_K\mathbf{P}_{>N}(\mathbf{P}_N u_N)^2 P_K\partial_x w_N\,dx\,dt'\Big|. \end{aligned} \tag{6.5}$$

For J_1 defined as follows

$$J_1 := \sum_{K>0}\langle K\rangle^{2(s-1)}\sup_{|t|\le T}\Big|\int_0^t\int_{\mathbb{T}} P_K(z_N y_N)P_K\partial_x w_N\,dx\,dt'\Big|, \tag{6.6}$$

proceeding as in the proof of Proposition 3.5 in [41], in particular, in estimating J defined in (3-25) by (3-31), we obtain the following estimate for J_1:

$$J_1 \lesssim \|z_N\|_{M^s_T} \|y_N\|_{M^{s-1}_T} \|w_N\|_{L^\infty_T H^{s-1}}. \tag{6.7}$$

We now estimate the norms for z_N, y_N on the RHS of (6.7). Applying Lemma 6.4 and (6.1), we get

$$\begin{aligned}
\|z_N\|_{M^s_T} &= \|u + \mathbf{P}_N u_N\|_{M^s_T} \lesssim \|u_0\|_{H^s}(1 + \|u_0\|_{H^s}),\\
\|y_N\|_{M^{s-1}_T} &\le \|u - u_N\|_{M^{s-1}_T} + \|\mathbf{P}_{>N} u_N\|_{M^{s-1}_T}\\
&\lesssim \|u - u_N\|_{L^\infty_T H^{s-1}_x} + \|u - u_N\|_{X^{s-2,1}_T} + N^{-1}\|u_0\|_{H^s},\\
\|u - u_N\|_{X^{s-2,1}_T} &\lesssim \|u^2 - \mathbf{P}_N(\mathbf{P}_N u_N)^2\|_{L^2_T H^{s-1}_x}\\
&\lesssim T^{\frac12}\big[\|(u - \mathbf{P}_N u_N)(u + \mathbf{P}_N u_N)\|_{L^\infty_T H^{s-1}_x} + N^{-1}\|(\mathbf{P}_N u_N)^2\|_{L^\infty_T H^s_x}\big]\\
&\lesssim T^{\frac12}\big[\|u - P_N u_N\|_{L^\infty_T H^{s-1}_x}\|u + \mathbf{P}_N u_N\|_{L^\infty_T H^{\frac12+}_x} + N^{-1}\|u_N\|^2_{L^\infty_T H^s_x}\big]\\
&\lesssim T^{\frac12}\big[\|u_0\|_{H^s_x}\|u - u_N\|_{L^\infty_T H^{s-1}_x} + N^{-1}\|u_0\|^2_{H^s_x}\big],
\end{aligned}$$

where we used the fact that $\mathbf{P}_{>N} u_N$ is the linear solution to ILW with initial data $\mathbf{P}_{>N} u_0$, the Duhamel formulation for $u - u_N$, and Sobolev product inequalities in [18, Lemma 3.4]. From (6.7) and Young's inequality, we get

$$\begin{aligned}
J_1 &\lesssim \|u - u_N\|^2_{L^\infty_T H^{s-1}_x}\|u_0\|_{H^s}(1 + \|u_0\|_{H^s})^2 + \|u - u_N\|_{L^\infty_T H^{s-1}_x}\|u_0\|^2_{H^s}(1 + \|u_0\|_{H^s})N^{-1}\\
&\lesssim \|u - u_N\|^2_{L^\infty_T H^{s-1}_x}\|u_0\|_{H^s}(1 + \|u_0\|_{H^s})^2 + N^{-2}\|u_0\|^2_{H^s}.
\end{aligned} \tag{6.8}$$

We now estimate the contribution coming from the second contribution on the RHS of (6.5). Let J_2 be given by

$$\begin{aligned}
J_2 &:= \sum_{K>0} \langle K\rangle^{2(s-1)} \sup_{|t|\le T}\Big|\int_0^t \int_{\mathbb{T}} P_K(\mathbf{P}_{>N}(\mathbf{P}_N u_N)^2))\partial_x P_K w_N\Big|\\
&= \sum_{\frac N2 \le K \le 4N} \langle K\rangle^{2(s-1)} \sup_{|t|\le T}\Big|\int_0^t \int_{\mathbb{T}} P_K \mathbf{P}_{>N}(\mathbf{P}_N u_N)^2 \partial_x P_K w_N\Big|,
\end{aligned} \tag{6.9}$$

since $(\mathbf{P}_N u_N)^2$ has Fourier support contained in $\{|n| \le 2N\}$, which only intersects with the support of ϕ_K with $K \le 4N$, and $P_K \mathbf{P}_{>N}$ is only nonzero if $K \ge \frac N2$.

Proceeding as in the proof of [41, Proposition 3.5] and [41, (3-14)], (3-26) becomes

$$\begin{aligned}
P_K(\mathbf{P}_N u_N)^2 &= P_K(v^2)\\
&= P_K(v_{\ll K} v) + P_K(v_{\sim K} v_{\lesssim K}) + \sum_{\frac N2 \le K \ll K_1} P_K(v_{\sim K_1} v_{\sim K_1})\\
&= v_{\ll K} P_K v + K^{-1}\Pi_\chi(\partial_x v_{\ll K}, v) + P_K(v_{\sim K} v_{\lesssim K}) + \sum_{\frac N2 \le K \ll K_1} P_K(v_{\sim K_1} v_{\sim K_1}),
\end{aligned}$$

where have used $v = \mathbf{P}_N u_N$ for simplicity, $v_{\sim K}, v_{\lesssim K}, v_{\ll K}$ to denote the restrictions of v via Litlewood-Paley decomposition to regions where the frequencies are $\sim K$, $\lesssim K$, or $\ll K$, respectively. Using we can also write

$$P_K(v_{\ll K} v) = v_{\ll K} P_K v + K^{-1}\Pi_\chi(\partial_x v_{\ll K}, v).$$

Proceeding as in [41, (3-27)], we get

$$\begin{aligned} J_2 \lesssim & \sum_{\frac{N}{2} \le K \le 4K} \sum_{K_1 \gtrsim K} K \langle K_1 \rangle^{2(s-1)} \sup_{|t| \le T} |I_t(v_K, v_{\sim K_1}, w_{N,K_1})| \\ & + \sum_{\frac{N}{2} \le K \le 4K} \sum_{K_1 \gtrsim K} K_1 \langle K_1 \rangle^{2(s-1)} \sup_{|t| \le T} |I_t(v_{\sim K_1}, v_K, w_{N,K_1})| \\ & + \sum_{\frac{N}{2} \le K \le 4K} \sum_{K_1 \gtrsim K} K \langle K \rangle^{2(s-1)} \sup_{|t| \le T} |I_t(v_{K_1}, v_{K_1}, w_{N,K})| \end{aligned} \tag{6.10}$$

From [41, (2-2)], we get that

$$J_2 \lesssim N^{-\theta} \|\mathbf{P}_N u_N\|^2_{M^s_T} \|w_N\|_{M^{s-1}_T}, \tag{6.11}$$

for some $\theta > 0$, due to the localization $K \sim N$ in the outer sum. Consequently, by (6.1) and Young's inequality, we get

$$J_2 \le \varepsilon \|u - u_N\|^2_{M^{s-1}_T} + C_\varepsilon N^{-\theta} \|u_0\|^2_{H^s},$$

for any $0 < \varepsilon < 1$ and some $C_\varepsilon > 0$.

Combining, (6.5), (6.8), and (6.11), we get

$$\|u - u_N\|^2_{L^\infty_T H^{s-1}_x} \le \|u - u_N\|^2_{L^\infty_T H^{s-1}_x} \big[\varepsilon + C \|u_0\|_{H^s} (1 + \|u_0\|_{H^s})^2\big] + N^{-\theta} C_\varepsilon \|u_0\|^2_{H^s}. \tag{6.12}$$

Using scaling to reduce the problem to small initial data, picking small ε, and undoing the scaling (see Remark 6.5 for details), we get that

$$\|u - u_N\|^2_{L^\infty_T H^{s-1}_x} \le C_\varepsilon N^{-\theta} \|u_0\|^2_{H^s}$$

from which it follows that

$$\sup_{\substack{|t| \le T \\ u_0 \in B^\sigma(R)}} \|\Phi_t(u_0) - \Phi^N_t(u_0)\|_{H^{s-1}_x} \le C_\varepsilon R^2 K^{-\theta},$$

proving (6.2), as intended. □

Remark 6.5. The proof of Proposition 6.3, as written, holds under a small data assumption, but this can be extended to general data via a scaling argument as in [41]. We detail the needed modifications to assure that the result holds for all $R > 0$ and $u_0 \in B^\sigma(R)$.

Given $\lambda \ge 1$, consider the scaling transformation S_λ defined by

$$S_\lambda(u)(t, x) := \delta^{-1} u(\lambda^{-2} t, \lambda^{-1} x).$$

Then, u is a solution to ILW (1.1) on $\mathbb{T}$ with initial data $u_0 \in B^\sigma(R)$ if and only if $u_\lambda := S_\lambda(u)$ is a solution to the following ILW equation on $\mathbb{T}_\lambda = \mathbb{R}/(2\pi\lambda\mathbb{Z})$ with scaled $\lambda\delta$ depth-parameter

$$\partial_t u_\lambda - \mathcal{G}_{\lambda\delta} \partial_x^2 u_\lambda = \partial_x (u_\lambda^2),$$

with initial $u_{0,\lambda}(x) := \lambda^{-1} u_0(\lambda^{-1} x)$. The same observation extends to solutions u_N to the truncated system (5.1). Note that the scaled initial data satisfies

$$\|u_{0,\lambda}\|_{H^s(\mathbb{T}_\lambda)} = \lambda^{-\frac{1}{2} - s} \|u_0\|_{H^s_\lambda(\mathbb{T})} \le \lambda^{-\frac{1}{2}} \|u_0\|_{H^s(\mathbb{T})},$$

where $\|f\|_{H^s_\lambda(\mathbb{T})} := \|\langle n \rangle_\lambda^s \widehat{f}(n)\|_{\ell^2_n}$ with $\langle n \rangle_\lambda = (\lambda^2 + n^2)^{\frac{1}{2}}$.

We can proceed in the proof of Proposition 6.3 for $S_\lambda(u), S_\lambda(u_N)$, where (6.12) becomes

$$\|S_\lambda(u-u_N)\|^2_{L^\infty_{\lambda^2T}H^{s-1}_x(\mathbb{T}_\lambda)} \le \|S_\lambda(u-u_N)\|^2_{L^\infty_{\delta^2T}H^{s-1}_x(\mathbb{T}_\lambda)}\big[\varepsilon + C\|u_{0,\lambda}\|_{H^s(\mathbb{T}_\lambda)}(1+\|u_{0,\lambda}\|_{H^s_{\mathbb{T}_\lambda}})^2\big] + C_\varepsilon N^{-\theta}\|u_{0,\lambda}\|^2_{H^s(\mathbb{T}_\lambda)}.$$

For $u_0 \in B^\sigma(R)$, by choosing $\varepsilon > 0$ small and $\lambda \ge 1$ sufficiently large depending on R, such that

$$\varepsilon + C\|u_{0,\lambda}\|^2_{H^s(\mathbb{T}_\lambda)}(1+\|u_{0,\lambda}\|_{H^s(\mathbb{T}_\lambda)})^2 \le \varepsilon + \lambda^{-1}CR^2(1+\lambda^{-\frac{1}{2}}R) \le \frac{1}{2},$$

we get that

$$\|S_\lambda(u-u_N)\|^2_{L^\infty_{\lambda^2T}H^{s-1}_x(\mathbb{T}_\lambda)} \le (2C_\varepsilon\lambda^{-1}R^2)N^{-\theta},$$

which gives the intended decay for the small data scaled solutions. To recover this result for the original solutions, note that

$$\begin{aligned}\|u-u_N\|_{L^\infty_T H^{s-1}_x(\mathbb{T})} &\le \frac{1}{\min(\lambda^{s-1},1)}\|u-u_N\|_{L^\infty_T H^{s-1}_{x,\lambda}(\mathbb{T})}\\ &= \frac{\lambda^{\frac{1}{2}+(s-1)}}{\min(\lambda^{s-1},1)}\|S_\lambda(u-u_N)\|_{L^\infty_{\lambda^2T}H^{s-1}_x(\mathbb{T}_\lambda)}\\ &\le \frac{\lambda^{s-1}\sqrt{2C_\varepsilon}R}{\min(\lambda^{s-1},1)}N^{-\frac{\theta}{2}},\end{aligned}$$

from which we get the needed decay in (6.2) for solutions with large data.

6.2. **Proof of Theorem 1.5.** The argument in the deep-water and shallow-water regime are analogous, thus we focus our discussion in this section in the deep-water regime. The truncated system (5.1) is still Hamiltonian and it was shown in [33] that the truncated Gibbs measure $\rho_{\delta,\frac{1}{2},N}$ associated with this system is still invariant. However, due to the lack of conservation of higher order quantities in Section 5, we do not expect the corresponding truncated measures $\rho_{\delta,\frac{k}{2},N,K}$ for $k \ge 3$ to be invariant. To bypass this issue, we proceed as in [55, 56]. We fix $0 < \delta < \infty$, $k \ge 3$, and $K > 0$, and let $\frac{1}{2} < s < \sigma < (k-1)/2$. We also use $\mathcal{B}(H^\sigma)$ for the Borel sets in $H^\sigma(\mathbb{T})$.

Recall that the dynamics of (5.1) decouple as follows

$$\Phi^N_t(u_0) = \mathbf{P}_N\Phi^N_t(u_0) + \mathbf{P}_{>N}\Phi^N_t(u_0) =: u^{\text{low}}_N(t) + u^{\text{high}}_N(t), \tag{6.13}$$

where $\mathbf{P}_{>N} = \text{Id} - \mathbf{P}_N$, where $u^{\text{high}}_N(t) = S(t)\mathbf{P}_{>N}u_0$ is the linear ILW evolution of the high frequency part of the initial data, while u^{low}_N solves

$$\partial_t u^{\text{low}}_N - \mathcal{G}_\delta\partial_x^2 u^{\text{low}}_N = \partial_x\mathbf{P}_N(u^{\text{low}}_N)^2, \tag{6.14}$$

with initial data $u^{\text{low}}_N|_{t=0} = \mathbf{P}_N u_0$. As mentioned in Section 5, the truncated system (5.1) is globally well-posed in $L^2(\mathbb{T})$.

We now consider the relevant measures associated with the truncated system. We recall that the measure $\rho_{\delta,\frac{k}{2},N,K}$ in (1.15) given by

$$\rho_{\delta,\frac{k}{2},N,K}(du) := Z^{-1}_{\delta,\frac{k}{2},N}F_{\delta,\frac{k}{2},N,K}(u)d\mu_{\delta,\frac{k}{2}}(u).$$

Note that we can write the Gaussian measure $\mu_{\delta,\frac{k}{2}}$ in (1.11) as product measure on $E_N\otimes E_N^\perp$, where E_N be the real vector space spanned by $\{\cos(nx),\sin(nx)\}_{1\le n\le N}$ and $E_N^\perp$ is the orthogonal complement of E_N in $H^\sigma(\mathbb{T})$. In particular, we have $\mu_{\delta,\frac{k}{2}}=\mu_N^{\text{low}}\otimes\mu_N^{\text{high}}$ where $\mu_N^{\text{low}}=(\mathbf{P}_N)_*\mu_{\delta,\frac{k}{2}}$ and $\mu_N^{\text{high}}=(\mathbf{P}_{>N})_*\mu_{\delta,\frac{k}{2}}$ are the push-forward images of the Gaussian measure $\mu_{\delta,\frac{k}{2}}$. Using the notation

$$\mathbf{P}_N u(x)=\frac{1}{\sqrt{2\pi}}\sum_{0<|n|\le N}\widehat{u}_n e^{inx},$$

we can write μ_N^{low} as

$$\mu_N^{\text{low}}(du)=Z_N^{-1}\exp\bigg(-\frac{1}{2}\sum_{\substack{\ell=0\\ \text{even}}}^{k}a_\ell\|\mathcal{G}_\delta^{\frac{k-\ell}{2}}\mathbf{P}_N u\|^2_{\dot{H}^{\frac{k}{2}}}\bigg)\prod_{0<|n|\le N}d\widehat{u}_n,$$

where $\prod_{0<|n|\le N}d\widehat{u}_n$ denotes the Lebesgue measure on E_N.

The main ingredient for the invariance argument is the following "change of variables" result, which allows us to use the decay estimates in Section 5 to control the time derivative of $\Phi_t^N(A)$ for a measurable set A.

Proposition 6.6. *For any $A\in\mathcal{B}(H^\sigma)$, we have the following identity*

$$\begin{aligned}&Z^{-1}_{\delta,\frac{k}{2},N}\int_{\Phi_t^N(A)}F_{\delta,\frac{k}{2},N,K}(u)d\mu(u)\\&=\int_A\eta_K(\|\mathbf{P}_N\Phi_t^N(u)\|^2_{L^2})\exp\big(-E_{\delta,\frac{k}{2}}(\mathbf{P}_N(\Phi_t^N(u))\big)\prod_{0<|n|\le N}d\widehat{u}_n\otimes d\mu_N^{\text{high}}(\mathbf{P}_{>N}u).\end{aligned}\tag{6.15}$$

To establish this result, we need some auxiliary results on the Lebesgue measure on E_N and the Gaussian measure μ_N^{high}.

Lemma 6.7. *The following results hold:*
(i) *The map $\Phi_t^N\mathbf{P}_N$ is measure preserving on E_N equipped with the Lebesgue measure $\prod_{0<|n|\le N}d\widehat{u}_n$.*
(ii) *The map $S(t)=e^{-t\widetilde{\mathcal{G}}_\delta\partial_x^2}$ is measures preserving on $E_N^\perp$ equipped with the Gaussian measure μ_N^{high}.*

Proof. The result in (i) follows from Liouville's theorem once we establish that the spatial Fourier coefficients of u_N^{low} evolve according to a Hamiltonian system of ODEs with a divergence free vector field. Let $v=u_N^{\text{low}}$ and $\widehat{v}(n)=a_n+ib_n$ for $|n|\le N$. Then,

$$\begin{aligned}&\partial_t\widehat{v}(n)=in(n\coth(\delta n)-\tfrac{1}{\delta})\widehat{v}(n)+\sum_{\substack{n=n_1+n_2\\|n_j|\le N}}in\widehat{v}(n_1)\widehat{v}(n_2)\\&\Longleftrightarrow\begin{cases}\partial_t a_n=-n(\coth(\delta n)-\tfrac{1}{\delta})b_n-\sum\limits_{\substack{n=n_1+n_2\\|n_j|\le N}}n(a_{n_1}b_{n_2}+a_{n_2}b_{n_1})\\ \partial_t b_n=n(n\coth(\delta n)-\tfrac{1}{\delta})a_n+\sum\limits_{\substack{n=n_1+n_2\\|n_j|\le N}}n(a_{n_1}a_{n_2}-b_{n_1}b_{n_2})\end{cases}\\&\Longleftrightarrow\partial_t(a_n,b_n)=(F_{a_n}(a,b),F_{b_n}(a,b)).\end{aligned}$$

Note that

$$-\frac{\partial F_{a_n}}{\partial b_n} = \frac{\partial F_{b_n}}{\partial a_n} = n(\coth(\delta n) - \tfrac{1}{\delta}),$$

from which we conclude that $\sum_{n=1}^{N}\Big(\frac{\partial F_{a_n}}{\partial b_n} + \frac{\partial F_{b_n}}{\partial a_n}\Big) = 0$, showing that the vector field F is divergence free.

For (ii), the same proof as that of [55, Lemma 5.3] applies, as $S(t)$ also induces a phase rotation on the Fourier side by $tn(n\coth(\delta n) - \frac{1}{\delta})$, analogous to the effect of the linear propagator for BO. In particular, by using that $in\widehat{\mathcal{G}_\delta}(n) = K_\delta(n)$, we get

$$\begin{aligned} S(t)(\cos(nx)) &= e^{itnK_\delta(n)}\cos(nx) = \cos(tnK_\delta(n) + nx)\\ &= \cos(tnK_\delta(n))\cos(nx) - \sin(tnK_\delta(n))\sin(nx), \end{aligned}$$

where $e^{-itnK_\delta(n)}$ acts like rotation and has effect of shifting the phase of the (positive) frequency components by $-nK_\delta(n)$. Similarly, we have

$$\begin{aligned} S(t)(\sin(nx)) &= \sin(tnK_\delta(n) + nx)\\ &= \sin(tnK_\delta(n))\cos(nx) + \cos(tnK_\delta(n))\sin(nx). \end{aligned}$$

Therefore for fixed t and n the map $S(t)$ acts as a rotation on the two dimensional real vector space spanned by $\cos(nx)$ and $\sin(nx)$. Hence by the invariance of the Lebesgue measure and the diagonal quadratic forms by rotations, any centered Gaussian measure on the two dimensional space $\text{span}\{\cos(nx), \sin(nx)\}$ is invariant by $S(t)$. The remaining of the proof is identical, thus we omit details. □

Proof of Proposition 6.6. By definition we have the identities

$$\mathbf{P}_N\Phi^N_{\delta,t} = \Phi^N_{\delta,t}\mathbf{P}_N, \qquad \mathbf{P}_{>N}\Phi^N_{\delta,t} = S(t)\mathbf{P}_{>N}, \tag{6.16}$$

and we introduce the notation $dL_N = \prod_{0<|n|\le N} d\widehat{u}_n$. Then, we can write

$$Z^{-1}_{\delta,\frac{k}{2},N}\int_{\Phi^N_t(A)} F_{\delta,\frac{k}{2},N,K}(u)d\mu_{\delta,\frac{k}{2}} \tag{6.17}$$

$$= \int_{\Phi^N_t(A)} \eta_K(\|\mathbf{P}_N u\|^2_{L^2})\exp(-E_{\delta,\frac{k}{2}}(\mathbf{P}_N u))\; dL_N\otimes d\mu^{\text{high}}_N \tag{6.18}$$

$$= \int_{E_N\otimes E_N^\perp} \mathbf{1}_{\Phi^N_t(A)}(u)\eta_K(\|\mathbf{P}_N u\|^2_{L^2})\exp(-E_{\delta,\frac{k}{2}}(\mathbf{P}_N u))\; dL_N\otimes d\mu^{\text{high}}_N. \tag{6.19}$$

Using Fubini's theorem and Lemma 6.7(ii), we get

$$\begin{aligned} &Z^{-1}_{\delta,\frac{k}{2},N}\int_{\Phi^N_t(A)} F_{\delta,\frac{k}{2},N,K}(u)d\mu\\ &= \int_{E_N}\eta_K(\|\mathbf{P}_N u\|^2_{L^2})e^{-E_{\delta,\frac{k}{2}}(\mathbf{P}_N u)}\Big(\int_{E_N^\perp}\mathbf{1}_{\Phi^N_t(A)}(\mathbf{P}_N u, S(t)\mathbf{P}_{>N}u)d\mu^{\text{high}}_N\Big)dL_N\\ &= \int_{E_N^\perp}\Big(\int_{E_N}\eta_K(\|\mathbf{P}_N u\|^2_{L^2})e^{-E_{\delta,\frac{k}{2}}(\mathbf{P}_N u)}\mathbf{1}_{\Phi^N_t(A)}(\mathbf{P}_N u, S(t)\mathbf{P}_{>N}u)dL_N\Big)d\mu^{\text{high}}_N. \end{aligned}$$

Now, Lemma 6.7(i) gives

$$Z^{-1}_{\delta,\frac{k}{2},N}\int_{\Phi^N_t(A)} F_{\delta,\frac{k}{2},N,K}(u)d\mu = \int_{E_N^\perp}\Big(\int_{E_N}\eta_K(\|\Phi^N_t(\mathbf{P}_N u)\|^2_{L^2})e^{-E_{\delta,\frac{k}{2}}(\Phi^N_t(\mathbf{P}_N u))}$$

$$\times \mathbf{1}_{\Phi_t^N(A)}(\Phi_t^N u)dL_N\Big)d\mu_N^{\text{high}}. \quad (6.20)$$

Since the solution map Φ_t^N is a bijection on $H^\sigma(\mathbb{T})$, we have that $\mathbf{1}_{\Phi_t^N(A)}(\Phi_t^N(u)) = \mathbf{1}_A(u)$, from which it follows that

$$Z_{\delta,\frac{k}{2},N}^{-1}\int_{\Phi_t^N(A)} F_{\delta,\frac{k}{2},N,K}(u)d\mu = \int_A \eta_K(\|\Phi_t^N\mathbf{P}_N u\|_{L^2}^2)e^{-E_{\delta,\frac{k}{2}}(\Phi_t^N\mathbf{P}_N u)}dL_N\otimes d\mu_N^{\text{high}}$$

and the result follows from (6.16). □

The following proposition establishes the almost invariance of the measure $\rho_{\delta,\frac{k}{2},N,K}$.

Proposition 6.8. *Let $\sigma \geq 0$ and $T > 0$. We have the following:*

$$\lim_{N\to\infty} \sup_{\substack{t\in[0,T]\\ A\in\mathcal{B}(H^\sigma)}} \Big|\frac{d}{dt}\int_{\Phi_t^N(A)} F_{\delta,\frac{k}{2},N,K}(u)d\mu_{\delta,\frac{k}{2}}\Big| = 0.$$

Proof. We proceed as in [55, Proposition 5.4].
Step 1: $t = 0$.
We first show that

$$\lim_{N\to\infty} \sup_{A\in\mathcal{B}(H^\sigma)} \Big|\frac{d}{dt}\Big(\int_{\Phi_t^N(A)} F_{\delta,\frac{k}{2},N,K}(u)d\mu_{\delta,\frac{k}{2}}\Big)_{t=0}\Big| = 0. \quad (6.21)$$

Using Proposition 6.6 and the conservation of L^2-norm for (5.1), we obtain

$$\begin{aligned}&\frac{d}{dt}\Big(\int_{\Phi_t^N(A)} F_{\delta,\frac{k}{2},N,K}(u)d\mu_{\delta,\frac{k}{2}}\Big)_{t=0}\\ &= -\int_A \frac{d}{dt}\Big(E_{\delta,\frac{k}{2}}(\Phi_t^N\mathbf{P}_N u)\Big)_{t=0} F_{\delta,\frac{k}{2},N,K}(u)d\mu_{\delta,\frac{k}{2}}.\end{aligned}$$

By Hölder's inequality, Proposition 3.2, and Proposition 5.1, we obtain the uniform convergence to zero.
Step 2: $t' \in (0, T]$.
We first use the definition of derivative to obtain

$$\begin{aligned}&\frac{d}{dt}\Big(\int_{\Phi_t^N(A)} F_{\delta,\frac{k}{2},N,K}(u)d\mu_{\delta,\frac{k}{2}}\Big)_{t=t'}\\ &= \lim_{h\to 0} h^{-1}\Big(\int_{\Phi_{t'+h}^N(A)} F_{\delta,\frac{k}{2},N,K}(u)d\mu_{\delta,\frac{k}{2}} - \int_{\Phi_{t'}^N(A)} F_{\delta,\frac{k}{2},N,K}(u)d\mu_{\delta,\frac{k}{2}}\Big).\end{aligned}$$

Then, using the fact that $\Phi_{t'+h}^N(A) = \Phi_h^N \circ \Phi_{t'}^N(A)$ we obtain that

$$\frac{d}{dt}\Big(\int_{\Phi_t^N(A)} F_{\delta,\frac{k}{2},N,K}(u)d\mu_{\delta,\frac{k}{2}}\Big)_{t=t'} = \frac{d}{dt}\Big(\int_{\Phi_t^N(\widetilde{A})} F_{\delta,\frac{k}{2},N,K}(u)d\mu_{\delta,\frac{k}{2}}\Big)_{t=0}$$

where $\widetilde{A} = \Phi_{t'}^N(A)$. Therefore, the result follows from Step 1 (since the limit holds uniformly in the integration set). □

From Proposition 6.8 and the Fundamental Theorem of Calculus, we can conclude that for all $\sigma \geq 0$, $T \in \mathbb{R}$, $A \in \mathcal{B}(H^\sigma)$, and $t \in [0, T]$, we have

$$\lim_{N\to\infty}\Big(\int_A F_{\delta,\frac{k}{2},N,K}(u)d\mu_{\delta,\frac{k}{2}} - \int_{\Phi_t^N(A)} F_{\delta,\frac{k}{2},N,K}(u)d\mu_{\delta,\frac{k}{2}}\Big) = 0. \quad (6.22)$$

We can now establish the following short-time result on the measure $\rho_{\delta,\frac{k}{2},K}$ transported by the full flow Φ_t of ILW (1.1), on compact sets.

Lemma 6.9. *Let $k \geq 3$, $\frac{1}{2} < \sigma < \frac{k-1}{2}$, and $R > 0$. Then, there exists $T = T(R) > 0$ such that for every compact set $\mathbf{K} \in \mathcal{B}(H^\sigma)$ with $\mathbf{K} \subset B^\sigma(R)$, we have*

$$\int_{\mathbf{K}} F_{\delta,\frac{k}{2},K}(u)d\mu_{\delta,\frac{k}{2}} \leq \int_{\Phi_t(\mathbf{K})} F(u)d\mu_{\delta,\frac{k}{2}}, \qquad \forall t \in (-T,T).$$

Proof. By (6.22) and (1.18), we get that

$$\begin{aligned}
&\lim_{N\to\infty} \int_{\Phi_t^N(\mathbf{K})} F_{\delta,\frac{k}{2},N,K}(u)d\mu_{\delta,\frac{k}{2}} \\
&= \lim_{N\to\infty} \bigg(\int_{\Phi_t^N(\mathbf{K})} F_{\delta,\frac{k}{2},N,K}(u)d\mu_{\delta,\frac{k}{2}} - \int_{\mathbf{K}} F_{\delta,\frac{k}{2},N,K}(u)d\mu_{\delta,\frac{k}{2}} \bigg) + \lim_{N\to\infty} \int_{\mathbf{K}} F_{\delta,\frac{k}{2},N,K}(u)d\mu_{\delta,\frac{k}{2}} \\
&= \int_{\mathbf{K}} F_{\delta,\frac{k}{2},K}(u)\, d\mu_{\delta,\frac{k}{2}}.
\end{aligned} \tag{6.23}$$

By Proposition 6.3, for $\frac{1}{2} < s < \sigma$, there exists $T = T(R) > 0$ such that for every $\varepsilon > 0$, there exists a suitable $N_0(\varepsilon)$ such that $\Phi_t^N(\mathbf{K}) \subset \Phi_t(\mathbf{K}) + B^s(\varepsilon)$ for all $N \geq N_0$ and $t \in (-T,T)$. Therefore,

$$\int_{\Phi_t^N(\mathbf{K})} F_{\delta,\frac{k}{2},N,K}(u)d\mu_{\delta,\frac{k}{2}} \leq \int_{\Phi_t(\mathbf{K})+B^s(\varepsilon)} F_{\delta,\frac{k}{2},N,K}(u)d\mu_{\delta,\frac{k}{2}} \tag{6.24}$$

for all $t \in (-T,T)$ and $N \geq N_0$. From the $L^1(d\mu_{\delta,\frac{k}{2}})$ convergence of the density in (1.18), taking limits as $N \to \infty$ in (6.24) and using (6.23), we get

$$\int_{\mathbf{K}} F_{\delta,\frac{k}{2},K}(u)\, d\mu_{\delta,\frac{k}{2}} = \lim_{N\to\infty} \int_{\Phi_t^N(\mathbf{K})} F_{\delta,\frac{k}{2},N,K}(u)\, d\mu_{\delta,\frac{k}{2}} \leq \int_{\Phi_t(\mathbf{K})+B^s(\varepsilon)} F_{\delta,\frac{k}{2},K}(u)\, d\mu_{\delta,\frac{k}{2}} \tag{6.25}$$

for all $t \in (-T,T)$. Since $\mathbf{K}$ is compact, then it is closed in $H^\sigma(\mathbb{T})$. Moreover, since Φ_t is a diffeomorphism on $H^\sigma(\mathbb{T})$, we have that $\Phi_t(\mathbf{K})$ is closed in $H^\sigma(\mathbb{T})$. As a consequence we deduce that

$$\bigcap_{\varepsilon>0} (\Phi_t(\mathbf{K}) + B^s(\varepsilon)) = \Phi_t(\mathbf{K}),$$

and by Lebesgue's theorem

$$\lim_{\varepsilon\to 0} \int_{\Phi_t(\mathbf{K})+B^s(\varepsilon)} F_{\delta,\frac{k}{2},K}(u)d\mu_{\delta,\frac{k}{2}} = \int_{\Phi_t(\mathbf{K})} F_{\delta,\frac{k}{2},K}(u)d\mu_{\delta,\frac{k}{2}}. \tag{6.26}$$

Lastly, the result follows from taking a limit as $\varepsilon \to 0$ in (6.25) and (6.26). ☐

To extend Lemma 6.9 globally-in-time, we need global-in-time control over the flow Φ_t on compact sets. From Lemma 6.1, we have that for all $\sigma \geq \frac{1}{2}$, $T > 0$ and $\mathbf{K} \subset H^\sigma(\mathbb{T})$ compact, then there exists $R > 0$ such that

$$\{\Phi_t(\mathbf{K}) :\ 0 \leq t \leq T\} \subset B^\sigma(R). \tag{6.27}$$

Lemma 6.10. *Let $k \geq 3$, $\frac{1}{2} < \sigma < \frac{k-1}{2}$, $t \in \mathbb{R}$ and $\mathbf{K} \subset H^\sigma(\mathbb{T})$ be a compact set. Then,*

$$\int_{\mathbf{K}} F_{\delta,\frac{k}{2},K}(u)d\mu_{\delta,\frac{k}{2}} \leq \int_{\Phi_t(\mathbf{K})} F_{\delta,\frac{k}{2},K}(u)d\mu_{\delta,\frac{k}{2}}.$$

Proof. We prove the result for any $t_0 \in \mathbb{R}$ positive, since the analysis for negative t_0 is analogous. By (6.27), there exists $R > 0$ such that

$$\{\Phi_t(\mathbf{K}) : \ 0 \le t \le t_0\} \subset B^\sigma(R). \tag{6.28}$$

Let $T = T(R) \in (0, t_0]$ be the one given in Lemma 6.9, and choose t_1 such that

$$t_1 \in (0, T] \quad \text{and} \quad t_0/t_1 \in \mathbb{N}.$$

Then,

$$\int_{\mathbf{K}} F_{\delta,\frac{k}{2},K}(u) d\mu_{\delta,\frac{k}{2}} \le \int_{\Phi_{t_1}(\mathbf{K})} F_{\delta,\frac{k}{2},K}(u) d\mu_{\delta,\frac{k}{2}}.$$

From (6.28), we have that $\Phi_{t_1}(\mathbf{K}) \subset B^\sigma(R)$, hence Lemma 6.9 can be iterated and we obtain

$$\int_{\Phi_{t_1}(\mathbf{K})} F_{\delta,\frac{k}{2},K}(u) d\mu_{\delta,\frac{k}{2}} \le \int_{\Phi_{t_1}(\Phi_{t_1}(\mathbf{K}))} F_{\delta,\frac{k}{2},K}(u) d\mu_{\delta,\frac{k}{2}} = \int_{\Phi_{2t_1}(\mathbf{K})} F_{\delta,\frac{k}{2},K}(u) d\mu_{\delta,\frac{k}{2}}.$$

By repeating this argument N_0 times such that $t_0 = N_0 t_1$, we cover the whole time interval $[0, t_0]$, from which we get

$$\int_{\mathbf{K}} F_{\delta,\frac{k}{2},K}(u) d\mu_{\delta,\frac{k}{2}} \le \int_{\Phi_{t_0}(\mathbf{K})} F_{\delta,\frac{k}{2},K}(u) d\mu_{\delta,\frac{k}{2}}.$$

This completes the proof of Lemma 6.10. □

Using the reversibility of the flow, we now obtain the statement.

Lemma 6.11. *Let $k \ge 3$, $\frac{1}{2} < \sigma < \frac{k-1}{2}$, and $t \in \mathbb{R}$. Then, for every compact $\mathbf{K} \subset H^\sigma$ we have*

$$\int_{\mathbf{K}} F_{\delta,\frac{k}{2},K}(u) d\mu_{\delta,\frac{k}{2}} = \int_{\Phi_t(\mathbf{K})} F_{\delta,\frac{k}{2},K}(u) d\mu_{\delta,\frac{k}{2}}.$$

Proof. Using Lemma 6.10, for every compact $\widetilde{\mathbf{K}} \subset H^\sigma(\mathbb{T})$, we can write

$$\int_{\widetilde{\mathbf{K}}} F_{\delta,\frac{k}{2},K}(u) d\mu_{\delta,\frac{k}{2}} \le \int_{\Phi_{-t}(\widetilde{\mathbf{K}})} F_{\delta,\frac{k}{2}}(u) d\mu_{\delta,\frac{k}{2}}.$$

By choosing $\widetilde{\mathbf{K}} = \Phi_t(\mathbf{K})$, and the fact that the flow Φ_t is a diffeomorphism, we get

$$\int_{\Phi_t(\mathbf{K})} F_{\delta,\frac{k}{2},K}(u) d\mu_{\delta,\frac{k}{2}} \le \int_{\mathbf{K}} F_{\delta,\frac{k}{2},K}(u) d\mu_{\delta,\frac{k}{2}}.$$

Combining this inequality with Lemma 6.10 completes the proof. □

Let us now complete the proof of Theorem 1.5. Let A be an arbitrary Borel set in $H^\sigma(\mathbb{T})$. It is well–known that there exists a sequence of compact sets $\mathbf{K}_n \subset A$ such that

$$\lim_{n\to\infty} \int_{\mathbf{K}_n} F_{\delta,\frac{k}{2},K}(u) d\mu_{\delta,\frac{k}{2}} = \int_A F_{\delta,\frac{k}{2},K}(u) d\mu_{\delta,\frac{k}{2}}.$$

On the other hand, by Lemma 6.11 we have

$$\int_{\mathbf{K}_n} F_{\delta,\frac{k}{2},K}(u) d\mu_{\delta,\frac{k}{2}} = \int_{\Phi_t(\mathbf{K}_n)} F_{\delta,\frac{k}{2},K}(u) d\mu_{\delta,\frac{k}{2}} \le \int_{\Phi_t(A)} F_{\delta,\frac{k}{2},K}(u) d\mu_{\delta,\frac{k}{2}}$$

since $\Phi_t(\mathbf{K}_n) \subset \Phi_t(A)$ and $F_{\delta,\frac{k}{2},K}(u) \geq 0$. As a consequence, we get

$$\int_A F_{\delta,\frac{k}{2},K}(u)d\mu_{\delta,\frac{k}{2}} \leq \int_{\Phi_t(A)} F_{\delta,\frac{k}{2},K}(u)d\mu_{\delta,\frac{k}{2}}.$$

The opposite inequality follows from the time reversibility of the flow. This completes the proof of Theorem 1.5.

Appendix A. Structure of the conserved quantities for ILW

A.1. **Deriving the conserved quantities for ILW and BO.** We follow the derivation of the conserved quantities for ILW in [50]. See also [25, 31, 39] for alternative derivations. Consider the following equations for u and V

$$2u = \mu(e^V - 1) + \mathcal{G}_\delta V_x + \delta^{-1}V - iV_x, \tag{A.1}$$

$$V_t = \mu(e^V - 1)V_x + \mathcal{G}_\delta V_{xx} + V_x\mathcal{G}_\delta V_x + \delta^{-1}VV_x, \tag{A.2}$$

where we are interested in u as a solution to ILW (1.1). The following lemma shows that if, for a given u, we can find $V = V(u)$ satisfying (A.1), then (A.2) holds.

Lemma A.1. *Let $u : \mathbb{R}^2 \to \mathbb{R}$ and $V : \mathbb{R}^2 \to \mathbb{C}$ satisfying* (A.1)*, with $V = \sum_{n\geq 1} \mu^{-n}\chi_n$, for functions χ_n independent of μ. Then, u solves ILW* (1.1) *if and only if V solves* (A.2).

Proof. Using (A.1), we have that

$$\begin{aligned} &2(\partial_t u - \mathcal{G}_\delta\partial_x^2 u - u\partial_x u) \\ &\quad = (\mu e^V + \delta^{-1} + \mathcal{G}_\delta\partial_x - i\partial_x)[V_t - \mu(e^V - 1)V_x - \mathcal{G}_\delta V_{xx} - V_x\mathcal{G}_\delta V_x - \delta^{-1}VV_x]. \end{aligned} \tag{A.3}$$

If V solves (A.2), from (A.3) we see that u solves ILW (1.1).

For the forward implication, since the expression in brackets on the RHS of (A.3) is of the form $f = \sum_{n\geq 1} \mu^{-n} f_n$, it suffices to show that the only such solution to $\mathcal{L}f = 0$, where $\mathcal{L} = \mu e^V + \delta^{-1} + \mathcal{G}_\delta\partial_x - i\partial_x$, is $f \equiv 0$. Let $\mathcal{L}_0 := \delta^{-1} + \mathcal{G}_\delta\partial_x - i\partial_x$ and f as above. Then,

$$0 = \langle \mathcal{L}f, \mathcal{L}_0 f\rangle = \int \overline{\mathcal{L}f}\mathcal{L}_0 f\, dx = -\mu^2 \int |e^V f|^2\, dx + \int |\mathcal{L}_0 f|^2\, dx.$$

Expanding V and f through its series representation, we have that

$$\begin{aligned} e^V f &= \bigg(1 + \sum_{\ell\geq 1} \frac{1}{\ell!}V^\ell\bigg)\bigg(\sum_{n\geq 1} \mu^{-n} f_n\bigg) \\ &= \bigg(1 + \sum_{n\geq 1} \mu^{-n} \sum_{\ell=1}^n \frac{1}{\ell!} \sum_{n_{1\cdots\ell}=n} \chi_{n_1}\cdots\chi_{n_\ell}\bigg)\bigg(\sum_{n\geq 1} \mu^{-n} f_n\bigg) \\ &= \mu^{-1} f_1 + \sum_{n\geq 2} \mu^{-n}\bigg(f_n + \sum_{n_{12}=n} f_{n_1}\widetilde{V}_{n_2}\bigg) \\ &=: \sum_{n\geq 1} \mu^{-n} A_n, \end{aligned}$$

where

$$\widetilde{V}_n = \sum_{\ell=1}^n \frac{1}{\ell!} \sum_{n_{1\cdots\ell}=n} \chi_{n_1}\cdots\chi_{n_\ell},$$

$$A_1 = f_1,$$
$$A_n = f_n + \sum_{n_{12}=n} f_{n_1}\widetilde{V}_{n_2}, \quad n \geq 2.$$

Then,

$$\begin{aligned}
0 &= -\sum_{n\geq 2}\mu^{-n+2}\sum_{n_{12}=n}\int \overline{A_{n_1}}A_{n_2}\,dx + \sum_{n\geq 2}\mu^{-n}\sum_{n_{12}=n}\int \overline{\mathcal{L}_0 f_{n_1}}\mathcal{L}_0 f_{n_2}\,dx \\
&= -\int |A_1|^2\,dx - \mu^{-1}\int [\overline{A_1}A_2 + \overline{A_2}A_1]\,dx \\
&\quad + \sum_{n\geq 2}\mu^{-n}\Big[-\sum_{n_{12}=n+2}\int \overline{A_{n_1}}A_{n_2}\,dx + \sum_{n_{12}=n}\int \overline{\mathcal{L}_0 f_{n_1}}\mathcal{L}_0 f_{n_2}\,dx\Big].
\end{aligned}$$

From the above, the coefficients of the powers of μ must be zero. For the coefficient of μ^0, we get

$$0 = -\int |A_1|^2\,dx = -\int |f_1|^2\,dx \implies A_1 = f_1 \equiv 0.$$

For the coefficient of μ^{-1} we get

$$0 = -\int [\overline{A_1}A_2 + \overline{A_2}A_1]\,dx$$

which is satisfied since $A_1 \equiv 0$. For the coefficient of μ^{-2}, we get

$$\begin{aligned}
0 &= -\int [\overline{A_1}A_3 + |A_2|^2 + \overline{A_3}A_1]\,dx + \int |\mathcal{L}_0 f_0|^2\,dx = -\int |A_2|^2\,dx \\
&\implies A_2 \equiv 0 \implies f_2 + f_1\widetilde{V}_1 \equiv 0 \implies f_2 \equiv 0.
\end{aligned}$$

For μ^{-3},

$$0 = -\int [\overline{A_1}A_4 + \overline{A_2}A_3 + \overline{A_3}A_2]\,dx + \int [\overline{\mathcal{L}_0 f_1}\mathcal{L}_0 f_2 + \overline{\mathcal{L}_0 f_2}\mathcal{L}_0 f_1]\,dx$$

which holds since $A_1 \equiv A_2 \equiv f_1 \equiv f_2 \equiv 0$.

To establish that $f_n \equiv 0$ for all $n \in \mathbb{N}$, we proceed by induction. Assume that $f_k \equiv 0$ for $1 \leq k \leq n-1$, from which it follows that $A_k \equiv 0$ for $1 \leq k \leq n-1$. Considering the coefficients of μ^{-2n+2} and μ^{-2n+1}, respectively, we get that

$$\begin{aligned}
0 &= -\sum_{n_{12}=2n}\int \overline{A_{n_1}}A_{n_2}\,dx + \sum_{n_{12}=2n-2}\int \overline{\mathcal{L}_0 f_{n_1}}\mathcal{L}_0 f_{n_2}\,dx = -\int |A_n|^2\,dx, \\
0 &= -\sum_{n_{12}=2n+1}\int \overline{A_{n_1}}A_{n_2}\,dx + \sum_{n_{12}=2n-1}\int \overline{\mathcal{L}_0 f_{n_1}}\mathcal{L}_0 f_{n_2}\,dx,
\end{aligned}$$

where we conclude from the first equality that

$$A_n \equiv 0 \implies f_n + \sum_{n_{12}=n} f_{n_1}\widetilde{V}_{n_2} = 0 \implies f_n \equiv 0,$$

while the second equality holds trivially from the assumptions and the conclusion that $f_n \equiv 0$. Combining all the results above, we conclude that the only solution $f = \sum_{n\geq 1}\mu^{-n}f_n$ to $\mathcal{L}f = 0$ is $f \equiv 0$. Consequently, the bracket on the RHS of (A.3) must be equal to 0. □

To derive the conserved quantities for ILW (1.1), let $\mu > 0$ and $V = \sum_{n\geq 1} \mu^{-n}\chi_n$. Replacing V in (A.1), we obtain the following recurrence for χ_n

$$\chi_1 = 2u, \tag{A.4}$$

$$\chi_n = -\sum_{j=2}^{n} \frac{1}{j!} \sum_{n_{1\cdots j}=n} \chi_{n_1}\cdots\chi_{n_j} - (-i+\mathcal{G}_\delta)\partial_x\chi_{n-1} - \frac{1}{\delta}\chi_{n-1}, \qquad n\geq 2. \tag{A.5}$$

From Lemma A.1, we know that since V satisfies (A.1), it also satisfies (A.2). Since the RHS of (A.2) has mean zero, V is a conserved quantity. To see that χ_n, for $n\in\mathbb{N}$, are conserved quantities, note that we can rewrite (A.2) as

$$V_t = 2\partial_x u_x - (\delta^{-1}+\mu)V_x + iV_{xx} + V_x\mathcal{G}_\delta V_x + \delta^{-1}VV_x$$

which implies that

$$\sum_{n\geq 1}\mu^{-n}\partial_t\chi_n = \partial_x\chi_1 + \sum_{n\geq 1}\mu^{-n}(-\delta+i\partial_x)\partial_x\chi_n - \sum_{n\geq 0}\mu^{-n}\partial_x\chi_{n+1} + \sum_{n\geq 2}\mu^{-n}\sum_{n_{12}=n}\partial_x\chi_{n_1}(\mathcal{G}_\delta\partial_x+\delta^{-1})\chi_{n_2}.$$

Collecting powers of μ, we get that

$$\begin{aligned}
\mu^0:&\qquad 0 = \partial_x\chi_1 - \partial_x\chi_1,\\
\mu^{-1}:&\qquad \partial_t\chi_1 = (-\delta+i\partial_x)\partial_x\chi_1 - \partial_x\chi_2,\\
\mu^{-n}:&\qquad \partial_t\chi_n = (-\delta+i\partial_x)\partial_x\chi_n - \partial_x\chi_{n+1} + \sum_{n_{12}=n}\partial_x\chi_{n_1}(\mathcal{G}_\delta\partial_x+\delta^{-1})\chi_{n_2},
\end{aligned}$$

for $n\geq 2$. From the second equation above, we see that $\int\chi_1\,dx$ is a conserved quantity, since the RHS is a full derivative. For $n\geq 2$, from the third equation, we get

$$\begin{aligned}
\partial_t\chi_{2m} &= \partial_x\big[(-\delta+i\partial_x)\chi_{2m} - \chi_{2m+1}\big] + \sum_{j=1}^{m-1}\big[\partial_x\chi_j(\mathcal{G}_\delta\partial_x\chi_{2m-j}) + (\mathcal{G}_\delta\partial_x\chi_j)\partial_x\chi_{2m-j}\big]\\
&\quad + \partial_x\chi_m(\mathcal{G}_\delta\partial_x\chi_m) + \delta^{-1}\sum_{j=1}^{m-1}\partial_x(\chi_j\chi_{2m-j}) + (2\delta)^{-1}\partial_x(\chi_m^2),\\
\partial_t\chi_{2m+1} &= \partial_x\big[(-\delta+i\partial_x)\chi_{2m+1} - \chi_{2m+2}\big]\\
&\quad + \sum_{j=1}^{m}[\partial_x\chi_j(\mathcal{G}_\delta\partial_x\chi_{2m+1-j}) + (\mathcal{G}_\delta\partial_x\chi_j)\partial_x\chi_{2m+1-j} + \partial_x(\chi_j\chi_{2m-j})],
\end{aligned}$$

when $n=2m$ and $n=2m+1$, respectively. Since $\mathcal{G}_\delta$ is anti-symmetric, the above allows us to conclude that $\int\chi_n\,dx$ is a conserved quantity for ILW (1.1) for all $n\in\mathbb{N}$. In fact, for all $n\in\mathbb{N}$, we have that

$$\frac{d}{dt}\int\chi_n\,dx = \frac{d}{dt}\int\operatorname{Re}\chi_n\,dx = \frac{d}{dt}\int\operatorname{Im}\chi_n\,dx = 0,$$

under the ILW dynamics[2].

[2]We believe that $\int\operatorname{Im}\chi_n\,dx = 0$ for all $n\in\mathbb{N}$, but establishing such a result would deviate from our focus. Instead, we will later define our conserved quantities only depending on $\int\operatorname{Re}\chi_n\,dx$.

Note that since $\mathcal{G}_\delta \to \mathcal{H}$ as $\delta \to \infty$, then (A.1)-(A.2) converge to

$$2u = \mu(e^V - 1) + \mathcal{H}V_x - iV_x, \tag{A.6}$$

$$V_t = \mu(e^V - 1)V_x + \mathcal{H}V_{xx} + V_x\mathcal{H}V_x, \tag{A.7}$$

and an analogue of Lemma A.1 follows for (A.6)-(A.7), i.e., if u and V solve (A.6) and it can be written as a formal series $V = \sum\limits_{n\geq 1} \mu^{-n}\chi_n^{\mathrm{BO}}$, then u solves BO (1.2) if and only if V solves (A.7). This is a consequence of the following identity for u and V satisfying (A.6):

$$\begin{aligned} &2(\partial_t u - \mathcal{H}\partial_x^2 u - 2u\partial_x u) \\ &= (\mu e^V + \mathcal{H}\partial_x - i\partial_x)\big[V_t - \mu(e^V - 1)V_x - \mathcal{H}V_{xx} - V_x\mathcal{H}V_x\big], \end{aligned}$$

and the fact that the only solution $f = \sum_{n\geq 1} \mu^{-n} f_n$ to $(\mu e^V + \mathcal{H}\partial_x - i\partial_x)f = 0$ is $f \equiv 0$. One can also derive recurrence formulas from (A.6) for the BO conserved quantities χ_n^{BO}:

$$\begin{aligned} \chi_1^{\mathrm{BO}} &= 2u, \\ \chi_n^{\mathrm{BO}} &= -\sum_{j=2}^{n} \frac{1}{j!} \sum_{n_{1\cdots j}=n} \chi_{n_1}^{\mathrm{BO}} \cdots \chi_{n_j}^{\mathrm{BO}} - (-i + \mathcal{H})\partial_x \chi_{n-1}^{\mathrm{BO}}, \quad n \geq 2. \end{aligned} \tag{A.8}$$

A.2. **Some useful properties of conserved quantities for ILW.** The following lemma describes the order of the contributions in $\operatorname{Re}\chi_n$ and $\operatorname{Im}\chi_n$ in terms of the number of u, ∂_x and δ^{-1}.

Lemma A.2. *For all $n \geq 2$, $\operatorname{Re}\chi_n$ and $\operatorname{Im}\chi_n$ can be written as polynomials in u and its derivatives with order n, where the order is defined as the sum of the number of u terms, the number of ∂_x operators, and the powers of δ^{-1}:*

$$n = \#u + \#\partial_x + \#\tfrac{1}{\delta}.$$

Proof. We prove this result by induction on n. For $n = 2$, we have

$$\chi_2 = -\tfrac{1}{2}\chi_1^2 + (i - \mathcal{G}_\delta)\partial_x\chi_1 - \delta^{-1}\chi_1 = [-2u^2 - 2\delta^{-1}u - 2\mathcal{G}_\delta\partial_x u] + iu_x$$

where we see that all terms have order 2. Now let $n \geq 3$ and assume that the claim holds for χ_m with $2 \leq m \leq n-1$. Therefore, all the terms in χ_m have order m, and from (A.5) we see that the contributions from the first sums have order $m_{1\cdots j} = n$ for $j = 2, \dots, n$, and the latter contributions have order $1 + (n-1) = n$, therefore satisfying the claim. $\square$

To construct the base Gaussian measures $\rho_{\delta,\frac{k}{2}}$, we require a better description of the quadratic terms in the conserved quantities $\operatorname{Re}\int \chi_n\, dx$. In particular, we find the correct linear combination of χ_k, $1 \leq k \leq n$, which guarantees that all quadratic in u terms have positive coefficients and exactly $n-2$ derivatives.

We first obtain a formula for the linear terms in u appearing in χ_n, which we call L_n.

Lemma A.3. *Let $n \in \mathbb{N}$, then*

$$L_n = (i\partial_x - \mathcal{G}_\delta\partial_x - \delta^{-1})^{n-1}L_1.$$

Proof. Since for all $n \in \mathbb{N}$, χ_n is at least linear in u, equation (A.5) shows that all the linear in u contributions to χ_n come from the second term in (A.5):

$$L_n = (i\partial_x - \mathcal{G}_\delta\partial_x - \delta^{-1})L_{n-1},$$

and the intended result follows from iterating the above expression. □

The following lemma gives a description of the quadratic in u terms in χ_n, which we denote by Q_n. We recall the following known results needed in the proof: for nonnegative integers q, l, m, n satisfying $n \geq q \geq 0$, we have that

$$\sum_{k=0}^{l} \binom{l-k}{m}\binom{q+k}{n} = \binom{l+q+1}{m+n+1}, \tag{A.9}$$

$$\sum_{k=0}^{l} \binom{k}{m} = \binom{l+1}{m+1}. \tag{A.10}$$

Lemma A.4. *We have that $Q_1 = 0$ and for $n \geq 2$*

$$\operatorname{Re}\int Q_n\,dx = 2(-1)^{n+1}\sum_{m=0}^{n-2}\frac{1}{\delta^{n-2-m}}\binom{n-1}{m+1}\sum_{\substack{\ell=0\\ \text{even}}}^{m}\binom{m+1}{\ell+1}\|\mathcal{G}_\delta^{\frac{m-\ell}{2}}u\|_{\dot H^{\frac{m}{2}}}^2 - \frac{1}{\delta}\operatorname{Re}\int Q_{n-1}\,dx \tag{A.11}$$

$$= (-1)^{n+1}\sum_{m=0}^{n-2}\frac{1}{\delta^{n-2-m}}\sum_{\substack{\ell=0\\ \ell\ \text{even}}}^{m} 2\binom{m+1}{\ell+1}\binom{n}{m+2}\|\mathcal{G}_\delta^{\frac{m-\ell}{2}}u\|_{\dot H^{\frac{m}{2}}}^2. \tag{A.12}$$

Proof. From (A.5), we see that

$$Q_n = -\frac{1}{2}\sum_{n_{12}=n} L_{n_1}L_{n_2} - (-i\partial_x + \mathcal{G}_\delta\partial_x + \delta^{-1})Q_{n-1}. \tag{A.13}$$

We start by focusing on the first contribution above. Integrating in x and doing integration be parts gives

$$\begin{aligned}
&-\frac{1}{2}\int \sum_{n_{12}=n} L_{n_1}L_{n_2}\,dx\\
&= -\frac{1}{2}\sum_{n_{12}=n}(-1)^{n_{12}-2}\int [(-i\partial_x + \mathcal{G}_\delta\partial_x + \tfrac{1}{\delta})^{n_1-1}2u][(-i\partial_x + \mathcal{G}_\delta\partial_x + \tfrac{1}{\delta})^{n_2-1}2u]\,dx\\
&= 2(-1)^{n+1}\sum_{n_{12}=n}\int u(i\partial_x + \mathcal{G}_\delta\partial_x + \tfrac{1}{\delta})^{n_1-1}(-i\partial_x + \mathcal{G}_\delta\partial_x + \tfrac{1}{\delta})^{n_2-1}u\,dx\\
&= 2(-1)^{n+1}\sum_{n_{12}=n}\sum_{m_1=0}^{n_1-1}\sum_{m_2=0}^{n_2-1}\frac{1}{\delta^{n_{12}-2-m_{12}}}\binom{n_1-1}{m_1}\binom{n_2-1}{m_2}\\
&\qquad\times\int u(i\partial_x + \mathcal{G}_\delta\partial_x)^{m_1}(-i\partial_x + \mathcal{G}_\delta\partial_x)^{m_2}u\,dx\\
&= 2(-1)^{n+1}\sum_{m=0}^{n-2}\frac{1}{\delta^{n-2-m}}\sum_{n_{12}=n}\sum_{\substack{0\le m_1\le n_1-1\\ 0\le m_2\le n_2-1\\ m=m_{12}}}\binom{n_1-1}{m_1}\binom{n_2-1}{m_2}\\
&\qquad\times\int u(i\partial_x + \mathcal{G}_\delta\partial_x)^{m_1}(-i\partial_x + \mathcal{G}_\delta\partial_x)^{m_2}u\,dx.
\end{aligned}$$

Focusing on the inner sums, for fixed $0 \le m \le n-2$, we have

$$\sum_{n_{12}=n} \sum_{\substack{0 \le m_1 \le n_1 - 1 \\ 0 \le m_2 \le n_2 - 1 \\ m = m_{12}}} \binom{n_1-1}{m_1}\binom{n_2-1}{m_2} \int u(i\partial_x + \mathcal{G}_\delta \partial_x)^{m_1}(-i\partial_x + \mathcal{G}_\delta\partial_x)^{m_2} u \, dx$$

$$= \sum_{n_{12}=n} \sum_{\substack{0 \le m_1 \le n_1 - 1 \\ 0 \le m_2 \le n_2 - 1 \\ m = m_{12}}} \sum_{\substack{0 \le \ell_1 \le m_1 \\ 0 \le \ell_2 \le m_2}} \binom{n_1-1}{m_1}\binom{n_2-1}{m_2}\binom{m_1}{\ell_1}\binom{m_2}{\ell_2}(-1)^{\ell_2}(i)^{\ell_{12}} \int u \mathcal{G}_\delta^{m-\ell_{12}} \partial_x^m u \, dx.$$

Since we care only about $\operatorname{Re} Q_n$, we can restrict the sum above to ℓ_{12} even:

$$\operatorname{Re} \cdots$$

$$= \sum_{n_{12}=n} \sum_{\substack{0 \le \ell \le m \\ \text{even}}} \sum_{\substack{0 \le m_1 \le n_1 - 1 \\ 0 \le m_2 \le n_2 - 1 \\ m = m_{12}}} \sum_{\substack{0 \le \ell_1 \le m_1 \\ 0 \le \ell_2 \le m_2 \\ \ell_{12} = \ell}} \binom{n_1-1}{m_1}\binom{n_2-1}{m_2}\binom{m_1}{\ell_1}\binom{m_2}{\ell_2}(-1)^{\ell_2}(-1)^{\ell/2} \int u \mathcal{G}_\delta^{m-\ell} \partial_x^m u \, dx$$

$$= \sum_{\substack{0 \le \ell \le m \\ \text{even}}} (-1)^{\ell/2} \int u \mathcal{G}_\delta^{m-\ell} \partial_x^m u \, dx \Bigg[\sum_{n_{12}=n} \sum_{\substack{0 \le m_1 \le n_1 - 1 \\ 0 \le m_2 \le n_2 - 1 \\ m = m_{12}}} \sum_{\substack{0 \le \ell_1 \le m_1 \\ 0 \le \ell_2 \le m_2 \\ \ell_{12} = \ell}} \binom{n_1-1}{m_1}\binom{n_2-1}{m_2}\binom{m_1}{\ell_1}\binom{m_2}{\ell_2}(-1)^{\ell_2} \Bigg].$$

We focus on simplifying the coefficient above. Note that

$$\sum_{n_{12}=n} \sum_{\substack{0 \le m_1 \le n_1 - 1 \\ 0 \le m_2 \le n_2 - 1 \\ m = m_{12}}} \sum_{\substack{0 \le \ell_1 \le m_1 \\ 0 \le \ell_2 \le m_2 \\ \ell_{12} = \ell}} \binom{n_1-1}{m_1}\binom{n_2-1}{m_2}\binom{m_1}{\ell_1}\binom{m_2}{\ell_2}(-1)^{\ell_2}$$

$$= \sum_{0 \le m_1 \le m} \sum_{\substack{0 \le \ell_1 \le m_1 \\ 0 \le \ell - \ell_1 \le m - m_1}} \sum_{n_1 = m_1 + 1}^{n-1+m_1-m} \binom{n_1-1}{m_1}\binom{n-n_1-1}{m-m_1}\binom{m_1}{\ell_1}\binom{m-m_1}{\ell-\ell_1}(-1)^{\ell-\ell_1}$$

$$= \sum_{0 \le m_1 \le m} \sum_{\substack{0 \le \ell_1 \le m_1 \\ 0 \le \ell - \ell_1 \le m - m_1}} \binom{m_1}{\ell_1}\binom{m-m_1}{\ell-\ell_1}(-1)^{\ell-\ell_1} \sum_{n_1=0}^{n-2} \binom{n_1}{m_1}\binom{n-2-n_1}{m-m_1}$$

$$= (-1)^\ell \binom{n-1}{m+1} \sum_{\ell_1=0}^{\ell} \sum_{m_1=\ell_1}^{m-(\ell-\ell_1)} \binom{m_1}{\ell_1}\binom{m-m_1}{\ell-\ell_1}(-1)^{\ell_1}$$

$$= (-1)^\ell \binom{n-1}{m+1}\binom{m+1}{\ell+1} \sum_{\ell_1=0}^{\ell} (-1)^{\ell_1}$$

$$= \binom{n-1}{m+1}\binom{m+1}{\ell+1}$$

where we used (A.9) for the sum in n_1 (with $q=0$, $\ell = n-2$, $m = m - m_1$, and $n = m_1$) and in m_1 (with $q = 0$, $\ell = m$, $n = \ell_1$, and $m = \ell - \ell_1$), and the fact that ℓ is even for the last equality. Consequently,

$$\operatorname{Re} \cdots = \sum_{\substack{\ell=0 \\ \text{even}}}^{m} (-1)^{\frac{\ell}{2}} \int u (\mathcal{G}_\delta \partial_x)^{m-\ell} \partial_x^\ell u \, dx \binom{n-1}{m+1}\binom{m+1}{\ell+1}$$

$$= \binom{n-1}{m+1} \sum_{\substack{\ell=0 \\ \text{even}}}^{m} \binom{m+1}{\ell+1} \|\mathcal{G}_\delta^{\frac{m-\ell}{2}} u\|_{\dot{H}^{\frac{m}{2}}}^2$$

by integration by parts, using the fact that $\mathcal{G}_\delta \partial_x$ is self-adjoint and ℓ is even.

Lastly, we conclude that

$$\begin{aligned}
\operatorname{Re} \int Q_n \, dx &= 2(-1)^{n+1} \sum_{m=0}^{n-2} \frac{1}{\delta^{n-2-m}} \binom{n-1}{m+1} \sum_{\substack{\ell=0 \\ \text{even}}}^{m} \binom{m+1}{\ell+1} \|\mathcal{G}_\delta^{\frac{m-\ell}{2}} u\|_{\dot{H}^{\frac{m}{2}}}^2 - \frac{1}{\delta} \operatorname{Re} \int Q_{n-1} \, dx \\
&=: A_n - \frac{1}{\delta} \operatorname{Re} \int Q_{n-1} \, dx,
\end{aligned}$$

proving (A.11). Let $a_{n,m,\ell} = 2\binom{n-1}{m+1}\binom{m+1}{\ell+1}$. To see (A.12), note that

$$\begin{aligned}
\operatorname{Re} \int Q_n \, dx &= \sum_{j=0}^{n-2} \frac{(-1)^j}{\delta^j} A_{n-j} \\
&= \sum_{j=0}^{n-2} (-1)^{n-j+1+j} \sum_{m=0}^{n-j-2} \frac{1}{\delta^{n-j-2-m+j}} \sum_{\substack{\ell=0 \\ \text{even}}}^{m} a_{n-j,m,\ell} \|\mathcal{G}_\delta^{\frac{m-\ell}{2}} u\|_{\dot{H}^{\frac{m}{2}}}^2 \\
&= (-1)^{n+1} \sum_{m=0}^{n-2} \frac{1}{\delta^{n-2-m}} \sum_{\substack{\ell=0 \\ \text{even}}}^{m} \bigg(\sum_{j=0}^{n-2-m} a_{n-j,m,\ell} \bigg) \|\mathcal{G}_\delta^{\frac{m-\ell}{2}} u\|_{\dot{H}^{\frac{m}{2}}}^2 \\
&= 2(-1)^{n+1} \sum_{m=0}^{n-2} \frac{1}{\delta^{n-2-m}} \sum_{\substack{\ell=0 \\ \text{even}}}^{m} \binom{m+1}{\ell+1} \binom{n}{m+2} \|\mathcal{G}_\delta^{\frac{m-\ell}{2}} u\|_{\dot{H}^{\frac{m}{2}}}^2,
\end{aligned}$$

since by (A.10), we have that

$$\begin{aligned}
\sum_{j=0}^{n-2-m} a_{n-j,m,\ell} &= 2\binom{m+1}{\ell+1} \sum_{j=0}^{n-2-m} \binom{n-1-j}{m+1} \\
&= 2\binom{m+1}{\ell+1} \sum_{j=m+1}^{n-1} \binom{j}{m+1} \\
&= 2\binom{m+1}{\ell+1} \binom{n}{m+2}.
\end{aligned}$$

□

We further want to rewrite the quadratic terms in (A.12) as the leading order term (when $m = n-2$) and a linear combination of $\operatorname{Re} \int Q_j \, dx$ for $j = 2, \dots, n-1$.

Lemma A.5. *For $n \geq 2$ there exist real numbers $b_{n,j}$, $j = 2, \dots, n-1$, such that*

$$\begin{aligned}
\int \operatorname{Re} Q_n \, dx &= B_n + \mathbf{1}_{n\geq 3} \sum_{m=2}^{n-1} \frac{1}{\delta^{n-m}} \binom{n}{m} (-1)^{n+m} B_m, \\
B_n &= (-1)^{n+1} \sum_{\substack{\ell=0 \\ \text{even}}}^{n-2} 2\binom{n-1}{\ell+1} \|\mathcal{G}_\delta^{\frac{n-2-\ell}{2}} u\|_{\dot{H}^{\frac{n-2}{2}}}^2.
\end{aligned} \tag{A.14}$$

Proof. For $n = 2$, from (A.12),

$$\operatorname{Re}\int Q_2\,dx = B_2 = -2\|u\|_{L^2}^2.$$

For $n = 3$, from (A.12), we have that

$$\int \operatorname{Re} Q_3\,dx = B_3 + \frac{3}{\delta}2\|u\|_{L^2}^2 = B_3 - \frac{3}{\delta}B_2.$$

For $n \geq 4$, from (A.12), note that

$$\begin{aligned}\operatorname{Re}\int Q_n\,dx &= B_n + (-1)^{n+1}\sum_{m=0}^{n-3}\frac{1}{\delta^{n-2-m}}\binom{n}{m+2}\sum_{\substack{\ell=0\\ \text{even}}}^{m}2\binom{m+1}{\ell+1}\|\mathcal{G}_\delta^{\frac{m-\ell}{2}}u\|_{\dot H^{\frac{m}{2}}}^2\\ &= B_n + (-1)^{n+1}\sum_{m=2}^{n-1}\frac{1}{\delta^{n-m}}\binom{n}{m}\sum_{\substack{\ell=0\\ \text{even}}}^{m-2}2\binom{m-1}{\ell+1}\|\mathcal{G}_\delta^{\frac{m-2-\ell}{2}}u\|_{\dot H^{\frac{m-2}{2}}}^2\\ &= B_n + (-1)^{n+1}\sum_{m=2}^{n-1}\frac{1}{\delta^{n-m}}\binom{n}{m}(-1)^{m+1}B_m\\ &= B_n + \sum_{m=2}^{n-1}\frac{1}{\delta^{n-m}}\binom{n}{m}(-1)^{n+m}B_m,\end{aligned}$$

as intended. □

We can finally define the final version of our conserved quantities.

Proposition A.6. *Let $k \in \mathbb{N}\cup\{0\}$. We define the k-th conserved quantity for ILW, $E_{\frac{k}{2}}(u)$ as follows*

$$E_0(u) := \frac{1}{2}\|u\|_{L^2}^2,$$

$$E_{\frac{k}{2}}(u) := (-1)^{k+1}\Bigg(4\sum_{\substack{\ell=0\\ \ell\text{ even}}}^{k}\binom{k+1}{\ell+1}\Bigg)^{-1}\Bigg[\int \operatorname{Re}\chi_{k+2}\,dx - \sum_{j=2}^{k+1}\frac{1}{\delta^{k+2-j}}\binom{k+2}{j}(-1)^{j+k}\int\operatorname{Re}\chi_j\,dx\Bigg],$$

These conserved quantities satisfy the structure in Lemma A.2, i.e., their terms are polynomials in u and its derivatives with order $k+2$, and have the following leading order quadratic terms

$$\Bigg(2\sum_{\substack{\ell=0\\ \ell\text{ even}}}^{k}\binom{k+1}{\ell+1}\Bigg)^{-1}\Bigg[\sum_{\substack{\ell=0\\ \ell\text{ even}}}^{k}\binom{k+1}{\ell+1}\|\mathcal{G}_\delta^{\frac{k-\ell}{2}}u\|_{\dot H^{\frac{k}{2}}}^2\Bigg]. \tag{A.15}$$

Proof. By definition of $E_{\frac{k}{2}}(u)$, since these are linear combinations of the conserved quantities $\operatorname{Re}\int\chi_j\,dx$, $j = 2,\ldots,k+2$, then they are also conserved quantities of ILW.

From Lemma A.2, we know that $\int\operatorname{Re}\chi_{k+2}\,dx$ has order $k+2$, while $\delta^{-(k+2-j)}\int\operatorname{Re}\chi_j\,dx$ has order $j+k+2-j = k+2$, for $2 \leq j \leq k+1$, from which we conclude that all the terms in $E_{\frac{k}{2}}(u)$ have order $k+2$.

Lastly, from Lemma A.5, we see that the quadratic in u terms of $E_{\frac{k}{2}}(u)$ are given by

$$
\begin{aligned}
&(-1)^{k+1}\Bigg(4\sum_{\substack{\ell=0\\ \ell \text{ even}}}^{k}\binom{k+1}{\ell+1}\Bigg)^{-1}\bigg[\int \operatorname{Re} Q_{k+2}\,dx-\sum_{j=2}^{k+1}\frac{1}{\delta^{k+2-j}}\binom{k+2}{j}(-1)^{j+k}\int \operatorname{Re} Q_j\,dx\bigg]\\
&=(-1)^{k+1}\Bigg(4\sum_{\substack{\ell=0\\ \ell \text{ even}}}^{k}\binom{k+1}{\ell+1}\Bigg)^{-1}B_{k+2}\\
&=(-1)^{k+1}\Bigg(4\sum_{\substack{\ell=0\\ \ell \text{ even}}}^{k}\binom{k+1}{\ell+1}\Bigg)^{-1}(-1)^{k+1}\sum_{\substack{\ell=0\\ \text{even}}}^{k}2\binom{k+1}{\ell+1}\|\mathcal{G}_\delta^{\frac{k-\ell}{2}}u\|^2_{\dot{H}^{\frac{k}{2}}}\\
&=\Bigg(2\sum_{\substack{\ell=0\\ \ell \text{ even}}}^{k}\binom{k+1}{\ell+1}\Bigg)^{-1}\bigg[\sum_{\substack{\ell=0\\ \text{even}}}^{k}\binom{k+1}{\ell+1}\|\mathcal{G}_\delta^{\frac{k-\ell}{2}}u\|^2_{\dot{H}^{\frac{k}{2}}}\bigg]
\end{aligned}
$$

as intended. □

The following lemma establishes the convergence of the ILW conserved quantities to the BO ones.

Lemma A.7. *Let $k\in\mathbb{N}\cup\{0\}$ and consider the k-th conserved quantity for BO given by*

$$
\begin{aligned}
E_0^{\mathrm{BO}}(u)&:=\frac{1}{2}\|u\|^2_{L^2},\\
E_{\frac{k}{2}}^{\mathrm{BO}}(u)&:=(-1)^{k+1}\Bigg(4\sum_{\substack{\ell=0\\ \ell \text{ even}}}^{k}\binom{k+1}{\ell+1}\Bigg)^{-1}\int \operatorname{Re}\chi_{k+2}^{\mathrm{BO}}\,dx,
\end{aligned}
$$

where χ_n^{BO} satisfies the recurrence (A.8)*. Moreover, we have that for $u\in H^{\frac{k}{2}}(\mathbb{T})$,*

$$
E_{\delta,\frac{k}{2}}(u)=E_{\frac{k}{2}}^{\mathrm{BO}}(u)+L_{\delta,\frac{k}{2}}(u)\quad \text{and}\quad \lim_{\delta\to\infty}E_{\delta,\frac{k}{2}}(u)=E_{\frac{k}{2}}^{\mathrm{BO}}(u),
$$

where $L_{\delta,\frac{k}{2}}(u)$ is defined implicitly from the expression above, and it contains all the terms in $E_{\delta,\frac{k}{2}}(u)$ with explicit powers of δ and those with no δ but $\mathcal{G}_\delta\partial_x$ replaced by $\mathcal{Q}_\delta$.

Proof. For $k\in\mathbb{N}$, by explicitly calculating the quadratic in u terms (or alternatively, replacing $\mathcal{G}_\delta$ by $\mathcal{H}$ in (A.15)) in $E_{\frac{k}{2}}^{\mathrm{BO}}(u)$, we see that these are given by

$$
\frac{1}{2}\|\mathcal{H}^{\frac{\alpha}{2}}u\|^2_{\dot{H}^{\frac{k}{2}}},\quad \alpha=\mathbf{1}_{k\text{ odd}}.
$$

To establish the last result, note that we can rewrite (A.5) as follows

$$
\begin{aligned}
\chi_1&=2u,\\
\chi_n&=-\sum_{j=2}^{n}\frac{1}{j!}\sum_{n_{1\cdots j}=n}\chi_{n_1}\cdots\chi_{n_j}-(-i+\mathcal{H})\partial_x\chi_{n-1}-(\tfrac{1}{\delta}+\mathcal{Q}_\delta)\chi_{n-1},\qquad n\geq 2,
\end{aligned}
$$

and we see that the terms with no dependence on δ, which we call $\chi_{n,0}$, satisfy the relation

$$
\chi_{1,0}=2u,
$$

$$\chi_{n,0} = -\sum_{j=2}^{n} \frac{1}{j!} \sum_{n_1 \cdots j = n} \chi_{n_1,0} \cdots \chi_{n_j,0} - (-i + \mathcal{H})\partial_x \chi_{n-1,0}, \quad n \geq 2,$$

from which we see that $\chi_{n,0} = \chi_n^{\mathrm{BO}}$. Then, the δ free and $\mathcal{Q}_\delta$ free terms in $E_{\delta,\frac{k}{2}}$ (after replacing $\mathcal{G}_\delta \partial_x$ by $\mathcal{H}\partial_x + \mathcal{Q}_\delta$) are given by

$$E^{[0]}_{\delta,\frac{k}{2}}(u) = (-1)^{k+1} \Bigg(4 \sum_{\substack{\ell=0 \\ \ell \text{ even}}}^{k} \binom{k+1}{\ell+1} \Bigg)^{-1} \bigg[\int \operatorname{Re} \chi_{k+2,0}\, dx \bigg] = E^{\mathrm{BO}}_{\frac{k}{2}}(u).$$

The remaining terms in $E_{\delta,\frac{k}{2}}(u)$ are as described in the lemma, and they all have at least one power of $\frac{1}{\delta}$ or a $\mathcal{Q}_\delta$ operator, which we call $L_{\delta,\frac{k}{2}}(u)$. It only remains to establish the convergence

$$\lim_{\delta \to \infty} L_{\delta,\frac{k}{2}}(u) = 0, \quad \forall u \in H^{\frac{k}{2}}(\mathbb{T}).$$

From Lemma A.2, all the contributions in $L_{\delta,\frac{k}{2}}(u)$ satisfy

$$\#u + \#\partial_x + \#\mathcal{Q}_\delta + \#\frac{1}{\delta} = k+2,$$

and they can be written as $\frac{1}{\delta^\alpha} \int p(u)\, dx$ where $p(u) \in \mathcal{P}_j(u)$ for some $0 \leq \alpha \leq k$ and $j \in \{2, \dots, k+2-\alpha\}$:

$$L_{\delta,\frac{k}{2}}(u) = \sum_{m=1}^{k} \Bigg[\sum_{j=2}^{k+2-m} \sum_{i=0}^{k+2-m-j} \sum_{\substack{p(u) \in \mathcal{P}_j(u) \\ ||p(u)||=i \\ |||p(u)|||=k-2-m-j-i \\ |p(u)| \leq \lceil \frac{k-1}{2} \rceil}} \frac{1}{\delta^m} \int p(u)\, dx$$

$$+ \sum_{j=2}^{k+2} \sum_{i=0}^{k+2-j} \sum_{\substack{p(u) \in \mathcal{P}_j(u) \\ \|p(u)\|=i \\ 1 \leq |||p(u)|||=k-2-j-i \\ |p(u)| \leq \lceil \frac{k-1}{2} \rceil}} \int p(u)\, dx \Bigg].$$

From the definition above, we can see that from Young's convolution inequality and (2.6), we have that for $\delta \geq 1$,

$$|L_{\delta,\frac{k}{2}}(u)| \lesssim \sum_{m=1}^{k} \sum_{j=2}^{k+2-m} \sum_{i=0}^{k+2-m-j} \sum_{\substack{p(u) \in \mathcal{P}_j(u) \\ ||p(u)||=i \\ |||p(u)|||=k-2-m-j-i \\ |p(u)| \leq \lceil \frac{k-1}{2} \rceil}} \frac{1}{\delta^{k-2-j-i}} \|u\|^j_{H^{\frac{k}{2}}}$$

$$+ \sum_{j=2}^{k+2} \sum_{i=0}^{k+2-j} \sum_{\substack{p(u) \in \mathcal{P}_j(u) \\ \|p(u)\|=i \\ 1 \leq |||p(u)|||=k-2-j-i \\ |p(u)| \leq \lceil \frac{k-1}{2} \rceil}} \frac{1}{\delta^{k-2-j-i}} \|u\|^j_{H^{\frac{k}{2}}}$$

$$\lesssim \frac{1}{\delta}(1 + \|u\|_{H^{\frac{k}{2}}})^{k+2} \to 0 \text{ as } \delta \to \infty. \qquad \square$$

In the remaining of this subsection, we establish some useful results on the leading order cubic terms of $E_{\frac{k}{2}}(u)$ and its general structure.

Lemma A.8. *For $k \in \mathbb{N}$, the cubic in u terms in $E_{\frac{k}{2}}(u)$ are a linear combination of terms of the form*

$$\frac{1}{\delta^\sigma}\int \prod_{j=1}^{3}(\mathcal{Q}_\delta^{\alpha_j}(\mathcal{H}\partial_x)^{\beta_j}\partial_x^{\gamma_j}u)\,dx$$

where $\sigma + \sum_{j=1}^3(\alpha_j+\beta_j+\gamma_j) = k-1$ and $\sigma, \alpha_j, \beta_j, \gamma_j \geq 0$, $\alpha_j + \beta_j \leq \gamma_j$.

Proof. From Proposition A.6, we know that the terms in $E_{\frac{k}{2}}(u)$ are of the form

$$\frac{1}{\delta^{k+2-j}}\int \operatorname{Re}\chi_j\,dx$$

for $j = 2, \dots, k+2$, and by Lemma A.2, the terms above which are cubic in u, have $\#\frac{1}{\delta} + \#\partial_x = (k+2-j)+j-3 = k-1$. Thus, it suffices to consider the structure of the cubic in u terms in $\int \operatorname{Re}\chi_j\,dx$ for $j \geq 2$.

By (possibly) doing integration by parts, all such terms can be written as

$$\frac{1}{\delta^\sigma}\int \prod_{j=1}^{3}[(\mathcal{G}_\delta\partial_x)^{\alpha_j}\partial_x^{\beta_j}u]\,dx$$

where $\sigma + \sum_{j=1}^3(\alpha_j+\beta_j) = k-1$. Replacing $\mathcal{G}_\delta\partial_x$ by $\mathcal{Q}_\delta + \mathcal{H}\partial_x$, we obtain that all the terms are of the form

$$\frac{1}{\delta^\sigma}\int \prod_{j=1}^{3}[\mathcal{Q}_\delta^{\widetilde{\alpha}_j}(\mathcal{H}\partial_x)^{\widetilde{\alpha}'_j}\partial_x^{\beta_j}u]\,dx,$$

where $\widetilde{\alpha}_j + \widetilde{\alpha}'_j = \alpha_j$, and thus $\sigma + \sum_{j=1}^3(\widetilde{\alpha}_j + \widetilde{\alpha}'_j + \beta_j) = k-1$ as intended. □

Using integration by parts, we can rewrite the cubic terms in $E_{\frac{k}{2}}(u)$ to guarantee that the cubic contributions have at most m derivatives on each term, when $k = 2m$ or $k = 2m+1$, $m \in \mathbb{N}$. Most contributions will have at most $m-1$ derivatives on each factor, while the remaining ones will be of the following form

$$B_{\frac{2m}{2}}(u) = \sum_{\substack{p(u)\in\mathcal{P}_3(u)\\ \widetilde{p}(u)=u\partial_x^{m-1}u\partial_x^m u\\ |||p(u)|||=0}} c_m(p)\int p(u)\,dx,$$

$$B_{\frac{2m+1}{2}}(u) = \sum_{\substack{p(u)\in\mathcal{P}_3(u)\\ \widetilde{p}(u)=u\partial_x^{m}u\partial_x^m u\\ |||p(u)|||=0}} c_m(p)\int p(u)\,dx,$$

when $k = 2m$ or $k = 2m+1$, respectively. Then, the following holds.

Lemma A.9. *Let $m \in \mathbb{N}$. Then, there exist constants $c_1, c_m(p)$ such that*

$$B_{\frac{2m}{2}}(u) = c_1\int u(\mathcal{H}\partial_x^{m-1}u)(\partial_x^m u)\,dx + \sum_{\substack{p(u)\in\mathcal{P}_3(u)\\ |p(u)|\leq m-1\\ |||p(u)|||=0}} c_m(p)\int p(u)\,dx.$$

Proof. Fix $m \in \mathbb{N}$ and let $p(u) \in \mathcal{P}_3(u)$ with $\widetilde{p}(u) = u\partial_x^{m-1}u\partial_x^m u$ and $|||p(u)||| = 0$. Then,

$$p(u) = (\mathcal{H}^{\alpha_1}u)(\mathcal{H}^{\alpha_2}\partial_x^{m-1}u)(\mathcal{H}^{\alpha_3}\partial_x^m u)$$

for $\alpha_1, \alpha_2, \alpha_3 \in \{0, 1\}$. The following equalities hold:

$$\begin{aligned}
\int u(\partial_x^{m-1}u)(\partial_x^m u)\,dx &= -\frac{1}{2}\int (\partial_x u)(\partial_x^{m-1}u)^2\,dx,\\
\int (\mathcal{H}u)(\partial_x^{m-1}u)(\partial_x^m u)\,dx &= -\frac{1}{2}\int (\mathcal{H}\partial_x u)(\partial_x^{m-1}u)^2\,dx\\
\int u(\partial_x^{m-1}u)(\mathcal{H}\partial_x^m u)\,dx &= -\int (\partial_x u)(\partial_x^{m-1}u)(\mathcal{H}\partial_x^{m-1}u)\,dx - \int u(\mathcal{H}\partial_x^{m-1}u)(\partial_x^m u)\,dx\\
\int (\mathcal{H}u)(\mathcal{H}\partial_x^{m-1}u)(\mathcal{H}\partial_x^m u)\,dx &= -\frac{1}{2}\int (\mathcal{H}\partial_x u)(\mathcal{H}\partial_x^{m-1}u)^2\,dx\\
\int (\mathcal{H}u_1)(\mathcal{H}u_2)u_3\,dx &= -\int u_1\mathcal{H}((\mathcal{H}u_2)u_3)\,dx\\
&= -\int u_1\big[(\mathcal{H}u_2)u_3 + \mathcal{H}(-u_2u_3 + (\mathcal{H}u_2)(\mathcal{H}u_3))\big]\,dx\\
&= -\int u_1(\mathcal{H}u_2)u_3 + (\mathcal{H}u_1)u_2u_3 - (\mathcal{H}u_1)(\mathcal{H}u_2)(\mathcal{H}u_3)\,dx
\end{aligned}$$

using the identity $\mathcal{H}(u\mathcal{H}v + v\mathcal{H}u) = \mathcal{H}(uv) - uv$. From the above identities, we conclude that all terms $\int p(u)\,dx$ can be written as intended. □

A similar description holds for $B_{\frac{2m+1}{2}}(u)$.

Lemma A.10. *Let $m \in \mathbb{N}$. Then, there exist constants $c_1, c_m(p)$ such that*

$$B_{\frac{2m+1}{2}}(u) = c_1\int u(\partial_x^m u)^2\,dx + c_2\int [\mathcal{H}u][\mathcal{H}\partial_x^m u][\partial_x^m u]\,dx + c_3\int u[\mathcal{H}\partial_x^m u]^2\,dx$$

for some constants c_1, c_2, c_3.

Proof. From Lemma A.8, we know that the terms in $B_{\frac{2m+1}{2}}(u)$ are of the form $\int p(u)\,dx$ where

$$p(u) = [\mathcal{H}^{\alpha_1}u][\mathcal{H}^{\alpha_2}\partial_x^m u][\mathcal{H}^{\alpha_3}\partial_x^m u]$$

and $0 \le \alpha_\ell \le 1$. The terms with $(\alpha_1, \alpha_2, \alpha_3) \in \{(0,0,0), (1,1,0), (1,0,1), (0,1,1)\}$ are already considered, thus we focus on the remaining choices, namely $(1,0,0), (0,1,0), (1,1,1)$:

$$\begin{aligned}
\int [\mathcal{H}u][\partial_x^m u]^2\,dx &= -\int u\mathcal{H}[\partial_x^m u]^2\,dx = -\int u[(\partial_x^m u)^2 + 2\mathcal{H}(\partial_x^m u\mathcal{H}\partial_x^m u)]\,dx\\
&= -\int u(\partial_x^m u)^2\,dx + 2\int [\mathcal{H}u][\mathcal{H}\partial_x^m u][\partial_x^m u]\,dx,\\
\int u\partial_x^m u\mathcal{H}\partial_x^m u\,dx &= -\int \mathcal{H}[u\partial_x^m u]\partial_x^m u\,dx = -\int [u\partial_x^m u + \mathcal{H}(u\mathcal{H}\partial_x^m u + \partial_x^m u\mathcal{H}u)]\partial_x^m u\,dx\\
&= -\int u(\partial_x^m u)^2\,dx + \int u[\mathcal{H}\partial_x^m u]^2\,dx + \int [\mathcal{H}u][\mathcal{H}\partial_x^m u][\partial_x^m u]\,dx,\\
\int [\mathcal{H}u][\mathcal{H}\partial_x^m u]^2\,dx &= -\int u\mathcal{H}[\mathcal{H}\partial_x^m u]^2\,dx = -\int u[(\mathcal{H}\partial_x^m u)^2 - 2\partial_x^m u\mathcal{H}\partial_x^m u]\,dx\\
&= -\int u[\mathcal{H}\partial_x^m u]^2\,dx + 2\int u\partial_x^m u\mathcal{H}\partial_x^m u\,dx
\end{aligned}$$

$$= \int u[\mathcal{H}\partial_x^m u]^2\,dx - 2\int u(\partial_x^m u)^2\,dx + 2\int [\mathcal{H}u][\mathcal{H}\partial_x^m u][\partial_x^m u]\,dx,$$

using the identity $\mathcal{H}(u\mathcal{H}v + v\mathcal{H}u) = \mathcal{H}(uv) - uv$. □

For completeness, we also include the proof of Lemma 5.4. See also [56].

Proof of Lemma 5.4. Let $\alpha_1, \alpha_2, \alpha_3 \in \{0,1\}$, then

$$\begin{aligned}
&\int [\mathcal{H}^{\alpha_1}u][\mathcal{H}^{\alpha_2}\partial_x^m u]\mathbf{P}_{>N}\mathcal{H}^{\alpha_3}\partial_x^m[u\partial_x u]\,dx\\
&= \sum_{j=1}^m c_j \int [\mathcal{H}^{\alpha_1}u][\mathcal{H}^{\alpha_2}\partial_x^m u]\mathbf{P}_{>N}\mathcal{H}^{\alpha_3}[\partial_x^j u\cdot \partial_x^{m+1-j}u]\,dx\\
&\qquad + \int [\mathcal{H}^{\alpha_1}u][\mathcal{H}^{\alpha_2}\partial_x^m u]\mathbf{P}_{>N}\mathcal{H}^{\alpha_3}[u\cdot \partial_x^{m+1}u]\,dx\\
&= \sum_{j=1}^m c_j \int [\mathcal{H}^{\alpha_1}u][\mathcal{H}^{\alpha_2}\partial_x^m u]\mathbf{P}_{>N}\mathcal{H}^{\alpha_3}[\partial_x^j u\cdot \partial_x^{m+1-j}u]\,dx\\
&\qquad - \int [\mathcal{H}^{\alpha_1}u][\mathcal{H}^{\alpha_2}\partial_x^m u]\mathbf{P}_{>N}\mathcal{H}^{\alpha_3}[\partial_x u\cdot \partial_x^{m}u]\,dx\\
&\qquad + \int [\mathcal{H}^{\alpha_1}u][\mathcal{H}^{\alpha_2}\partial_x^m u]\mathbf{P}_{>N}\mathcal{H}^{\alpha_3}\partial_x[u\cdot \partial_x^{m}u]\,dx
\end{aligned} \tag{A.16}$$

for constants $c_j \in \mathbb{R}$. We focus on the last contribution and denote it by $\mathrm{I}(u)$.

If $\alpha_1 = \alpha_2 = \alpha_3 = 0$, then

$$\mathrm{I}(u) = \int \mathbf{P}_{>N}[u\partial_x^m u]\mathbf{P}_{>N}\partial_x[u\partial_x^m u]\,dx = 0.$$

If $\alpha_1 = 1$, $\alpha_2 = \alpha_3 = 0$, then using the fact that $\mathcal{H} = -i(\mathbf{P}_+ - \mathbf{P}_-)$ and $\mathbf{P}_{>N}(\mathbf{P}_+\mathbf{P}_N f\cdot \mathbf{P}_-\mathbf{P}_N g) = 0$, we get that

$$\begin{aligned}
\mathrm{I}(u_N) &= \int \mathbf{P}_{>N}[\mathcal{H}u_N\cdot \partial_x^m u_N]\mathbf{P}_{>N}\partial_x[u_N\partial_x^m u_N]\,dx\\
&= -i\int \mathbf{P}_{>N}[(u_N^+ - u_N^-)\partial_x^m u_N]\mathbf{P}_{>N}\partial_x[(u_N^+ + u_N^-)\partial_x^m u_N]\,dx\\
&= -i\int \mathbf{P}_{>N}[u_N^+\partial_x^m u_N^+ - u_N^-\partial_x^m u_N^-]\mathbf{P}_{>N}\partial_x[u_N^+\partial_x^m u_N^+ + u_N^-\partial_x^m u_N^-]\,dx\\
&= -i\int \mathbf{P}_{>N}[u_N^+\partial_x^m u_N^+]\mathbf{P}_{>N}\partial_x[u_N^-\partial_x^m u_N^-]\,dx + i\int \mathbf{P}_{>N}[u_N^-\partial_x^m u_N^-]\mathbf{P}_{>N}\partial_x[u_N^+\partial_x^m u_N^+]\,dx\\
&= -2i\int \mathbf{P}_{>N}[u_N^+\partial_x^m u_N^+]\mathbf{P}_{>N}\partial_x[u_N^-\partial_x^m u_N^-]\,dx.
\end{aligned}$$

If $\alpha_1 = 0$, $\alpha_2 = \alpha_3 = 1$, then

$$\begin{aligned}
\mathrm{I}(u_N) &= \int \mathbf{P}_{>N}[u_N\cdot \mathcal{H}\partial_x^m u_N]\mathbf{P}_{>N}\partial_x\mathcal{H}[u_N\partial_x^m u_N]\,dx\\
&= -\int \mathbf{P}_{>N}[u_N\partial_x^m u_N^+ - u_N\partial_x^m u_N^-]\mathbf{P}_{>N}\partial_x(\mathbf{P}_+ - \mathbf{P}_-)[u_N\partial_x^m u_N]\,dx\\
&= -\int \mathbf{P}_{>N}\mathbf{P}_-[-u_N^-\partial_x^m u_N^-]\mathbf{P}_{>N}\partial_x\mathbf{P}_+[u_N^+\partial_x^m u_N^+]\,dx
\end{aligned}$$

$$- \int \mathbf{P}_{>N}\mathbf{P}_+[u_N^+\partial_x^m u_N^+]\mathbf{P}_{>N}\partial_x\mathbf{P}_-[u_N^-\partial_x^m u_N^-]\,dx = 0.$$

Lastly, if $\alpha_1 = \alpha_2 = \alpha_3 = 1$, then

$$\begin{aligned}
\mathrm{I}_N(u_N) &= \int \mathbf{P}_{>N}[\mathcal{H}u_N \cdot \mathcal{H}\partial_x^m u_N]\mathbf{P}_{>N}\partial_x\mathcal{H}[u_N\partial_x^m u_N]\,dx \\
&= -\int \mathbf{P}_{>N}[u_N^+\partial_x^m u_N^+ + u_N^-\partial_x^m u_N^- - u_N^+\partial_x^m u_N^- - u_N^-\partial_x^m u_N^+]\mathbf{P}_{>N}\partial_x\mathcal{H}[u_N\partial_x^m u_N]\,dx \\
&= i\int \mathbf{P}_{>N}[u_N^+\partial_x^m u_N^+ + u_N^-\partial_x^m u_N^-]\mathbf{P}_{>N}\partial_x(\mathbf{P}_+ - \mathbf{P}_-)[u_N\partial_x^m u_N]\,dx \\
&= i\int \mathbf{P}_{>N}\mathbf{P}_-[u_N^-\partial_x^m u_N^-]\mathbf{P}_{>N}\partial_x\mathbf{P}_+[u_N^+\partial_x^m u_N^+]\,dx \\
&\quad + i\int \mathbf{P}_{>N}\mathbf{P}_+[u_N^+\partial_x^m u_N^+]\mathbf{P}_{>N}\partial_x\mathbf{P}_-[-u_N^-\partial_x^m u_N^-]\,dx \\
&= -2i\int \mathbf{P}_{>N}\mathbf{P}_+[u_N^+\partial_x^m u_N^+]\mathbf{P}_{>N}\partial_x\mathbf{P}_-[u_N^-\partial_x^m u_N^-]\,dx.
\end{aligned}$$

The first result follows by replacing these identities in (A.16).

For the second result, note that if $\alpha_1 = \alpha_3 = 0, \alpha_2 = 1$, then

$$\begin{aligned}
&\int \mathbf{P}_{>N}[u_N \cdot \mathcal{H}\partial_x^m u_N]\mathbf{P}_{>N}\partial_x[u_N\partial_x^m u_N]\,dx + \int \mathbf{P}_{>N}[u_N\partial_x^m u_N]\mathbf{P}_{>N}\partial_x\mathcal{H}[u_N\partial_x^m u_N]\,dx \\
&= \int \mathbf{P}_{>N}[u_N\partial_x^m u_N]\mathbf{P}_{>N}\partial_x[\mathcal{H}(u_N\partial_x^m u_N) - u_N\mathcal{H}\partial_x^m u_N]\,dx
\end{aligned}$$

and proceeding as before, we have that

$$\begin{aligned}
&\mathbf{P}_{>N}[\mathcal{H}(u_N\partial_x^m u_N) - u_N\mathcal{H}\partial_x^m u_N] \\
&= -i\mathbf{P}_{>N}[(\mathbf{P}_+ - \mathbf{P}_-)(u_N\partial_x^m u_N) - u_N\partial_x^m(u_N^+ - u_N^-)] \\
&= -i\mathbf{P}_{>N}[\mathbf{P}_+(u_N\partial_x^m u_N - u_N\partial_x^m u_N^+ + u_N\partial_x^m u_N^-) + \mathbf{P}_-(-u_N\partial_x^m u_N - u_N\partial_x^m u_N^+ + u_N\partial_x^m u_N^-)] \\
&= -2i\mathbf{P}_{>N}[\mathbf{P}_+(u_N^-\partial_x^m u_N^-) - \mathbf{P}_-(u_N^+\partial_x^m u_N^+)] = 0
\end{aligned}$$

from which we conclude that the earlier contribution is also 0. If $\alpha_1 = \alpha_2 = 0, \alpha_3 = 1$, the contribution is the same as above, and therefore 0. If $\alpha_1 = \alpha_2 = 1, \alpha_3 = 0$, then

$$\begin{aligned}
&\int \mathbf{P}_{>N}[\mathcal{H}u_N \cdot \mathcal{H}\partial_x^m u_N]\mathbf{P}_{>N}\partial_x[u_N\partial_x^m u_N]\,dx + \int \mathbf{P}_{>N}[\mathcal{H}u_N\partial_x^m u_N]\mathbf{P}_{>N}\partial_x\mathcal{H}[u_N\partial_x^m u_N]\,dx \\
&= \int \mathbf{P}_{>N}\partial_x[u_N\partial_x^m u_N]\mathbf{P}_{>N}\partial_x[\mathcal{H}u_N\mathcal{H}\partial_x^m u_N - \mathcal{H}(\mathcal{H}u_N \cdot \partial_x^m u_N)]\,dx.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\mathbf{P}_{>N}[\mathcal{H}u_N\mathcal{H}\partial_x^m u_N - \mathcal{H}(\mathcal{H}u_N \cdot \partial_x^m u_N)] \\
&= -\mathbf{P}_{>N}[(u_N^+ - u_N^-)\partial_x^m(u_N^+ - u_N^-) - (\mathbf{P}_+ - \mathbf{P}_-)(u_N^+ \cdot \partial_x^m u_N - u_N^- \cdot \partial_x^m u_N)] \\
&= -\mathbf{P}_{>N}[\mathbf{P}_+(u_N^+\partial_x^m u_N^+ - u_N^+\partial_x^m u_N^+) + \mathbf{P}_-(u_N^-\partial_x^m u_N^- - u_N^-\partial_x^m u_N^-)] = 0
\end{aligned}$$

from which we conclude that the earlier contribution is also 0. If $\alpha_1 = \alpha_3 = 1, \alpha_2 = 0$, the quantity is the same as above and therefore 0. □

Appendix B. Conserved quantities for sILW

We follow the derivation of the conserved quantities for sILW in [16, 30]. See also [50, 10, 11] for alternative approaches.

Consider the following equations for u, q:

$$u = \frac{1}{2\delta^2}\big[2i\delta\varepsilon q - (1 - \tfrac{i\delta}{\varepsilon})(e^{2i\delta\varepsilon q} - 1)\big] + \varepsilon\partial_x q + i\delta\varepsilon\widetilde{\mathcal{G}}_\delta\partial_x q, \tag{B.1}$$

$$\partial_t q = \frac{1}{\delta^2}\partial_x q\big[2i\delta\varepsilon q - (1 - \tfrac{i\delta}{\varepsilon})(e^{2i\delta\varepsilon q} - 1)\big] + \widetilde{\mathcal{G}}_\delta\partial_x^2 q + 2i\delta\varepsilon(\partial_x q)(\widetilde{\mathcal{G}}_\delta\partial_x q). \tag{B.2}$$

Then, we can show that for $q = \sum\limits_{n\geq 0}\varepsilon^n h_n$ and u satisfying (B.1), then u solves sILW (1.4) if and only q solves (B.2). This follows from the fact that for u, q as above, we can write

$$\begin{aligned}
&\partial_t u - \widetilde{\mathcal{G}}_\delta\partial_x^2 u - 2u\partial_x u \\
&\qquad = \mathcal{L}\Big\{\partial_t q - \frac{1}{\delta^2}\partial_x q\big[2i\delta\varepsilon q - (1 - \tfrac{i\delta}{\varepsilon})(e^{2i\delta\varepsilon q} - 1)\big] - \widetilde{\mathcal{G}}_\delta\partial_x^2 q - 2i\delta\varepsilon(\partial_x q)(\widetilde{\mathcal{G}}_\delta\partial_x q)\Big\}
\end{aligned}$$

where the operator $\mathcal{L}$ is given by

$$\mathcal{L} := (i - \tfrac{\varepsilon}{\delta})e^{2i\delta\varepsilon q} + \varepsilon(\delta^{-1} - i\partial_x + \delta\widetilde{\mathcal{G}}_\delta\partial_x)$$

and we can show that if $f = \sum\limits_{n\geq 0}\varepsilon^n f_n$ and $\mathcal{L}f = 0$, then $f \equiv 0$, from which the equivalence above follows.

From (B.2), we see that $\int q\,dx$ is conserved under (1.4), thus replacing $q = \sum_{n\geq 0}\varepsilon^n h_n$ and replacing in (B.1) shows that h_n is conserved for all $n \geq 0$ and we derive the following recurrence relation for $n \geq 2$:

$$\begin{aligned}
h_0 &= -u, \\
h_1 &= -i\delta u^2 - (1 + i\delta\widetilde{\mathcal{G}}_\delta)\partial_x u, \\
h_n &= -\frac{1}{2\delta^2}\sum_{k=2}^{n}\frac{(2i\delta)^k}{k!}\sum_{n_{1\cdots k} = n-k} h_{n_1}\cdots h_{n_k} + \frac{i}{2\delta}\sum_{k=2}^{n+1}\frac{(2i\delta)^k}{k!}\sum_{n_{1\cdots k} = n+1-k} h_{n_1}\cdots h_{n_k} \\
&\qquad + (1 + i\delta\widetilde{\mathcal{G}}_\delta)\partial_x h_{n-1}.
\end{aligned} \tag{B.3}$$

As written, half of the conserved quantities, namely $\int h_{2n+1}\,dx$, have trivial limits as $\delta \to 0$. To avoid this, we instead consider other quantities which can be written as a linear combination of the h_n ones, but which all have non-trivial limits as $\delta \to 0$. In particular, for $n \in \mathbb{N}$, we define $\widetilde{h}_n$ as

$$\widetilde{h}_n := \Big(\frac{\mathbf{1}_{n\text{ odd}}}{i\delta} + \frac{\mathbf{1}_{n\text{ even}}}{\delta^2}\Big)\Big(h_n + \frac{1}{i\delta}h_{n-1} + \cdots + \frac{1}{(i\delta)^{n-1}}h_1\Big) \tag{B.4}$$

$$= \Big(\frac{\mathbf{1}_{n\text{ odd}}}{i\delta} + \frac{\mathbf{1}_{n\text{ even}}}{\delta^2}\Big)\sum_{j=1}^{n}\frac{1}{(i\delta)^{n-j}}h_j \tag{B.5}$$

The following lemma gives a simplified expression for $\widetilde{h}_n$ and further information on their quadratic in u terms which we denote by $\widetilde{Q}_n$. We will need the following version of the Rothe-Hagen identity for the proof: let $x, y \in \mathbb{C}$ and $z \in \mathbb{N}$, then

$$\sum_{k=0}^{z}\binom{x-1+k}{k}\binom{y+z-1-k}{z-k} = \binom{x+y+z-1}{z}. \tag{B.6}$$

Lemma B.1. *For $n \in \mathbb{N}$, we have that*

$$\Big(\mathbf{1}_{n \text{ odd}} i\delta + \mathbf{1}_{n \text{ even}}\delta^2\Big)\widetilde{h}_n = \frac{i}{2\delta}\sum_{k=2}^{n+1}\frac{(2i\delta)^k}{k!}\sum_{n_{1\cdots k}=n+1-k} h_{n_1}\cdots h_{n_k} + \sum_{j=1}^{n}\frac{1}{(i\delta)^{n-j}}(1+i\delta\widetilde{\mathcal{G}}_\delta)\partial_x h_{j-1}. \tag{B.7}$$

Proof. From the (B.5) and (B.3), we have that

$$\begin{aligned}
&\Big(\mathbf{1}_{n \text{ odd}} i\delta + \mathbf{1}_{n \text{ even}}\delta^2\Big)\widetilde{h}_n \\
&= \sum_{j=2}^{n}\frac{1}{2(i\delta)^{n-j+2}}\sum_{k=2}^{j}\frac{(2i\delta)^k}{k!}\sum_{n_{1\cdots k}=j-k} h_{n_1}\cdots h_{n_k} \\
&\quad - \sum_{j=1}^{n}\frac{1}{2(i\delta)^{n-j+1}}\sum_{k=2}^{j+1}\frac{(2i\delta)^k}{k!}\sum_{n_{1\cdots k}=j+1-k} h_{n_1}\cdots h_{n_k} + \sum_{j=1}^{n}\frac{1}{(i\delta)^{n-j}}(1+i\delta\widetilde{\mathcal{G}}_\delta)\partial_x h_{j-1} \\
&= \frac{i}{2\delta}\sum_{k=2}^{n+1}\frac{(2i\delta)^k}{k!}\sum_{n_{1\cdots k}=n+1-k} h_{n_1}\cdots h_{n_k} + \sum_{j=1}^{n}\frac{1}{(i\delta)^{n-j}}(1+i\delta\widetilde{\mathcal{G}}_\delta)\partial_x h_{j-1},
\end{aligned}$$

as intended.

□

B.1. **Structure of the sILW conserved quantities.** We first show that $\int \widetilde{h}_n\, dx$ only has non-negative powers of δ, which guarantees that there are no divergent terms in δ as $\delta \to 0$.

Lemma B.2. *For $n \geq 0$, h_n has only non-negative powers of δ. Therefore, for $n \in \mathbb{N}$, $\int \widetilde{h}_{2n-1}\, dx, \int \widetilde{h}_{2n}\, dx$ can be written as polynomials in δ with only non-negative powers of δ.*

Proof. Since $h_0 = -u$ and from (B.3), it follows that h_n for $n \geq 0$ only has non-negative powers of δ. Similarly, from the definition (B.7), we have

$$\begin{aligned}
\int \widetilde{h}_{2n-1}\, dx &= \frac{1}{2\delta^2}\sum_{k=2}^{2n}\frac{(2i\delta)^k}{k!}\sum_{n_{1\cdots k}=2n-k}\int h_{n_1}\cdots h_{n_k}\, dx, \\
\int \widetilde{h}_{2n}\, dx &= \frac{i}{2\delta^3}\sum_{k=2}^{2n+1}\frac{(2i\delta)^k}{k!}\sum_{n_{1\cdots k}=2n+1-k}\int h_{n_1}\cdots h_{n_k}\, dx,
\end{aligned}$$

where the result follows for the odd-indexed quantities from the result on h_j. For the even-indexed quantities, the contributions from $3 \leq k \leq 2n+1$ have only non-negative powers of δ, so we focus on $k = 2$:

$$\frac{i}{2\delta^3}\frac{(2i\delta)^2}{2}\sum_{n_{12}=2n-1}\int h_{n_1}h_{n_2}\, dx = -\frac{i}{\delta}\sum_{n_{12}=2n-1}\int h_{n_1}h_{n_2}\, dx.$$

We can write $h_j = \sum_{k=0}^{j}\delta^k h_{j,k}$, and the only problematic terms above come from the δ^0 coefficients of h_{n_1}, h_{n_2}, namely

$$-\frac{i}{\delta}\sum_{n_{12}=2n-1}\int h_{n_1,0}h_{n_2,0}\, dx.$$

From Lemma B.4, we can see that $h_{n,0}$ satisfies the same recurrence relation as the KdV conserved quantities, and from (B.16) and Lemma B.6(i), we can see that the above corresponds to $-\frac{i}{\delta}\int h_{2n+1}^{\text{KdV}}\,dx$ which is zero. Consequently, all the contributions have non-negative powers of δ. $\square$

In the shallow-water setting, we must change our definition of degree for the conserved quantities, since the scaling by δ breaks the previous result that all terms have the same degree $\#u + \#\partial_x + \#\frac{1}{\delta}$. In this case, we can only establish an upper bound for the number of powers of u and operators ∂_x, ignoring the powers of $\frac{1}{\delta}$.

Lemma B.3. *For $n \geq 0$, all the terms in h_n satisfy*

$$\#u + \#(\widetilde{\mathcal{G}}_\delta \partial_x) + \#\partial_x \leq n+1,$$

and there are at least as many powers of δ as operators $\widetilde{\mathcal{G}}_\delta$ in each term. Moreover, the terms in $\int \widetilde{h}_n\,dx$ have order at most $n+1$.

Proof. Since $h_0 = -u$ and $h_1 = -i\delta(u^2 + \widetilde{\mathcal{G}}_\delta u_x) - u_x$, we see that h_0 has degree 1, while all the terms in h_1 have degree 2. Now, let $n \in \mathbb{N}$ and assume that the result holds for h_k with $0 \leq k \leq n-1$. Then, recall (B.3):

$$\begin{aligned} h_n = &-\frac{1}{2\delta^2}\sum_{k=2}^{n}\frac{(2i\delta)^k}{k!}\sum_{n_{1\cdots k}=n-k} h_{n_1}\cdots h_{n_k} + \frac{i}{2\delta}\sum_{k=2}^{n+1}\frac{(2i\delta)^k}{k!}\sum_{n_{1\cdots k}=n+1-k} h_{n_1}\cdots h_{n_k} \\ &+ (1+i\delta\widetilde{\mathcal{G}}_\delta)\partial_x h_{n-1}. \end{aligned}$$

The first terms above have order at most $(n_1+1)+\cdots+(n_k+1) = n-k+k = n$, by assumption. Similarly, the second terms have order at most $(n_1+1)+\cdots+(n_k+1) = n+1-k+k = n+1$. Lastly, the terms arising from the last contribution have order at most $1+(n-1+1) = n+1$, since h_{n-1} has terms with order at most n and ∂_x and $i\delta\widetilde{\mathcal{G}}_\delta\partial_x$ increase the order by 1.

For the second result regarding the number of $\widetilde{\mathcal{G}}_\delta$ operators and powers of δ, note that h_0 and h_1 both satisfy the claim. As before, let $n \in \mathbb{N}$ and assume that the claim holds for h_k with $0 \leq k \leq n-1$. Then, from (B.3), we see that the number of $\widetilde{\mathcal{G}}_\delta$ operators stays the same from the h_j contributions while there could be an increase in the power of δ in the first and second group of contributions. The only term with an extra $\widetilde{\mathcal{G}}_\delta$ operator is the last one, which also comes with an extra power of δ, thus the result follows from the inductive hypothesis.

For the result on $\int \widetilde{h}_n$, from (B.7), we note that

$$(\mathbf{1}_{n\ \text{odd}} i\delta + \mathbf{1}_{n\ \text{even}}\delta^2)\int \widetilde{h}_n = \frac{i}{2\delta}\sum_{k=2}^{n+1}\frac{(2i\delta)^k}{k!}\sum_{n_{1\cdots k}=n+1-k}\int h_{n_1}\cdots h_{n_k}$$

and from the result on h_j, we see that all the contributions have order at most $(n_1+1)+\cdots+(n_k+1) = n+1-k+k = n+1$, as intended. $\square$

We need a more explicit description of the coefficients of δ^0 and δ^1 in $h_n, \int \widetilde{h}_n$, namely $h_{n,0}, h_{n,1}, \int \widetilde{h}_{n,0}, \int \widetilde{h}_{n,1}$ as in

$$h_n = \sum_{k=0}^{n} \delta^k h_{n,k}, \qquad \int \widetilde{h}_n \, dx = \sum_{k=0}^{n} \delta^k \int \widetilde{h}_{n,k} \, dx.$$

Then, we have the following recurrence for the coefficients of δ^0 and δ.

Lemma B.4. *The following holds for* $h_n = \sum_{k=0}^{n} \delta^k h_{n,k}$

$$\begin{aligned} &h_{0,0} = -u, && h_{1,0} = -u_x, && h_{2,0} = u^2 - u_{xx}, \\ &h_{0,1} = 0, && h_{1,1} = -iu^2 - i\widetilde{\mathcal{G}}_\delta u_x, && h_{2,1} = 2iu^3 + 2iu\widetilde{\mathcal{G}}_\delta u_x - 6iuu_x - 2i\widetilde{\mathcal{G}}_\delta u_{xx}, \end{aligned}$$

$$h_{n,0} = \sum_{n_1+n_2=n-2} h_{n_1,0} h_{n_2,0} + \partial_x h_{n-1,0}, \tag{B.8}$$

$$\begin{aligned} h_{n,1} = 2 \sum_{n_{12}=n-2} h_{n_1,0} h_{n_2,1} + i\frac{2}{3} \sum_{n_{123}=n-3} h_{n_1,0} h_{n_2,0} h_{n_3,0} \qquad &\text{(B.9)} \\ - i \sum_{n_{12}=n-1} h_{n_1,0} h_{n_2,0} + \partial_x h_{n-1,1} + i\widetilde{\mathcal{G}}_\delta \partial_x h_{n-1,0}. \qquad &\text{(B.10)} \end{aligned}$$

Moreover, the degree $(\#u, \#\partial_x, \#\widetilde{\mathcal{G}}_\delta)$ *of all the terms is of the form*

$$\begin{cases} (n+1-j, 2j, 0), & \textit{for } h_{2n,0}, \\ (n+1-j, 2j+1, 0), & \textit{for } h_{2n+1,0}, \end{cases} \tag{B.11}$$

for $0 \le j \le n$ *and* $n \ge 0$.

Proof. The recurrences for $h_{n,0}, h_{n,1}$ follow from (B.3). For the first few elements $h_{n,0}$, $(\# \, u, \#\partial_x)$ is of the following type

$$\begin{aligned} (1-j, 2j), &\qquad 0 \le j \le 0 \qquad \text{for } h_{0,0}, \\ (1-j, 2j+1), &\qquad 0 \le j \le 0 \qquad \text{for } h_{1,0}, \\ (2-j, 2j), &\qquad 0 \le j \le 1 \qquad \text{for } h_{2,0}, \\ (2-j, 2j+1), &\qquad 0 \le j \le 1 \qquad \text{for } h_{3,0}, \end{aligned}$$

where the latter is true since

$$h_{3,0} = 2h_{0,0}h_{1,0} + \partial_x h_{2,0} = 2uu_x + 2uu_x - u_{xxx} = 4uu_x - u_{xxx}.$$

Assume that the result holds for $0 \le k \le n$. By the recurrence, we have

$$\begin{aligned} h_{2n+2,0} &= \sum_{n_{12}=2n} h_{n_1,0} h_{n_2,0} + \partial_x h_{2n+1,0}, \\ h_{2n+3,0} &= \sum_{n_{12}=2n+1} h_{n_1,0} h_{n_2,0} + \partial_x h_{2n+2,0}. \end{aligned}$$

For $h_{2n+2,0}$, we have

$$(\#u, \#\partial_x) \in \bigcup_{2n_{12}=2n} \bigcup_{0 \le j_i \le n_i} \{(n_1+1-j_1, 2j_1) + (n_2+1-j_2, 2j_2)\}$$

$$
\begin{aligned}
&\cup \bigcup_{2n_{12}+2=2n} \bigcup_{0\le j_i\le n_i} \{(n_1+1-j_1, 2j_1+1)+(n_2+1-j_2,2j_2+1)\}\\
&\cup \bigcup_{0\le j\le n} \{(n+1-j,2j+2)\}\\
= &\bigcup_{0\le j\le n} \{(n+2-j,2j)\} \cup \bigcup_{0\le j\le n-1} \{(n-1+2-j,2j+2)\}\\
&\cup \bigcup_{0\le j\le n} \{(n+1-j,2j+2)\}\\
= &\bigcup_{0\le j\le n+1} \{n+2-j,2j\}
\end{aligned}
$$

as intended. Similarly, for $h_{2n+3,0}$, we have that

$$
\begin{aligned}
(\#u,\#\partial_x) \in &\bigcup_{2n_{12}+1=2n+1} \bigcup_{0\le j_i\le n_i} \{(n_1+1-j_1,2j_1)+(n_2+1-j_2,2j_2+1)\}\\
&\cup \bigcup_{0\le j\le n+1} \{(n+2-j,2j+1)\}\\
&= \bigcup_{0\le j\le n} \{n+2-j,2j+1\} \cup \bigcup_{0\le j\le n+1} \{n+2-j,2j+1\}\\
&= \bigcup_{0\le j\le n+1} \{n+2-j,2j+1\},
\end{aligned}
$$

which shows (B.11). □

In the following, we present more explicit descriptions of the quadratic, cubic, and quartic in u terms, as in the following lemmas.

Lemma B.5. *For $n\in\mathbb{N}$ we have that the quadratic terms of $\int \widetilde{h}_n\,dx$ can be written as*

$$
\Big(\mathbf{1}_{n\text{ odd}} i\delta + \mathbf{1}_{n\text{ even}}\delta^2\Big)\int \widetilde{Q}_n\,dx = (-1)^{\frac{n+2}{2}}\delta \sum_{\substack{m=0\\ m\equiv (n-1)\bmod 2}}^{n-1} \delta^m \binom{n}{m} \|\widetilde{\mathcal{G}}_\delta^{\frac{m}{2}} u\|^2_{\dot H^{\frac{n-1}{2}}}. \tag{B.12}
$$

Proof. We first need a description of the linear in u contributions L_n of h_n: $L_0=-u$, $\widetilde{Q}_0=0$ and for $n\in\mathbb{N}$

$$
\begin{aligned}
L_n &= (1+i\delta\widetilde{\mathcal{G}}_\delta)\partial_x L_{n-1} = (1+i\delta\widetilde{\mathcal{G}}_\delta)^n\partial_x^n(-u),\\
\Big(\mathbf{1}_{n\text{ odd}} i\delta + \mathbf{1}_{n\text{ even}}\delta^2\Big)\widetilde{Q}_n &= \frac{i}{2\delta}\frac{(2i\delta)^2}{2}\sum_{n_{12}=n-1} L_{n_1}L_{n_2} + \sum_{j=1}^n \frac{1}{(i\delta)^{n-j}}(1+i\delta\widetilde{\mathcal{G}}_\delta)\partial_x Q_{j-1},
\end{aligned}
$$

where Q_j denotes the quadratic terms in h_j. Note that we are only left with the first terms when we consider $\int \widetilde{Q}_n\,dx$, since the last contributions are full derivatives,

$$
\begin{aligned}
&\Big(\mathbf{1}_{n\text{ odd}} i\delta + \mathbf{1}_{n\text{ even}}\delta^2\Big)\int \widetilde{Q}_n\,dx\\
&= -i\delta\sum_{\ell=0}^{n-1}\int (1+i\delta\widetilde{\mathcal{G}}_\delta)^{n-1-\ell}\partial_x^{n-1-\ell}u(1+i\delta\widetilde{\mathcal{G}}_\delta)^{\ell}\partial_x^{\ell}u\,dx
\end{aligned}
$$

$$= -i\delta \sum_{\ell=0}^{n-1} \sum_{\ell_1=0}^{n-1-\ell} \sum_{\ell_2=0}^{\ell} \binom{n-1-\ell}{\ell_1}\binom{\ell}{\ell_2}(i\delta)^{\ell_1+\ell_2} \int (\widetilde{\mathcal{G}}_\delta^{\ell_1}\partial_x^{n-1-\ell}u)(\widetilde{\mathcal{G}}_\delta^{\ell_2}\partial_x^{\ell}u)\,dx$$
$$= -i\delta \sum_{\ell=0}^{n-1} \sum_{\ell_1=0}^{n-1-\ell} \sum_{\ell_2=0}^{\ell} \binom{n-1-\ell}{\ell_1}\binom{\ell}{\ell_2}(i\delta)^{\ell_1+\ell_2}(-1)^{\ell_1+n-1-\ell} \int u\widetilde{\mathcal{G}}_\delta^{\ell_1+\ell_2}\partial_x^{n-1}u\,dx,$$

using integration by parts. Note that the integrals above are zero when $\ell_1+\ell_2+n-1$ is odd. We now consider two cases depending on the parity of n to simplify the above expression.

<u>Case 1: n even</u>

If n is even, then

$$\delta^2 \int \widetilde{Q}_n\,dx$$
$$= -i\delta \sum_{\substack{m=1\\ m \text{ odd}}}^{n-1} (i\delta)^m (-1)^{\frac{(m-1)+(n-2)}{2}+n-1} \sum_{\substack{0\le\ell\le n-1\\ 0\le\ell_1\le n-1-\ell\\ 0\le m-\ell_1\le\ell}} \binom{n-1-\ell}{\ell_1}\binom{\ell}{m-\ell_1}(-1)^{\ell_1-\ell}\|\widetilde{\mathcal{G}}_\delta^{\frac{m}{2}}u\|^2_{\dot{H}^{\frac{n-1}{2}}}$$
$$= i\delta(-1)^{\frac{n-2}{2}} \sum_{\substack{m=1\\ m \text{ odd}}}^{n-1} \delta^m i^{2m-1}\|\widetilde{\mathcal{G}}_\delta^{\frac{m}{2}}u\|^2_{\dot{H}^{\frac{n-1}{2}}} \sum_{\substack{0\le\ell\le n-1\\ m\le\ell+\ell_1\le n-1}} \binom{n-1-\ell}{\ell_1}\binom{\ell}{m-\ell_1}(-1)^{\ell_1-\ell}$$
$$= (-1)^{\frac{n}{2}} \sum_{\substack{m=1\\ m \text{ odd}}}^{n-1} \delta^{m+1}\|\widetilde{\mathcal{G}}_\delta^{\frac{m}{2}}u\|^2_{\dot{H}^{\frac{n-1}{2}}} \sum_{k=m}^{n-1}\sum_{\ell=0}^{n-1}(-1)^{k-2\ell}\binom{n-1-\ell}{k-\ell}\binom{\ell}{m-k+\ell}$$
$$= (-1)^{\frac{n}{2}} \sum_{\substack{m=1\\ m \text{ odd}}}^{n-1} \delta^{m+1}\|\widetilde{\mathcal{G}}_\delta^{\frac{m}{2}}u\|^2_{\dot{H}^{\frac{n-1}{2}}} \sum_{k=m}^{n-1}(-1)^k\binom{n}{m}$$
$$= (-1)^{\frac{n}{2}+1} \sum_{\substack{m=1\\ m \text{ odd}}}^{n-1} \delta^{m+1}\binom{n}{m}\|\widetilde{\mathcal{G}}_\delta^{\frac{m}{2}}u\|^2_{\dot{H}^{\frac{n-1}{2}}}$$

where we used the Rothe-Hagen identity (B.6) and the fact that m and $n-1$ are odd.

<u>Case 2: n odd</u>

If n is odd, then

$$i\delta \int \widetilde{Q}_n\,dx$$
$$= -i\delta \sum_{\substack{m=0\\ \text{even}}}^{n-1} (i\delta)^m (-1)^{\frac{m}{2}+n-1+\frac{n-1}{2}} \sum_{\substack{0\le\ell\le n-1\\ 0\le\ell_1\le n-1-\ell\\ 0\le m-\ell_1\le\ell}} \binom{n-1-\ell}{\ell_1}\binom{\ell}{m-\ell_1}(-1)^{\ell_1-\ell}\|\widetilde{\mathcal{G}}_\delta^{\frac{m}{2}}u\|^2_{\dot{H}^{\frac{n-1}{2}}}$$
$$= -i\delta(-1)^{\frac{n-1}{2}} \sum_{\substack{m=0\\ \text{even}}}^{n-1} \delta^m i^{2m}\|\widetilde{\mathcal{G}}_\delta^{\frac{m}{2}}u\|^2_{\dot{H}^{\frac{n-1}{2}}} \sum_{\substack{0\le\ell\le n-1\\ 0\le\ell_1\le n-1-\ell\\ 0\le m-\ell_1\le\ell}} \binom{n-1-\ell}{\ell_1}\binom{\ell}{m-\ell_1}(-1)^{\ell_1-\ell}$$

$$= -i\delta(-1)^{\frac{n-1}{2}} \sum_{\substack{m=0\\ \text{even}}}^{n-1} \delta^m \|\widetilde{\mathcal{G}}_\delta^{\frac{m}{2}} u\|^2_{\dot{H}^{\frac{n-1}{2}}} \sum_{k=m}^{n-1}\sum_{\ell=0}^{n-1}(-1)^{k-2\ell}\binom{n-1-\ell}{k-\ell}\binom{\ell}{m-k+\ell}$$

$$= i\delta(-1)^{\frac{n+1}{2}} \sum_{\substack{m=0\\ \text{even}}}^{n-1} \delta^m \binom{n}{m} \|\widetilde{\mathcal{G}}_\delta^{\frac{m}{2}} u\|^2_{\dot{H}^{\frac{n-1}{2}}}$$

by proceeding as before, using the Rothe-Hagen identity (B.6) and the fact that m and $n-1$ are even. The result follows from combining the two cases. $\square$

B.2. **Convergence of sILW conserved quantities.** By taking a formal limit as $\delta \to 0$ of (B.1)-(B.2) we obtain the following

$$\begin{aligned} u &= -q + \varepsilon\partial_x q + \varepsilon^2 q^2, \\ \partial_t q &= -2(\partial_x q)(q - \varepsilon^2 q^2) - \frac{1}{3}\partial_x^3 q, \end{aligned} \tag{B.13}$$

from which, proceeding as before, we can obtain the conserved quantities for the KdV equation

$$\partial_t u + \frac{1}{3}\partial_x^3 u = 2u\partial_x u.$$

We can see that for $q = \sum_{n\geq 0} \varepsilon^n h_n^{\mathrm{KdV}}$, then h_n^{KdV} are conserved quantities of KdV for $n \geq 0$ which satisfy

$$u = -\sum_{n\geq 0}\varepsilon^n h_n^{\mathrm{KdV}} + \sum_{n\geq 1}\varepsilon^n \partial_x h_{n-1}^{\mathrm{KdV}} + \sum_{n\geq 2}\varepsilon^n \sum_{n_{12}=n-2} h_{n_1}^{\mathrm{KdV}} h_{n_2}^{\mathrm{KdV}}.$$

We can then obtain the following recurrence relation

$$h_0^{\mathrm{KdV}} = -u, \tag{B.14}$$

$$h_1^{\mathrm{KdV}} = -\partial_x u, \tag{B.15}$$

$$h_n^{\mathrm{KdV}} = \partial_x h_{n-1}^{\mathrm{KdV}} + \sum_{\ell=0}^{n-2} h_{n-2-\ell}^{\mathrm{KdV}} h_\ell^{\mathrm{KdV}}. \tag{B.16}$$

In the following lemma, we focus on the quadratic in u terms of h_{2n}^{KdV}, denoted by Q_{2n}^{KdV}, and recall that the odd-indexed conserved quantities are trivial.

Lemma B.6. (i) For $n \geq 0$, $\int h_{2n+1}^{\mathrm{KdV}}\, dx = 0$.
(ii) For $n \geq 1$, the quadratic in u terms in h_{2n}^{KdV} satisfy the following

$$\int Q_{2n}^{\mathrm{KdV}}\, dx = (-1)^{n-1}\|u\|^2_{\dot{H}^{n-1}}.$$

Proof. Part (i) was proven in Proposition 2.9 in [30]. For (ii), note that

$$\int Q_{2n}^{\mathrm{KdV}}\, dx = \sum_{\ell=0}^{2n-2}\int L_{2n-2-\ell}^{\mathrm{KdV}} L_\ell^{\mathrm{KdV}}\, dx,$$

where L_j^{KdV} denotes the linear in u terms in h_j^{KdV}, which satisfy

$$L_0^{\mathrm{KdV}} = -u, \qquad L_1^{\mathrm{KdV}} = -\partial_x u, \qquad L_n^{\mathrm{KdV}} = \partial_x L_{n-1}^{\mathrm{KdV}} = \partial_x^n L_0^{\mathrm{KdV}} = -\partial_x^n u.$$

Replacing the above in the expression for the quadratic terms gives

$$\begin{aligned}\int Q_{2n}^{\mathrm{KdV}}\,dx &= \sum_{\ell=0}^{2n-2}\int \partial_x^{2n-2-\ell}u\partial_x^{\ell}u\,dx = \sum_{\ell=0}^{2n-2}(-1)^{2n-2-\ell}\int u\partial_x^{2n-2}u\,dx\\ &= \sum_{\ell=0}^{2n-2}(-1)^{-\ell+n-1}\|u\|^2_{\dot H^{n-1}} = (-1)^{n-1}\|u\|^2_{\dot H^{n-1}},\end{aligned}$$

as intended. □

Let us introduce some notation. Let $\widetilde{E}_{\delta,\frac{k}{2}}(u)$ and $E_{\mathrm{KdV},k}(u)$ denote the sILW and KdV conserved quantities, respectively, which we define as follows:

$$\begin{aligned}\widetilde{E}_{\delta,\frac{2k-1}{2}}(u) &:= (-1)^{k+1}\frac{3}{4k}\int \widetilde{h}_{2k}(u)\,dx,\\ \widetilde{E}_{\delta,\frac{2k}{2}}(u) &:= (-1)^{k+1}\frac{1}{2}\int \widetilde{h}_{2k+1}(u)\,dx,\\ E_{\mathrm{KdV},k}(u) &:= (-1)^{k}\frac{1}{2}\int h_{2k+2}^{\mathrm{KdV}}(u)\,dx.\end{aligned}\tag{B.17}$$

Moreover, we also introduce the δ-free contributions in $\widetilde{E}_{\delta,\frac{2k-1}{2}}, \widetilde{E}_{\delta,\frac{2k}{2}}$ denoted by $\widetilde{E}^{[0]}_{\delta,\frac{2k-1}{2}}, \widetilde{E}^{[0]}_{\delta,\frac{2k}{2}}$, respectively, and defined as follows

$$\begin{aligned}\widetilde{E}^{[0]}_{\delta,\frac{2k-1}{2}}(u) &:= (-1)^{k+1}\frac{3}{4k}\int \widetilde{h}^{*}_{2k}(u)\,dx,\\ \widetilde{E}^{[0]}_{\delta,\frac{2k}{2}}(u) &:= (-1)^{k+1}\frac{1}{2}\int \widetilde{h}_{2k+1,0}(u)\,dx\end{aligned}\tag{B.18}$$

where $\widetilde{h}_{2k+1,0}$ denotes the coefficient of δ^0 in $\widetilde{h}_{2k+1}$ and $\widetilde{h}^{*}_{2k,0}$ can be obtained from $\widetilde{h}_{2k,0}$ by replacing $\mathcal{G}_\delta$ by $-\frac{1}{3}\partial_x$. Then, note that

$$\begin{aligned}\widetilde{E}_{\delta,\frac{2k-1}{2}}(u) &= \widetilde{E}^{[0]}_{\delta,\frac{2k-1}{2}}(u) + \widetilde{L}_{\delta,\frac{2k-1}{2}}(u),\\ \widetilde{E}_{\delta,\frac{2k}{2}}(u) &= \widetilde{E}^{[0]}_{\delta,\frac{2k}{2}}(u) + \widetilde{L}_{\delta,\frac{2k}{2}}(u),\end{aligned}\tag{B.19}$$

where $\widetilde{L}_{\delta,k}$ is implicitly defined by the relations above.

Our main goal is to establish the following proposition on the convergence of the sILW conserved quantities $\widetilde{E}_{\delta,\frac{2k-1}{2}}(u), \widetilde{E}_{\delta,\frac{2k}{2}}(u)$ to the KdV conserved quantity $E_{\mathrm{KdV},k}(u)$ as $\delta\to 0$.

Proposition B.7. *Let $k\geq 1$. Then, for $u\in H^k(\mathbb{T})$, we have that*

$$\lim_{\delta\to 0}\widetilde{E}_{\delta,\frac{2k-1}{2}}(u) = \lim_{\delta\to 0}\widetilde{E}_{\delta,\frac{2k}{2}}(u) = E_{\mathrm{KdV},k}(u).$$

Proof.
Part 1: convergence of $\widetilde{E}_{\delta,\frac{2k}{2}}(u)$
From (B.18), (B.7), (B.8), (B.16), and (B.17), note that

$$\widetilde{E}^{[0]}_{\delta,\frac{2k}{2}}(u) = (-1)^{k+1}\frac{1}{2}\int \widetilde{h}_{2k+1,0}\,dx$$

$$
\begin{aligned}
&= (-1)^k \frac{1}{2} \sum_{n_{12}=2k} \int h_{n_1,0} h_{n_2,0}\, dx \\
&= (-1)^k \frac{1}{2} \sum_{n_{12}=2k} \int h^{\mathrm{KdV}}_{n_1} h^{\mathrm{KdV}}_{n_2}\, dx \\
&= (-1)^k \frac{1}{2} \int h^{\mathrm{KdV}}_{2k+2}\, dx \\
&= E_{\mathrm{KdV},k}(u)
\end{aligned}
$$

since $h_{j,0} = h^{\mathrm{KdV}}_j$ for $j \geq 0$. Consequently, by (B.19), establishing the convergence of $\widetilde{E}_{\delta,\frac{2k}{2}}(u)$ reduces to showing that

$$
\lim_{\delta\to 0} \widetilde{L}_{\delta,\frac{2k}{2}}(u) = 0 \tag{B.20}
$$

for $u \in H^k(\mathbb{T})$.

From Lemma B.3 and (B.7), we can easily see that all the terms in $\widetilde{L}_{\delta,\frac{2k}{2}}(u)$ satisfy that

$$
\#u + \#\, \partial_x \leq 2k+2, \qquad \#\delta \geq 1, \qquad \#\delta \geq \#\widetilde{\mathcal{G}}_\delta,
$$

since we removed the terms in $\widetilde{E}_{0,\frac{2k}{2}}(u)$ which have no powers of δ. We consider two types of terms: $\#\delta \geq \#\widetilde{\mathcal{G}}_\delta + 1$ and $\#\delta = \#\widetilde{\mathcal{G}}_\delta$.

Case 1: $\#\delta \geq \#\widetilde{\mathcal{G}}_\delta + 1$ The terms in $\widetilde{L}_{\delta,\frac{2k}{2}}(u)$ can be written as $\delta^\beta \int p(u)\, dx$ where $p(u) \in \mathcal{P}_j(u)$ for some $j \in \{2, \dots, 2k+2\}$ and

$$
\widetilde{p}(u) = \prod_{\ell=1}^{j} \partial_x^{\alpha_\ell} u
$$

for $\alpha_{1\cdots j} \leq 2k+2-j \leq 2k$, and with $\beta \geq |||p(u)||| + 1$. Moreover, by using IBP, we can guarantee that we only see polynomials $p(u)$ as the above with the added restriction that $0 \leq \alpha_\ell \leq k$ for $\ell = 1, \dots, j$. Using Lemma 2.7, we have that $|\widehat{\delta\widetilde{\mathcal{G}}_\delta(n)}| \leq 1$ for all $n \in \mathbb{Z}^*$, and by Cauchy-Schwarz inequality we can estimate these terms as follows for $0 < \delta \leq \delta_0$

$$
\begin{aligned}
\delta^\beta \bigg| \int p(u)\, dx \bigg| &\lesssim_{\delta_0} \delta \sum_{n_{1\cdots j}=0} \prod_{\ell=1}^{j} |n_\ell|^{\alpha_\ell} |\widehat{u}(n_\ell)| \\
&\lesssim_{\delta_0} \delta \bigg(\sum_{n_{1\cdots j}=0} \frac{|n_1|^{2k} |\widehat{u}(n_1)|}{\prod_{\ell=1}^{j} |n_\ell|^{2(k-\alpha_\ell)}} \bigg)^{\frac{1}{2}} \|u\|_{H^k}^{j-1} \\
&\lesssim_{\delta_0} \delta \|u\|_{H^k}^{j}
\end{aligned}
$$

where the finiteness of the sum follows from the assumptions on α_ℓ.

Case 2: $\#\delta = \#\widetilde{\mathcal{G}}_\delta$ These contributions can be written as $\delta^\beta \int p(u)\, dx$ with $p(u)$ defined as in Case 1 but with $\beta = |||p(u)|||$. For $j \geq 3$, note that $\alpha_{1\cdots j} \leq 2k+2-j \leq 2k-1$ and from Lemma 2.3 we have that $|\widehat{\widetilde{\mathcal{G}}_\delta(n)}| \leq \min(\frac{1}{3}|n|, \frac{1}{\delta})$. For all operators $\widetilde{\mathcal{G}}_\delta$ apart from one, we use upper bound by $\frac{1}{\delta}$, while for one of the terms we use the upper bound by $\frac{1}{3}|n|$. This means that these terms can be estimated analogously to $\delta^\beta \int p(u)\, dx$ where $p(u) \in \widetilde{\mathcal{P}}_j(u)$ for $j \in \{3, \dots, 2k+2\}$ with $1 \geq \beta = |||p(u)||| + 1$, $\alpha_{1\cdots j} \leq 2k+3-j$, and $0 \leq \alpha_\ell \leq k$. All these terms can be estimated as in Case 1. It only remains to show the estimate for the quadratic terms. The remaining terms are quadratic in u and, by (B.12), can be written as

$$\frac{1}{2}\sum_{\substack{m=2\\ \text{even}}}^{2k}\binom{2k+1}{m}\delta^m\|\widetilde{\mathcal{G}}_\delta^{\frac{m}{2}}u\|_{\dot{H}^k}^2.$$

Note that

$$\delta^m\|\widetilde{\mathcal{G}}_\delta^{\frac{m}{2}}u\|_{\dot{H}^k}^2=\sum_{n\neq 0}|\widehat{u}(n)|^2|n|^{2k}\delta^m(\widehat{\widetilde{\mathcal{G}}_\delta}(n))^m=\sum_{n\neq 0}|\widehat{u}(n)|^2|n|^{2k-m}\delta^m\mathrm{L}_\delta(n)^m.$$

From Lemma 2.3(i), we know that for any $\eta>0$ and $n\in\mathbb{Z}^*$,

$$\delta|n|\le\eta \qquad\Longrightarrow\qquad (\delta\mathrm{L}_\delta(n))^m\le\eta^m|n|^{2m}\le\eta^m|n|^m,$$
$$\delta|n|>\eta \qquad\Longrightarrow\qquad (\delta\mathrm{L}_\delta(n))^m\le\delta^m\frac{1}{\delta^m}|n|^m=|n|^m,$$

consequently

$$\begin{aligned}\delta^m\|\widetilde{\mathcal{G}}_\delta^{\frac{m}{2}}u\|_{\dot{H}^k}^2&\le\eta^m\sum_{\delta|n|\le\eta}|\widehat{u}(n)|^2|\xi|^{2k-m+m}+\sum_{\delta|n|\ge\eta}|\widehat{u}(n)|^2|n|^{2k-m+m}\\&\le\eta^m\|u\|_{\dot{H}^k}^2+\sum_{|n|\ge\eta/\delta}|\widehat{u}(n)|^2|n|^{2k}.\end{aligned}$$

By picking $\eta=\sqrt{\delta}$, for $u\in\dot{H}^n$, we get that

$$\frac{1}{2}\sum_{\substack{m=2\\ \text{even}}}^{2k}\binom{2k+1}{m}\delta^m\|\widetilde{\mathcal{G}}_\delta^{\frac{m}{2}}u\|_{\dot{H}^k}^2\lesssim_{\delta_0}\delta\|u\|_{H^k}^2+\|\mathbf{P}_{>\delta^{-\frac{1}{2}}}u\|_{H^k}^2.$$

Combining the results from both cases, we get that for $0<\delta<\delta_0$

$$|\widetilde{L}_{\delta,\frac{2k}{2}}(u)|\lesssim_{\delta_0}\delta(1+\|u\|_{H^k})^{2k+2}+\|\mathbf{P}_{>\delta^{-\frac{1}{2}}}u\|_{H^k}^2\to 0$$

as $\delta\to 0$ for $u\in H^k$, from which (B.20) follows.

Part 2: convergence of $\widetilde{E}_{\delta,\frac{2k-1}{2}}(u)$

From (B.18) and (B.19), we see that $\widetilde{E}^{[0]}_{\delta,\frac{2k-1}{2}}(u)$ is independent of δ, so we must show that for $u\in H^k(\mathbb{T})$

$$\widetilde{E}^{[0]}_{\delta,\frac{2k-1}{2}}(u)=E_{\mathrm{KdV},k}(u)\qquad\text{and}\qquad\lim_{\delta\to 0}\widetilde{L}_{\delta,\frac{2k-1}{2}}(u)=0.$$

We first focus on the convergence of the leftover terms in $\widetilde{L}_{\delta,\frac{2k-1}{2}}(u)$. From (B.17), (B.7), and Lemma B.3, we see that all the terms in $\widetilde{E}_{\delta,\frac{2k-1}{2}}(u)$ can be written as $\delta^\beta\int p(u)\,dx$ where $p(u)\in\widetilde{\mathcal{P}}_j(u)$ for some $j\in\{2,\dots,2k+1\}$ with

$$\widetilde{p}(u)=\prod_{\ell=1}^{j}\partial_x^{\alpha_\ell}u,$$

$\alpha_{1\cdots j}\le 2k+1-j$, and with one of the three conditions on β and the number of $\widetilde{\mathcal{G}}_\delta$ operators $|||p(u)|||$:

(i) $\beta\ge|||p(u)|||+1$ (ii) $\beta=|||p(u)|||\ge 1$ (iii) $\beta=0,\ |||p(u)|||=1.$

We consider different cases depending on the choices for $\beta, |||p(u)|||$ above.
Case (i): $\#\delta \geq \#\widetilde{\mathcal{G}}_\delta + 1$ In this case, we have that $\#\partial_x \leq 2k+1-j \leq 2k-1$, so we can always use IBP to consider polynomials with $0 \leq \alpha_\ell \leq k$. We can then proceed as in Case 1 in Part 1, to obtain the estimate

$$\cdots \lesssim_{\delta_0} \delta \|u\|_{H^k}^j.$$

Case (ii): $\#\delta = \#\widetilde{\mathcal{G}}_\delta \geq 1$ Here, we proceed as in Case 2 in Part 1. For all $\widetilde{\mathcal{G}}_\delta$ operators apart from 1, we use the fact that $|\delta\widehat{\mathcal{G}_\delta}(n)| \leq 1$ for all $n \in \mathbb{Z}^*$. For the remaining one we use $|\widehat{\widetilde{\mathcal{G}}_\delta}(n)| \leq |n|$, which amounts to the loss of one derivative, while keeping one power of δ. These terms can be estimated in a similar way to terms written as $\delta \int p(u)\, dx$ where $p(u) \in \widetilde{\mathcal{P}}_j(u)$ for $j \in \{2, \dots, 2k+1\}$, $\widetilde{p}(u) = \prod_{\ell=1}^j \partial_x^{\alpha_\ell} u$ with $\alpha_{1\cdots j} \leq 2k+2-j \leq 2k$ and $0 \leq \alpha_\ell \leq k$. In particular, they can be controlled by

$$\cdots \lesssim \delta \|u\|_{H^k}^j.$$

Case (iii): $\#\delta = 0, \#\widetilde{\mathcal{G}}_\delta = 1$ From the definition of $\widetilde{E}^{[0]}_{\delta, \frac{2k-2}{2}}(u)$, these terms which appear in $\widetilde{L}_{\delta, \frac{2k-1}{2}}(u)$ are obtained from those in $\int \widetilde{h}_{2k,0}\, dx$ with one $\widetilde{\mathcal{G}}_\delta$ operator replaced by $(\widetilde{\mathcal{G}}_\delta + \frac{1}{3}\partial_x)$. From Lemma 2.3, we know that for fixed $n \in \mathbb{Z}^*$, $|\mathcal{F}_x(\widetilde{\mathcal{G}}_\delta + \frac{1}{3}\partial_x)(n)| \leq |n|$ and $\lim_{\delta\to 0} |\mathcal{F}_x(\widetilde{\mathcal{G}}_\delta + \frac{1}{3}\partial_x)(n)| = 0$. By IBP, we can assume that $k \geq \alpha_1 \geq \cdots \geq \alpha_\ell \geq 0$, $\alpha_{1\cdots j} \leq 2k+1-j \leq 2k-1$, and that the $\widetilde{\mathcal{G}}_\delta$ is acting on the factor with α_2 derivatives (analogous in all other cases)

$$\begin{aligned}
\bigg| \int p(u)\, dx \bigg| &\lesssim \sum_{n_{1\cdots j}=0} |n_2||h(\delta, n_2)| \prod_{\ell=1}^j |n_\ell|^{\alpha_\ell} |\widehat{u}(n_\ell)| \\
&\lesssim \bigg(\sum_{n_{1\cdots j}=0} \frac{|n_2|^2}{\prod_{\ell=1}^j |n_\ell|^{2(k-\alpha_\ell)}} |n_1|^{2k} |\widehat{u}(n_1)|^2 \bigg)^{\frac{1}{2}} \|u\|_{H^k}^{j-2} \|h(\delta,n)|n|^k \widehat{u}(n)\|_{\ell^2_n} \\
&\lesssim \||n|^k h(\delta,n)\widehat{u}(n)\|_{\ell^2_n} \|u\|_{H^k}^{j-1}
\end{aligned}$$

where h is given in Lemma 2.3, which converges to 0 pointwise in n as $\delta \to 0$. Also, since $|h(\delta,n)| \leq 1$ uniformly, we see that the above quantity is bounded by $\|u\|_{H^k}^j$.

Combining the estimates from all the terms above, we have that for $0 < \delta \leq \delta_0$

$$|\widetilde{L}_{\delta, \frac{2k-1}{2}}(u)| \lesssim_{\delta_0} \delta(1 + \|u\|_{H^k})^{2k+1} + \|h(\delta,n)|n|^k \widehat{u}(n)\|_{\ell^2_n} (1 + \|u\|_{H^k})^{2k} \to 0 \tag{B.21}$$

as $\delta \to 0$ from the decay in $\delta \to 0$ of the first term, the pointwise convergence $\lim_{\delta\to 0} h(\delta,n) \to 0$, and the dominated convergence theorem.

The above establishes that for $u \in H^k(\mathbb{T})$

$$\lim_{\delta\to 0} \widetilde{E}_{\delta, \frac{2k-1}{2}}(u) = \widetilde{E}^{[0]}_{\delta, \frac{2k-1}{2}}(u),$$

thus it remains to show that this limit agrees with $E_{\text{KdV},k}(u)$. It is known that all polynomial conserved quantities of KdV with quadratic terms of order $H^k(\mathbb{T})$ can be written as a linear combination of $E_{\text{KdV},k}(u)$; see Theorem 2.16 in [30], for example. From Lemma B.6, (B.12), and (B.17), we see that the quadratic in u terms in $\widetilde{E}^{[0]}_{\delta, \frac{2k-1}{2}}(u)$ agree with those in

$E_{\text{KdV},k}(u)$. Thus, if we show that $\widetilde{E}^{[0]}_{\delta,\frac{2k-1}{2}}(u)$ is conserved for KdV, then we must have that $\widetilde{E}^{[0]}_{\delta,\frac{2k-1}{2}}(u) = E_{\text{KdV},k}(u)$ for all $u \in H^k(\mathbb{T})$.

Fix $u_0 \in H^k(\mathbb{T})$, $\delta > 0$, and let $0 < \delta < \delta_0$. From Lemma **??**, there exists $T > 0$ independent of δ, such that there exist unique solutions $u^\delta, u^{\text{KdV}} \in C([0,T]; H^k(\mathbb{T}))$ of sILW and KdV, respectively, with initial data u_0, satisfying

$$\lim_{\delta\to 0} \|u^\delta - u^{\text{KdV}}\|_{C([0,T];H^k)} = 0. \tag{B.22}$$

Then, using the fact that $\widetilde{E}_{\delta,\frac{2k-1}{2}}$ is conserved for sILW and (B.19), we get that

$$\begin{aligned}
&\widetilde{E}^{[0]}_{\delta,\frac{2k-1}{2}}(u_t^{\text{KdV}}) - \widetilde{E}^{[0]}_{\delta,\frac{2k-1}{2}}(u_0)\\
&= \big[\widetilde{E}^{[0]}_{\delta,\frac{2k-1}{2}}(u_t^{\text{KdV}}) - \widetilde{E}_{\delta,\frac{2k-1}{2}}(u_t^\delta)\big] + \big[\widetilde{E}_{\delta,\frac{2k-1}{2}}(u_0) - \widetilde{E}^{[0]}_{\delta,\frac{2k-1}{2}}(u_0)\big]\\
&= \big[\widetilde{E}^{[0]}_{\delta,\frac{2k-1}{2}}(u_t^{\text{KdV}}) - \widetilde{E}^{[0]}_{\delta,\frac{2k-1}{2}}(u_t^\delta)\big] - \widetilde{L}_{\delta,\frac{2k-1}{2}}(u_t^\delta) + \widetilde{L}_{\delta,\frac{2k-1}{2}}(u_0)
\end{aligned}$$

where $u_t^{\text{KdV}}, u_t^\delta$ denote the solutions evaluated at time t. From (B.22), there exists $\delta_0 > 0$ such that for $0 < \delta \le \delta_0$ we have

$$\|u^\delta\|_{L^\infty_T H^k} \le \|u^{\text{KdV}}\|_{L^\infty_T H^k} + 1.$$

Using the multilinearity of $\widetilde{E}^{[0]}_{\delta,\frac{2k-1}{2}}$ and using the fact that all the terms here can be written as $\int p(u)\,dx$ where $p(u) \in \widetilde{\mathcal{P}}_j(u)$ for $j \in \{2,\dots,2k+1\}$ with $\|p(u)\| \le 2k+2-j \le 2k$, $|||p(u)||| = 0$, and at most k derivatives on each term, from Hölder's inequality and Sobolev's inequality, we get that

$$\begin{aligned}
\big|\widetilde{E}^{[0]}_{\delta,\frac{2k-1}{2}}(u_t^{\text{KdV}}) - \widetilde{E}^{[0]}_{\delta,\frac{2k-1}{2}}(u_t^\delta)\big| &\lesssim (1 + \|u^\delta\|_{L^\infty_T H^k} + \|u^{\text{KdV}}\|_{L^\infty_T H^k})^{2k}\|u^\delta - u^{\text{KdV}}\|_{L^\infty_T H^k}\\
&\lesssim (1 + \|u^{\text{KdV}}\|_{L^\infty_T H^k})\|u^\delta - u^{\text{KdV}}\|_{L^\infty_T H^k} \to 0
\end{aligned}$$

as $\delta \to 0$ from the convergence in (B.22). Moreover, from (B.21), we have that

$$\begin{aligned}
|\widetilde{L}_{\delta,\frac{2k-1}{2}}(u_t^\delta)| &\lesssim_{\delta_0} (1 + \|u^\delta\|_{L^\infty_T H^k})^{2k+1}\big(\delta + \|h(\delta,n)|n|^k\widehat{u}_t^\delta(n)\|_{L^\infty_T \ell^2_n}\big)\\
&\lesssim_{\delta_0} (1 + \|u^{\text{KdV}}\|_{L^\infty_T H^k})^{2k+1}\big(\delta + \|h(\delta,n)|n|^k\widehat{u}_t^\delta(n)\|_{L^\infty_T \ell^2_n}\big)\\
&\lesssim_{\delta_0} (1 + \|u^{\text{KdV}}\|_{L^\infty_T H^k})^{2k+1}\big(\delta + \|u^{\text{KdV}}\|_{L^\infty_T \ell^2_n}\big),\\
|\widetilde{L}_{\delta,\frac{2k-1}{2}}(u_0)| &\lesssim_{\delta_0} (1 + \|u_0\|_{H^k})^{2k+1}\big(\delta + \|h(\delta,n)|n|^k\widehat{u}_0(n)\|_{\ell^2_n}\big),
\end{aligned}$$

from which we see that both contributions vanish as $\delta \to 0$, from the decay in δ, the pointwise convergence $\lim_{\delta\to 0} h(\delta,n) = 0$, and the dominated convergence theorem. Therefore, we have that

$$\widetilde{E}^{[0]}_{\delta,\frac{2k-1}{2}}(u_t^{\text{KdV}}) = \widetilde{E}^{[0]}_{\delta,\frac{2k-1}{2}}(u_0),$$

i.e., $\widetilde{E}^{[0]}_{\delta,\frac{2k-1}{2}}$ is a conserved quantity under the KdV dynamics, which completes our proof of convergence. □

B.3. **Structure of remainder $\widetilde{R}_{\delta,\frac{k}{2}}(u)$ of conserved quantities $\widetilde{E}_{\delta,\frac{k}{2}}(u)$.** The following lemma provides a more detailed description of the cubic and quartic in u terms in $\widetilde{E}_{\delta,k-\frac{1}{2}}(u)$, i.e., the even indexed remainders.

Lemma B.8. *Let $k \in \mathbb{N}$. The cubic in u contributions in $\widetilde{E}_{\delta,k-\frac{1}{2}}(u)$ are of one of the following type:*
(i) $\#\delta \geq \#\widetilde{\mathcal{G}}_\delta$ *and* $\#\partial_x \leq 2k-2$
(ii) $\#\delta - 1 = \#\widetilde{\mathcal{G}}_\delta$ *and the terms are of the form,*

$$\frac{1}{\delta}\int [(-1+i\delta\widetilde{\mathcal{G}}_\delta)^\alpha(1+i\delta\widetilde{\mathcal{G}}_\delta)^{n_1}\partial_x^{\alpha+n_1}u][(1+i\delta\widetilde{\mathcal{G}}_\delta)^{m_1}\partial_x^{m_1}u][(1+i\delta\widetilde{\mathcal{G}}_\delta)^{m_2}\partial_x^{m_2}u]\,dx$$

where L_j denote the linear in u terms in h_j and

$$\begin{aligned} n_1+m_1+m_2 &= n_1+n_2-2-\alpha,\\ n_1+n_2 &= 2k-1,\\ \alpha &\geq 0, \end{aligned}$$

(iii) $\#\delta = \#\widetilde{\mathcal{G}}_\delta$ *and the terms are of the form*

$$\int [(1+i\delta\widetilde{\mathcal{G}}_\delta)^{n_1}\partial_x^{n_1}u](1+i\delta\widetilde{\mathcal{G}}_\delta)^{n_2-1}\partial_x^{n_2-1}(u^2)\,dx$$

where $n_1+n_2 = 2k-1$.
Moreover, $\widetilde{E}_{\delta,2}(u)$ has no quartic in u terms, while for $k \geq 2$, these terms in $\widetilde{E}_{\delta,2k}(u)$, are of the form:
(i) $\#\delta \geq \#\widetilde{\mathcal{G}}_\delta$ *and* $\#\partial_x \leq 2k-3$;
(ii) $\#\delta - 1 \geq \#\widetilde{\mathcal{G}}_\delta$ *and* $\#\partial_x \leq 2k-5$ *for* $k \geq 3$.
In particular, for $k=2$, all the quartic terms are of type (i).
Also, we have that the cubic in u terms of $\widetilde{E}_{\delta,2k}(u)$ are of one of the form $\int p(u)\,dx$, where the polynomial $p(u) \in \widetilde{\mathcal{P}}_3(u)$ is of one of the following types:
(i) $\|p(u)\| \leq 2k-2$, $|p(u)| \leq k-1$, *and* $\widetilde{p}(u) \neq u\partial_x^{k-1}u\partial_x^{k-1}u$
(ii) $\|p(u)\| = 2k-2$ *and*

$$\begin{aligned} p(u) &= \delta^{m_{12}}\int u(\widetilde{\mathcal{G}}_\delta^{m_1}\partial_x^{k-1}u)(\widetilde{\mathcal{G}}_\delta^{m_2}\partial_x^{k-1}u)\,dx \quad \text{or}\\ p(u) &= \delta^m\int [\widetilde{\mathcal{G}}_\delta^m\partial_x^{\alpha_1}u][\partial_x^{\alpha_2}u][\partial_x^{\alpha_3}u]\,dx, \end{aligned}$$

where $0 \leq m_{12}, m \leq 2n-2$ even, and $\{\alpha_1,\alpha_2,\alpha_3\} = \{0, k-1\}$ with $\alpha_{123} = 2k-2$.

Proof. From (B.7), we see

$$\widetilde{E}_{\delta,2n}(u) \sim \frac{i}{2\delta^3}\sum_{k=2}^{2n+1}\frac{(2i\delta)^k}{k!}\sum_{n_{1\cdots k}=2n+1-k}\int h_{n_1}\cdots h_{n_k}\,dx \tag{B.23}$$

and Lemma B.3 guarantees that all h_{n_j} terms have $\#\delta \geq \#\widetilde{\mathcal{G}}_\delta$ and the cubic terms have $\#\partial_x \leq 2n+1-3 = 2n-2$. Therefore, the cubic terms arising from $k \geq 3$ in the above sum are of type (i).

The cubic terms of type (ii) and (iii) come from contributions with $k = 2$ in (B.23). We can see that the cubic terms coming from $k = 2$ are of the following form

$$\mathrm{I} := \frac{i}{2\delta^3}\frac{(2i\delta)^2}{2}\sum_{n_{12}=2n-1}\int L_{n_1}Q_{n_2}\,dx,$$

where L_j, Q_j denote the linear and quadratic in u terms in h_j, respectively. Note that from (B.3), we have

$$\begin{aligned}
&L_0 = -u, && Q_0 = 0,\\
&L_1 = -(1+i\delta\widetilde{\mathcal{G}}_\delta)\partial_x u, && Q_1 = -i\delta u^2,\\
&L_n = -(1+i\delta\widetilde{\mathcal{G}}_\delta)^n\partial_x^n u, && Q_n = \sum_{n_{12}=n-2} L_{n_1}L_{n_2} - i\delta\sum_{n_{12}=n-1} L_{n_1}L_{n_2} + (1+i\delta\widetilde{\mathcal{G}}_\delta)\partial_x Q_{n-1}.
\end{aligned} \tag{B.24}$$

Replacing these expressions above, we obtain that

$$\begin{aligned}
\mathrm{I} = &-\frac{i}{\delta}\sum_{n_{12}=2n-1}\int [(1+i\delta\widetilde{\mathcal{G}}_\delta)^{n_1}\partial_x^{n_1}u]\sum_{m_{12}=n_2-2} L_{m_1}L_{m_2}\,dx\\
&-\sum_{n_{12}=2n-1}\int [(1+i\delta\widetilde{\mathcal{G}}_\delta)^{n_1}\partial_x^{n_1}u]\sum_{m_{12}=n_2-1} L_{m_1}L_{m_2}\,dx\\
&-\frac{i}{\delta}\sum_{n_{12}=2n-1}\int [(1+i\delta\widetilde{\mathcal{G}}_\delta)^{n_1}\partial_x^{n_1}u](1+i\delta\widetilde{\mathcal{G}}_\delta)\partial_x Q_{n_2-1}\,dx\\
&=: \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3.
\end{aligned} \tag{B.25}$$

The contributions in I_1 are of type (ii) with $\alpha = 0$, while those in II_2 are of type (i) since $\#\delta = \#\widetilde{\mathcal{G}}_\delta$ and $\#\partial_x = n_1 + m_1 + m_2 = n_1 + n_2 - 1 = 2n - 1 - 1 = 2n - 2$. To decide on the type of the terms in I_3, we must further replace the expression for Q_{n_2-1}.

If $n_2 - 1 = 1$, then $\mathrm{I}_3 = 0$. If $n_2 - 1 = 2$, then I_3 is of type (iii) because $Q_2 = -i\delta u^2$, thus this extra power of δ combined with $\frac{1}{\delta}$ at the front guarantee that $\#\delta = \#\widetilde{\mathcal{G}}_\delta$. If $n_2 - 1 \geq 3$, then we replace Q_{n_2-1} by its expression to get

$$\begin{aligned}
\mathrm{I}_3 = &\frac{1}{\delta}\sum_{n_{12}=2n-1}\int [(1+i\delta\widetilde{\mathcal{G}}_\delta)^{n_1}\partial_x^{n_1}u](1+i\delta\widetilde{\mathcal{G}}_\delta)\partial_x\sum_{n_{12}=n_2-3} L_{m_1}L_{m_2}\,dx\\
&- i\sum_{n_{12}=2n-1}\int [(1+i\delta\widetilde{\mathcal{G}}_\delta)^{n_1}\partial_x^{n_1}u](1+i\delta\widetilde{\mathcal{G}}_\delta)\partial_x\sum_{m_{12}=n_2-2} L_{m_1}L_{m_2}\,dx\\
&+\frac{1}{\delta}\sum_{n_{12}=2n-1}\int [(1+i\delta\widetilde{\mathcal{G}}_\delta)^{n_1}\partial_x^{n_1}u](1+i\delta\widetilde{\mathcal{G}}_\delta)^2\partial_x^2 Q_{n_2-2}\,dx\\
&=: \mathrm{I}_{3,1} + \mathrm{I}_{3,2} + \mathrm{I}_{3,3}.
\end{aligned}$$

As before, we see that terms in $\mathrm{I}_{3,1}$ are of type (ii) with $\alpha = 1$ (after integration by parts) and those in $\mathrm{I}_{3,2}$ are of type (i). As before, if $n_2 - 2 = 2$, then this term is of type (iii), otherwise we iterate the above process and note that the contributions coming from the first term in the recurrence of Q_{n_2-j} always give rise to type (ii) terms, those coming from the second terms are of type (i), and the process terminates when $n_2 - j = 2$ when we obtain a term of type (iii).

Proceeding as for the cubic in u terms, we know that the terms arising from $k \geq 3$ in (B.23), have at most $2n+1-4 = 2n-3$ derivatives and they have $\#\delta \geq \#\widetilde{\mathcal{G}}_\delta$, thus they are of type (i). Note that the quartic terms arising from $k = 2$ are of the form

$$\mathrm{II} := \frac{i}{2\delta^3}\frac{(2i\delta)^2}{2}\sum_{n_{12}=2n-1}(L_{n_1}C_{n_2} + Q_{n_1}Q_{n_2}),$$

where C_n denotes the cubic in u terms in h_n, which satisfy the following

$$\begin{aligned}
&C_0 = 0,\\
&C_1 = 0,\\
&C_2 = \frac{4}{3}\delta^2 u^3,\\
&C_n = \sum_{n_{12}=n-2} L_{n_1}Q_{n_2} + i\frac{2}{3}\delta\sum_{n_{123}=n-3} L_{n_1}L_{n_2}L_{n_3} - i\delta\sum_{n_{12}=n-1} L_{n_1}Q_{n_2}\\
&\qquad + \frac{2}{3}\delta^2\sum_{n_{123}=n-2} L_{n_1}L_{n_2}L_{n_3} + (1 + i\delta\widetilde{\mathcal{G}}_\delta)\partial_x C_{n-1}.
\end{aligned}$$

From before, we know that

- L_n has no extra powers of δ and $\#\partial_x = n$,
- Q_n has terms with (1) no extra powers of δ and $\#\partial_x = n-2$; (ii) one extra power of δ and $\#\partial_x = n-1$.

Consequently, the terms in C_n for $n \geq 3$ are of the form

$$\begin{aligned}
\sum_{n_{12}=n-2} L_{n_1}Q_{n_2} &\sim \mathbf{1}_{n\geq 4}[\delta^0 + \partial_x^{n-4}] + [\delta^1 + \partial_x^{n-3}],\\
i\frac{2}{3}\delta\sum_{n_{123}=n-3} L_{n_1}L_{n_2}L_{n_3} &\sim [\delta^1 + \partial_x^{n-3}],\\
-i\delta\sum_{n_{12}=n-1} L_{n_1}Q_{n_2} &\sim [\delta^1 + \partial_x^{n-2}],\\
\frac{2}{3}\delta^2\sum_{n_{123}=n-2} L_{n_1}L_{n_2}L_{n_3} &\sim [\delta^1 + \partial_x^{n-2}],
\end{aligned}$$

from which we see that the terms excluding the last one in the expression for C_n are of the form

$$\sim [\delta^0 + \partial_x^{n-4}]\mathbf{1}_{n\geq 4} + [\delta^1 + \partial_x^{n-2}].$$

Note that if $n-1 = 2$, then, the term $(1+i\delta\widetilde{\mathcal{G}}_\delta)\partial_x C_{n-1}$ is of the form $[\delta^1 + \partial_x^1] = [\delta^1 + \partial_x^{n-2}]$. Otherwise, $n-1 \geq 3$ and we can replace the expression to obtain more terms of the form $\sim [\delta^0 + \partial_x^{n-4}]\mathbf{1}_{n\geq 4} + [\delta^1 + \partial_x^{n-2}]$. Iterating this process, until we reach the contribution

$$(1 + i\delta\widetilde{\mathcal{G}}_\delta)^{n-2}\partial_x^{n-2}C_2,$$

we conclude that all of the contributions in C_n are of this given form.

Now, looking back at II, we have that

$$\mathrm{II} = -\frac{1}{\delta}\sum_{n_{12}=2n-1}(L_{n_1}C_{n_2} + Q_{n_1}Q_{n_2})$$

$$\begin{aligned}
&\sim \sum_{n_{12}=2n-1} \delta^{-1}[\delta^0 + \partial_x^{n_1}]\big\{[\delta^0 + \partial_x^{n_2-4}]\mathbf{1}_{n_2\geq 4} + [\delta^1 + \partial_x^{n_2-2}]\big\} \\
&\quad + \sum_{n_{12}=2n-1} \delta^{-1}\prod_{j=1}^{2}\big\{[\delta^0 + \partial_x^{n_j-2}] + [\delta^1 + \partial_x^{n_j-1}]\big\} \\
&\sim \sum_{n_{12}=2n-1} \big\{[\delta^{-1} + \partial_x^{n_{12}-4}] + [\delta^0 + \partial_x^{n_{12}-2}] + [\delta^1 + \partial_x^{n_{12}-3}]\big\} \\
&\sim [\delta^{-1} + \partial_x^{2n-5}]\mathbf{1}_{n\geq 3} + [\delta^0 + \partial_x^{2n-3}]\mathbf{1}_{n\geq 2}.
\end{aligned}$$

Note that $n = 1$ has no quartic terms. For $n = 2$, all of the terms are of type (i). For $n \geq 3$, II also contributes with terms of type (ii) which satisfy that $\#\delta - 1 \geq \#\widetilde{\mathcal{G}}_\delta$ and $\#\partial_x = 2n - 5$.

The cubic in u contributions of $\widetilde{E}_{\delta,2n}(u)$ are the following

$$\frac{i}{2\delta^3}\frac{(2i\delta)^2}{2}\sum_{n_{12}=2n-1}\int L_{n_1}Q_{n_2}\,dx + \frac{i}{2\delta^3}\frac{(2i\delta)^3}{3!}\sum_{n_{123}=2n-2}\int L_{n_1}L_{n_2}L_{n_3}\,dx. \tag{B.26}$$

For the second group of terms above, we have that

$$\frac{2}{3}\sum_{n_{123}=2n-2}\int\prod_{j=1}^{3}(1 + i\delta\widetilde{\mathcal{G}}_\delta)^{n_j}\partial_x^{n_j}u\,dx$$

and all the terms can be written as one of type (i) (possibly after integration by parts) as long as $n_1, n_2, n_3 \geq 1$. The remaining terms have that $n_j = 0$ for some $j \in \{1, 2, 3\}$, and we assume $j = 3$ by symmetry. Then, these can be written as

$$\sum_{0\leq m_{12}\leq 2n-2}\int (i\delta)^{m_{12}}(\widetilde{\mathcal{G}}_\delta^{m_1}\partial_x^{k-1}u)(\widetilde{\mathcal{G}}_\delta^{m_2}\partial_x^{k-1}u)u\,dx,$$

and the above must vanish when m_{12} is odd, since they would be purely imaginary. Therefore, we are only left with terms of type (ii) as intended. We now consider the first contribution in (B.26), which can be written as in (B.25). Then, I_1 has terms with $2n-3$ derivatives, which are therefore of type (i), while the terms in I_2 are as the second ones in (B.26) discussed above. It only remains to consider I_3. If $n_2 \geq 2$ in I_3, then we use (B.24) to get

$$\begin{aligned}
\mathrm{I}_3 &= \frac{i}{\delta}\sum_{n_{12}=2n-1}\int L_{n_1}(1 + i\delta\widetilde{\mathcal{G}}_\delta)\partial_x Q_{n_2-1}\,dx \\
&= \frac{i}{\delta}\sum_{m_{123}=2n-4}\int(-1 + i\delta\widetilde{\mathcal{G}}_\delta)\partial_x L_{m_1}L_{m_2}L_{m_3}\,dx \\
&\quad + \sum_{m_{123}=2n-3}\int(-1 + i\delta\widetilde{\mathcal{G}}_\delta)\partial_x L_{m_1}L_{m_2}L_{m_3}\,dx \\
&\quad + \frac{i}{\delta}\sum_{n_{12}=2n-1}\int L_{n_1}(1 + i\delta\widetilde{\mathcal{G}}_\delta)^2\partial_x^2 Q_{n_2-2}\,dx
\end{aligned}$$

where the first contribution is as I_1 and the second as in II_2. We can continue to iterate as above, adding terms like $\mathrm{I}_1, \mathrm{II}_2$ which are of the intended type, until we reach the terms of the form

$$\begin{aligned}
&\frac{i}{\delta}\sum_{n_{12}=2n-1}\int L_{n_1}(1+i\delta\widetilde{\mathcal{G}}_\delta)^{n_2-1}\partial_x^{n_2-1}Q_1\,dx\\
&= -\sum_{n_{12}=2n-1}\int[(1+i\delta\widetilde{\mathcal{G}}_\delta)^{n_1}\partial_x^{n_1}u][(1+i\delta\widetilde{\mathcal{G}}_\delta)^{n_2-1}\partial_x^{n_2-1}(u^2)]\,dx\\
&= -\sum_{n_{12}=2n-2}\sum_{\ell=0}^{n_2}c_\ell\int[(1+i\delta\widetilde{\mathcal{G}}_\delta)^{n_1}\partial_x^{n_1}u][(1+i\delta\widetilde{\mathcal{G}}_\delta)^{n_2}(\partial_x^\ell u\cdot\partial_x^{n_2-\ell}u)]\,dx.
\end{aligned}$$

By doing integration by parts, the terms above are of type (i) if $n_1, \ell, n_2-\ell \geq 1$. Otherwise, we have at least one of them equal to 0, where all the terms can be written as one of the following:

$$\begin{aligned}
&\int[(1+i\delta\widetilde{\mathcal{G}}_\delta)^{n_1}(1-i\delta\widetilde{\mathcal{G}}_\delta)^{n_2}u][\partial_x^{n-1}u]^2\,dx\\
&= \sum_{m_1=0}^{n_1}\sum_{m_2=0}^{n_2}(-1)^{m_2}(i\delta)^{m_{12}}[\widetilde{\mathcal{G}}_\delta^{m_{12}}u][\partial_x^{n-1}u]^2\,dx,\\
&\int[(1+i\delta\widetilde{\mathcal{G}}_\delta)^{n_1}(1-i\delta\widetilde{\mathcal{G}}_\delta)^{n_2}\partial_x^{n-1}u][u\partial_x^{n-1}u]\,dx\\
&= \sum_{m_1=0}^{n_1}\sum_{m_2=0}^{n_2}(-1)^{m_2}(i\delta)^{m_{12}}[\widetilde{\mathcal{G}}_\delta^{m_{12}}\partial_x^{n-1}u][u\partial_x^{n-1}u]\,dx,
\end{aligned}$$

where we can further restrict the sums above to m_{12} even, from which we see that these terms are of type (ii). □

The following lemma will be relevant when establishing invariance.

Lemma B.9. *Let $k\in\mathbb{N}$. The cubic in u terms in $\widetilde{E}_{\delta,2k+1}(u)$ are of the form $\int p(u)\,dx$ where the polynomial $p(u)\in\widetilde{\mathcal{P}}_3(u)$ is of one of the following types:*
(i) $\|p(u)\|\leq 2k-1$ *and* $|p(u)|\leq k-1$*;*
(ii) $\|p(u)\| = 2k-1$ *and*

$$p(u) = \delta^{m_{12}+1}u(\widetilde{\mathcal{G}}_\delta^{m_2}\partial_x^{k-1}u)(\widetilde{\mathcal{G}}_\delta^{m_3}\partial_x^k u) \qquad \text{or} \qquad p(u)=\delta^{m_3+1}(\widetilde{\mathcal{G}}_\delta^{m_3}u)(\partial_x^{k-1}u)(\partial_x^k u)$$

where $0\leq m_{12}\leq 2k-1$, $0\leq m_3\leq 2k-1$ odd.

Proof. For $k=1$, the only cubic in u contributions in $\widetilde{E}_{\delta,3}(u)$ are of the form

$$p(u)=u^3 \qquad \text{and} \qquad p(u)=\delta^2u^2\widetilde{\mathcal{G}}_\delta u_x,$$

which are of type (i) and (ii), respectively.

From (B.7) and (B.24), we see that the only cubic contributions in $\int\widetilde{h}_{2k+1}$ are

$$\begin{aligned}
&-\sum_{n_{12}=2k}\int L_{n_1}Q_{n_2}\,dx - i\frac{2}{3}\delta\sum_{n_{123}=2k-1}\int L_{n_1}L_{n_2}L_{n_3}\,dx\\
&= -\sum_{n_{12}=2k}\int L_{n_1}\mathbf{1}_{n_2\geq1}\bigg\{\sum_{m_{12}=n_2-2}L_{m_1}L_{m_2} - i\delta\sum_{m_{12}=n_2-1}L_{m_1}L_{m_2}\\
&\qquad + (1+i\delta\widetilde{\mathcal{G}}_\delta)\partial_x Q_{n_2-1}\bigg\}\,dx - i\frac{2}{3}\delta\sum_{n_{123}=2k-1}\int L_{n_1}L_{n_2}L_{n_3}\,dx
\end{aligned}$$

$$= -\mathbf{1}_{k\geq 2} \sum_{m_{123}=2k-2} \int L_{m_1} L_{m_2} L_{m_3}\, dx + \frac{i\delta}{3} \sum_{m_{123}=2k-1} \int L_{m_1} L_{m_2} L_{m_3}\, dx$$
$$- \sum_{m_{12}=2k-1} \int L_{m_1} (1 + i\delta \widetilde{\mathcal{G}}_\delta) \partial_x Q_{m_2}\, dx. \tag{B.27}$$

Since L_j has j derivatives, the first contribution has $m_{123} = 2k-2$ derivatives and by integration by parts can be written as the integral of a polynomial of type (i). For the second contribution, there are $2k-1$ derivatives, and these contributions can be written as type (i) if $m_1, m_2, m_3 \leq k-1$ or $m_{j_1} \geq k$ while $m_{j_2}, m_{j_3} \leq k-2$, for $\{j_1, j_2, j_3\} = \{1, 2, 3\}$. Thus, the only terms left to consider have $\{m_1, m_2, m_3\} = \{0, k-1, k\}$, after integration by parts. We can assume $m_1 = 0, m_2 = k-1, m_3 = k$ by symmetry, to get

$$- \frac{i\delta}{3} \sum_{\ell_1=0}^{k-1} \sum_{\ell_2=0}^{k} (i\delta)^{\ell_{12}} \int u (\widetilde{\mathcal{G}}_\delta^{\ell_1} \partial_x^{k-1} u)(\widetilde{\mathcal{G}}_\delta^{\ell_2} \partial_x^k u)\, dx$$
$$= -\frac{i\delta}{3} \sum_{\substack{0\leq \ell_1 \leq k-1 \\ 0 \leq \ell_2 \leq k \\ \ell_{12} \text{ odd}}} (i\delta)^{\ell_{12}} \int u (\widetilde{\mathcal{G}}_\delta^{\ell_1} \partial_x^{k-1} u)(\widetilde{\mathcal{G}}_\delta^{\ell_2} \partial_x^k u)\, dx,$$

since if ℓ_{12} is even, the contributions would be purely imaginary and they must vanish since the conserved quantities are real, where we also see that $0 \leq \ell_{12} \leq 2k-1$, so these terms are of type (ii). Lastly, for the third contribution,

$$- \sum_{m_{12}=2k-1} \int L_{m_1} (1 + i\delta \widetilde{\mathcal{G}}_\delta) \partial_x Q_{m_2}\, dx,$$

if $m_2 \geq 2$, then we replace (B.24) to obtain

$$- \sum_{m_{12}=2k-1} \int L_{m_1} (1 + i\delta \widetilde{\mathcal{G}}_\delta) \bigg\{ \sum_{n_{12}=m_2-2} L_{n_1} L_{n_2} - i\delta \sum_{n_{12}=m_2-1} L_{n_1} L_{n_2} + (1 + i\delta \widetilde{\mathcal{G}}_\delta) \partial_x Q_{m_2-1} \bigg\}\, dx$$
$$= - \sum_{n_{123}=2k-3} \int [(-1 + i\delta \widetilde{\mathcal{G}}_\delta) \partial_x L_{n_1}] L_{n_2} L_{n_3}\, dx + i\delta \sum_{n_{123}=2k-2} \int [(-1 + i\delta \widetilde{\mathcal{G}}_\delta) \partial_x L_{n_1}] L_{n_2} L_{n_3}\, dx$$
$$- \sum_{n_{12}=2k-2} \int L_{n_1} (1 + i\delta \widetilde{\mathcal{G}}_\delta)^2 \partial_x^2 Q_{n_2}\, dx,$$

where again we see that the first contribution is of type (i), the second contribution has only terms of type (i) and (ii) by the same argument as for the second contribution in (B.27). We repeat this process until we reach the contribution

$$\sum_{n_{12}=2k} \int L_{n_1} (1 + i\delta \widetilde{\mathcal{G}}_\delta)^{n_2-1} \partial_x^{n_2-1} Q_1\, dx$$
$$= \sum_{n_{12}=2k} \int L_{n_1} (1 + i\delta \widetilde{\mathcal{G}}_\delta)^{n_2-1} \partial_x^{n_2-1} (-i\delta u^2)\, dx$$
$$= i\delta \sum_{n_{12}=2k} \int [(1 + i\delta \widetilde{\mathcal{G}}_\delta)^{n_1} \partial_x^{n_1} u][(1 + i\delta \widetilde{\mathcal{G}}_\delta)^{n_2-1} \partial_x^{n_2-1} (u^2)]\, dx$$

$$= -i\delta \sum_{n_{12}=2k-1} \sum_{\ell=0}^{n_2} c_\ell \int L_{n_1}(1+i\delta\widetilde{\mathcal{G}}_\delta)^{n_2}(\partial_x^\ell u \partial_x^{n_2-\ell}u)\,dx$$

for some constants c_ℓ. The terms above are of type (i) (possibly after integration by parts) if $n_1, \ell, n_2-\ell \le k-1$ or $\max(n_1,\ell,n_2-\ell) \ge k$ and $\min(n_1,\ell,n_2-\ell), \operatorname{med}(n_1,\ell,n_2-\ell) \le k-2$. The remaining terms have that $\max(n_1,\ell,n_2-\ell) \ge k$ and $\operatorname{med}(n_1,\ell,n_2\ell) \ge k-1$, which implies equality since the sum of three indices is $2k-1$, i.e., we must have $\{n_1,\ell,n_2-\ell\} = \{0,k-1,k\}$. These terms can be written as

$$\begin{aligned}
&- i\delta c_0 \int L_0(1+i\delta\widetilde{\mathcal{G}}_\delta)^{2k-1}(\partial_x^{k-1}u\partial_x^k u)\,dx - i\delta c_k \int L_k(1+i\delta\widetilde{\mathcal{G}}_\delta)^{k-1}(u\partial_x^{k-1}u)\,dx \\
&\quad - i\delta c_{k-1} \int L_{k-1}(1+i\delta\widetilde{\mathcal{G}}_\delta)^{k}(u\partial_x^{k}u)\,dx \\
&= i\delta c_0 \sum_{m_1=0}^{2k-1} \int (-1)^{2k-1-m_1}(i\delta)^{m_1}(\widetilde{\mathcal{G}}_\delta^{m_1}u)(\partial_x^{k-1}u)(\partial_x^k u)\,dx \\
&\quad + i\delta c_k \sum_{\substack{0\le m_1\le k\\ 0\le m_2\le k-1}} \int (-1)^{k-m_1}(i\delta)^{m_{12}} u(\partial_x^{k-1}u)(\widetilde{\mathcal{G}}_\delta^{m_{12}}\partial_x^k u)\,dx \\
&\quad + i\delta c_{k-1} \sum_{\substack{0\le m_1\le k-1\\ 0\le m_2\le k}} \int (-1)^{k-1-m_1}(i\delta)^{m_{12}} u(\partial_x^{k}u)(\widetilde{\mathcal{G}}_\delta^{m_{12}}\partial_x^{k-1} u)\,dx \\
&= i\delta c_0 \sum_{\substack{0\le m_1\le 2k-1\\ \text{odd}}} \int (i\delta)^{m_1}(\widetilde{\mathcal{G}}_\delta^{m_1}u)(\partial_x^{k-1}u)(\partial_x^k u)\,dx \\
&\quad + i\delta c_k \sum_{\substack{0\le m_1\le k\\ 0\le m_2\le k-1\\ m_{12}\text{ odd}}} (-1)^{k-m_1}(i\delta)^{m_{12}} u(\partial_x^{k-1}u)(\widetilde{\mathcal{G}}_\delta^{m_{12}}\partial_x^k u)\,dx \\
&\quad + i\delta c_{k-1} \sum_{\substack{0\le m_1\le k-1\\ 0\le m_2\le k\\ m_{12}\text{ odd}}} \int (-1)^{k-1-m_1}(i\delta)^{m_{12}} u(\partial_x^{k}u)(\widetilde{\mathcal{G}}_\delta^{m_{12}}\partial_x^{k-1} u)\,dx
\end{aligned}$$

where we see that all the terms are of type (ii), as intended. This completes the proof. $\square$

Acknowledgements. G.L. was supported by the Maxwell Institute Graduate School in Analysis and its Applications, a Centre for Doctoral Training funded by the UK Engineering and Physical Sciences Research Council (Grant EP/L016508/01), the Scottish Funding Council, Heriot-Watt University and the University of Edinburgh. A.C., G.L, T.O., and G.Z. were supported by the European Research Council (grant no. 864138 "SingStochDispDyn").

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