

Semisimple Quantum Cohomology, deformations of stability conditions and the derived category

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Contents

ACKNOWLEDGEMENTS	v
Chapter 1. Introduction	1
Chapter 2. Semisimple quantum cohomology and blow-ups	3
1. Introduction	3
2. Definitions and Notations	4
3. Semisimple quantum cohomology and blow-ups	6
4. Exceptional systems and Dubrovin's conjecture	12
Chapter 3. Polynomial Bridgeland stability conditions	15
1. Introduction	15
2. Polynomial stability conditions	17
3. The moduli space of polynomial stability conditions	20
4. Canonical stability conditions	25
Chapter 4. Moduli spaces of weighted stable maps and Gromov-Witten invariants	29
1. Introduction	29
2. Geometry of moduli spaces of weighted stable maps	30
3. Elementary morphisms	34
4. Birational behaviour under weight changes	36
5. Virtual fundamental classes and Gromov-Witten invariants	38
6. Graph-language	41
7. Graph-level description of virtual fundamental classes.	47
Bibliography	55

CHAPTER 1

Introduction

The different parts of this thesis are all devoted to developments in algebraic geometry that are, at least indirectly, motivated by mirror symmetry and physics.

Mirror symmetry was born with the prediction that numbers of rational curves in a projective variety can be computed from period integrals on the space of complex deformations of its mirror variety ([CdlOGP92]). The introduction of quantum cohomology [KM94] and Frobenius manifolds [Dub93] led to both precise mathematical definitions on the side of counting rational curves [BM96, BF97, Beh97], and a conceptual framework to the enumerative conjectures of mirror symmetry [Bar02].

At the ICM 1994, Kontsevich's conjecture of homological mirror symmetry [Kon95] added a new perspective. Four years later, again at the ICM, Dubrovin suggested a new conjecture [Dub98] that could most likely be derived by combining a good understanding of both classical and Kontsevich's homological mirror symmetry. However, it is a statement relating the Frobenius manifold of the quantum cohomology of a projective variety V to properties of the derived category $D^b(V)$ of coherent sheaves on V , without any reference to its mirror partner: It claims that quantum multiplication in $H^*(V)$ becomes semisimple for generic parameters if and only if $D^b(V)$ has a so-called exceptional collection, and makes more predictions comparing structure invariants of quantum cohomology and the exceptional collection. Chapter 2 is devoted to this conjecture; more specifically, it is shown to hold for the blow-up of X at points if it holds for X .

Dubrovin's conjecture is not alone in claiming a relationship between the derived category and quantum cohomology. For example, Ruan conjectured that birationally K -equivalent varieties have isomorphic quantum cohomology [Rua99]—according to a conjecture by Kawamata, they also have equivalent derived categories. Recently, this has motivated the *crepant resolution conjecture* by Bryan and Graber, which is a quantum cohomology analogue of the derived category formulation of McKay correspondence [BKR01].

This relationship seems so far more coincidental than systematic. Bridgeland's introduction of stability conditions on the derived category might, among many other things, eventually lead to a better understanding of this coincidence. Since their introduction in [Bri02b], they have created interest from various different perspectives. As proven by Bridgeland, the set of stability conditions is a smooth manifold; in the above context, it is particularly interesting that, conjecturally, this moduli space has the structure of a Frobenius manifold. At this point, its relation to the Frobenius manifold

of quantum cohomology remains unexplored beyond conjectures in [Bri02a] for the case of \mathbb{P}^2 .

Many questions about Bridgeland's stability conditions are open. Chapter 3 is a contribution towards the problem of constructing stability conditions in the sense of Bridgeland. Apart from a few examples, even the existence of a stability condition on the derived category $D^b(V)$ of a smooth projective variety V is unknown. We introduce the notion of a *polynomial stability condition* that should be seen as a natural limit of stability conditions in the sense of Bridgeland. Our theorem 3.2.5 shows that their moduli space is again a smooth manifold, generalizing the main result of [Bri02b]; further, theorem 4.2 shows the existence of a family of canonical polynomial stability conditions for any projective variety V .

Finally, in chapter 4 we study a non-linear example of families of stability conditions; it deforms Kontsevich's notion of a stable map, which is fundamental for the definition of quantum cohomology and Gromov-Witten invariants in algebraic geometry. By Kontsevich's definition, a stable map to a projective variety V is a map $f: C \rightarrow V$ from a curve C with distinct marked points (x_1, \dots, x_n) in the smooth locus of C , such that f has finitely many automorphisms that leave every marking fixed. In the case of *weighted stable maps*, each marking is assigned a weighting between 0 and 1, and marked points are allowed to collide as long as the sum of their weights does not exceed one.

We show the existence of moduli space of weighted stable maps as proper Deligne-Mumford stacks of finite type. We study in detail the birational behaviour of the moduli spaces under changes of weights. We introduce a category of weighted marked graphs to keep track of their boundary components and the natural morphisms between them. By constructing virtual fundamental classes, Gromov-Witten invariants are defined. We show that they satisfy all properties one might naturally expect. In particular, weighted Gromov-Witten invariants without gravitational descendants do not depend on the choice of weights; on the other hand, their behaviour when including gravitational descendants promises to be interesting, in particular for the case of semisimple quantum cohomology as studied in chapter 2.

Semisimple quantum cohomology and blow-ups

1. Introduction

This chapter is motivated by a conjecture proposed by Boris Dubrovin in his talk at the International Congress of Mathematicians (ICM) in Berlin 1998. It claims that the quantum cohomology of a projective variety X is generically semisimple if and only if its bounded derived category $D^b(X)$ of coherent sheaves is generated by an exceptional collection. We discuss here a modification of this conjecture proposed in [BM04] and show its compatibility with blowing up at a point.

Quantum multiplication gives (roughly speaking) a commutative associative multiplication $\circ_\omega: H^*(X) \otimes H^*(X) \rightarrow H^*(X)$ depending on a parameter $\omega \in H^*(X)$. Semisimplicity of quantum cohomology means that for generic parameters ω , the resulting algebra is semisimple. More precisely, quantum cohomology produces a formal Frobenius supermanifold whose underlying manifold is the completion at the point zero of $H^*(X)$. We call a Frobenius manifold generically semisimple if it is purely even and the spectral cover map $\text{Spec}(\mathcal{T}\mathcal{M}, \circ) \rightarrow \mathcal{M}$ is unramified over a general fibre. Generically semisimple Frobenius manifolds are particularly well understood. There exist two independent classifications of their germs, due to Dubrovin and Manin. Both identify a germ via a finite set of invariants. As mirror symmetry statements include an isomorphism of Frobenius manifolds, this means that in the semisimple case one will have to control only this finite set of invariants.

In [BM04], it was proven that the even-dimensional part H^{ev} of quantum cohomology cannot be semisimple unless $h^{p,q} = 0$ for all $p \neq q, p + q \equiv 0 \pmod{2}$. On the other hand, the subspace $\bigoplus_p H^{p,p}(X)$ gives rise to a Frobenius submanifold. This suggested the following modification of Dubrovin's conjecture: The Frobenius submanifold of (p, p) -cohomology is semisimple if and only if there exists an exceptional collection of length $\text{rk} \bigoplus_p H^{p,p}(X)$.

A consequence of this modified conjecture is the following: *Whenever X has semisimple (p, p) -quantum cohomology, the same is true for the blow-up of X at any number of points.* We prove this in Theorem 3.1.1.

We would like to point out that our result suggests another small change of the formulation of Dubrovin's conjecture. Dubrovin assumed that being Fano is an additional necessary condition for semisimple quantum cohomology. However, as our result holds for the blow-up at an arbitrary number of points, it yields many non-Fano

counter-examples. We suggest to just drop any reference to X being Fano from the conjecture.

2. Definitions and Notations

Let X be a smooth projective variety over \mathbb{C} . By $H_X := \bigoplus H^{p,p}(X, \mathbb{C})$, we denote the space of (p, p) -cohomology. Let $\Delta_0, \dots, \Delta_m, \Delta_{m+1}, \dots, \Delta_r$ be a homogeneous basis of H_X , such that Δ_0 is the unit, and $\Delta_{m+1}, \dots, \Delta_r$ are a basis of $H^{1,1}(X)$.

We denote the correlators in the quantum cohomology of X by

$$\langle \Delta_{i_1} \dots \Delta_{i_n} \rangle_\beta.$$

This is the number of appropriately counted stable maps

$$f: (\mathcal{C}, y_1, \dots, y_n) \rightarrow X$$

where \mathcal{C} is a semi-stable curve of genus zero, y_1, \dots, y_n are marked points on \mathcal{C} , the fundamental class of \mathcal{C} is mapped to β under f , and $\Delta_{i_1}, \dots, \Delta_{i_n}$ are cohomology classes representing conditions for the images of the marked points. In the case of $\beta = 0$ it is artificially defined to be zero if $n \neq 3$, and equal to $\int_X \Delta_{i_1} \cup \Delta_{i_2} \cup \Delta_{i_3}$ if $n = 3$.

Such a correlator vanishes unless

$$(1) \quad k(\beta) := (c_1(X), \beta) = 3 - \dim X + \sum \left(\frac{|\Delta_{i_j}|}{2} - 1 \right)$$

where $|\Delta_{i_j}|$ are the degrees of the cohomology classes.

Before writing down the potential of quantum cohomology and the resulting product, we will define the ring that it lives in. Let $\{x_k | k \leq m\}$ be the dual coordinates of $H_X/H^{1,1}(X)$ corresponding to the homogeneous basis $\{\Delta_k\}$. Instead of dual coordinates in $H^{1,1}(X)$, we want to consider exponentiated coordinates. This is done most elegantly by adjoining a formal coordinate q^β for effective classes $\beta \in H_2(X, \mathbb{Z})/\text{torsion}$ with $q^{\beta_1+\beta_2} = q^{\beta_1}q^{\beta_2}$. Now let

$$F_X = \mathbb{Q}[[x_k, q]]$$

be the completion of the polynomial ring generated by x_k and monomials q^β with β effective.

We consider F_X as the structure ring of the formal Frobenius manifold associated to H_X . The vector space H_X acts on F_X as a space of derivations: $\Delta_k, k \leq m$ acts as $\frac{\partial}{\partial x_k}$, and the divisorial classes $\Delta_k, k > m$ act via $q^\beta \mapsto (\Delta_k, \beta)q^\beta$. Hence we can formally consider H_X as the space of horizontal tangent fields of the formal Frobenius manifold \mathcal{M} , and $F_X \otimes H_X$ as its tangent bundle $\mathcal{T}\mathcal{M}$.

The flat structure of this formal manifold is given by the Poincaré pairing g on H_X . Given the flat metric, the whole structure of a formal Frobenius manifold is an algebra structure on $F_X \otimes H_X$ over F_X given by the third partial derivatives of a potential $\Phi \in F_X$:

$$g(\Delta_i \circ \Delta_j, \Delta_k) = \Delta_i \Delta_j \Delta_k \Phi$$

To be able to consistently work only with exponentiated coordinates on $H^{1,1}$, we slightly deviate from this definition: We use only the non-classical part

$$\Phi_X = \sum_{\beta \neq 0} \langle e^{\sum_{k \leq m} x_k \Delta_k} \rangle_{\beta} q^{\beta}.$$

of the Gromov-Witten potential (it is a consequence of the divisor axiom that it makes sense to write Φ_X in this way), and define the product via $g(\Delta_i \circ \Delta_j, \Delta_k) = g(\Delta_i \cup \Delta_j, \Delta_k) + \Delta_i \Delta_j \Delta_k \Phi_X$. The choice of the ring F_X is governed by the two properties that it has to contain Φ_X , and that H_X has to act on it as a vector space of derivations. This is enough to ensure that all standard constructions associated to a Frobenius manifold are defined over F_X .

Explicitly, the multiplication is given by

$$(2) \quad \Delta_i \circ \Delta_j = \Delta_i \cup \Delta_j + \sum_{\beta \neq 0} \sum_{k \neq 0} \langle \Delta_i \Delta_j \Delta_k e^{\sum_{k \leq m} x_k \Delta_k} \rangle_{\beta} \Delta^k q^{\beta}$$

where Δ^k are the elements of the basis dual to (Δ_k) with respect to the Poincaré pairing. The multiplication endows $F_X \otimes H_X$ with the structure of a commutative, associative algebra with $1 \otimes \Delta_0$ being the unit.

We call the whole structure of the formal Frobenius manifold on F_X and H_X *reduced quantum cohomology*. The map of rings

$$(3) \quad F_X \rightarrow F_X \otimes H_X, \quad f \mapsto f \otimes \Delta_0$$

is the formal replacement of the spectral cover map $\text{Spec}(\mathcal{T}\mathcal{M}, \circ) \rightarrow \mathcal{M}$ of a non-formal Frobenius manifold.

Definition 2.1. *X has semisimple reduced quantum cohomology if the spectral cover map (3) is generically unramified.*

More concretely, semisimplicity over a geometric point $F_X \rightarrow k$ of F_X means that after base change to k , the ring $k \otimes H_X$ with the quantum product is isomorphic to k^{r+1} with component-wise multiplication. Generic semisimplicity means that this is true for a dense open subset in the set of k -valued points of F_X .

Finally, we recall the definition of the Euler field of quantum cohomology. It is given by

$$\mathcal{E} = -c_1(X) + \sum_{k \leq m} \left(1 - \frac{|\Delta_k|}{2} \right) x_k \Delta_k.$$

It induces a grading on F_X and $F_X \otimes H_X$ by its Lie derivative. E. g., a vector field is homogeneous of degree d if $\text{Lie}_{\mathcal{E}}(X) = [\mathcal{E}, X] = dX$. It is clear that the Poincaré pairing is homogeneous of degree $(2 - \dim X)$ by the induced Lie derivative on $(H_X^*)^{\otimes 2}$. Further, from the dimension axiom (1) it follows that Φ_X is homogeneous of degree $(3 - \dim X)$. It is a purely formal consequence of these two facts that the multiplication \circ is homogeneous of degree 1 with respect to \mathcal{E} (see [Man99, section I.2]).

3. Semisimple quantum cohomology and blow-ups

3.1. Motivation. So let us now assume that the variety X satisfies the modified version of Dubrovin's conjecture, i.e. that it has both an exceptional collection of length $\text{rk} \bigoplus_p H^{p,p}(X)$, and semisimple reduced quantum cohomology. Let \tilde{X} be its blow-up at some points. By remark 4.4.2, this is a test for the modified version of Dubrovin's conjecture 4.2.2: We know that \tilde{X} has an exceptional system of desired length, so it should have semisimple reduced quantum cohomology as well:

Theorem 3.1.1. *Let $\tilde{X} \rightarrow X$ be the blow-up of a smooth projective variety X at any number of closed points.*

If the reduced quantum cohomology of X is generically semisimple, then the same is true for \tilde{X} .

In the case of dimension two, Del Pezzo surfaces were treated in [BM04], where the results of [GP98] on their quantum cohomology were used. The generalization presented here uses instead the results in Andreas Gathmann's paper [Gat01], with an improvement from the later paper [Hu00] by J. Hu. The essential idea is a variant of the idea used in [BM04]: a partial compactification of the spectral cover map where the exponentiated coordinate of an exceptional class vanishes. However, in our case, this is only possible after base change to a finite cover of the spectral cover map.

3.2. More notations. We want to compare the reduced quantum cohomology of \tilde{X} with that of X . We may and will restrict ourselves to the blow-up $j: \tilde{X} \rightarrow X$ of a single point. For the pull-back $j^*: H^*(X) \rightarrow H^*(\tilde{X})$ and the push-forward $j_*: H^*(\tilde{X}) \rightarrow H^*(X)$ we have the identity $j_*j^* = \text{id}_{H^*(X)}$. Hence $H^*(\tilde{X}) = j^*(H^*(X)) \oplus \ker j_*$ canonically. We will identify $j^*(H^*(X))$ with $H^*(X)$ from now on and get a canonical decomposition $H_{\tilde{X}} = H_X \oplus H_E$ with $H_E = \bigoplus_{1 \leq k \leq n-1} \mathbb{C} \cdot E^k$, where E is the exceptional divisor of j . The dual coordinates $(x_k) =: \underline{x}$ on $H_X/H^{1,1}(X)$ get extended via coordinates $(x_2^E, \dots, x_{n-1}^E) =: \underline{x}^E$ to dual coordinates of $H_{\tilde{X}}/H^{1,1}(\tilde{X})$. Let $E' \in H_2(\tilde{X})$ be the class of a line in the exceptional divisor $E \cong \mathbb{P}^{n-1}$. From Poincaré duality and the decomposition of $H^*(\tilde{X})$, we get a corresponding decomposition $H_2(\tilde{X}, \mathbb{Z}) = H_2(X, \mathbb{Z}) \oplus \mathbb{Z} \cdot E'$ in homology, where we have identified $H_2(X)$ with its image via the dual of j_* . With this identification, the cone of effective curves in X is a subcone of the effective cone in \tilde{X} . Hence F_X is a subring of $F_{\tilde{X}}$. We will call elements $\beta \in H_2(X) \subset H_2(\tilde{X})$ non-exceptional, and $\beta \in \mathbb{Z}E'$ purely exceptional.

We can also view F_X as a quotient of $F_{\tilde{X}}$: Let I be the completion of the subspace in $F_{\tilde{X}}$ generated by monomials $\underline{x}^a \cdot (\underline{x}^E)^b q^{\tilde{\beta}}$ with $\underline{b} \neq (0, \dots, 0)$ or $\tilde{\beta} \notin H_2(X)$. Then evidently $F_X = F_{\tilde{X}}/I$. But note that I is *not* an ideal, as there are effective classes $\tilde{\beta}_1, \tilde{\beta}_2 \in H_2(\tilde{X}) \setminus H_2(X)$ whose sum $\tilde{\beta}_1 + \tilde{\beta}_2$ is in $H_2(X)$.

Also, it is not true that $F_X \otimes H_X$ is a subring of $F_{\tilde{X}} \otimes H_{\tilde{X}}$. The next section will summarize the results of [Gat01] that will enable us to study the relation between the two reduced quantum cohomology rings.

3.3. Gathmann's results.

Theorem 3.3.1. *The following assertions relate the Gromov-Witten invariants of \tilde{X} to those of X (which we will denote by $\langle \dots \rangle_{\tilde{\beta}}^{\tilde{X}}$ and $\langle \dots \rangle_{\beta}^X$, respectively):*

- (1) (a) *Let $\beta \in H_2(\tilde{X})$ be any non-exceptional homology class—so β is any element of $H_2(X)$ —, and let T_1, \dots, T_m be any number of non-exceptional classes in $H^*(X) \subset H^*(\tilde{X})$, which we can identify with their preimages in $H^*(X)$. Then it does not matter whether we compute their Gromov-Witten invariants with respect to \tilde{X} or X :*

$$\langle T_1 \otimes \dots \otimes T_m \rangle_{\beta}^{\tilde{X}} = \langle T_1 \otimes \dots \otimes T_m \rangle_{\beta}^X.$$

- (b) *Consider the Gromov-Witten invariants $\langle T_1 \otimes \dots \otimes T_m \rangle_{\tilde{\beta}}^{\tilde{X}}$ with $\tilde{\beta}$ being purely exceptional, i. e. $\tilde{\beta} = d \cdot E'$.*

If any of the cohomology classes T_1, \dots, T_m are non-exceptional, the invariant is zero. All invariants involving only exceptional cohomology classes can be computed recursively from the following:

$$\langle E^{n-1} \otimes E^{n-1} \rangle_{E'}^{\tilde{X}} = 1.$$

They depend only on n .

- (2) (a) *Using the associativity relations, it is possible to compute all Gromov-Witten invariants of \tilde{X} from those mentioned above in 1a and 1b.*
 (b) *Vanishing of mixed classes: Write $\tilde{\beta} \in H_2(\tilde{X})$ in the form $\tilde{\beta} = \beta + d \cdot E'$ where β is the non-exceptional part; assume that $\beta \neq 0$. Let T_1, \dots, T_m be non-exceptional cohomology classes. Let l be a non-negative integer, and let $2 \leq k_1, \dots, k_l \leq n - 1$ be integers satisfying*

$$(k_1 - 1) + \dots + (k_l - 1) < (d + 1)(n - 1).$$

Unless we have both $d = 0$ and $l = 0$, this implies the vanishing of

$$\langle T_1 \otimes \dots \otimes T_m \otimes E^{k_1} \otimes \dots \otimes E^{k_l} \rangle_{\beta} = 0.$$

Proof. The statement in no. 1a is proven by J. Hu in [Hu00, Theorem 1.2]. This is lemma 2.2 in [Gat01]; since the proof of this lemma is the only place where Gathmann uses the convexity of X (see remark 2.3 in that paper), we can drop this assumption from his theorems.

The other equations follow trivially from statements in lemma 2.4 and proposition 3.1 in [Gat01]. \square

3.4. Proof of Theorem 3.1.1. Let us first restate Gathmann's results in terms of the potentials Φ_X and $\Phi_{\tilde{X}}$: We can write $\Phi_{\tilde{X}}$ as

$$(4) \quad \Phi_{\tilde{X}} = \Phi_X + \Phi_{\text{pure}} + \Phi_{\text{mixed}}$$

where Φ_X is the sum coming from all non-exceptional $\tilde{\beta} = \beta$ and non-exceptional cohomology classes (coinciding with the potential of X by no. 1a), Φ_{pure} is the sum coming from all correlators with $\tilde{\beta}$ being purely exceptional (i. e. a positive multiple

of E'), and Φ_{mixed} the sum from correlators for mixed homology classes $\tilde{\beta} = \beta + d \cdot E'$ with $0 \neq \beta \in H_2(X)$ and $d \neq 0$.

Now let $\tilde{\mathcal{E}}$ and \mathcal{E} be the Euler fields of \tilde{X} and X , respectively. Let us consider the grading induced by

$$\tilde{\mathcal{E}} - \mathcal{E} = (n-1)E + \sum_{2 \leq k \leq n-1} (1-k)x_k^E E^k.$$

Lemma 3.4.1. *With respect to $\tilde{\mathcal{E}} - \mathcal{E}$, the potential Φ_{pure} is homogeneous of degree $3-n$, and Φ_{mixed} only has summands of degree less than or equal to $1-n$.*

Proof. The assertion about Φ_{pure} is just the dimension axiom (1) of \tilde{X} , as $\mathcal{E}\Phi_{\text{pure}} = 0$. The statement about Φ_{mixed} is equivalent to Gathmann's vanishing result, theorem 3.3.1 no. 2b. \square

Let $J \triangleleft F_{\tilde{X}}$ be the ideal generated by x_2^E, \dots, x_{n-1}^E . We will show that the spectral cover map of \tilde{X} is already generically semisimple when restricted to the fibre

$$(5) \quad F_{\tilde{X}}/J \rightarrow H_{\tilde{X}} \otimes F_{\tilde{X}}/J.$$

Write a monomial $q^{\tilde{\beta}}$ in $F_{\tilde{X}}$ as $q^{\tilde{\beta}} = Q^{-d}q^{\beta}$ if $\tilde{\beta} = \beta + d \cdot E'$ with $\beta \in H_2(X)$. We make the base change to the cover given by adjoining $Z := \sqrt[n]{Q}$. More precisely, we first localize¹ at Q^{-1} and adjoin an $(n-1)$ -th root of Q : We consider $R := (F_{\tilde{X}}/J)[Q][Z]/(Z^{n-1} - Q)$.

On the other hand, consider the subring B of R that consists of power series in which Z only appears with non-negative degrees.² Then R is a completion of the localization $B[Z^{-1}]$ of B . We claim that the quantum product “extends” to a product over B .

We define M as the free B -submodule of $B \otimes H^*(\tilde{X})$ generated by

$$\langle H^*(X), ZE, Z^2E^2, \dots, Z^{n-1}E^{n-1} = QE^{n-1} \rangle.$$

More invariantly, B is the completed subspace of R generated by monomials with non-positive degree with respect to $\tilde{\mathcal{E}} - \mathcal{E}$. And M is the submodule of $B \otimes H_{\tilde{X}}$ generated by $B \otimes H_X$ and all elements of strictly negative degree in $B \otimes H_E$.

Lemma 3.4.2. *• The quantum product restricts to M , i. e. $M \circ M \subseteq M$, and there is the following cartesian diagram:*

$$\begin{array}{ccc} B & \longrightarrow & M \\ \downarrow & & \downarrow \\ R & \longrightarrow & R \otimes H^*(\tilde{X}) \end{array}$$

¹Note that Q itself is not an element of $F_{\tilde{X}}$.

²The ring B is neither $F_X[[Z]]$ nor $F_X[Z]$; it is a different completion of $F_X[Z]$.

- Consider the push-out with respect to $B \rightarrow B/(Z) = F_X$. Then the spectral cover map decomposes as

$$\begin{array}{ccc} B & \longrightarrow & M \\ \downarrow & & \downarrow \\ F_X & \longrightarrow & (F_X \otimes H_X) \oplus F_X[z]/(z^{n-1} - (-1)^{n-1}) \end{array}$$

where the product on $F_X \otimes H_X$ is the quantum product of X .

First, we show how to derive Theorem 3.1.1 from the above lemma. By the induction hypothesis, $F_X \rightarrow F_X \otimes H_X$ is generically semisimple. The second part of the lemma then tells us that the map $B \rightarrow M$ is generically semisimple over the fibre of $Z = 0$.

E. g. by the criterion [EGA, IV, Proposition 17.3.6] of unramifiedness, it is clear that semisimplicity is an open condition for finite flat maps. Hence, also $B \rightarrow M$ is generically semisimple. The same is then true for its completed localization $(F_{\tilde{X}}/J)[Q][Z]/(Z^{n-1} - Q)$. It is also evident that the finite extension $(F_{\tilde{X}}/J)[Q] \rightarrow (F_{\tilde{X}}/J)[Q][Z]/(Z^{n-1} - Q)$ cannot change generic semisimplicity. Hence the spectral cover map (5) must be generically semisimple (as its localization at Q is). And again by openness of semisimplicity, it also holds for the full reduced quantum cohomology of \tilde{X} .

Proof.[of the lemma] We want to analyze the behaviour of multiplication with respect to the grading of $\tilde{\mathcal{E}} - \mathcal{E}$. We decompose the quantum product $\circ_{\tilde{X}}$, understood as a bilinear map $(B \otimes H_{\tilde{X}}) \otimes (B \otimes H_{\tilde{X}}) \rightarrow B \otimes H_{\tilde{X}}$, into a sum $\circ_{\tilde{X}} = \circ_X + \circ_{\text{class}}^E + \circ_{\text{pure}} + \circ_{\text{mixed}}$ according to the decomposition of $\Phi_{\tilde{X}}$ in (4); we have written \circ_{class}^E for the classical cup product of exceptional classes $E^i \circ_{\text{class}}^E E^j = E^{i+j}$ for $0 \leq i, j \leq n-1$ and $i > 0$ or $j > 0$. So for example \circ_{pure} is defined by $\tilde{g}(U \circ_{\text{pure}} V, W) = UVW\Phi_{\text{pure}}$ with \tilde{g} as the Poincaré pairing on \tilde{X} .

We claim that \circ_X , \circ_{pure} and \circ_{mixed} are of degree 0, 1 and ≤ -1 , respectively.

This is clear for \circ_X and follows by standard Euler field computations from the assertions in lemma 3.4.1 (compare with the computations in [Man99, section I.2]):

Take a homogeneous component Φ_d of degree d of any of the two relevant potentials, and \circ_d the corresponding component of the multiplication. Let U, V and W be vector fields of degree u, v and w , respectively:

$$\begin{aligned} (\tilde{\mathcal{E}} - \mathcal{E})\tilde{g}(U \circ_d V, W) &= (\tilde{\mathcal{E}} - \mathcal{E})UVW\Phi_d \\ &= [\tilde{\mathcal{E}} - \mathcal{E}, U]VW\Phi_d + U[\tilde{\mathcal{E}} - \mathcal{E}, V]W\Phi_d \\ &\quad + UV[\tilde{\mathcal{E}} - \mathcal{E}, W]\Phi_d + UVW(\tilde{\mathcal{E}} - \mathcal{E})\Phi_d \\ (6) \quad &= (u + v + w + d)UVW\Phi_d \\ &= (u + v + w + d)\tilde{g}(U \circ_d V, W) \end{aligned}$$

Now write $\tilde{g} = g + g^E$ where g is the Poincaré pairing of X and g^E the pairing of exceptional classes $g^E(E^i, E^k) = \delta_{i+j, n}(-1)^{n-1}$. Then g is of degree zero, and

g^E of degree $2 - n$ with respect to $\tilde{\mathcal{E}} - \mathcal{E}$. Let $\circ_d = \circ_d^0 + \circ_d^E$ accordingly. Then $U \circ_d V = U \circ_d^0 V + U \circ_d^E V$ is just the decomposition of $U \circ_d V$ in the orthogonal sum $H_{\tilde{X}} = H_X \oplus H_E$; in particular, $U \circ_d V$ is homogeneous if and only if $U \circ_d^0 V$ and $U \circ_d^E V$ are. So we have:

$$\begin{aligned}
(\tilde{\mathcal{E}} - \mathcal{E})\tilde{g}(U \circ_d^0 V, W) &= (\tilde{\mathcal{E}} - \mathcal{E})g(U \circ_d^0 V, W) \\
&= g([\tilde{\mathcal{E}} - \mathcal{E}, U \circ_d^0 V], W) + g(U \circ_d^0 V, [\tilde{\mathcal{E}} - \mathcal{E}, W]) \\
&= g([\tilde{\mathcal{E}} - \mathcal{E}, U \circ_d^0 V], W) + wg(U \circ_d^0 V, W) \\
(\tilde{\mathcal{E}} - \mathcal{E})\tilde{g}(U \circ_d^E V, W) &= (\tilde{\mathcal{E}} - \mathcal{E})g^E(U \circ_d^E V, W) \\
&= \text{Lie}_{\tilde{\mathcal{E}} - \mathcal{E}}(g^E)(U \circ_d^E V, W) \\
&\quad + g([\tilde{\mathcal{E}} - \mathcal{E}, U \circ_d^E V], W) + g(U \circ_d^E V, [\tilde{\mathcal{E}} - \mathcal{E}, W]) \\
&= g((\tilde{\mathcal{E}} - \mathcal{E})(U \circ_d^E V), W) + (2 - n + w)\tilde{g}(U \circ_d^E V, W).
\end{aligned}$$

Comparing with (6), we see that $U \circ_d^0 V$ is of degree $u + v + d$, and $U \circ_d^E V$ of degree $u + v + d + n - 2$, in other words, \circ_d^0 has degree d and \circ_d^E degree $d + n - 2$. Hence, the claim about the degree of \circ_{mixed} is obvious, and the one about \circ_{pure} follows from the additional fact that the derivative of Φ_{pure} in H_X -direction is zero, so that \circ_{pure}^0 is zero.

It is clear that M is closed with respect to \circ_X and \circ_{class}^E . That it is also closed under the multiplication \circ_{mixed} follows directly by degree reasons from the description of M in terms of degrees. With respect to \circ_{pure} we can argue via degrees if we additionally note that $H_X \circ_{\text{pure}} H_{\tilde{X}} = 0$.

So we have proven $M \circ M \subseteq M$, and it remains to analyze the product on $M/ZM \cong F_X \otimes H_{\tilde{X}}$. Note that all elements in M of degree ≤ -2 are mapped to zero in this quotient.

It is clear that \circ_X induces the quantum product of X on the subspace $F_X \otimes H_X$ and is zero on H_E . We already noted that $H_X \circ_{\text{pure}} H_{\tilde{X}} = 0$. Also, $Y_1 \circ_{\text{mixed}} Y_2$ is always zero if Y_1 or Y_2 is in $M \cap B \otimes H_E$ for degree reasons.

We investigate the product with ZE . For this we can ignore \circ_X and \circ_{mixed} . The classical part contributes $ZE \circ_{\text{class}}^E (ZE)^i = (ZE)^{i+1}$ for $0 \leq i \leq n - 1$. For \circ_{pure} we finally have to use the explicit multiplication formula:

$$ZE \circ_{\text{pure}} (ZE)^i = (-1)^{n-1} \sum_{d>0} \sum_j \langle EE^i E^j \rangle_{dE'} Z^{i+1} E^{n-j} Q^{-d}.$$

By the dimension axiom, this can only be non-zero if $(n - 1)d = 3 - n + (1 - 1) + (i - 1) + (j - 1)$, or, equivalently, $(n - 1)(d + 1) = i + j$. This is only possible for $d = 1$ and $i = j = n - 1$, where we have $\langle EE^{n-1} E^{n-1} \rangle_{E'} = -\langle E^{n-1} E^{n-1} \rangle_{E'} = -1$. We thus get

$$ZE \circ (ZE)^i = \begin{cases} Z^{i+1} E^{i+1} & \text{if } i \leq n - 2 \\ (-1)^n ZE & \text{if } i = n - 1. \end{cases}$$

Let $Y := (-1)^n Q E^{n-1} = (-1)^n Z^{n-1} E^{n-1}$. As a consequence of the last equation, multiplication by Y in the ring M/ZM is the identity on $(M \cap B \otimes H_E)/ZM \cong F_X \otimes H_E$. In particular, Y is an idempotent and gives a splitting of $M/ZM \cong F_X \otimes H_E \oplus K$ into the image $F_X \otimes H_E$ and the kernel K of $Y \circ$. The algebra structure on $F_X \otimes H_E$ is isomorphic to $F_X[z]/(z^{n-1} - (-1)^{n-1})$ via $z \mapsto ZE$.

The kernel is generated by $\Delta_1, \dots, \Delta_m, \Delta_0 - Y$, and $\Delta_0 - Y$ is its unit. Mapping each element in K to its degree zero component, we get an isomorphism $K \rightarrow F_X \otimes H_X$ that maps the multiplication on K isomorphically to its degree zero component \circ_X , and the lemma is proven. \square

3.5. Further Questions. The first example where our theorem applies is the case of $X = \mathbb{P}^n$. For $n = 2$, this yields the semisimplicity of quantum cohomology for all Del Pezzo surfaces as proven earlier in [BM04]. Further, semisimplicity has been established in [TX97] by Tian and Xu, using results of Beauville (see [Bea97]), for low degree complete intersections in \mathbb{P}^n .

Generally speaking, once the three-point Gromov-Witten correlators are known, and thus generators and relations for the small quantum cohomology ring, it is an exercise purely in commutative algebra to check generic semisimplicity in small quantum cohomology. For example, using Batyrev's formula for Fano toric varieties [Bat93] and its explicit version for the projectivization of splitting bundles over \mathbb{P}^n given in [AM04], semisimplicity can be shown to hold for these bundles. In the recent preprint [Cio05], generic semisimplicity was systematically studied for three-dimensional Fano manifolds, thus verifying Dubrovin's conjecture for 36 Fano threefolds out of 59 having purely even-dimensional cohomology.

Of course, our theorem 3.1.1 covers only the first part of Dubrovin's conjecture. It would be very encouraging if it was possible to show his statement on Stokes matrices in a similar way. To my knowledge, the only case where this part has been checked is the case of projective spaces (cf. [Guz99]).

Revisiting Gathmann's algorithm to compute the invariants of \tilde{X} (Theorem 3.3.1, no. 2a), we notice that all the initial data it uses is already determined by the multiplication in the special fibre $Z = 0$ of our partially compactified spectral cover map. In other words, the Frobenius manifold on $F_{\tilde{X}}$ and $H_{\tilde{X}}$ is already determined by the structure at $Z = 0$.

Yet our construction does not yield a Frobenius structure at the divisor $Z = 0$. If there was a formalism of Frobenius manifolds with singularities along divisors, and if there was a way to extend Dubrovin's Stokes matrices to these divisorial Frobenius manifolds, this might also lead to an elegant treatment of Stokes matrices of blow-ups.

Also, one would like to extend the method to the case of the blow-up along a subvariety, analogously to Orlov's Theorem 4.4.1. The next-trivial case of the blow-up along a fibre $\{x_0\} \times Y$ in a product $X \times Y$ follows from our result and the discussion of products in section 4.3.

4. Exceptional systems and Dubrovin's conjecture

In this section, we briefly review Dubrovin's conjecture and its modified version, and explain how our theorem fits into this context.

4.1. Exceptional systems in triangulated categories. We consider a triangulated category \mathcal{C} . We assume that it is linear over a ground field \mathbb{C} .

Definition 4.1.1. • *An exceptional object in \mathcal{C} is an object \mathcal{E} such that the endomorphism complex of \mathcal{E} is concentrated in degree zero and equal to \mathbb{C} :*

$$\mathrm{RHom}^\bullet(\mathcal{E}, \mathcal{E}) = \mathbb{C}[0]$$

- *An exceptional collection is a sequence $\mathcal{E}_0, \dots, \mathcal{E}_m$ of exceptional objects, such that for all $i > j$ we have no morphisms from \mathcal{E}_i to \mathcal{E}_j :*

$$\mathrm{RHom}^\bullet(\mathcal{E}_i, \mathcal{E}_j) = 0 \quad \text{if } i > j$$

- *An exceptional collection of objects is called a complete exceptional collection (or exceptional system), if the objects $\mathcal{E}_0, \dots, \mathcal{E}_m$ generate \mathcal{C} as a triangulated category: The smallest subcategory of \mathcal{C} that contains all \mathcal{E}_i , and is closed under isomorphisms, shifts and cones, is \mathcal{C} itself.*

The first example is the bounded derived category $D^b(\mathbb{P}^n)$ on a projective space with the series of sheaves $\mathcal{O}(i), \mathcal{O}(i+1), \dots, \mathcal{O}(i+n)$ (for any i). Exceptional systems were studied extensively by a group at the Moscow University, see e. g. the collection of papers in [Rud90].

More generally, exceptional systems exist on flag varieties; other examples include quadrics in \mathbb{P}^n and projective bundles over a variety for which the existence of an exceptional system is already known.

4.2. Dubrovin's conjecture. On the other side of Dubrovin's conjecture we consider the Frobenius manifold M associated (as in [Man99] or [Dub99]) to the quantum cohomology of X . As already mentioned in the introduction, Dubrovin's conjecture relates generic semisimplicity of M to the existence of an exceptional system:

Conjecture 4.2.1. [Dub98] *Let X be a projective variety.*

The quantum cohomology of X is generically semisimple if and only if there exists an exceptional system in its derived category $D^b(X)$.

In further claims of his conjecture, he relates invariants of M to characteristics of the exceptional system: The so-called *Stokes matrix* S of the Frobenius manifold should have entries $S_{ij} = \chi(\mathcal{E}_i, \mathcal{E}_j)$. We almost completely omit these parts of his conjecture in our discussion.

An expectation underlying Dubrovin's conjecture is that the mirror partner of such a variety X will be the unfolding of a function with isolated singularities. The quantum cohomology should be isomorphic to a Frobenius manifold structure on the base space of the unfolding, as established by Barannikov for projective spaces, cf. [Bar01].

If X has cohomology with Hodge indices other than (p, p) , it can neither have an exceptional system, nor can the Frobenius manifold of its quantum cohomology be semisimple:

- To make sense of all parts of Dubrovin's conjecture, an exceptional collection should have length $\text{rk } H^{\text{ev}}(X)$. But the length of an exceptional collection is bounded by the rank of $N^*(X)$, the group of algebraic cycles modulo numerical equivalence.³ And this is bounded by $\text{rk } N^*(X) \leq \text{rk } H_X$.
- The subspace $\bigoplus_p H^{p,p}(X) \subset H^*(X)$ gives rise to a Frobenius submanifold M' of M ; this is the Frobenius manifold we constructed in section 2. This is a maximal Frobenius submanifold of M that has a chance of being semisimple ([BM04, Theorem 1.8.1]).

This suggested the following modification:

Conjecture 4.2.2. [BM04] *The variety X has generically semisimple reduced quantum cohomology (i. e. M' is generically semisimple) if and only if there exists an exceptional collection of length $\text{rk } \bigoplus_p H^{p,p}(X)$ in $D^b(X)$.*

4.3. Products. It follows easily from well-known facts that Dubrovin's conjecture is compatible with products, i. e. when it is true for two varieties X, Y , it will also hold for their product $X \times Y$.

Theorem 4.3.1. *Let $\mathcal{E}_0, \dots, \mathcal{E}_m$ be an exceptional system on X , and $\mathcal{F}_0, \dots, \mathcal{F}_{m'}$ one on Y . Then $(\mathcal{E}_{i_k} \boxtimes \mathcal{F}_{j_k})_k$ forms an exceptional system on $X \times Y$ if $(i_k, j_k)_k$ is an indexing of the set $\{1, \dots, m\} \times \{1, \dots, m'\}$ which satisfies $i_k < i_{k'}$ or $j_k < j_{k'}$ for all $k < k'$.*

This follows from the Leray spectral sequence computing the Ext-groups on $X \times Y$. It also shows that the Stokes matrix of the exceptional system on $X \times Y$ is the tensor product of the Stokes matrices on X and Y :

$$\chi(\mathcal{E}_{i_k} \boxtimes \mathcal{F}_{j_k}, \mathcal{E}_{i_{k'}} \boxtimes \mathcal{F}_{j_{k'}}) = \chi(\mathcal{E}_{i_k}, \mathcal{E}_{i_{k'}}) \cdot \chi(\mathcal{F}_{j_k}, \mathcal{F}_{j_{k'}})$$

The corresponding statements hold for quantum cohomology: Let M and M' be the Frobenius manifolds associated to the quantum cohomology of X and Y , respectively. The Frobenius manifold of the quantum cohomology of $X \times Y$ is the tensor product $M \otimes M'$ ([KM96], [Beh99], [Kau96]). A pair of semisimple points in M and M' yields a semisimple point in $M \otimes M'$, and the Stokes matrix of the tensor product is the tensor product of the Stokes matrices of M and M' ([Dub99, Lemma 4.10]). It is also clear that the same holds for the reduced quantum cohomology on H_X, H_Y and $H_X \otimes H_Y$.

Hence, Dubrovin's conjecture follows for the product if it holds for $X \times Y$. And in cases where $H_X \otimes H_Y = H_{X \times Y}$, i. e. $\bigoplus_p H^{p,p}(X) \otimes \bigoplus_p H^{p,p}(Y) = \bigoplus_p H^{p,p}(X \times Y)$, the same holds for the modified conjecture 4.2.2.

³From the Hirzebruch-Riemann-Roch theorem, it follows easily that the Chern characters of the exceptional objects are linearly independent.

4.4. Complete exceptional systems and blow-ups.

Theorem 4.4.1. [Ori92] *Let Y be a smooth subvariety of the smooth projective variety X . Let $\rho: \tilde{X} \rightarrow X$ be the blow-up of X along Y .*

If both Y and X have an exceptional system, then the same is true for \tilde{X} .

Consider the case where Y is a point: Let $\mathbb{P}^{n-1} \cong E \subset \tilde{X}$ be the exceptional divisor (n is the dimension of X). If $\mathcal{E}_0, \dots, \mathcal{E}_r$ is a given exceptional system in $D^b(X)$, then $\mathcal{O}_E(-n+1), \dots, \mathcal{O}_E(-2), \mathcal{O}_E(-1), \rho^*(\mathcal{E}_0), \dots, \rho^*(\mathcal{E}_r)$ is an exceptional system in $D^b(\tilde{X})$. Hence, the following holds:

Remark 4.4.2. *If X has an exceptional collection of length $\mathrm{rk} H_X$, then the analogous statement is true for the blow-up of X at any number of points.*

Polynomial Bridgeland stability conditions

1. Introduction

This chapter introduces polynomial stability conditions, a generalization of Bridgeland’s notion of a stability condition on a triangulated category [Bri02b]. We show that it has the same deformation properties, and that every projective variety has a canonical family of polynomial stability conditions.

1.1. Bridgeland’s stability conditions. Since their introduction in [Bri02b], stability conditions for triangulated categories have drawn an increasing amount of interest from various perspectives. Stability has been an important tool in studying abelian categories for a long time; in algebraic geometry, the study of semistable sheaves and their moduli spaces has drawn much attention. For an abstract study of the notion of stability in an abelian category, see [Rud97].

Generalizing this notion of stability to triangulated categories can already be considered a breakthrough; this is the point of view adopted by [GKR04]. However, the original motivation by Bridgeland is somewhat different, and twofold.

Originally, it developed as an attempt to understand Douglas’ construction [Dou02] of Π -stability of D -branes mathematically. Following Douglas’ ideas, Bridgeland showed that the set of stability conditions has a natural structure as a smooth manifold. In the case of the bounded derived category $D^b(X)$ for a smooth projective variety X , this moduli space of stability conditions is a fibre of the moduli space of superconformal field theories (SCFTs); further deformations of a SCFT are given by deformations of X .

Further, the analysis of stability conditions for K3 surfaces [Bri03] was an attempt at studying the automorphism group of their derived categories by understanding its action on the space of stability conditions.

Another motivation, suggested to me by Yuri I. Manin, is the following: It is well-known that given ampleness of the canonical class (or its inverse) of X , the variety can be reconstructed from its bounded derived category [BO01]. Without this assumption, the statement becomes fundamentally wrong, and the proof breaks down already at its first step, the intrinsic characterization of point-like objects in $D^b(X)$ (the shifts $k(x)[j]$ of skyscraper sheaves for closed points $x \in X$).

However, proposals by Aspinwall [Asp03] suggest that a stability condition provides exactly the missing data to characterize the point-like objects. Inside Bridgeland’s moduli space, there should be a special chamber of stability conditions that are

essentially determined by two classes $\beta, \omega \in \text{NS}_{\mathbb{R}}(X)$ in the Néron-Severi group, with ω being ample; see the next section for a more detailed description. Then point-like objects are simply semi-stable objects (for such a stability condition) of correct class in the K -group, and X should be reconstructed as their moduli space.

Moving to a different chamber of the moduli space of stability conditions, the moduli space \tilde{X} of semi-stable objects of the same class is expected to be a birational transformation of X , and its bounded derived category $D^b(\tilde{X})$ will often be isomorphic to $D^b(X)$.

1.2. Constructing stability conditions. Let X be a smooth, n -dimensional projective variety, $D^b(X)$ its bounded derived category of coherent sheaves, and $\mathcal{N}(X)$ the numerical Grothendieck group of $D^b(X)$ (i.e. the quotient of the Grothendieck group by the nullspace of the bilinear form $\chi(A, B) = \chi(\mathbf{R}\text{Hom}(A, B))$). A *numerical stability condition* on $D^b(X)$ can be given by a bounded t-structure (see definition 2.1.2) on $D^b(X)$ with heart $\mathcal{A} \subset D^b(X)$, and a so-called central charge: a group homomorphism $Z: \mathcal{N}(D^b(X)) \rightarrow \mathbb{C}$ such that any object $A \in \mathcal{A}$ is mapped to a semi-closed half-plane: $Z(A) = re^{i\pi\phi}$ for $r > 0$, $\phi_0 \leq \phi < \phi_0 + 1$ and fixed $\phi_0 \in \mathbb{R}$.

This positivity condition on the cone $\mathcal{N}^+(\mathcal{A}) \subset \mathcal{N}(X)$ is highly non-trivial. For any X , classes $\beta, \omega \in \text{NS}(X) \otimes \mathbb{R}$ in the Néron-Severi group with β arbitrary and ω ample, one would expect to have a numerical stability condition with central charge $Z_{\beta, \omega}$ given by the following formula:

$$(7) \quad Z_{\beta, \omega}(E) = (e^{\beta+i\omega}, v(E))$$

Here $v: \mathcal{N}(X) \rightarrow \text{Num}_*(X)$ is the Mukai vector given by $v(E) = \text{ch}(E)\sqrt{\text{td}(X)}$, and the pairing (A, B) on $\text{Num}_*(X)$ determined by $(v(E), v(F)) = -\chi(\mathbf{R}\text{Hom}(E, F))$ for $E, F \in \mathcal{N}(X)$. By Hirzebruch-Riemann-Roch, the pairing is given by $(A, B) = -\int_X P(A) \cdot B$, where $P(A)$ is the parity operator that acts as $(-1)^k$ on classes of codimension k .

This is the class of stability conditions we alluded to in the previous section when describing Aspinwall’s picture. However, except for the case of $K3$ surfaces [Bri03], and varieties that admit an exceptional collection in $D^b(X)$ with a strong additional property [Mac04], the existence of a matching t-structure is unknown.

1.3. Polynomial stability conditions. In this chapter, we will show that it is possible to determine the limit of these t-structures as $\omega \rightarrow \infty$: Replacing ω by $N\omega$, the central charge $Z_{\beta, N\omega}$ given by equation (7) becomes a polynomial in N . We introduce a notion of polynomial stability condition (Definition 2.1.4) where the central charge has values in polynomials $\mathbb{C}[N]$ instead of \mathbb{C} ; this gives a precise meaning to a “stability condition in the limit of $N \rightarrow \infty$ ”. With theorem 3.2.5, we show that polynomial stability conditions satisfy the same deformation properties as in Bridgeland’s situation. Further, in section 4 we show the existence of a canonical t-structure whose central charge is $Z_{\beta, N\omega}$ defined above. The t-structure is given by a category of perverse coherent sheaves [Bez00]; in Bezrukavnikov’s language, the heart consists of “perverse sheaves of middle perversity”.

1.4. Notation. If Σ is a set of objects in a triangulated category \mathcal{D} (resp. a set of subcategories of \mathcal{D}), we write $\langle \Sigma \rangle$ as the full subcategory generated by Σ and extensions; i.e. the smallest full subcategory of \mathcal{D} that is closed under extensions and contains Σ (resp. contains all subcategories in Σ).

By a *semi-metric* on a set Σ we denote a function $d: \Sigma \times \Sigma \rightarrow [0, \infty]$ that satisfies the triangle inequality and $d(x, x) = 0$, but is not necessarily finite or non-zero for two distinct elements. Similarly, we call a function $\| \cdot \|: V \rightarrow [0, \infty]$ on a vector space a *semi-norm* if it satisfies subadditivity and linearity with respect to multiplication with scalars.

2. Polynomial stability conditions

2.1. Slicings.

Definition 2.1.1. Let (S, \succeq) be a linearly ordered set, equipped with an order-preserving bijection $\tau: S \rightarrow S$ (called the *shift*) satisfying $\tau(\phi) \succeq \phi$. An S -valued *slicing* of a triangulated category \mathcal{D} is given by full additive extension-closed subcategories $\mathcal{P}(\phi)$ for all $\phi \in S$, such that the following properties are satisfied:

- (a) For all $\phi \in S$, we have $\mathcal{P}(\tau(\phi)) = \mathcal{P}(\phi)[1]$.
- (b) If $\phi \succ \psi$ for $\phi, \psi \in S$, and $A \in \mathcal{P}(\phi), B \in \mathcal{P}(\psi)$, then $\text{Hom}(A, B) = 0$
- (c) For all non-zero objects $X \in \mathcal{D}$, there is a finite sequence $\phi_0 \succ \phi_1 \succ \dots \succ \phi_n$ of elements in S , and a Postnikov tower of exact triangles

$$(8) \quad \begin{array}{ccccccc} F^0 X & \longrightarrow & F^1 X & \longrightarrow & F^2 X & \longrightarrow & \dots & F^{n-1} X & \longrightarrow & F^n X \\ & & \swarrow & & \swarrow & & & \swarrow & & \swarrow \\ & & X_1 & & X_2 & & & X_n & & \end{array}$$

with $X_i \in \mathcal{P}(\phi_i)$.

This was called a “slicing” in the case of $S = \mathbb{R}$ in [Bri02b], and “stability data” or “t-stability” in [GKR04]. The objects in $\mathcal{P}(\phi)$ are called *semistable of phase ϕ* . We follow [GKR04] in calling the Postnikov tower (8) the *Harder-Narasimhan filtration* of X , and writing $X \rightsquigarrow (X_1, \dots, X_n)$ if we are only interested in the quotients of the filtration. Elements of S are also called *phases*.

For immediate comparison, recall the definition of a bounded t-structure:

Definition 2.1.2. A bounded t-structure on a triangulated category \mathcal{D} is a pair of full subcategories $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ with the following properties:

- (a) $\mathcal{D}^{\leq 0}[1] \subset \mathcal{D}^{\leq 0}$ and $\mathcal{D}^{\geq 0}[-1] \subset \mathcal{D}^{\geq 0}$.
- (b) We write $\mathcal{D}^{\leq i} = \mathcal{D}^{\leq 0}[-i]$. Then for any $X \in \mathcal{D}^{\leq -1}, Y \in \mathcal{D}^{\geq 0}$ we have

$$\text{Hom}(X, Y) = 0.$$

- (c) For any object $X \in \mathcal{D}$, there is an exact triangle

$$\tau_{\leq -1} X \rightarrow X \rightarrow \tau_{\geq 0} X \rightarrow^{[1]}$$

with $\tau_{\leq -1} X \in \mathcal{D}^{\leq -1}$ and $\tau_{\geq 0} X \in \mathcal{D}^{\geq 0}$.

- (d) (*Boundedness.*) Any object $X \in \mathcal{D}$ is contained in $\mathcal{D}^{\leq N}$ and $\mathcal{D}^{\geq M}$ for some $N \gg 0$ and $M \ll 0$.

From condition (b) it follows that $\tau_{\leq -1}$ is functorial and a right adjoint to the inclusion $\mathcal{D}^{\leq -1} \subset \mathcal{D}$. The subcategory $\mathcal{A} = \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$ is called the *heart*. It is automatically abelian. If $\mathcal{D} = \mathcal{D}^b(\mathcal{A})$ is the derived category of an abelian category \mathcal{A} , then \mathcal{A} can be recovered as the heart of the standard t-structure $\mathcal{D}^{\leq 0} = \{X^\bullet \in \mathcal{D}^b(\mathcal{A}) \mid H^k(X^\bullet) = 0 \text{ for all } k > 0\}$.

Every object X has a finite set of non-zero cohomology objects $H^k(X) := \tau_{\leq k} \tau_{\geq k}(X)[k] \in \mathcal{A}$. They are part of exact triangles $H^k(X^\bullet)[-k] \rightarrow \tau_{\geq k}(X^\bullet) \rightarrow \tau_{\geq k+1}(X)$ which yield a finite filtration of X . It follows that a bounded t-structure yields a \mathbb{Z} -valued slicing with $\mathcal{P}(n) = \mathcal{A}[n]$, and in fact the two notions are equivalent (see [Bri03, Lemma 3.1]).

From now on, let S be the set of continuous function germs $\phi: (\mathbb{R} \cup \{+\infty\}, +\infty) \rightarrow \mathbb{R}$ such that there exists a polynomial $z(N) \in \mathbb{C}[N]$ and a positive function germ $m: (\mathbb{R} \cup \{+\infty\}, +\infty) \rightarrow \mathbb{R}_{>0}$ with $z(N) = m(N)e^{\pi i \phi(N)}$ for $N \gg 0$. The set S is linearly ordered by setting

$$(9) \quad \phi \prec \psi \Leftrightarrow \phi(x) < \psi(x) \quad \text{for } x \gg 0.$$

(The condition that ϕ, ψ can be written as arguments of polynomial functions guarantees that either $\phi \prec \psi$ or $\psi \preceq \phi$.) It is equipped with a shift τ defined as $\tau(\phi) = \phi + 1$. In our construction, S -valued slicings will play the role of \mathbb{R} -valued slicings in Bridgeland's construction.

The following easy lemma is implicitly assumed in [GKR04], but we will need it explicitly:

Lemma 2.1.3. *Let S_1, S_2 be two linearly ordered sets equipped with shifts τ_1, τ_2 , and let $\pi: S_1 \rightarrow S_2$ be a morphism of ordered sets commuting with τ_1, τ_2 . Then π induces a push-forward of stability conditions as follows: If \mathcal{P} is an S_1 -valued slicing, then $\pi_* \mathcal{P}(\phi_2)$ for some $\phi_2 \in S_2$ is defined as $\langle \{\mathcal{P}(\phi_1) \mid \pi(\phi_1) = \phi_2\} \rangle$.*

Proof. Conditions (a) and (b) of definition 2.1.1 are evident. To prove (c) for an object X , we start with its Harder-Narasimhan filtration $X \rightsquigarrow (X_{1,1}, \dots, X_{k-1, n_{k-1}}, X_{k,1}, \dots, X_{k, n_k}, X_{k+1, n_{k+1}}, \dots, X_{n, n_n})$ given by \mathcal{P} , where $X_{k,j}$ is of phase $\phi_{k,j} \in S_1$ such that $\phi_k := \pi(\phi_{k,1}) = \pi(\phi_{k,2}) = \dots = \pi(\phi_{k, n_k})$ and $\pi(\phi_{k, n_k}) \succ \pi(\phi_{k+1,1})$. Repeated use of the octahedral axiom then yields objects Y_k with Harder-Narasimhan filtrations $Y_k \rightsquigarrow (X_{k,1}, \dots, X_{k, n_k})$ and $X \rightsquigarrow (Y_1, \dots, Y_n)$; see [GKR04, Proposition 4.3 no. 2] for a complete proof. Since $Y_k \in \pi_* \mathcal{P}(\phi_k)$, the assertion follows. \square

In the language of [GKR04], \mathcal{P} is a “finer t-stability” than $\pi_* \mathcal{P}$. We will make use of the following push-forwards: By the projection $\pi: S \rightarrow \mathbb{R}, \phi \mapsto \phi(\infty)$, we obtain an \mathbb{R} -valued slicing from every S -valued slicing. Further, for each $\phi_0 \in S$ we get a projection $\pi^{\phi_0}: S \rightarrow \mathbb{Z}, \phi \mapsto \max_{n \in \mathbb{Z}} \phi_0 + n \preceq \phi$ (we could also choose $\phi \mapsto \max_{n \in \mathbb{Z}} \phi_0 + n \prec \phi$). This produces a bounded t-structure from every S -valued

slicing; in other words, an S -valued slicing is a refinement of a bounded t-structure, breaking up the category into even smaller slices.

For any interval I in the set of phases, we get an extension-closed subcategory $\mathcal{P}(I) = \langle \{\mathcal{P}(\phi) \mid \phi \in I\} \rangle$. In the case of an S -valued slicing, the categories $\mathcal{P}([\phi, \phi + 1])$ and $\mathcal{P}((\phi, \phi + 1])$ are abelian, as they are the hearts of the t-structures constructed in the last paragraph. If $\phi \prec \psi \prec \phi + 1$, the subcategories $\mathcal{P}([\phi, \psi])$, $\mathcal{P}((\phi, \psi))$ etc. are quasi-abelian:¹ The proof for these statements carries over literally from the one given by Bridgeland, because we can include these categories into the abelian category $\mathcal{P}([\phi, \phi + 1])$. The slices $\mathcal{P}(\phi)$ are abelian.

Definition 2.1.4. A polynomial stability condition on a triangulated category \mathcal{D} is given by a pair (Z, \mathcal{P}) , where \mathcal{P} is an S -valued slicing of \mathcal{D} and Z is a group homomorphism $Z: K(\mathcal{D}) \rightarrow \mathbb{C}[N]$, with the following property: if $0 \neq E \in \mathcal{P}(\phi)$, then $Z(E)(N) = m(E)(N)e^{\pi i \phi(N)}$ for $N \gg 0$ and some function germ $m(E): (\mathbb{R}_{+\infty}, +\infty) \rightarrow \mathbb{R}_{>0}$.

If Z actually maps to the constant polynomials $\mathbb{C} \subset \mathbb{C}[N]$, this is exactly a stability condition as defined in [Bri02b, Definition 5.1].

2.2. Centered slope function. The following definition and proposition shows how a polynomial stability condition can be seen as a refinement of a \mathbb{Z} -valued slicing induced by a compatible central charge Z .

Definition 2.2.1. A polynomial stability function on a quasi-abelian category \mathcal{A} is a group homomorphism $Z: K(\mathcal{A}) \rightarrow \mathbb{C}[N]$ such that there exists a function $\phi_0 \in S$ with the following property:

For any $0 \neq E \in \mathcal{A}$, we can write $Z(E)(N) = r_E(N)e^{2\pi i \phi_E(N)}$ with $r_E(N) > 0$ and $\phi_0 \preceq \phi_E \prec \phi_0 + 1$.

Approximately, this means that Z is mapping the “effective cone” in $K(\mathcal{A})$ to one half-plane for $N \gg 0$. We call ϕ_E the phase of E ; the function $\text{Ob } \mathcal{A} \rightarrow S, E \mapsto \phi_E$ is a slope function in the sense that it satisfies the see-saw property on short exact sequences (cf. [Rud97]). An object $0 \neq E$ is said to be *semistable with respect to Z* if for all subobjects $0 \neq A \subset E$, we have $\phi_A \preceq \phi_E$ (equivalently, if for every quotient $E \twoheadrightarrow B$ in \mathcal{A} we have $\phi_B \succeq \phi_E$). We say that a stability function has the Harder-Narasimhan property if for all $E \in \mathcal{A}$, there is a finite filtration $0 = E_0 \subset E_1 \subset \dots \subset E_n = E$ such that E_i/E_{i-1} are semistable with slopes $\phi_{E_1/E_0} \succ \phi_{E_2/E_1} \succ \dots \succ \phi_{E_n/E_{n-1}}$. Proposition 2.4 in [Bri02b] shows that the Harder-Narasimhan property can be deduced from rather weak assumptions on Z .

Finally, note that the set of polynomials for which functions r_E and ϕ_E as in the above definitions exist forms a convex cone in $\mathbb{C}[N]$. Its only extremal ray is the set of polynomials with $\phi_E = \phi$. This is an important reason why many of the proofs of [Bri02b] carry over automatically to our situation.

¹We refer to [Bri02b, section 4] for an introduction to quasi-abelian categories.

Proposition 2.2.2. *Giving a polynomial stability condition is equivalent to giving a bounded t-structure on \mathcal{D} and a polynomial stability function on its heart with the Harder-Narasimhan property.*

We first spell out how to go from one side to another: Given a polynomial stability condition (Z, \mathcal{P}) , we choose any $\phi_0 \in S$. The projection $\pi^{\phi_0} : S \rightarrow \mathbb{Z}$ defined in the last section yields a bounded t-structure $\pi_*^{\phi_0} \mathcal{P}$ with heart $\mathcal{P}([\phi_0, \phi_0 + 1])$, for which Z is a stability function. From a stability function on the heart \mathcal{A} of a bounded t-structure, we only need to produce an S -valued slicing: For any ϕ with $\phi_0 \preceq \phi \prec \phi_0 + 1$, we let $\mathcal{P}(\phi)$ be the subcategory of \mathcal{A} of objects semistable with respect to Z of phase ϕ , and $\mathcal{P}(\phi + n) = \mathcal{P}(\phi)[n]$.

The only thing left to check is that starting with a polynomial stability condition (Z, \mathcal{P}) , the semistable objects of slope ϕ with respect to the stability function Z on $\mathcal{P}([\phi_0, \phi_0 + 1])$ are identical to the objects in the original $\mathcal{P}(\phi)$, and vice versa. This is easily verified from two facts: For $A \in \mathcal{P}((\phi, \phi_0 + 1))$, the function ϕ_A (cf. definition 2.2.1) satisfies $\phi \prec \phi_A \prec \phi_0 + 1$, and given $B \in \mathcal{P}(\phi)$, the homomorphisms $\text{Hom}(A, B) = 0$ vanish.

3. The moduli space of polynomial stability conditions

3.1. The topology. We continue with the following translations of definitions of [Bri02b] to our situation:

Definition 3.1.1. *A polynomial stability condition (Z, \mathcal{P}) is called locally finite if there exists a real number $\epsilon > 0$ such that for all $\phi \in S$, the quasi-abelian category $\mathcal{P}((\phi - \epsilon, \phi + \epsilon))$ is of finite length.*

Definition 3.1.2. *If the triangulated category \mathcal{D} is linear over a field, a polynomial stability condition (Z, \mathcal{P}) on \mathcal{D} is called numerical if $Z : K(\mathcal{D}) \rightarrow \mathbb{C}[N]$ factors via $\mathcal{N}(\mathcal{D})$, the numerical Grothendieck group.*

Let $\text{Stab}_{\text{Pol}}(\mathcal{D})$ be the set of locally finite polynomial stability conditions on \mathcal{D} . Our next goal is to define a topology on $\text{Stab}_{\text{Pol}}(\mathcal{D})$. Bridgeland introduced the following generalized metric on the space of \mathbb{R} -valued slicings:

For any $X \in \mathcal{D}$ and an \mathbb{R} -valued slicing, let $\phi_{\mathcal{P}}^-(X)$ and $\phi_{\mathcal{P}}^+(X)$ be the smallest and highest phase appearing in the Harder-Narasimhan filtration of X according to 2.1.1(c), respectively. Equivalently, $\phi_{\mathcal{P}}^+(X)$ is the largest ϕ such that there exists a stable object E of phase ϕ and a non-zero morphism $E \rightarrow X$; similarly, $\phi_{\mathcal{P}}^-(X)$ is the smallest phase ϕ with a stable object E and a non-trivial morphism $X \rightarrow E$. Then $d(\mathcal{P}, \mathcal{Q}) \in [0, \infty]$ is defined as

$$(10) \quad d(\mathcal{P}, \mathcal{Q}) = \sup_{0 \neq X \in \mathcal{D}} \{ |\phi_{\mathcal{P}}^-(X) - \phi_{\mathcal{Q}}^-|, |\phi_{\mathcal{P}}^+(X) - \phi_{\mathcal{Q}}^+| \}.$$

Via π_* , we pull this back to a semi-metric d_S on the space of S -valued slicings.

Following [Bri02b, section 6], we introduce a semi-norm on the infinite-dimensional linear space $\text{Hom}(\mathcal{K}(\mathcal{D}), \mathbb{C}[N])$ for all $\sigma = (Z, \mathcal{P}) \in \text{Stab}_{\text{Pol}}(\mathcal{D})$:

$$(11) \quad \begin{aligned} & \|\cdot\|_{\sigma}: \text{Hom}(\mathcal{K}(\mathcal{D}), \mathbb{C}[N]) \rightarrow [0, \infty] \\ & \|U\|_{\sigma} = \sup \left\{ \limsup_{N \rightarrow \infty} \frac{|U(E)(N)|}{|Z(E)(N)|} \mid E \text{ semistable in } \sigma \right\} \end{aligned}$$

The next step is to show that [Bri02b, Lemma 6.2] carries over: For $0 < \epsilon < \frac{1}{4}$, and $\sigma = (Z, \mathcal{P}) \in \text{Stab}_{\text{Pol}}(\mathcal{D})$ define $B_{\epsilon}(\sigma) \subset \text{Stab}_{\text{Pol}}(\mathcal{D})$ as

$$B_{\epsilon}(\sigma) = \{\tau = (\mathcal{Q}, W) \mid \|W - Z\|_{\sigma} < \sin(\pi\epsilon) \text{ and } d_S(\mathcal{P}, \mathcal{Q}) < \epsilon\}.$$

Lemma 3.1.3. *If $\tau = (\mathcal{Q}, W) \in B_{\epsilon}(\sigma)$, then the semi-norms $\|\cdot\|_{\sigma}, \|\cdot\|_{\tau}$ of σ and τ are equivalent, i.e. there are constants k_1, k_2 such that $k_1\|U\|_{\sigma} < \|U\|_{\tau} < k_2\|U\|_{\sigma}$ for all $U \in \text{Hom}(\mathcal{K}(\mathcal{D}), \mathbb{C}[N])$.*

Proof. Let X be an object with $\phi_{\mathcal{P}}^{+}(X) - \phi_{\mathcal{P}}^{-}(X) < \eta$ for some $0 \leq \eta < \frac{1}{2}$ and the given stability condition $\sigma = (Z, \mathcal{P})$. Then the following inequality holds:

$$(12) \quad \limsup_{N \rightarrow \infty} \frac{|U(E)(N)|}{|Z(E)(N)|} < \frac{\|U\|_{\sigma}}{\cos(\pi\eta)}$$

The reason is the same as for the corresponding inequality (*) in the proof of [Bri02b, Lemma 6.2]: Consider the filtration $X \rightsquigarrow (X_1, \dots, X_n)$ of X into its semistable factors, and apply (11) to all X_i . Since the points $Z(X_i)(N)$ lie in a sector with an angle smaller than $\pi\eta$ for $N \gg 0$, we have $|Z(X)(N)| \geq \cos(\pi\eta) \sum_i |Z(X_i)(N)|$ for $N \gg 0$, and the lemma follows.

Now assume E is semistable in τ ; due to $d_S(\mathcal{P}, \mathcal{Q}) < \epsilon < \frac{1}{4}$, the inequality (12) holds with $\eta = 2\epsilon$. We first apply it with $U = Z - W$ to obtain an inequality of the form $|Z(E)(N)| < k|W(E)(N)|$ for all E semistable in τ , all $N \gg 0$ and a fixed constant k . Then we apply (12) for an arbitrary U to conclude $\|U\|_{\tau} < \frac{k}{\cos(\pi 2\epsilon)}\|U\|_{\sigma}$. The other inequality follows with the same argument. \square

On $\text{Hom}(\mathcal{K}(\mathcal{D}), \mathbb{C}[N])$ we have the natural topology of point-wise convergence; via the forgetful map $(Z, \mathcal{P}) \mapsto Z$ we can pull this back to get a system of open sets in $\text{Stab}_{\text{Pol}}(\mathcal{D})$. Now equip $\text{Stab}_{\text{Pol}}(\mathcal{D})$ with the topology generated, in the sense of a subbasis², by this system of open sets and the sets $B_{\epsilon}(\sigma)$ defined above.

Let E be stable in some polynomial stability condition $\sigma = (Z, \mathcal{P})$. Then the definition of the topology implies that the degree of the polynomial $Z(E)(N)$ is constant on the connected component of σ . In particular, Bridgeland's moduli space $\text{Stab}(\mathcal{D})$ is just the union of the connected components of $\text{Stab}_{\text{Pol}}(\mathcal{D})$ where the image of the central charge lies in $\mathbb{C} \subset \mathbb{C}[N]$.

As in [Bri02b], the subspace

$$\{U \in \text{Hom}(\mathcal{K}(\mathcal{D}), \mathbb{C}[N]) \mid \|U\|_{\sigma} < \infty\}$$

²A topology T on a set S is generated by a subbasis Π of subsets of S if open sets in T are exactly the (infinite) unions of finite intersections of sets in Π .

is locally constant in $\text{Stab}_{\text{Pol}}(\mathcal{D})$ and hence constant on a connected component Σ , denoted by $V(\Sigma)$. It is equipped with the topology generated by the topology of point-wise convergence and the semi-norms $\|\cdot\|_\sigma$ for $\sigma \in \Sigma$ (which are equivalent by lemma 3.1.3); we have obtained:

Proposition 3.1.4. *For each connected component of $\Sigma \subset \text{Stab}_{\text{Pol}}(\mathcal{D})$ there is a topological vector space $V(\Sigma)$, which is a subspace of $\text{Hom}(\mathcal{K}(\mathcal{D}), \mathbb{C}[N])$ such that the forgetful map $\Sigma \rightarrow V(\Sigma)$ given by $(Z, \mathcal{P}) \mapsto Z$ is continuous.*

The next proposition shows that this map is locally injective:

Proposition 3.1.5. *Suppose that $\sigma = (Z, \mathcal{P})$ and $\tau = (Z, \mathcal{Q})$ are polynomial stability conditions with identical central charge Z and $d_S(\mathcal{P}, \mathcal{Q}) < 1$. Then they are identical.*

Again, the proof of [Bri02b, Lemma 6.4] carries over literally. The main reason for this is that the following argument works in our situation, too: Given an abelian category \mathcal{A} and a stability function Z , an object $E \in \mathcal{A}$ is stable of slope ϕ_0 if and only if $\phi_E = \phi_0$ (where ϕ_E and ϕ_0 are as defined in definition 2.2.1).

3.2. Deformations of a polynomial stability condition. We will prove that the forgetful map $(Z, \mathcal{P}) \mapsto Z$ is a local homeomorphism. In other words, a polynomial stability condition can be deformed uniquely by deforming its central charge:

Theorem 3.2.1. *Let $\sigma = (Z, \mathcal{P})$ be a locally finite polynomial stability condition. Then there is an $\epsilon > 0$ such that if a group homomorphism $W : \mathcal{K}(\mathcal{D}) \rightarrow \mathbb{C}[N]$ satisfies $\|W - Z\|_\sigma < \sin(\pi\epsilon)$, there is a locally finite stability condition $\tau = (W, \mathcal{Q})$ with $d_S(\mathcal{P}, \mathcal{Q}) < \epsilon$.*

We will follow Bridgeland's arguments in [Bri02b, section 7] closely. We won't repeat the proof of most of the preparatory lemmas, but we will present here the (slightly modified) proof of the main step, the existence of Harder-Narasimhan filtrations for the new stability condition (lemma 3.2.4).

Consider any interval (a, b) of length $b - a < 1 - 2\epsilon$. As mentioned previously, the category $\mathcal{P}((a, b))$ is quasi-abelian; it is embedded into the abelian category $\mathcal{P}([a, a + 1))$. We say that A with a morphism $A \rightarrow B$ is a *strict subobject* of B , if it is a subobject in $\mathcal{P}([a, a + 1))$, and the quotient is an element of $\mathcal{P}((a, b))$; analogously for *strict quotient*.

Since the phases of W and Z differ by at most ϵ for σ -semistable objects, it follows that W is a stability function for the category $\mathcal{P}((a, b))$. For any object $E \in \mathcal{P}((a, b))$, we will write ψ_E and ϕ_E for its phase (in the sense of definition 2.2.1) with respect to W and Z , respectively. We say that an object E of $\mathcal{P}((a, b))$ is W -semistable if $\psi_A \preceq \psi_E$ holds for all strict subobjects $A \subset E$.

We choose $\epsilon < \frac{1}{10}$ such that $\mathcal{P}([\phi - 4\epsilon, \phi + 4\epsilon])$ is of finite length for every $\phi \in S$. Let $\mathcal{Q}(\phi)$ be the full subcategory of W -semistable objects in $\mathcal{P}((\phi - \epsilon, \phi + \epsilon))$.

Lemma 3.2.2. *Let $(a, b) \subset S$ be an interval containing ϕ of length $b - a \prec 1 - 2\epsilon$. Then every object in $\mathcal{Q}(\phi)$ is W -semistable in the quasi-abelian category $\mathcal{P}((a, b))$; conversely, every W -semistable object E of phase ϕ is an object of $\mathcal{P}((\phi - \epsilon, \phi + \epsilon))$; in particular, if (a, b) contains $(\phi - \epsilon, \phi + \epsilon)$, then E is an object in $\mathcal{Q}(\phi)$.*

The proofs of [Bri02b, Lemma 7.5 and Lemma 7.3] carry over with no changes.

Lemma 3.2.3. *If $\phi \succ \psi$, and $A \in \mathcal{Q}(\phi)$, $B \in \mathcal{Q}(\psi)$, then $\text{Hom}(A, B) = 0$.*

See [Bri02b, Lemma 7.6] for the proof. As a consequence, the pair of subcategories $\mathcal{Q}([\phi, +\infty))$ and $\mathcal{Q}((-\infty, \phi + 1))$ is semiorthogonal (definition 2.1.2, (b)).

Fix an interval (a, b) as above with the additional property that $\mathcal{A} := \mathcal{P}((a, b))$ has finite length.

Lemma 3.2.4. *Every object $E \in \mathcal{P}([a+2\epsilon, b-4\epsilon])$ has a finite Harder-Narasimhan filtration by W -semistable objects in $\mathcal{P}((a, b - 2\epsilon))$; the filtration quotients E_i satisfy $a + \epsilon \prec \psi_{E_i} \prec b - 3\epsilon$.*

Proof. Let \mathcal{G} be the set of objects E in $\mathcal{A} = \mathcal{P}((a, b - 2\epsilon))$ that satisfy the following two properties:

- (a) Every W -semistable strict subobject $A \subset E$ of E in \mathcal{A} satisfies $\psi_A \prec b - 3\epsilon$.
- (b) Every W -semistable strict quotient $E \twoheadrightarrow B$ of E in \mathcal{A} satisfies $\psi_B \succ a + \epsilon$.

All objects of $\mathcal{P}([a + 2\epsilon, b - 4\epsilon])$ are contained in \mathcal{G} : for $E \twoheadrightarrow B$, we have $\psi_B + \epsilon \succ \phi_B^- \succeq a + 2\epsilon$, and similarly for a subobject $A \subset E$.

We will show that all elements of \mathcal{G} have a Harder-Narasimhan filtration as desired. We say that $E \twoheadrightarrow B$ is a *maximal destabilizing quotient* (mdq) of E (with respect to the slope function ψ), if every quotient $E \twoheadrightarrow B'$ satisfies $\psi_{B'} \succeq \psi_B$, with equality holding only if $E \twoheadrightarrow B'$ factors via $E \twoheadrightarrow B$. Both conditions may be tested only for W -semistable B' , and if $E \twoheadrightarrow B$ is a mdq, then B is automatically W -semistable.

Assume that $E \in \mathcal{G}$ has a mdq $E \twoheadrightarrow B$. Consider the short exact sequence $0 \rightarrow E' \rightarrow E \rightarrow B \rightarrow 0$ with $E' \neq 0$ (otherwise, E is already semistable). By assumption, $\psi_B \succ a + \epsilon$. Now consider any quotient $E' \twoheadrightarrow B'$, define K as its kernel and Q via the following diagram of short exact sequences:

$$\begin{array}{ccccccc}
 & & & 0 & & 0 & \\
 & & & \downarrow & & \downarrow & \\
 0 & \longrightarrow & K & \longrightarrow & E' & \longrightarrow & B' \longrightarrow 0 \\
 & & \downarrow = & & \downarrow & & \downarrow \\
 0 & \longrightarrow & K & \longrightarrow & E & \longrightarrow & Q \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & B & \xrightarrow{=} & B \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0
 \end{array}$$

By definition of mdq, $\psi_Q \succ \psi_B$ and thus $\psi_{B'} \succ \psi_Q \succ \psi_B \succ a + \epsilon$ by the see-saw property; in particular, E' is an element of \mathcal{G} . By induction on the length of E , we may assume that E' has a Harder-Narasimhan filtration $0 = E'_0 \subset E'_1 \subset \cdots \subset E'_n = E'$. We have just shown that its last filtration quotient E'/E'_{n-1} satisfies $\psi_{E'/E'_{n-1}} \succ \psi_B$, and thus this sequence extends to a Harder-Narasimhan filtration of E .

It remains to show that every object $E \in \mathcal{G}$ has a mdq. We will show this for the larger class \mathcal{H} of objects such that

- (a) $\psi_E \prec b - 3\epsilon$.
- (b) Every W -semistable strict quotient $E \twoheadrightarrow B$ of E in \mathcal{A} satisfies $\psi_B \succ a + \epsilon$.

By induction on length, it is sufficient to construct a non-trivial strict subobject $A \subset E$ with $\psi_A \succ \psi_E$ such that

- (*) for every W -semistable quotient $E \twoheadrightarrow B$ with $\psi_B \preceq \psi_E$, we have $\text{Hom}(A, B) = 0$.

In that case, the quotient $E' = E/A$ satisfies $\psi_{E'} \prec \psi_E \prec b - 3\epsilon$, so it is an element of \mathcal{H} ; every quotient $E \twoheadrightarrow B$ as above factors via $E \twoheadrightarrow E'$, and thus a mdq of E' is also a mdq for E .

If E is not contained in $\mathcal{P}((a, b - 2\epsilon))$, there is a non-trivial short exact sequence $0 \rightarrow A \rightarrow E \rightarrow E' \rightarrow 0$ with $A \in \mathcal{P}([b - 2\epsilon, b))$ and $E' \in \mathcal{P}((a, b - 2\epsilon))$. The condition (*) holds by applying lemma 3.2.2 to B . Otherwise, and if E is not semistable, it has a W -semistable strict subobject $A \subset E$ with $\psi_A \succ \psi_E$. Then $b - \epsilon \succ \psi_A$ by lemma 3.2.2; again by the same lemma, it follows $A \in \mathcal{Q}(\psi_A)$. Hence we can use lemma 3.2.3 to show condition (*). \square

In particular $\mathcal{P}(\phi) \subset \mathcal{Q}((\phi - \epsilon, \phi + \epsilon))$, and thus $\mathcal{P}((-\infty, \phi - \epsilon]) \subset \mathcal{Q}((-\infty, \phi))$ as well as $\mathcal{P}([\phi + \epsilon, +\infty)) \subset \mathcal{Q}((\phi, +\infty))$. Given an arbitrary object E of \mathcal{D} , we can first construct a three-step filtration $E \rightsquigarrow (E_0, E_1, E_2)$ with $E_0 \in \mathcal{P}([\phi + \epsilon, +\infty))$, $E_1 \in \mathcal{P}((\phi - \epsilon, \phi + \epsilon))$ and $E_2 \in \mathcal{P}((-\infty, \phi - \epsilon])$. Since we assumed $\epsilon < \frac{1}{10}$, lemma 3.2.4 gives a Harder-Narasimhan filtration of E_1 ; altogether we obtain a filtration of E that we can collapse into an exact triangle $E' \rightarrow E \rightarrow E''$ with $E' \in \mathcal{Q}([\phi, +\infty))$ and $E'' \in \mathcal{Q}((-\infty, \phi))$.

Hence we have shown that $\mathcal{Q}([\phi, +\infty))$, $\mathcal{Q}((-\infty, \phi))$ is a bounded t-structure, for which W is a stability function with the Harder-Narasimhan property. By proposition 2.2.2, the pair (W, \mathcal{Q}) is a polynomial stability condition, finishing the proof of theorem 3.2.1.

Combining propositions 3.1.4, 3.1.5 and theorem 3.2.1, we obtain the following generalization of [Bri02b, Theorem 1.2]:

Theorem 3.2.5. *The set $\text{Stab}_{\text{Pol}}(\mathcal{D})$ of locally finite polynomial stability conditions on a triangulated category \mathcal{D} is a smooth manifold. For each connected component $\Sigma \subset \text{Stab}_{\text{Pol}}(\mathcal{D})$ there is a topological vector space $V(\Sigma)$, a subspace of $\text{Hom}(\mathcal{K}(\mathcal{D}), \mathbb{C}[N])$, such that the forgetful map $\sigma = (Z, \mathcal{P}) \mapsto Z$ gives local coordinates $\Sigma \rightarrow V(\Sigma)$ at every point of Σ .*

Restricted to the subset $\text{Stab}_{\text{Pol}}^{\mathbb{N}}(\mathcal{D})$ of numerical stability conditions, the same statements hold with $\mathcal{K}(\mathcal{D})$ replaced by $\mathcal{N}(\mathcal{D})$. Additionally, if $\mathcal{N}(\mathcal{D})$ is finite dimensional, then every connected component of $\text{Stab}_{\text{Pol}}^{\mathbb{N}}(\mathcal{D})$ is finite dimensional, too.

4. Canonical stability conditions

4.1. Perverse coherent sheaves. In this section we construct a stability condition with a slope function as given in the introduction. The t-structure is given by a construction of perverse coherent sheaves; their theory was apparently sketched, but not published, by Deligne, and worked out by Bezrukavnikov [Bez00]. For convenience, we use the somewhat more explicit description [Kas04] by Kashiwara.

Let X be a projective variety, and $D^b(X)$ its bounded derived category of complexes of quasi-coherent sheaves with coherent cohomology. Consider the following increasing filtration of $\text{Coh } X$ by the dimension of support:

$$\mathcal{A}^{\leq k} = \{\mathcal{F} \in \text{Coh } X \mid \dim \text{supp } \mathcal{F} \leq 2k + 1\}$$

Note that all $\mathcal{A}^{\leq k}$ are closed under subobjects and quotients, i.e. they are abelian subcategories of $\text{Coh } X$.

Theorem 4.1.1. [Bez00, Kas04] *The following pair defines a t-structure on $D^b(X)$:*

$$(13) \quad \mathcal{D}^{\# \leq n} = \{X \in \mathcal{D} \mid H^{-k}(X) \in \mathcal{A}^{\leq k+n} \text{ for all } k \in \mathbb{Z}\}$$

$$(14) \quad \mathcal{D}^{\# \geq n} = \{X \in \mathcal{D} \mid \text{Hom}(A, X) = 0 \text{ for all } k \in \mathbb{Z} \text{ and } A \in \mathcal{A}^{\leq k+n}[k+1]\}$$

In the language of [Bez00], this is the t-structure of perverse coherent sheaves of middle perversity (i.e. for the perversity function $p(x) = \lceil -\frac{\dim(x)}{2} \rceil$ in his notation). In the notation of [Kas04], this t-structure is obtained from the filtration of supports given by $\Phi^{-i} = \{Z \subset X \mid \dim Z \leq 2i + 1\} = \mathfrak{S}^{m-2i}$, which yields a t-structure by theorem 3.5 and 5.9 *ibid*.

Our description of $\mathcal{D}^{\# \geq n}$ differs from the one given by Kashiwara. However, $\mathcal{D}^{\# \geq n}$ is uniquely determined as the right orthogonal to $\mathcal{D}^{\# \leq n-1}$; the latter is generated by all $A \in \mathcal{A}^{\leq k+n}[k+1]$ and extensions, and hence its right-orthogonal must be the intersection of the right-orthogonals of the given objects A . The reason for our choice of notation of $\mathcal{A}^{\leq k}$ as an increasing filtration is that these subcategories can be recovered from the new t-structure as $\mathcal{A}^{\leq k} = \mathcal{A} \cap \mathcal{D}^{\# \leq k}$.

Theorem 4.2. *Let β, ω be classes in the real Néron-Severi group $NS(X) \otimes \mathbb{R}$ such that ω lies in the ample cone. Let $Z_{\beta, \omega}: K(X) \rightarrow \mathbb{C}[N]$ be defined via*

$$Z_{\beta, \omega}(E)(N) = (e^{\beta+iN\omega}, v(E))$$

Then $Z_{\beta, \omega}(E)$ is a polynomial stability function for the heart $\mathcal{A}^{\#}$ of the bounded t-structure defined in Theorem 4.1.1.

Proof. We will prove that $Z_{\beta,\omega}$ satisfies the conditions of definition 2.2.1 for any constant function ϕ with $0 < \phi < \frac{1}{2}$.

Let E be any non-zero object in \mathcal{A}^\sharp . Let k be maximal such that $H^{-k}(E) \neq 0$. We know $H^{-k}(E) \in \mathcal{A}^{\leq k}$ by (13) and $E \in \mathcal{D}^{\sharp \leq 0}$. If $H^{-k}(E) \in \mathcal{A}^{\leq k-1}$, then $\text{Hom}(H^{-k}(E)[k], E) = 0$ by (14) and $E \in \mathcal{D}^{\sharp \geq 0}$, which is a contradiction. Hence $\dim \text{supp } H^{-k}E \in \{2k, 2k+1\}$.

Since $\text{ch}(E) = \sum_i (-1)^i \text{ch}(H^i(E))$, this means that the only contribution to $\text{ch}(E)$ (and thus, to $v(E)$) of dimension $2k+1$ and $2k$ comes from $H^{-k}(E)$. Thus

$$\begin{aligned} Z_{\beta,\omega}(E)(N) &= (-1)^k Z_{\beta,\omega}(H^{-k}(E)) + O(N^{2k-1}) \\ &\approx (-1)^k (e^{\beta+iN\omega}, v(H^{-k}(E))) \approx (-1)^{k+1} \int_X e^{-iN\omega} \text{ch}(H^{-k}(E)); \end{aligned}$$

now if $\text{ch}_{2k+1}(H^{-k}(E)) \neq 0$, we get

$$\begin{aligned} Z_{\beta,\omega}(E)(N) &\approx (-1)^{k+1} \int_X (-i)^{2k+1} \frac{N^{2k+1}}{(2k+1)!} \omega^{2k+1} \text{ch}_{2k+1}(H^{-k}(E)) \\ &= i \frac{N^{2k+1}}{(2k+1)!} \int_X \omega^{2k+1} \text{ch}_{2k+1}(H^{-k}(E)) \end{aligned}$$

otherwise, $\text{ch}_{2k}(H^{-k}(E)) \neq 0$, and we obtain

$$\begin{aligned} Z_{\beta,\omega}(E)(N) &\approx (-1)^{k+1} \int_X (-i)^{2k} \frac{N^{2k}}{(2k)!} \omega^{2k} \text{ch}_{2k}(H^{-k}(E)) \\ &= -\frac{N^{2k}}{(2k)!} \int_X \omega^{2k} \text{ch}_{2k}(H^{-k}(E)). \end{aligned}$$

Since ω is an ample class, and $\text{ch}_{2k+1}(H^{-k}(E))$ respectively $\text{ch}_{2k}(H^{-k}(E))$ is effective, this shows $Z_{\beta,\omega}(E)(N) \rightarrow +i\infty$ respectively $Z_{\beta,\omega}(E)(N) \rightarrow -\infty$ as $N \rightarrow \infty$, and the claim follows. \square

The following remark goes in the direction of recovering X from $D^b(X)$ with a stability condition:

Remark 4.2.1. *The set of stable objects E in $\mathcal{P}(1)$ with $Z_{\beta,\omega} = -1$ is the set of skyscraper sheaves $k(x)$ for closed points $x \in X$.*

Proof. This is almost tautological. E must be an element in \mathcal{A}^\sharp . From $Z(E) = 1$ and the proof of theorem 4.2 it follows that E is concentrated in degree zero, that the support of E is zero-dimensional, and that $\text{ch}(E) = [\text{pt}]$. Hence $E = k(x)$.

Conversely, we need to show that all $k(x)$ are stable. It is immediate that $k(x) \in \mathcal{A}^\sharp$. It is sufficient to show that $k(x)$ has no subobjects in \mathcal{A}^\sharp . Otherwise, there would be an exact triangle $A \rightarrow k(x) \rightarrow A' \rightarrow^{[1]}$ with $A, A' \in \mathcal{A}^\sharp$. The long exact sequence in cohomology (with respect to the standard t-structure) yields $0 \rightarrow H^{-1}(A') \rightarrow H^0(A) \rightarrow k(x) \rightarrow H^0(A') \rightarrow 0$ exact and $H^{-k}(A') \cong H^{-k+1}(A)$ for $k > 1$. Hence $H^{-k}(A') \in \mathcal{A}^{\leq k-1}$ for all $k > 0$. Thus $\tau_{<0}(A') \in \mathcal{D}^{\sharp < 0}$. Since

$A' \in \mathcal{D}^{\# \geq 0}$, the canonical morphism $\tau_{<0}(A') \rightarrow A'$ is zero, so $\tau_{<0}(A') = 0$. Thus both A and A' are concentrated in degree zero, and we have an impossible non-trivial short exact sequence of coherent sheaves $0 \rightarrow A \rightarrow k(x) \rightarrow A' \rightarrow 0$. \square

Of course, instead of a set-theoretic identification, we would like to exhibit X as the moduli space of stable objects. This would need a work-around for the "gluing problem" of objects in the derived category.

We would like to remark that this stability condition is *not* equivalent to Gieseker- or Rudakov-stability (the latter in the sense of [Rud97]). While our stability condition is also a refinement of Mumford-stability, it is a different refinement. This was already observed in [Bri03]: Proposition 12.2 *ibid.* would not be true for objects E with $(c_1(E) - r(E)B) \cdot \omega < 0$ (using Bridgeland's notation).

Moduli spaces of weighted stable maps and Gromov-Witten invariants

1. Introduction

In this chapter, we generalize Hassett's notion of weighted stable curves to the case of weighted stable maps, study their moduli spaces, and introduce Gromov-Witten invariants based on weighted stable maps.

When constructing moduli spaces via geometric invariant theory, it is well understood how different choices of a stability condition (and thus different polarizations in the GIT setting) yield birationally different compactifications of the moduli space (see e.g. [Tha96]). As discussed in the previous chapter, this should extend to the case of derived categories in the setting of Bridgeland's stability conditions. Hassett's study of the moduli spaces of weighted stable curves [Has03] yields a non-linear example of the same phenomenon. By introducing weights to the markings, it yields different compactifications of the Deligne-Mumford moduli space of curves with marked points. We generalize his approach to the case of weighted stable maps, and study the birational behaviour under weight changes in detail: the moduli spaces are constant in chambers of a finite chamber decomposition of the space of weights, and change birationally via blowups when crossing a wall of the chamber decomposition.

A further motivation of this study is the work by Losev and Manin on painted stable curves [LM00, LM04, Man04], which are a special case of weighted stable curves. They introduced the notion of an \mathcal{L} -algebra as an extension of the notion of a cohomological field theory of [KM94]. By constructing virtual fundamental classes, we introduce Gromov-Witten invariants based on weighted stable maps. Including gravitational descendants, we obtain an \mathcal{L} -algebra in the sense of [LM04].

1.1. Plan. In section 2, we define the precise moduli problem and construct its moduli space as a proper Deligne-Mumford stack. We show the existence of birational contraction morphism for any reduction of the weights; in particular, all moduli spaces of weighted stable maps are birational contractions of the Kontsevich moduli space.

We establish the existence of all basic morphisms (gluing, changing the target, forgetting markings etc.) between them in section 3. Section 4 exhibits the reduction morphisms as explicit blow-ups, and describes the chamber decompositions of the set of admissible weights.

In section 5, we postulate a list of basic properties for virtual fundamental classes, and discuss consequences for the weighted Gromov-Witten invariants. After introducing the language of weighted graphs in section 6, we prove a more complete graph-level list of properties of the virtual fundamental classes in section 7.

2. Geometry of moduli spaces of weighted stable maps

2.1. The moduli problem. Let k be a field of any characteristic, V/k a projective variety, and $\beta \in \text{CH}^1(V)$ an effective one-dimensional class in the Chow ring. Let S be a finite set with weights $\mathcal{A}: S \rightarrow \mathbb{Q} \cap [0, 1]$, and let $g \geq 0$ be any genus.

Definition 2.1.1. A nodal curve of genus g over a scheme T/k is a proper, flat morphism $\pi: C \rightarrow T$ of finite type such that for every geometric point $\text{Spec } \eta$ of T , the fibre over $\text{Spec } \eta$ is a connected curve of genus g with only nodes as singularities.

Given $(g, S, \mathcal{A}, \beta)$ as above, a prestable map of type (g, \mathcal{A}, β) over T is a tuple (C, π, s, f) where $\pi: C \rightarrow T$ is a nodal curve of genus g , $s = (s_i)_{i \in S}$ is an S -tuple of sections $s_i: T \rightarrow C$, and f is a map $f: C \rightarrow V$ with $f_*([C]) = \beta$, such that

- (1) the image of any section s_i with positive weight $\mathcal{A}(i) > 0$ lies in the smooth locus of C/T ,
- (2) for any subset $I \subset S$ such that the intersection $\bigcap_{i \in I} s_i(T)$ of the corresponding sections is non-empty, we have $\sum_{i \in I} \mathcal{A}(i) \leq 1$.

Definition 2.1.2. A stable map of type (g, \mathcal{A}, β) over T is a prestable map (C, π, s, f) of the same type such that $K_\pi + \sum_{i \in S} \mathcal{A}(i)s_i + 3f^*(M)$ is π -relatively ample for some ample divisor M on V .

We will often call such a curve (g, \mathcal{A}) -stable when the homology class β is irrelevant.

Remark 2.1.3. Assume that (C, π, s, f) is a (g, \mathcal{A}) -prestability map over T . Then it is (g, \mathcal{A}) -stable if and only if it is (g, \mathcal{A}) -stable over geometric points of T .

Over an algebraically closed field, ampleness of $K_\pi + \sum_{i \in S} \mathcal{A}(i)s_i + 3f^*(M)$ can only fail on irreducible components C that are of genus 0 and get mapped to a point by f . Precisely, if n_C is the number of nodal points (counted with multiplicity), then ampleness is equivalent to $n_C + \sum_{i: s_i \in C} \mathcal{A}(i) > 2$.

In particular, stability can be checked with an arbitrary ample divisor M ; if all sections have weight 1 (we will write this as $\mathcal{A} = \underline{1}_S$), weighted stability agrees with the definition of a stable map by Kontsevich.

We consider the data g, S, \mathcal{A}, β *admissible*, if $\beta \neq 0$ or $2g - 2 + \sum_{i \in S} \mathcal{A}(i) > 0$, and if β is bounded by the characteristic (cf. [BM96, Theorem 3.14]: this means that k has characteristic zero, or that $\beta \cdot L < \text{char } k$ for some very ample line bundle L on V).

Theorem 2.1.4. Given admissible data g, S, \mathcal{A}, β , let $\overline{M}_{g, \mathcal{A}}(V, \beta)$ be the category of stable maps of type (g, \mathcal{A}, β) and their isomorphisms, with the standard structure as a groupoid over schemes over $\text{Spec } k$.

This category is a proper algebraic Deligne-Mumford stack of finite type.

2.2. Reduction morphisms for weight changes. If $\beta \neq 0$, consider the configuration space $C_{g,S}(V, \beta)$: the open substack of $\overline{M}_{g,\mathcal{A}}(V, \beta)$ of maps that do not contract any irreducible component, and for which all markings are distinct. Obviously, $C_{g,S}(V, \beta)$ does not depend on \mathcal{A} , every $\overline{M}_{g,\mathcal{A}}(V, \beta)$ is a compactification of $C_{g,S}(V, \beta)$, and thus all the moduli stacks for different \mathcal{A} are birational. The following proposition gives actual morphisms, provided that the weights are comparable. They will be analyzed in more detail in section 4.

Consider two weights $\mathcal{A}, \mathcal{B}: S \rightarrow \mathbb{Q} \cap [0, 1]$ such that $\mathcal{A}(i) \geq \mathcal{B}(i)$ for all $i \in S$; we will just write $\mathcal{A} \geq \mathcal{B}$ from now on. Any (g, \mathcal{A}) -stable map is obviously (g, \mathcal{B}) -prestable, but it may not be (g, \mathcal{B}) -stable. However, we can stabilize the curve with respect to \mathcal{B} :

Proposition 2.2.1. *If $g, S, \beta, \mathcal{A} \geq \mathcal{B}$ are as above, there is a natural reduction morphism*

$$\rho_{\mathcal{B},\mathcal{A}}: \overline{M}_{g,\mathcal{A}}(V, \beta) \rightarrow \overline{M}_{g,\mathcal{B}}(V, \beta).$$

It is surjective and birational.¹ Over an algebraically closed field η , it is given by adjusting the weights and then successively contracting all (g, \mathcal{B}) -unstable components.

Given three weight data $\mathcal{A} \geq \mathcal{B} \geq \mathcal{C}$, the reduction morphisms respect composition: $\rho_{\mathcal{C},\mathcal{A}} = \rho_{\mathcal{C},\mathcal{B}} \circ \rho_{\mathcal{B},\mathcal{A}}$.

In particular, every moduli space $\overline{M}_{g,\mathcal{A}}(V, \beta)$ is a birational contraction of the Kontsevich moduli space $\overline{M}_{g,S}(V, \beta) = \overline{M}_{g,1_S}(V, \beta)$.

2.3. Proofs of the geometric properties. As in the case of (g, \mathcal{A}) -stable curves, the following vanishing result is essential to ensure that all constructions are compatible with base change:

Proposition 2.3.1. [Has03, Proposition 3.3] *Let C be a connected nodal curve of genus g over an algebraically closed field, D an effective divisor supported in the smooth locus of C , and L an invertible sheaf with $L \cong \omega_C^k(D)$ for $k > 0$.*

1. *If L is nef, and $L \neq \omega_C$, then L has vanishing higher cohomology.*
2. *If L is nef and has positive degree, then L^N is basepoint free for $N \geq 2$.*
3. *If L is ample, then L^N is very ample when $N \geq 3$.*
4. *Assume L is nef and has positive degree, and let C' denote the image of C under L^N with $N \geq 3$. Then C' is a nodal curve with the same arithmetic genus as C , obtained by collapsing the irreducible components of C on which L has degree zero. Components on which L has positive degree are mapped birationally onto their images.*

¹an isomorphism over a scheme-theoretically dense subset

2.3.2. *Stability and geometric points.* We will first show how remark 2.1.3 follows from this proposition: Consider the line bundle

$$L = \omega_C^k (k \sum_{i \in S} \mathcal{A}(i) s_i) \otimes f^*(\mathcal{O}(M))^{3k},$$

where k is such that all numbers $k\mathcal{A}(i)$ are integer. Then by the proposition and the base change theorems, formation of $P := \text{Proj}(\pi_*(L^N))$ commutes with base change. By definition, L is relatively ample iff the induced morphism $p: C \rightarrow P$ is defined everywhere and an open immersion. By [SGA1, exposé I, Théorème 5.1], this is the case if and only if p is everywhere defined, radical, flat and unramified. All these conditions can be checked on geometric fibers (for flatness, this follows from [EGA, IV, Théorème 11.3.10], for unramifiedness from the conormal sequence).

2.3.3. *Reduction morphisms.* By Grothendieck's descent theory, $\overline{M}_{g,\mathcal{A}}(V, \beta)$ is a stack in the étale topology, i. e. the Isom functors are sheaves and any étale descent datum is effective. We first show the existence of the natural reduction morphisms $\rho_{\mathcal{B},\mathcal{A}}$ as maps between these abstract stacks. This will enable us to use the results of [BM96] on $\overline{M}_{g,S}(V, \beta)$ to shorten our proofs.

Using the vanishing result 2.3.1, the proof of proposition 2.2.1 is completely analogous to that of theorem 4.1 in [Has03]: Let $\mathcal{B}_\lambda = \lambda\mathcal{A} + (1 - \lambda)\mathcal{B}$, and let $1 = \lambda_0 > \lambda_1 > \dots > \lambda_N = 0$ be a finite set such that for all $\lambda \notin \{\lambda_0, \dots, \lambda_N\}$, the following condition holds: *There is no subset $I \subset S$ such that $\sum_{i \in I} \mathcal{B}_\lambda(i) = 1$ and $\sum_{i \in I} \mathcal{B}_1(i) \neq 1$.* (*)

We will construct $\rho_{\mathcal{B},\mathcal{A}}$ as the composition $\rho_{\mathcal{B},\mathcal{A}} = \rho_{\mathcal{B}(\lambda_N), \mathcal{B}(\lambda_{N-1})} \circ \dots \circ \rho_{\mathcal{B}(\lambda_1), \mathcal{B}(\lambda_0)}$. This means we can assume that the condition (*) holds for all $0 < \lambda < 1$.

Fix an ample divisor M on V , and fix a natural number k so that $k\mathcal{B}(i)$ is an integer for all i . Let L be the invertible sheaf $L := \omega_C^k (k \sum_{i \in S} \mathcal{B}(i) s_i) \otimes f^*(M)^3$ for any (g, \mathcal{A}) -stable map $f: C \rightarrow V$ over T . Due to condition (*), it is nef; also it has positive degree. Let C' be the image of C under the map induced by L^N for some $N \geq 3$, i.e. $C' = \text{Proj } \mathcal{R}$ where \mathcal{R} is the graded sheaf of rings on T given by $\mathcal{R}_l = \pi_*((L^N)^l)$. Let $t: C \rightarrow C'$ be the natural map, and let $s'_i = t \circ s_i$. By the same arguments as in the non-weighted case, C' is a nodal curve of genus g , and s'_i lie in the smooth locus whenever $\mathcal{B}(i) > 0$. By proposition 2.3.1, L has vanishing higher cohomology; so the formation of $\pi_*((L^N)^l)$ and hence that of C' commutes with base change. Over an algebraically closed field, this morphism agrees with the description via contraction of unstable components. In particular, C' is (b, \mathcal{B}) -prestable.

The original f factors via the induced morphism $f': C' \rightarrow V$. Let L' be the line bundle $L' := \omega_{C'}^k (k \sum_{i \in S} \mathcal{B}(i) s_i) \otimes f'^*(M)^3$. Then $t_*L = L'$; hence L' is ample and (C', π', s', f') is a (g, \mathcal{B}) -stable map. The induced morphism $T \rightarrow \overline{M}_{g,\mathcal{B}}(V, \beta)$ commutes with base change and thus yields the map $\rho_{\mathcal{B},\mathcal{A}}$ between stacks as claimed.

To prove surjectivity, it is sufficient to show that every (g, \mathcal{B}) -stable map (C, s, f) over an algebraically closed field K is the image of some (g, \mathcal{A}) -stable map (C', s', f') over K . It is obvious how to construct C' : If $I \subset S$ is a subset of the markings such that condition (2) of definition 2.1.1 is violated for the weighting \mathcal{A} , i.e. the marked

points $s_i, i \in I$ coincide and $\sum_{i \in I} \mathcal{A}(i) > 1$, we can attach a copy of $\mathbb{P}^1(K)$ at this point, move the marked points to arbitrary but different points on \mathbb{P}^1 , and extend the map constantly along \mathbb{P}^1 .

Birationality (for $\beta \neq 0$) follows from the fact that $\rho_{\mathcal{B}, \mathcal{A}}$ is an isomorphism over the configuration space $C_{g, S}(V, \beta)$, which is a dense and open subset. The compatibility with composition follows immediately once we have shown the the moduli spaces are separate: the two morphisms $\rho_{\mathcal{C}, \mathcal{A}}$ and $\rho_{\mathcal{C}, \mathcal{B}} \circ \rho_{\mathcal{B}, \mathcal{A}}$ agree on the configuration space.

Proposition 2.3.4. *The diagonal $\Delta: \overline{M}_{g, \mathcal{A}}(V, \beta) \rightarrow \overline{M}_{g, \mathcal{A}}(V, \beta) \times \overline{M}_{g, \mathcal{A}}(V, \beta)$ is representable, separated and finite.*

Let (C_1, π_1, s_1, f_1) and (C_2, π_2, s_2, f_2) be two families of (g, \mathcal{A}) -stable maps to V over a scheme T . We have to show that $\text{Isom}((C_1, \pi_1, s_1, f_1), (C_2, \pi_2, s_2, f_2))$ is represented by a scheme finite and separated over T . Since V is projective and β is bounded by the characteristic, we can use exactly the same argument as in the proof of [BM96, Lemma 4.2]: one shows that étale locally on T , one can extend the set of markings to $S \cup S'$ and find additional S' -tuples of sections $(s_1)'$ and $(s_2)'$, such that $(C_1, \pi_1, s_1 \cup s_1')$ and $(C_2, \pi_2, s_2 \cup s_2')$ are $(g, \mathcal{A} \cup \underline{1}_{S'})$ -stable curves, and that there is a natural closed immersion

$$\text{Isom}((C_1, \pi_1, s_1, f_1), (C_2, \pi_2, s_2, f_2)) \rightarrow \text{Isom}((C_1, \pi_1, (s_1, s_1')), (C_2, \pi_2, (s_2, s_2'))).$$

Since $\overline{M}_{g, \mathcal{A} \cup \underline{1}_{S'}}$ has a representable, separated and finite diagonal by [Has03], the claim of the proposition follows.

2.3.5. Existence as Deligne-Mumford stacks. In particular, the diagonal is proper and thus the moduli stack separated. As $\overline{M}_{g, \underline{1}_S}(V, \beta)$ is proper and the reduction morphism $\rho_{\mathcal{A}, \underline{1}_S}: \overline{M}_{g, S}(V, \beta) \rightarrow \overline{M}_{g, \mathcal{A}}(V, \beta)$ is surjective, $\overline{M}_{g, \mathcal{A}}(V, \beta)$ is also proper.

Finally, the existence of a flat covering of finite type follows with almost the same argument as the one in [BM96], following Proposition 4.7 there. However, some changes are required, so we spell it out in detail: We write $\mathcal{A}_n = \mathcal{A} \cup \underline{1}_{\{1, \dots, n\}}$ for the weight data obtained from \mathcal{A} by adding n weights of 1. Let $\overline{M}_{g, \mathcal{A}_n}^o(V, \beta)$ be the open substack of $\overline{M}_{g, \mathcal{A}_n}(V, \beta)$ where the additional sections of weight one lie in the smooth locus of $C_{g, \mathcal{A}}(V, \beta)$ and away from the existing sections (in other words, the open substack where the map is already (g, \mathcal{A}) -stable after forgetting the additional sections). The obvious forgetful map

$$\phi_{\mathcal{A}, \mathcal{A}_n}^o: \overline{M}_{g, \mathcal{A}_n}^o(V, \beta) \rightarrow \overline{M}_{g, \mathcal{A}}(V, \beta)$$

is smooth and in particular flat. Let $U_{g, \mathcal{A}_n}^o(V, \beta)$ be the open substack of $\overline{M}_{g, \mathcal{A}_n}^o(V, \beta)$ where the curve is already (g, \mathcal{A}_n) -stable as a curve. Then for high enough n , the restriction $\phi_{\mathcal{A}, \mathcal{A}_n}^o|_{U_{g, \mathcal{A}_n}^o(V, \beta)}$ to this substack is surjective. On the other hand, $U_{g, \mathcal{A}_n}^o(V, \beta)$ is an open substack of the relative morphism space $\text{Mor}_{\overline{M}_{g, \mathcal{A}_n}}(V, \beta)$ (parametrizing maps $T \rightarrow \overline{M}_{g, \mathcal{A}_n}$ together with a map of the pull-back of the universal curve C_{g, \mathcal{A}_n} to V). So a flat presentation of the morphism space induces one for $\overline{M}_{g, \mathcal{A}}(V, \beta)$.

3. Elementary morphisms

3.1. Gluing morphisms. As in the non-weighted case, we can glue curves at marked points, but to guarantee that the resulting curves is prestable, we have to assume that both markings have weight 1:

Let $g_1, S_1, \mathcal{A}_1, \beta_1$ and $g_2, S_2, \mathcal{A}_2, \beta_2$ be weight data, such that the extensions $g_i, S_i \cup \{0\}, \mathcal{A}_i \cup \{0 \mapsto 1\}, \beta_i$ by an additional marking of weight 1 are admissible. Denote by ev_0 be the evaluation morphisms $\text{ev}_0: \overline{M}_{g_i, \mathcal{A}_i \cup \{1\}}(V, \beta_i) \rightarrow V$ given by evaluating the additional section: $\text{ev}_0 = f \circ s_0$. Similarly, let g, S, \mathcal{A}, β be weight data such that $g, S \cup \{0, 1\}, \mathcal{A} \cup \{1, 1\}, \beta$ is admissible, and let ev_0, ev_1 be the additional evaluation morphisms.

Proposition 3.1.1. *There are natural gluing morphisms*

$$\left(\overline{M}_{g_1, \mathcal{A}_1 \cup \{1\}}(V, \beta_1) \times \overline{M}_{g_2, \mathcal{A}_2 \cup \{1\}}(V, \beta_2)\right) \times_{V \times V} V \rightarrow \overline{M}_{g_1+g_2, \mathcal{A}_1 \cup \mathcal{A}_2}(V, \beta_1 + \beta_2)$$

and

$$\overline{M}_{g, \mathcal{A} \cup \{1, 1\}}(V, \beta) \times_{V \times V} V \rightarrow \overline{M}_{g+1, \mathcal{A}}(V, \beta).$$

The product over $V \times V$ is taken via the morphism $(\text{ev}_0, \text{ev}_0)$ respectively $(\text{ev}_0, \text{ev}_1)$ on the left, and the diagonal $\Delta: V \rightarrow V \times V$ on the right.

There is nothing new to prove here, except to note that the weight of 1 guarantees that the markings (of positive weight) do not meet the additional node on the glued curve.

Proposition 3.2. *Let $\mu: V \rightarrow W$ be a morphism, and $(g, S, \mathcal{A}, \beta)$ be admissible data for V , such that $(g, S, \mathcal{A}, \mu_*(\beta))$ is also admissible. Then there is a natural push-forward*

$$\overline{M}_{g, \mathcal{A}}(V, \beta) \rightarrow \overline{M}_{g, \mathcal{A}}(W, \mu_*(\beta))$$

that is obtained by composing the maps with μ , followed by stabilization.

One could adapt the proof of [BM96] to the weighted case; instead, we give a proof analogous to the one in section 2.3.3.

Let $f: C \rightarrow V$ be the universal map over $\overline{M}_{g, \mathcal{A}}(V, \beta)$, let $f' = \mu \circ f$ be the induced map to W , and let M' be an ample divisor on V' . By the assumptions, the divisor $D' = K_\pi + \sum_{i \in S} \mathcal{A}(i)s_i + 3f'^*M'$ has positive degree; however, it need not be nef. Hence we consider $D = K_\pi + \sum_{i \in S} \mathcal{A}(i)s_i + 3f^*M$ and $D(\lambda) = \lambda D + (1 - \lambda)D'$ for $0 \leq \lambda \leq 1$. Let $\{\lambda_1, \dots, \lambda_N\}$ be the set of λ for which the degree of $D(\lambda)$ is zero on any irreducible component of C , and let $k_r, r = 1 \dots N$ be an integer such that $k_r \lambda_r$ and $k_r \mathcal{A}(i), i \in S$ is integer.

Then $L_1 = \omega^{k_1}(k_1 \sum_{i \in S} \mathcal{A}(i)s_i + k_1(3f^*M\lambda_1 + (1 - \lambda_1)3f'^*M'))$ is a nef invertible sheaf on C for which proposition 2.3.1 applies. Hence C_1 defined by $C_1 := \text{Proj } \mathcal{R}_1$ and $(\mathcal{R}_1)_l = \pi_*(L_1^{3l})$ is again a flat nodal curve of genus g , contracting all components of C on which L_1 fails to be ample, and f' factors via a unique morphism $f_1: C_1 \rightarrow W$. We proceed inductively to obtain $f_N: C_N \rightarrow W$ on which D' is ample; this induces

the map of moduli stacks. Note that $C \rightarrow C_N \rightarrow W$ is the universal factorization of f' such that $f_N: C_N \rightarrow W$ is a (g, \mathcal{A}) -stable map.

Proposition 3.3. *Given admissible weight data $(g, S, \mathcal{A}, \beta)$, let $(g, S \cup \{*\}, \mathcal{A} \cup \{a\} = \mathcal{A} \coprod \{*\} \mapsto a, \beta)$ be the weight data obtained by adding a marking $\{*\}$ of arbitrary weight $a \in \mathbb{Q} \cap [0, 1]$. There is a natural forgetful map*

$$\phi_{\mathcal{A}, \mathcal{A} \cup \{a\}}: \overline{M}_{g, \mathcal{A} \cup \{a\}}(V, \beta) \rightarrow \overline{M}_{g, \mathcal{A}}(V, \beta)$$

obtained by forgetting the additional marking and stabilization. If $a = 0$, then $\phi_{\mathcal{A}, \mathcal{A} \cup \{0\}}$ is the universal curve over $\overline{M}_{g, \mathcal{A}}(V, \beta)$.

We can construct this map as the composition

$$\phi_{\mathcal{A}, \mathcal{A} \cup \{0\}} \circ \rho_{\mathcal{A} \cup \{0\}, \mathcal{A} \cup \{a\}}: \overline{M}_{g, \mathcal{A} \cup \{a\}}(V, \beta) \rightarrow \overline{M}_{g, \mathcal{A} \cup \{0\}}(V, \beta) \rightarrow \overline{M}_{g, \mathcal{A}}(V, \beta).$$

The second morphism $\phi_{\mathcal{A}, \mathcal{A} \cup \{0\}}$ is the naive forgetful morphism, as a map is $(g, \mathcal{A} \cup \{0\})$ -stable if and only if it is (g, \mathcal{A}) -stable.

Proposition 3.4. *Let $S' \coprod S'' = S$ be a partition of the markings such that $\mathcal{A}(S'') = \sum_{i \in S''} \mathcal{A}(i) \leq 1$. Then there is a natural map*

$$\overline{M}_{g, \mathcal{A}|_{S' \cup \{\mathcal{A}(S'')\}}}(V, \beta) \rightarrow \overline{M}_{g, \mathcal{A}}(V, \beta).$$

It is given by setting $s_i = s_*$ for all $i \in S'$. It identifies $\overline{M}_{g, \mathcal{A}|_{S' \cup \{\mathcal{A}(S'')\}}}(V, \beta)$ with the locus of $\overline{M}_{g, \mathcal{A}}(V, \beta)$ where all $s_i, i \in S''$ agree.

3.5. Weighted marked graphs. A graph was defined in [BM96] as a quadruple $\tau = (V_\tau, F_\tau, \partial_\tau, j_\tau)$ of a set of vertices V_τ , a set of flags F_τ , a morphism $\partial_\tau: F_\tau \rightarrow V_\tau$ and an involution $j_\tau: F_\tau \rightarrow F_\tau$. We think of a graph in terms of its geometric realization: it is obtained by identifying in the disjoint union $\coprod_{f \in F_\tau} [0, 1]$ the points 0 for all flags f attached to the same vertex via $v = \partial_\tau(f)$, and the points 1 for all orbits of j_τ . A flag f with $j_\tau(f) = f$ is called a *tail* of the vertex $\partial_\tau(f)$, whereas a pair $\{f, j_\tau(f)\}$ for $f \neq j_\tau(f)$ is called an *edge*, connecting the (not necessarily distinct) vertices $\partial_\tau(f)$ and $\partial_\tau(j_\tau(f))$.

Given a projective variety V , a weighted modular V -graph is a graph τ together with a genus $g: V_\tau \rightarrow \mathbb{Z}_{\geq 0}$, a weighting $\mathcal{A}: F_\tau \rightarrow \mathbb{Q} \cap [0, 1]$ such that $\mathcal{A}(f) = 1$ for all flags that are part of an edge, and a marking $\beta: V_\tau \rightarrow H_2^+(V)$. To any weighted stable map we can associate its dual graph: a vertex for every irreducible component, an edge for every node, and a tail for every marking. Conversely, to every weighted modular graph we can associate the moduli space of tuples of weighted stable maps $f_v: C_v \rightarrow V$ of type $(g(v), S_v = \{f \in F_\tau: \partial(f) = v\}, \mathcal{A}|_{S_v}, \beta(v))$, such that for every edge $\{f, f' = j_\tau(f)\}$ connecting the vertices $v = \partial_\tau(f)$ and $v' = \partial_\tau(f')$, the corresponding evaluation morphisms are identical: $f_v \circ s_f = f_{v'} \circ s_{f'}$. Via gluing, this gives a single weighted stable map $f: C \rightarrow V$; if all C_v are smooth, its dual graph will give back τ .

The moduli space $\overline{M}_{g,\mathcal{A}}(V, \beta)$ corresponds to the one-vertex graphs with the set S of tails. The morphisms constructed in this section correspond to elementary morphisms between graphs with one and two vertices. Extending this set of morphisms to higher codimension boundary strata, indexed by graphs with more vertices, naturally leads to a category of weighted stable marked graphs. We will adopt this viewpoint in section 6, and show that we get a functor \overline{M} from the graph category to Deligne-Mumford stacks over k .

4. Birational behaviour under weight changes

For this section, we will fix g, S, β , and analyze more systematically the reduction morphisms $\rho_{\mathcal{A},\mathcal{B}}$ of proposition 2.2.1 for varying weight data \mathcal{A}, \mathcal{B} .

4.1. Exceptional locus and reduction morphism as blow-up.

Proposition 4.1.1. [Has03, Proposition 4.5] *Assume we have weight data $\mathcal{A} \geq \mathcal{B} > 0$. The reduction morphism $\rho_{\mathcal{B},\mathcal{A}}$ contracts the boundary divisors $D_{I,J}$ given as the image of the gluing morphism*

$$\overline{M}_{0,\mathcal{A}|_I \cup \{1\}}(V, 0) \times_V \overline{M}_{g,\mathcal{A}|_J \cup \{1\}}(V, \beta) \rightarrow \overline{M}_{g,\mathcal{A}}(V, \beta)$$

for all partitions $I \coprod J = S$ of S with

$$\sum_{i \in I} \mathcal{A}(i) > 1 \quad \text{and} \quad b_I := \sum_{i \in I} \mathcal{B}(i) \leq 1.$$

There is a factorization of $\rho_{\mathcal{B},\mathcal{A}}|_{D_{I,J}}$ via

$$\overline{M}_{0,\mathcal{A}|_I \cup \{1\}}(V, 0) \times_V \overline{M}_{g,\mathcal{A}|_J \cup \{1\}}(V, \beta) \rightarrow \overline{M}_{g,\mathcal{A}|_J \cup \{1\}}(V, \beta) \rightarrow \overline{M}_{g,\mathcal{A}|_J \cup \{b_I\}}(V, \beta).$$

We may assume that there is just one such I and that $b_I = 1$. The stabilization contracts components on which $\omega_C^k(k \sum_{i \in S} \mathcal{B}(i)s_i) \otimes f^*(M)^3$ has degree zero. Such a component can only be a genus zero irreducible component mapped to a point that has a single node and markings given exactly by s_i for $i \in I$.

In particular, the exceptional locus of $\rho_{\mathcal{B},\mathcal{A}}$ is given by all $D_{I,J}$ for partitions $I \cap J = S$ as above with the additional condition $|I| > 2$. When all sets $I \subset S$ such that $\sum_{i \in I} \mathcal{A}(i) > 1$ and $\sum_{i \in I} \mathcal{B}(i) \leq 1$ satisfy $|I| = 2$, then $\rho_{\mathcal{B},\mathcal{A}}$ is an isomorphism.

Remark 4.1.2. *Assume that for $\mathcal{A} > \mathcal{B} > 0$, there is exactly one partition $I \coprod J = S$ of S as in the proposition. Then $\rho_{\mathcal{B},\mathcal{A}}$ is the blowup of $\overline{M}_{g,\mathcal{B}}(V, \beta)$ along the locus $C_{IJ} \cong \overline{M}_{g,\mathcal{B}|_J \cup \{b_I\}}(V, \beta)$ of weighted stable curves where all section s_i for $i \in I$ are identical.*

The divisor $D_{I,J}$ is the scheme-theoretic inverse image of C_{IJ} . By deformation theory of singular curves, $D_{I,J}$ is a cartier divisor. By the universal property of blow-ups, this shows that $\rho_{\mathcal{B},\mathcal{A}}$ factors via the blow-up $\rho' : M \rightarrow \overline{M}_{g,\mathcal{B}}(V, \beta)$ of C_{IJ} .

For the converse, let C' be the pull-back of the universal curve along ρ' , let E be the exceptional divisor of ρ' , and write $\rho'^{-1}s_i : M \rightarrow C'$ for the pull-back of the sections s_i

over $\overline{M}_{g,\mathcal{B}}(V, \beta)$. Let C_0 be the common image $(\rho'^{-1}s_i)(E)$ of the exceptional divisor for any $i \in I$, and let C be blow-up of C' at C_0 . The center $C_0 \subset C'$ is a codimension two regular embedding, and embeds as a Cartier divisor in both $(\rho'^{-1}s_i)(M)$ for any $i \in I$, and in the restriction of C' to E . Thus the fibers of C over E are obtained from that of the universal curve over $\overline{M}_{g,\mathcal{B}}(V, \beta)$ by attaching a projective line at the marked point given by any s_i for $i \in I$, and every section $\rho'^{-1}s_i$ lifts to a section $s_i: C \rightarrow M$.

Over E , the image is contained in the attached projective line, away from the node, as $s_i(M)$ and the fibre over E meet transversely in C' . Also, since the images of $s_i, i \in I$ intersect transversely in the universal curve over $\overline{M}_{g,\mathcal{B}}(V, \beta)$, any tangent vector at a point of C_0 tangent to all the images of $(\rho'^{-1}s_i)(M), i \in I$ is already tangent to C_0 ; thus the sections $s_i: M \rightarrow C$ cannot all be mapped to the same point of the projective line.

Hence, with the induced map to V , we have constructed a (g, \mathcal{A}) -stable map, and so a map $M \rightarrow \overline{M}_{g,\mathcal{A}}(V, \beta)$; it is an inverse to the map in the opposite direction constructed above, as this is true over $C_{g,S}(V, \beta)$ and both stacks are separated.

Proposition 4.1.3. *Let \mathcal{A}, \mathcal{B} as in proposition 4.1.1, except we allow some weights of \mathcal{B} to be zero. Let $i \in S$ be a marking with $\mathcal{A}(i) > \mathcal{B}(i) = 0$. Then $\rho_{\mathcal{A},\mathcal{B}}$ additionally contracts the boundary components $C_{(g_1,0,g_2),(I_1,I_0,I_2),(\beta_1,0,\beta_2)}$ which are defined as the image of the gluing morphisms*

$$\begin{aligned} \overline{M}_{g_1,\mathcal{A}|_{I_1 \cup \{1\}}}(V, \beta_1) \times_V \overline{M}_{0,\mathcal{A}|_{I_0 \cup \{i\}} \cup \{1,1\}}(V, 0) \times_V \overline{M}_{g_2,\mathcal{A}|_{I_2 \cup \{1\}}}(V, \beta_1) \\ \rightarrow \overline{M}_{g,\mathcal{A}}(V, \beta) \end{aligned}$$

for all $g_1 + g_2 = g, \beta_1 + \beta_2 = \beta$ and disjoint partitions $I_1 \cup I_0 \cup \{i\} \cup I_2 = S$ such that $\mathcal{A}(j) = 0$ for $j \in I_0$.

The restriction $\rho_{\mathcal{B},\mathcal{A}}$ factors via the projection of the second component to a point.

In other words, this is the boundary component of singular curves such that the section s_i is contained in a node after stabilization.

4.2. Chamber decomposition. We now assume $\beta \neq 0$, and consider the set of positive weights $\mathcal{D}_n = (0, 1]^S \subset \mathbb{R}^S$. The walls \mathcal{W}_c and \mathcal{W}_f of the coarse and fine chamber decomposition, respectively, are given by²

$$\mathcal{W}_c = \left\{ \sum_{i \in I} \mathcal{A}(i) = 1 \mid I \subset S, 2 < |I| \right\}$$

$$\mathcal{W}_f = \left\{ \sum_{i \in I} \mathcal{A}(i) = 1 \mid I \subset S, 2 \leq |I| \right\}.$$

Coarse and fine chambers are connected component of the complements $\mathcal{D}_n \setminus \mathcal{W}_c$ and $\mathcal{D}_n \setminus \mathcal{W}_f$, respectively.

²The conditions $|S| < n - 2$ and $|S| \leq n - 2$ for the coarse and fine chamber decompositions, respectively, in [Has03, section 5] are correct only when $g = 0$ and don't apply in our case as we assumed $\beta \neq 0$.

Proposition 4.2.1. (cf. [Has03, Proposition 5.1]) *The coarse chamber decomposition is the coarsest decomposition such that $\overline{M}_{g,\mathcal{A}}(V, \beta)$ is constant in each chamber. The fine chamber decomposition is the coarsest decomposition such that the universal curve $\mathcal{C}_{g,\mathcal{A}}(V, \beta)$ is constant in each chamber.*

Corollary 4.2.2. *Let \mathcal{A} be positive weight data in the interior of a fine open chamber. Then for small $\epsilon > 0$, the forgetful morphism $\phi_{\mathcal{A},\mathcal{A}\cup\{\epsilon\}}$ identifies $\overline{M}_{g,\mathcal{A}\cup\{\epsilon\}}(V, \beta)$ with the universal curve $\mathcal{C}_{g,\mathcal{A}}(V, \beta) \rightarrow \overline{M}_{g,\mathcal{A}}(V, \beta)$.*

This holds by definition for $\epsilon = 0$, and it follows easily from the earlier propositions that $\rho_{\mathcal{A}\cup\{0\},\mathcal{A}\cup\{\epsilon\}}$ is an isomorphism.

5. Virtual fundamental classes and Gromov-Witten invariants

5.1. Expected properties. The crucial step in the construction of Gromov-Witten invariants is the construction of virtual fundamental classes of expected dimension:

$$[\overline{M}_{g,\mathcal{A}}(V, \beta)]^{\text{virt}} \in A_{(1-g)(\dim V - 3) - K_V \cdot \beta + |S|} \overline{M}_{g,\mathcal{A}}(V, \beta)$$

We will provide now a basic list of properties that such a construction should satisfy, and proceed to draw some conclusions about Gromov-Witten invariants in the remainder of the section.

(1) *Mapping to a point.* If $\beta = 0$, then

$$[\overline{M}_{g,\mathcal{A}}(V, 0)]^{\text{virt}} = c_{g \dim V}(R^1 \pi_* f^* TV)$$

(2) *Forgetting a tail.* Assume \mathcal{A} and ϵ are as in corollary 4.2.2, so that $\phi_{\mathcal{A},\mathcal{A}\cup\{\epsilon\}}$ is the universal curve over $\overline{M}_{g,\mathcal{A}}(V, \beta)$. In particular, this implies that $\phi_{\mathcal{A},\mathcal{A}\cup\{\epsilon\}}$ is flat, and thus defines a pull-back in intersection theory. We require

$$\phi_{\mathcal{A},\mathcal{A}\cup\{\epsilon\}}(V, \beta)^* [\overline{M}_{g,\mathcal{A}}(V, \beta)]^{\text{virt}} = [\overline{M}_{g,\mathcal{A}\cup\{\epsilon\}}(V, \beta)]^{\text{virt}}.$$

(3) *Combining tails.* Assume we are in the situation of proposition 3.4. Since all sections lie in the smooth locus of the curve, $\mu_{S/S'}$ is a regular embedding, and we require that

$$\mu_{S/S'}^! [\overline{M}_{g,\mathcal{A}}(V, \beta)]^{\text{virt}} = [\overline{M}_{g,\mathcal{A}|_{S' \cup \{\mathcal{A}(S'')\}}}(V, \beta)].$$

(4) *Gluing.* We fix $g_1, S_1, \mathcal{A}_1, g_2, S_2, \mathcal{A}_2$ and some $\beta \in H_2^+(V)$. Set $g = g_1 + g_2$ and $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$. Consider the gluing morphisms

$$\begin{aligned} \mu_{\beta_1, \beta_2} : \overline{M}_{g_1, \mathcal{A}_1 \cup \{1\}}(V, \beta_1) \times \overline{M}_{g_2, \mathcal{A}_2 \cup \{1\}}(V, \beta_2) \times_{V \times V} V \\ \rightarrow \overline{M}_{g, \mathcal{A}}(V, \beta) \end{aligned}$$

of proposition 3.1.1 for all β_1, β_2 with $\beta_1 + \beta_2 = \beta$. The union of their images is the boundary component in $\overline{M}_{g,\mathcal{A}}(V, \beta)$ given as the pull-back

$$\begin{array}{ccc} \overline{M}_{(g_1, \mathcal{A}_1)|(g_2, \mathcal{A}_2)}(V, \beta) & \longrightarrow & \overline{M}_{g, \mathcal{A}}(V, \beta) \\ \downarrow & & \downarrow \\ \overline{M}_{g_1, \mathcal{A}_1 \cup \{1\}} \times \overline{M}_{g_2, \mathcal{A}_2 \cup \{1\}} & \xrightarrow{\mu} & \overline{M}_{g, \mathcal{A}} \end{array}$$

Since the moduli spaces of weighted stable curves are smooth, μ is a l.c.i. morphism and defines a pull-back $\mu^! [\overline{M}_{g, \mathcal{A}}(V, \beta)]^{\text{virt}}$. On the other hand, via the diagonal $\Delta: V \rightarrow V \times V$, we can pull-back the virtual fundamental class on the product $\overline{M}_{g_1, \mathcal{A}_1 \cup \{1\}}(V, \beta_1) \times \overline{M}_{g_2, \mathcal{A}_2 \cup \{1\}}(V, \beta_2)$ to the fibre product that is the source of μ_{β_1, β_2} . We require

$$\begin{aligned} \sum_{\beta_1 + \beta_2 = \beta} \mu_{\beta_1, \beta_2} \Delta^! \left([\overline{M}_{g_1, \mathcal{A}_1 \cup \{1\}}(V, \beta_1)]^{\text{virt}} \times [\overline{M}_{g_2, \mathcal{A}_2 \cup \{1\}}(V, \beta_2)]^{\text{virt}} \right) \\ = \mu^! [\overline{M}_{g, \mathcal{A}}(V, \beta)]^{\text{virt}}. \end{aligned}$$

- (5) *Kontsevich-stable maps.* If all weights are 1, then $[\overline{M}_{g, \mathcal{A}}(V, \beta)]^{\text{virt}}$ agrees with the definition of virtual fundamental classes of [BF97, Beh97].
- (6) *Reducing weights.* Given two set of weights $\mathcal{A} > \mathcal{B}$, we require compatibility with the reduction morphism $\rho_{\mathcal{B}, \mathcal{A}}$:

$$\rho_{\mathcal{B}, \mathcal{A}*} [\overline{M}_{g, \mathcal{A}}(V, \beta)]^{\text{virt}} = [\overline{M}_{g, \mathcal{B}}(V, \beta)]^{\text{virt}}$$

Evidently, properties (1), (2) and (4) are direct generalizations of properties satisfied by the virtual fundamental classes of the non-weighted moduli spaces, while (3) and (6) are new.

Theorem 5.1.1. *There is a system of virtual fundamental classes satisfying all of the above properties.*

While the Behrend-Fantechi construction can be applied to our situation and provides virtual fundamental classes, we instead use (5) and (6) as a definition, and prove that these classes automatically satisfy the desired properties.

We postpone the proof of the above properties to section 7, after having generalized them to a bigger class of morphisms labelled by a category of weighted stable graphs. In the remainder of the section we will instead proceed to give some consequences of theorem 5.1.1.

5.2. Gromov-Witten invariants. As in the non-weighted case, one defines the n -point Gromov-Witten invariants of V depending on weights $\mathcal{A}: \{1, \dots, n\} \rightarrow [0, 1] \cap \mathbb{Q}$ via

$$\begin{aligned} \langle \rangle_{g, \mathcal{A}, \beta}: H^*(V)^{\otimes n} &\rightarrow \mathbb{C} \\ \langle \gamma_1 \otimes \dots \otimes \gamma_n \rangle_{g, \mathcal{A}, \beta} &= \int_{[\overline{M}_{g, \mathcal{A}}(V, \beta)]^{\text{virt}}} \text{ev}_1^*(\gamma_1) \cup \dots \cup \text{ev}_n^*(\gamma_n) \end{aligned}$$

and Gromov-Witten invariants including gravitational descendants via

$$\langle \tau_1^{k_1} \gamma_1 \cdots \tau_n^{k_n} \gamma_n \rangle_{g, \mathcal{A}, \beta} = \int_{[\overline{M}_{g, \mathcal{A}}(V, \beta)]^{\text{virt}}} \psi_1^{k_1} \text{ev}_1^*(\gamma_1) \cup \cdots \cup \psi_n^{k_n} \text{ev}_n^*(\gamma_n)$$

where ψ_i is the tautological class associated to the section s_i : $\psi_i = c_1(s_i^* \Omega_C)$ where Ω_C is the relative cotangent bundle of the universal curve C over $\overline{M}_{g, \mathcal{A}}(V, \beta)$.

Proposition 5.2.1. *Gromov-Witten invariants without gravitational descendants do not depend on the choice of weights \mathcal{A} .*

It is enough to prove this for two weights $\mathcal{A} > \mathcal{B}$. The evaluation morphisms $\text{ev}_i: \overline{M}_{g, \mathcal{A}}(V, \beta) \rightarrow V$ factor via the reduction morphism $\rho_{\mathcal{B}, \mathcal{A}}$. Hence the claim follows from property (6) and the projection formula.

5.3. Extended modular operad. Let $\mathcal{A}_{m, n}$ be the weight data consisting of m weights of one, and n weights of $\epsilon \leq \frac{1}{n}$. The moduli spaces $\overline{M}_{g, \mathcal{A}_{m, n}}$ were called $L_{g, m, n}$ in [LM04] and studied more closely in [Man04]. Markings with weight one and ϵ are white and black points in the language of [LM04], respectively: white points may not coincide with any other point, whereas any number of black points are allowed to coincide. Similarly, we write $L_{g, m, n}(V, \beta)$ for the moduli spaces of weighted stable maps $L_{g, m, n}(V, \beta) = \overline{M}_{g, \mathcal{A}_{m, n}}(V, \beta)$.

In [LM04], the notion of an \mathcal{L} -algebra was introduced by a combinatorial description. It is an extension of the graph-level description of the genus zero-part of a cohomological field theory in the sense of [KM94]. By the results of [Man04], the "economy class description" of [LM04, section 4.2.2] can be translated into the following geometric description:

Let $(T; F, (,))$ be a triple consisting of two \mathbb{Z}_2 -graded \mathbb{Q} -vector spaces T, F , where the latter is equipped with a (super)symmetric non-degenerate scalar product $(,)$. An \mathcal{L} -algebra on $(T; F, (,))$ over a \mathbb{Q} -algebra R can be given as a collection of maps

$$I_{0; m, n}: T^{\otimes n} \otimes F^{\otimes m} \rightarrow H_*(L_{0; m, n}) \otimes_{\mathbb{Q}} R$$

being compatible with gluing of black points and the trace on F .

We obtain the \mathcal{L} -algebra of quantum cohomology of V including gravitational descendants as follows: Let $F = H^*(V, \mathbb{Q})$, equipped with the Poincaré pairing, and let $T = \bigoplus_{k \geq 0} z^k F$. We denote by $\text{ev}_1^W, \dots, \text{ev}_m^W$ and $\text{ev}_1^B, \dots, \text{ev}_n^B$ the evaluation maps $L_{0; m, n}(V, \beta) \rightarrow V$ induced by the markings of weight one and ϵ , respectively, and by $\pi: L_{0; m, n}(V, \beta) \rightarrow L_{0; m, n}$ the forgetful map. Let $\psi_i, i = 1 \dots n$ be the tautological classes associated to the section s_i^B of weight ϵ . Let $\mathbb{Q}[[q]]$ be the Novikov ring of V , i.e. the formal completion of the polynomial ring over the semigroup of effective classes in $H_2(V)/\text{torsion}$.

Then we define $I_{0; m, n}$ as

$$I_{0; m, n} (z^{k_1} \gamma_1 \otimes \cdots \otimes z^{k_n} \gamma_n \otimes \delta_1 \otimes \cdots \otimes \delta_m)$$

$$= \sum_{\beta \in H_2^+(V)} q^\beta \mathcal{P} \left(\pi_* \left(\prod_{i=1}^n (\text{ev}_i^B)^* \gamma_i \psi_i^{k_i} \prod_{i=1}^n (\text{ev}_i^W)^* \delta_i \cap [L_{0,m,n}(V, \beta)]^{\text{virt}} \right) \right)$$

where $\pi: L_{0,m,n}(V, \beta) \rightarrow L_{0,m,n}$ is the forgetful map, and $\mathcal{P}(s) \in H^* L_{0,m,n}$ is the Poincaré dual of $s \in H_* L_{0,m,n}$.

Theorem 5.3.1. *The above definition of $I_{0,m,n}$ yields a cyclic \mathcal{L} -algebra (in the sense of the economy class description in [LM04, section 4.2.2]).*

The only thing to check is the compatibility with gluing, in the formal sense of [LM04, diagram (4.8)]. This holds due to property (4) of section 5.1.

5.4. Comments. In [LM04], it was shown that the datum of an \mathcal{L} -algebra is equivalent to a geometric structure, a solution of the so-called commutativity equation. However, the structure of an \mathcal{L} -algebra does not capture the complete structure we have available:

- (1) By property (6), the inclusion $F = z^0 F \subset T$ is compatible with the reduction morphisms $L_{0,m,n} \rightarrow L_{0,m-1,n+1}$ in the obvious sense.
- (2) Relating the gravitational descendants to intersection numbers in $L_{0,m,n}$ by an analysis analogous to the one in [KM98] will, of course, lead to many more relations among the correlators.

One might hope that these can be integrated in the geometric picture of [LM04].

As a side remark, it is worth pointing out that the tautological classes $\psi_i, i = 1 \dots n$ in $L_{0,m,n}(V, \beta)$ are compatible with pull-back along the forgetful morphism $L_{0,m,n+1}(V, \beta)$; this is not true in the non-weighted case.

6. Graph-language

6.1. Weighted marked graphs. The elementary morphism described in section 3 generate a larger system of morphisms. They are best modelled over a category of weighted marked graphs; this category generalizes the category of marked graphs introduced in [BM96] by introducing weights of tails. We follow [BM96, section 1] closely.

We recall from section 3.5 the definition of a graph:

Definition 6.1.1. [BM96, Definition 1.1] *A graph τ is a quadruple $(F_\tau, V_\tau, j_\tau, \partial_\tau)$ of a finite set V_τ of vertices, a finite set F_τ of flags, an involution $j_\tau: F_\tau \rightarrow F_\tau$ and a map $\partial_\tau: F_\tau \rightarrow V_\tau$. We call $S_\tau = \{f \in F_\tau \mid j_\tau f = f\}$ the set of tails, and $E_\tau = \{\{f, j_\tau f\} \mid f \in F_\tau \text{ and } j_\tau f \neq f\}$ the set of edges.*

Definition 6.1.2. *A weighted modular graph is a graph $\tau = (F_\tau, V_\tau, j_\tau, \partial_\tau)$ endowed with two maps $g_\tau: V_\tau \rightarrow \mathbb{Z}_{\geq 0}$ and $\mathcal{A}_\tau: F_\tau \rightarrow \mathbb{Q} \cap (0, 1]$ such that $\mathcal{A}_\tau(f) = 1$ for all flags f that are part of an edge, i.e. for which $j_\tau(f) \neq f$.*

The number $g_\tau(v)$ is called the genus of a vertex, and $\mathcal{A}_\tau(f)$ the weight of a flag.

Definition 6.1.3. *Given a semigroup A with indecomposable zero, a weighted A -graph (τ, α) is a weighted modular graph τ with a map $\alpha: V_\tau \rightarrow A$. A weighted marked graph is a pair $(A, (\tau, \alpha))$ where A is a semigroup with indecomposable zero, and (τ, α) is an A -graph.*

We will often omit α from the notation and call τ an A -graph.

Morphisms in the category of weighted marked graphs are generated by two different types, *combinatorial morphisms* and *contractions*. More precisely, since the associated geometric morphisms are contravariant with respect to the combinatorial morphisms, and covariant with respect to contractions, the morphisms will be generated by contractions and formal inverses of the combinatorial morphisms.

Only condition (2) of the definition of a combinatorial morphism of modular graphs ([BM96, Definition 1.7]) needs to be adopted to our situation:

Definition 6.1.4. *Let (σ, α) and (τ, β) be weighted A -graphs. A combinatorial morphism $a: (\sigma, \alpha) \rightarrow (\tau, \beta)$ is a pair of maps $a_F: F_\sigma \rightarrow F_\tau$ and $a_V: V_\sigma \rightarrow V_\tau$, satisfying the following conditions:*

- (1) *The morphisms commute with ∂ , i.e. we have $a_V \circ \partial_\sigma = \partial_\tau \circ a_F$. In particular, for any $v \in V_\sigma$ and $w = a_V(v) \in V_\tau$, we get an induced map $a_{V,v}: F_\sigma(v) \rightarrow F_\tau(w)$.*
- (2) *Consider the above map $a_{V,v}$. Then for any $f \in F_\tau(w)$, the inequality*

$$\sum_{f' \in F_\sigma(v): a_{V,v}(f')=f} \mathcal{A}_\sigma(f') \leq \mathcal{A}_\tau(f)$$

is satisfied.

- (3) *Let $\{f, \bar{f}\}$ be an edge of σ , i.e. $f \in F_\sigma$, $\bar{f} = j_\sigma(f) \neq f$. Then there exist $n \geq 1$ and n edges $\{f_1, \bar{f}_1\}, \dots, \{f_n, \bar{f}_n\}$ of τ such that $v_i := \partial_\tau(\bar{f}_i) = \partial_\tau(f_{i+1})$ and $\beta(v_i) = 0$ for all $1 \leq i < n$.*
- (4) *For every $v \in V_\sigma$ we have $\alpha(v) = \beta(a_V(v))$.*
- (5) *For every $v \in V_\sigma$ we have $g(v) = g(a_V(v))$.*

A combinatorial morphism of weighted marked graphs $(B, \sigma, \beta) \rightarrow (A, \tau, \alpha)$ is a pair (ξ, a) where $\xi: A \rightarrow B$ is a homomorphism of semigroups, and $a: (\sigma, \beta) \rightarrow (\tau, \xi \circ \alpha)$ is a combinatorial morphism of B -graphs.

Note that we do not require that j_σ and j_τ commute with a_F and a_V ; in particular, σ could be obtained from τ by cutting an edge into two tails. Other examples of combinatorial morphisms are morphisms adding tails or adding connected components. There are essentially two new types of morphisms compared to the non-weighted case:

- (1) *(Combining tails.)* Consider a subset $\{t_1, \dots, t_n\} \in F_\sigma(v)$ of tails attached to a vertex v , and assume that its sum of weights satisfies $\sum_i \mathcal{A}_\sigma(t_i) \leq 1$. Then we can form a new graph τ by replacing the tails $\{t_1, \dots, t_n\}$ with a single tail \bar{t} of weight $\mathcal{A}_\tau(\bar{t}) := \sum_i \mathcal{A}_\sigma(t_i)$.
- (2) *(Increasing the weights.)* This means that (τ, β) are identical to (σ, α) as modular graphs, but the weighting \mathcal{A}_τ satisfies $\mathcal{A}_\tau \geq \mathcal{A}_\sigma$.

We refer to [BM96, Definition 1.3] for the definition of a contraction $\phi: \tau \rightarrow \sigma$ of graphs. It is obtained by collapsing a subgraph consisting entirely of edges (and the adjoining vertices) to one vertex for every connected component of the subgraph. It is given by an injective map $\phi^F: F_\sigma \rightarrow F_\tau$ (which is bijective on tails) and a surjective map $\phi_V: V_\tau \rightarrow V_\sigma$.

Definition 6.1.5. *A contraction of weighted marked graphs $\phi: (\tau, \beta) \rightarrow (\sigma, \alpha)$ is a contraction $\phi: \tau \rightarrow \sigma$ of graphs such that*

- (1) $\alpha(v) = \sum_{w \in \phi_V^{-1}(v)} \alpha(w)$ for all $v \in V_\sigma$,
- (2) $g(v) = \sum_{w \in \phi_V^{-1}(v)} \alpha(w) + H^1(|\tau_v|)$ for all $v \in V_\sigma$ and τ_v being the subgraph of τ being collapsed onto v , and
- (3) $\mathcal{A}_\tau(\phi^F(f)) = \mathcal{A}_\sigma(f)$ for all tails $f \in S_\sigma$.

Definition 6.1.6. *A vertex v of a weighted modular A -graph (τ, α) is called stable if $\alpha(v) \neq 0$ or $2g(v) - 2 + \sum_{f \in F_\tau: \partial_\tau(f)=v} \mathcal{A}_\tau(f) > 0$. A graph is stable if all its vertices are stable.*

Remark 6.1.7. *Let (τ, α) be a weighted A -graph. There is a unique weighted stable A -graph (τ^s, α^s) and a combinatorial morphism $(\tau^s, \alpha^s) \rightarrow (\tau, \alpha)$, such that every combinatorial morphism $(\sigma, \beta) \rightarrow (\tau, \alpha)$ from a stable A -graph (σ, β) factors uniquely through (τ^s, α^s) .*

The graph (τ^s, α^s) is called the *stabilization* of (τ, α) . Similarly, there is a stabilization of weighted modular graphs. The stabilization τ^s of the underlying modular graph τ of an A -graph (τ, α) is also called the *absolute stabilization*.

The stabilization (τ^s, α^s) can be constructed via a sequence of steps as below, following [BM96, Proposition 1.13]:

- (1) If there is a connected component of τ that has only one vertex, and this vertex is unstable, we remove this connected component from τ .
- (2) If there is an unstable vertex v attached to one edge $\{f_0, \bar{f}_0 = j_\tau(f_0)\}$ with $\partial_\tau(f_0) = v$, $\partial_\tau(\bar{f}_0) \neq v$ and $n \geq 0$ tails f_1, \dots, f_n , we remove the vertex v and the flags f_0, \dots, f_n from the graph and modify j such that $j(\bar{f}_0) = \bar{f}_0$, i.e. the edge becomes a tail at the vertex $\partial_\tau(\bar{f}_0)$ with weight one.
- (3) If there is an unstable vertex v attached to two edges $\{f_1, \bar{f}_1 = j_\tau(f_1)\}$ and $\{f_2, \bar{f}_2 = j_\tau(f_2)\}$ with $\partial_\tau(f_i) = v$ and $\partial_\tau(\bar{f}_i) \neq v$, we remove v and the tails f_i from the graph, and modify j such that $j(\bar{f}_1) = \bar{f}_2$. In other words, we combine the tails \bar{f}_1, \bar{f}_2 to form a new edge.

At every step, any combinatorial morphism $(\sigma, \beta) \rightarrow (\tau, \alpha)$, where (σ, β) is a stable V -graph, factors uniquely through the new graph, and the claim of the remark follows by induction on the number of unstable vertices.

Definition 6.1.8. *Let (A, τ) and (B, σ) be weighted stable marked graphs. A morphism $(A, \tau) \rightarrow (B, \sigma)$ is quadruple (ξ, a, τ', ϕ) where $\xi: A \rightarrow B$ is a homomorphism*

of semigroups, τ' is a weighted stable B -graph, $a: \tau' \rightarrow \tau$ makes (ξ, a) into a combinatorial morphism of weighted marked graphs, and $\phi: \tau' \rightarrow \sigma$ is a contraction of B -graphs.

$$\begin{array}{ccc} B & & \tau' \xrightarrow{\phi} \sigma \\ \xi \uparrow & & \downarrow a \\ A & & \tau \end{array}$$

We think of this morphism as the composition of ϕ with the inverse of (ξ, a) , except that (ξ, a) itself is not a morphism in the category of weighted stable marked graphs. As explained earlier, this construction is motivated by the fact that the geometric morphism are covariant with respect to isogenies, but contravariant with respect to combinatorial morphisms.

To define compositions, we need the definition of stable pullback; the construction of [BM96] applies with minor changes. Given a combinatorial morphism of weighted marked graphs $(a, \xi): (B, \rho) \rightarrow (A, \tau)$ and a contraction of weighted A -graphs $\phi: \sigma \rightarrow \tau$, it canonically constructs a weighted stable B -graph π , together with a contraction of B -graphs $\psi: \pi \rightarrow \rho$ and a combinatorial morphism of weighted marked graphs $b: \pi \rightarrow \sigma$:

$$\begin{array}{ccc} B & & \pi \xrightarrow{\psi} \rho \\ \xi \uparrow & & \downarrow b \quad \downarrow a \\ A & & \sigma \xrightarrow{\phi} \tau \end{array}$$

We call π the *stable pullback of ρ under ϕ* . We will describe how to obtain π from ρ , assuming that ϕ is an elementary isogeny.

If ϕ contracts a loop adjacent to a vertex $v \in V_\tau$, we simply readd a loop at every preimage $v' \in a_V^{-1}(v)$ (and decrease its genus by one). If ϕ contracts an edge $\{f, \bar{f}\}$ connecting the vertices $v_1 = \partial_\sigma(f), v_2 = \partial_\sigma(\bar{f})$, let $v = \phi_V(v_1) = \phi_V(v_2)$ their common image in τ , and let $v' \in a_V^{-1}(v)$ be any vertex in the preimage of v in ρ . There can be two cases:

- (1) Replace v' by two vertices v'_1, v'_2 connected by an edge $\{f', \bar{f}'\}$; their class and genus are determined by the corresponding vertex in σ : $\alpha_\pi(v'_i) = \xi(\alpha_\sigma(v_i))$ and $g_\pi(v'_i) = g_\sigma(v_i)$. A flag f_1 of v is moved to v'_1 or v'_2 according to its position in σ , i.e. according to whether $\phi^F(a_F(f_1))$ is attached to v_1 or v_2 ; its weight remains unchanged. Now if either v'_1 or v'_2 is unstable, we undo this construction and skip to case (2). Otherwise, it remains to define the maps: ψ is the map contracting $\{f', \bar{f}'\}$; the combinatorial morphism b is given by sending v'_i to v_i , and by sending a flag $f_1 \neq f'$ of v'_i to $(\phi^F \circ a \circ (\psi^F)^{-1})(f_1)$. Other than that, b agrees with a .
- (2) Assume that in the above construction, the vertex v'_2 was unstable. We leave ρ unchanged, and let b_V send v' to v_1 . Let f_1 be a flag of v' ; we set $b_F(f_1) =$

$\phi^F(a_V(f_1))$ if that is a flag attached to v_1 , otherwise $b_F(f_1) = f$, where f defined above is part of the edge connection v_1 and v_2 .

The same construction is iteratively applied to every such vertex v to obtain π .

Geometrically, the isogeny ϕ corresponds to the inclusion of a boundary component $\overline{M}(\sigma)$ of the moduli space $\overline{M}(\tau)$ associated to τ , and the stable pull-back constructs the boundary component of $\overline{M}(\rho)$ upon which the boundary component $\overline{M}(\sigma)$ is naturally mapped by morphism $\overline{M}(\tau) \rightarrow \overline{M}(\rho)$ associated to a .

Proposition and Definition 6.1.9. *Let $(\xi, a, \tau', \phi): (A, \tau) \rightarrow (B, \sigma)$ and $(\eta, b, \sigma', \psi): (B, \sigma) \rightarrow (C, \rho)$ be morphisms of weighted stable marked graphs. Then we define the composition $(\eta, b, \sigma', \psi) \circ (\xi, a, \tau', \phi): (A, \tau) \rightarrow (C, \rho)$ to be $(\eta\xi, ac, \tau'', \psi\xi)$ where (c, τ'', ξ) is the stable pullback of σ' under ϕ .*

This composition is associative, defining the category of weighted stable marked graphs.

$$\begin{array}{ccccc}
 C & & \tau'' & \xrightarrow{\xi} & \sigma' & \xrightarrow{\psi} & \rho \\
 \eta \uparrow & & \downarrow c & & \downarrow b & & \\
 B & & \tau' & \xrightarrow{\phi} & \sigma & & \\
 \xi \uparrow & & \downarrow a & & & & \\
 A & & \tau & & & &
 \end{array}$$

We denote by \mathfrak{G}_s^w the category of weighted stable marked graphs, and by \mathfrak{A} the category of semigroups with indecomposable zeros.

6.2. Weighted stable maps indexed by graphs. As in [BM96, section 3], let \mathfrak{V} be the category of smooth projective varieties over a field k . Consider the fibered product $\mathfrak{V}\mathfrak{G}_s^w$ of categories

$$\begin{array}{ccc}
 \mathfrak{V}\mathfrak{G}_s^w & \longrightarrow & \mathfrak{G}_s^w \\
 \downarrow & & \downarrow \\
 \mathfrak{V} & \xrightarrow{H_2^+} & \mathfrak{A}
 \end{array}$$

where H_2^+ is the functor that associates to V the semigroup of effective classes in $\text{CH}^1(V)$. Objects of $\mathfrak{V}\mathfrak{G}_s^w$ are pairs (V, τ) where V is a smooth projective variety over k and τ is a weighted stable $H_2^+(V)$ -graph.

For any weighted graph τ and any vertex $v \in V_\tau$, let $F_v = \{f \in F_\tau \mid \partial_\tau(f) = v\}$ be the set of flags attached to v , and $\mathcal{A}_v = \mathcal{A}|_{F_v}$ be their weight data.

Definition 6.2.1. *A stable map of type (V, τ) for an object (V, τ) in $\mathfrak{V}\mathfrak{G}_s^w$ is a collection of stable maps (C_v, x_v, f_v) to V of type $(g(v), \mathcal{A}_v, \alpha(v))$ for every $v \in V_\tau$, such that $f_{\partial_\tau(i)}(x_i) = f_{\partial_\tau(j_\tau(i))}(x_{j_\tau(i)})$ for all flags i .*

For a scheme T and $(V, \tau) \in \mathfrak{WG}_s^w$, let $\overline{M}(T)(V, \tau)$ be the groupoid of families of weighted stable maps of type (V, τ) over T , and let $\overline{M}(T)$ be the groupoid of arbitrary weighted stable maps.

Theorem 6.2.2. *For a fixed scheme T , $\overline{M}(T)$ defines a 2-functor*

$$\overline{M}(T)(__): \mathfrak{WG}_s^w \rightarrow \overline{M}(T).$$

For every base change $u: T' \rightarrow T$, the pullback $u^: \overline{M}(T) \rightarrow \overline{M}(T')$ commutes with the functors $\overline{M}(T)(__)$ and $\overline{M}(T')(__)$.*

Finally, for fixed (V, τ, α) , the category of weighted stable maps of type (V, τ, α) is a proper algebraic Deligne-Mumford stack $\overline{M}(V, \tau, \alpha)$ of finite type.

Of course, the compatibility with base change in particular implies that that $\overline{M}(\Phi)$ for some morphism Φ in \mathfrak{WG}_s^w induces a morphism between the stacks associated by \overline{M} to the source and target; i.e. \overline{M} is a 2-functor from \mathfrak{WG}_s^w to the 2-category of Deligne-Mumford stacks.³

The last claim of the theorem immediately follows from theorem 2.1.4 and the fact that by definition it is a closed substack of $\prod_{v \in V_\tau} \overline{M}_{g(v), \mathcal{A}(v)}(V, \alpha(v))$.

To prove the first and second claim of the theorem, we need to prove the existence of a functorial push-forward in $\overline{M}(T)$ associated to every morphism $(\xi, a, \tau', \phi): (V, \tau) \rightarrow (W, \sigma)$ in \mathfrak{WG}_s^w , and show that they are compatible with base change. Every morphism in \mathfrak{WG}_s^w can be written as a composition of elementary morphisms of one of the following types: changing the target (I), increasing the weights (II), forgetting a tail (III), combining tails (IV), complete combinatorial morphisms (V), contracting an edge (VI) and contracting a loop (VII). For complete combinatorial morphisms this is immediate (and there is nothing to add to the discussion in [BM96, section 3, case IV]). All other cases have already been treated in 3 in the case where the target is a one-vertex graph; the general case follows immediately from this.

What is left to prove is that the associated morphism are compatible with composition in the category of weighted stable marked graphs, i.e. that it does not depend on the way we break up a morphism into a composition of elementary morphisms.

For compositions of contractions with contractions, respectively of the (inverses of) combinatorial morphisms with combinatorial morphisms this is immediate, and the only interesting case to prove is the case of the composition $(\xi, a)^{-1} \circ \phi$ of (the formal inverse of) a combinatorial morphism $(\xi, a): (B, \rho) \rightarrow (A, \tau)$ and a contraction of A -graphs $\phi: \sigma \rightarrow \tau$. In fact, the formation of stable pull-back exactly makes sure that this compatibility holds, and the claim follows easily by following every step of the stable pull-back construction.

³Implicitly, we passed from the description of a stack as a category fibered in groupoids to the description as a 2-functor to the 2-category of groupoids. See e.g. [Man99, Chapter V] for a discussion of both viewpoints.

7. Graph-level description of virtual fundamental classes.

To define Gromov-Witten invariants based on weighted stable maps, we need to define virtual fundamental classes in the Chow ring $A_* (\overline{M}(V, \tau, \alpha))$ of the moduli spaces. To formulate the required behaviour with respect to restriction to boundary components of the moduli space, we need to introduce the notion of isogenies of weighted stable graphs and their cartesian isogeny diagrams. (We won't introduce the complete cartesian extended isogeny category as in [BM96].)

7.1. Isogenies of graphs. For our purposes, we need to refine the definition of an isogeny as given in [BM96, Definition 5.4].

Definition 7.1.1. *We say that the one-vertex V -graph σ is a contraction of small tails of the one-vertex V -graph τ if it is obtained from τ by a sequence of steps, each forgetting a single tail, such that in every step we are in the situation of corollary 4.2.2 (the weight data of τ is contained in a fine open chamber, and the weight of the additional tail in σ is small enough that changing it to zero would not cross a wall of the fine chamber decomposition).*

This implies that the associated map $\overline{M}(\tau) \rightarrow \overline{M}(\sigma)$ is flat, as it is a sequence of projection maps of the universal curve.

Definition 7.1.2. *An isogeny $\Phi: \tau \rightarrow \sigma$ of weighted stable A -graphs is given by an injective map $\Phi^F: F_\sigma \rightarrow F_\tau$ of flags and a surjective map $\Phi_V: V_\tau \rightarrow V_\sigma$ of vertices such that the following conditions hold:*

- (1) Φ^F commutes with the boundary maps $\partial_\tau, \partial_\sigma$, i. e. for any flag $f \in F_\sigma$, we have $\Phi_V(\partial_\tau(\Phi^F(f))) = \partial_\sigma(f)$.
- (2) For any vertex $v \in V_\sigma$, let τ_v be the subgraph of τ that consists of all vertices send to v by Φ_V , and all edges joining them. We require that
 - (a) $g(v) = \sum_{w \in V_{\tau_v}} g(w) + \dim H^1(|\tau_v|)$ and
 - (b) $\alpha(v) = \sum_{w \in V_{\tau_v}} \alpha(w)$
- (3) Φ^F respects the weights, i.e. $\mathcal{A}_\tau \circ \Phi^F = \mathcal{A}_\sigma$.
- (4) For any $v \in V_\tau$, let τ_v be the one-vertex graph obtained from τ by removing all other vertices, and cutting off the edges starting from v into a tail of weight 1; let σ_v be the graph obtained from τ_v by removing all tails not in the image of Φ^F . The condition is that σ_v is a contraction of small tails of τ_v .

In the geometric realizations of the graphs, an isogeny is given by collapsing a set of disjoint closed connected subgraphs $|\tau_v| \subset |\tau|$ consisting of edges and small tails to a single vertex $v \in V_\sigma$. It can be written as the composition of a morphism contracting small tails, and a contraction as in definition 6.1.5.

7.2. Cartesian isogeny diagrams. Consider a stable V -graph σ and its absolute stabilization $a: \sigma^s \rightarrow \sigma$, as well as an isogeny of weighted modular graphs $\Phi: \tau^s \rightarrow \sigma^s$. In [BM96, section 5] the pull-back $\tau = (\tau_i)_{i \in I}$ of σ along Φ is constructed. For

each $i \in I$, the stable V -graph τ_i comes with a stabilization morphism $a_i: \tau^s \rightarrow \tau_i$ and an isogeny $\Phi_i: \tau_i \rightarrow \sigma$ such that the diagram

$$\begin{array}{ccc} \tau_i & \xrightarrow{\Phi_i} & \sigma \\ a_i \uparrow & & \uparrow b \\ \tau^s & \xrightarrow{\Phi} & \sigma^s \end{array}$$

commutes.

Its construction is as follows:⁴ To every edge $\{f, \bar{f}\}$ of σ^s there is a long edge in σ consisting of edges $\{f_1, \bar{f}_1\}, \dots, \{f_n, \bar{f}_n\}$ and vertices $v_i = \partial_\sigma(f_i) = \partial_\sigma(\bar{f}_{i+1})$ such that $b^F(f) = f_1$, $b^F(\bar{f}) = \bar{f}_n$ and the vertices v_i are of genus 0 and have no further flags. We replace the edge $\{\Phi^F(f), \Phi^F(\bar{f})\}$ of τ^s by the same long edge $\{f_1, \bar{f}_1\}, \dots, \{f_n, \bar{f}_n\}$. Similarly, to every tail $f \in S_{\sigma^s}$ there is a long tail $\{f_1, \bar{f}_1\}, \dots, \{f_n, \bar{f}_n\}$ of edges as above and some number $k \geq 0$ of additional tails f_{n+1}, \dots, f_{n+k} . The additional tails are attached to the last vertex v_n of the tail, $\partial_\sigma(f_{n+i}) = v_n = \partial_\sigma(\bar{f}_n)$ for $1 \leq i \leq k$, and the sum of weights is bounded as $\sum_{1 \leq i \leq k} \mathcal{A}(f_{n+i}) \leq 1$. Again we replace the tail $\Phi^F(f) \in S_{\sigma^s}$ with the same long tail, preserving the weights.

We thus obtain a weighted graph τ' with a combinatorial morphism $a: \tau^s \rightarrow \tau'$ and an isogeny of graphs $\Phi': \tau' \rightarrow \sigma$. Now let I be the set of V -structures on τ' such that Φ' is an isogeny of weighted V -graphs. We get a set $(\tau_i)_{i \in I}$ of V -graphs such that the induced morphism $a_i: \tau^s \rightarrow \tau_i$ is an absolute stabilization, and $\Phi_i: \tau_i \rightarrow \sigma$ is an isogeny of V -graphs.

The same construction can be made for a tuple $(\sigma_j)_{j \in J}$ of V -graphs with absolute stabilization morphisms $b_j: \sigma \rightarrow \sigma_j$. The formation of pull-back then becomes compatible with composition.

7.3. Expected properties.

Definition 7.3.1. *Let τ be a weighted stable V -graph, where V is of pure dimension $\dim V$, and has canonical class ω_V . We define the class $\beta(\tau)$, the Euler characteristic $\chi(\tau)$, the genus $g(\tau)$ and the dimension $\dim(\tau)$ of τ as*

$$\begin{aligned} \beta(\tau) &= \sum_{v \in V_\tau} \beta(v) \\ \chi(\tau) &= \chi(|\tau|) - \sum_{v \in V_\tau} g(v) \\ g(\tau) &= 1 - \chi(\tau) \\ \dim(\tau) &= \chi(\tau)(\dim V - 3) - \beta(\tau) \cdot \omega_V + |S_\tau| - |E_\tau| \end{aligned}$$

We now fix V . An orientation will be a system of virtual fundamental classes $J(V, \tau) \subset A_{\dim(V, \tau)}(\overline{M}(V, \tau))$ for all stable V -graphs τ bounded by the characteristic, satisfying the list of properties given below.

⁴Unlike [BM96, section 5], we omit the orbit map as well as the notion of an extended isogeny.

- (1) (*Mapping to a point*). If τ is a graph of class zero, and $|\tau|$ is nonempty and connected, then

$$J(V, \tau) = c_{g(\tau) \dim V} (R^1 \pi_* f^* TV).$$

- (2) (*Forgetting tails*). Let $\Phi: \sigma \rightarrow \tau$ be a morphism of stable V -graphs given by forgetting a small tail of σ , i.e. such that τ is obtained from σ by a contraction of a small tail. Then $\overline{M}(\Phi)$ is flat, and we require

$$J(V, \sigma) = \overline{M}(\Phi)^* J(V, \tau).$$

- (3) (*Combining tails*). Let $\Phi: \sigma \rightarrow \tau$ be a morphism splitting up a tail into several of them, i. e. one that is induced by a combinatorial morphism $a: \tau \rightarrow \sigma$ combining several tails $f_1, \dots, f_k \in S_\tau$ to a single tail $f \in S_\sigma$ with weight $\mathcal{A}_\sigma(f) = \sum_{i=1}^k \mathcal{A}_\tau(f_i)$. Then $\overline{M}(\Phi)$ is a regular closed embedding, and the condition is

$$J(V, \sigma) = \overline{M}(\Phi)^! J(V, \tau).$$

- (4a) (*Products*). For any two stable V -graphs σ, τ , let $\sigma \times \tau$ be the disjoint union of the graphs of σ and τ with the obvious structure as a stable V -graph. Then

$$J(V, \sigma \times \tau) = J(V, \sigma) \times J(V, \tau).$$

- (4b) (*Cutting edges*). Let $\Phi: \sigma \rightarrow \tau$ be a morphism obtained by cutting an edge $\{f, \bar{f}\}$ of σ into two tails. By abuse of notation, we identify the flags $f, \bar{f} \subset F_\sigma$ with the corresponding tails $f, \bar{f} \subset S_\tau$. We obtain a cartesian square

$$\begin{array}{ccc} \overline{M}(V, \sigma) & \xrightarrow{\overline{M}(\Phi)} & \overline{M}(V, \tau) \\ \downarrow \text{ev}_f = \text{ev}_{\bar{f}} & & \downarrow \text{ev}_f \times \text{ev}_{\bar{f}} \\ V & \xrightarrow{\Delta} & V \times V \end{array}$$

and require that

$$J(V, \sigma) = \Delta^! J(V, \tau).$$

- (4c) (*Isogenies*). Let $(\sigma_j)_{j \in J}$ be a tuple of V -graphs with absolute stabilization σ^s and $\tau^s \rightarrow \sigma^s$ an isogeny. Let $(\tau_i)_{i \in I}$ be the tuple of V -graphs completing this to a cartesian isogeny diagram. We obtain an induced commutative, but not cartesian diagram

$$\begin{array}{ccc} \prod_{i \in I} \overline{M}(\tau_i) & \longrightarrow & \prod_{j \in J} \overline{M}(\sigma_j) \\ \downarrow & & \downarrow \\ \overline{M}(\tau^s) & \xrightarrow{\overline{M}(\Phi)} & \overline{M}(\sigma^s) \end{array}$$

and thus an induced map

$$h: \prod_{i \in I} \overline{M}(\tau_i) \rightarrow \overline{M}(\tau^s) \times_{\overline{M}(\sigma^s)} \prod_{j \in J} \overline{M}(\sigma_j).$$

We require that

$$h_* \left(\sum_{i \in I} J(V, \tau_i) \right) = \sum_{j \in J} \overline{M}(\Phi)^! J(V, \sigma_j).$$

- (5) *Kontsevich-stable maps.* Assume that all weights satisfy $\mathcal{A}(s) = 1$. Then $J(V, \tau)$ agrees with the definition of the virtual fundamental class $J(V, \tilde{\tau})$ for the underlying stable V -graphs $\tilde{\tau}$ according to [Beh97, BF97].
- (6) *Reducing weights.* Let $\Phi: \sigma \rightarrow \tau$ be a morphism of weighted stable V -graphs obtained by reducing weights, i. e. such that Φ is induced by a combinatorial morphism $\tau \rightarrow \sigma$ that is the identity on the modular graph structure, but such that $\mathcal{A}_\sigma(f) \geq \mathcal{A}_\tau(f)$ for all flags $f \in F_\tau = F_\sigma$. Then $\overline{M}(\Phi)$ is a reduction morphism, and we require that

$$\overline{M}(\Phi)_* (J(V, \sigma)) = (J(V, \tau)).$$

Theorem 7.3.2. *There is a system of virtual fundamental classes satisfying all properties listed in the previous section.*

Note that (4a), (4b) and (4c) imply condition (4) of theorem 5.1.1, whereas the other conditions for one-vertex graphs are identical to the corresponding condition *ibid.*

Of course, (1), (2) and (4a-c) are direct generalizations of properties of the virtual fundamental classes in the non-weighted setting. The only caveat is that for morphisms contracting or forgetting a tail, we always have to assume the situation of corollary 4.2.2. This is to be expected: if we forget a tail of bigger weight, the forgetful map factorizes via a non-trivial reduction morphism ρ . However, there is no reason to assume that the virtual fundamental class is a pull-back of a class via ρ .

As we already explained, we use (5) and (6) as the definition:

Definition and Remark 7.3.3. *For any weighted stable V -graph τ , let τ^1 be the weighted stable V -graph obtained by setting all weights to 1, let $w(\tau): \tau \rightarrow \tau^1$ be the combinatorial morphism increasing the weights, and $W(\tau): \tau^1 \rightarrow \tau$ the induced morphism in the category of weighted marked graphs. Then any combinatorial morphism $\tau \rightarrow \sigma$ to a V -graph σ with all weights equal to 1 factors uniquely via $w(\tau)$.*

By abuse of notation, we write $W(\tau): \overline{M}(V, \tau^1) \rightarrow \overline{M}(V, \tau)$ also for the induced map on moduli spaces, and define $J(V, \tau)$ as

$$J(V, \tau) := W(\tau)_* J(V, \tau^1)$$

where the latter is as defined in [Beh97, BF97].

We will now show how to obtain these properties from those listed in Definition 7.1 in [BM96], which have been verified for the Behrend-Fantechi construction of the virtual fundamental class in [Beh97]. As a preparation, we need the following lemma:

Lemma 7.3.4. *Let $\Phi: \sigma \rightarrow \tau$ be an isogeny of V -graphs, and let $\Phi^1: \sigma^1 \rightarrow \tau^1$ be the same morphism for the graphs with weight 1. Consider the commutative (but not necessarily cartesian) square*

$$\begin{array}{ccc} \overline{M}(V, \sigma^1) & \xrightarrow{\overline{M}(\Phi^1)} & \overline{M}(V, \tau^1) \\ \downarrow \overline{M}(W(\sigma)) & & \downarrow \overline{M}(W(\tau)) \\ \overline{M}(V, \sigma) & \xrightarrow{\overline{M}(\Phi)} & \overline{M}(V, \tau) \end{array}$$

and the induced morphism $h: \overline{M}(V, \sigma^1) \rightarrow \overline{M}(V, \sigma) \times_{\overline{M}(V, \tau)} \overline{M}(V, \tau^1)$. Then $\overline{M}(\Phi)^!$ and $h_* \circ \overline{M}(\Phi^1)^!$ yield the same orientation to the projection

$$\overline{M}(V, \sigma) \times_{\overline{M}(V, \tau)} \overline{M}(V, \tau^1) \rightarrow \overline{M}(V, \tau^1).$$

(By definition, an orientation of a morphism $f: X \rightarrow Y$ is an element of the bivariant intersection theory $A^*(Y \rightarrow X)$, i.e. in particular a morphism $A_*(X') \rightarrow A_*(Y')$ for every pull-back $f': X' \rightarrow Y'$ of f .)

We may assume that Φ is an elementary isogeny, so we have one of the following two cases:

- *Contraction of an edge.* It is sufficient to consider the case where τ has only one vertex, so both $\overline{M}(\Phi)$ and $\overline{M}(\Phi^1)$ are a gluing morphism as in proposition 3.1.1. Consider the first case, where Φ contracts a non-looping edge (the other case follows similarly). An object in the product consists of a pair of weighted stable maps $((C_1, f_1), (C_2, f_2))$ of type σ and τ^1 , respectively, together with an isomorphism the reduction of C_2 to type τ with the curve obtained by gluing the two components of C_1 . Since the sections cannot meet the node, this is only possible if C_2 already consists of two components, which together form a weighted stable maps of type σ^1 . The induced map to $\overline{M}(V, \sigma^1)$ is an inverse to h , i.e. the above diagram is a cartesian square.

Both $\overline{M}(\Phi)$ and $\overline{M}(\Phi^1)$ are a codimension one regular embedding with compatible normal bundle, and the claim follows by standard intersection theory.

- *Contraction of a small tail.* In this case, both $\overline{M}(\Phi)$ and $\overline{M}(\Phi^1)$ are flat. The orientation given by $\overline{M}(\Phi)$ is the same as that of the projection to the second factor of the product. Since h is a blow-up at a regularly embedded substack, we have $h_* \circ h^* = \text{id}$, and the assertion follows.

We proceed with the proof of theorem 7.3.2.

- (1) This follows from the same property [BM96, Definition 7.1, (1)] in the non-weighted case and projection formula.

(2) Consider the diagram of lemma 7.3.4:

$$\begin{aligned}
\overline{M}(\Phi)^! J(V, \tau) &= \overline{M}(\Phi)^! \overline{M}(W(\tau))_* J(V, \tau^1) && \text{(by definition)} \\
&= p_{1*} \overline{M}(\Phi)^! J(V, \tau^1) && \text{(push-forward)} \\
&= p_{1*} h_* \overline{M}(\Phi^1)^! J(V, \tau^1) && \text{(by lemma 7.3.4)} \\
&= \overline{M}(W(\sigma))_* J(V, \sigma^1) && (*) \\
&= J(V, \sigma) && \text{(by definition)}
\end{aligned}$$

Here (*) holds by [BM96, Definition 7.1, (4)].

(4a) This is obvious from the same property for non-weighted graphs [BM96, Definition 7.1, (2)].

(4b) The natural map $\overline{M}(V, \sigma^1) \rightarrow \overline{M}(V, \tau^1)$ fits as an additional row on top of diagram given in condition (4b), so that all squares are cartesian. Thus the claim follows from property [BM96, Definition 7.1, (3)] and push-forward.

(4c) We may assume that $|J| = 1$, so we are just dealing with a single V -graph σ and its absolute stabilization σ^s .

Consider σ^1 and its absolute stabilization $(\sigma^1)^s$. By the universal property of stabilization, the composition of the combinatorial morphisms of weighted graphs $\sigma^s \rightarrow \sigma \rightarrow \sigma^1$ factors uniquely via $(\sigma^1)^s$. Similarly, for each $i \in I$ let τ_i^1 be the corresponding graphs with weights 1, and let, by some abuse of notation, $(\tau^1)^s$ be their common absolute stabilization; we obtain a combinatorial morphism $\tau^s \rightarrow (\tau^1)^s$.

These morphisms can be completed to the following diagram of a cube:

$$\begin{array}{ccccc}
& & \coprod_i \tau_i & \xrightarrow{\coprod_i \Phi_i} & \sigma \\
& \nearrow W(\tau_i) & \downarrow & & \nearrow W(\sigma) \\
\coprod_i \tau_i^1 & \xrightarrow{\coprod_i \Phi_i^1} & \sigma^1 & & \\
\downarrow & & \downarrow & & \downarrow \\
& & \tau^s & \xrightarrow{\Phi} & \sigma^s \\
\downarrow & \nearrow & \downarrow & & \downarrow \\
(\tau^1)^s & \xrightarrow{\Phi^1} & (\sigma^1)^s & &
\end{array}$$

More precisely, there exist unique contractions $\Phi^1: (\tau^1)^s \rightarrow (\sigma^1)^s$ and $\Phi_i^1: \tau_i^1 \rightarrow \sigma^1$ such that

- (I) the top and bottom square are commutative in the category of weighted marked graphs, and
- (II) the square in front is a cartesian isogeny diagram.

Assuming these claims, the desired property can be deduced from the corresponding property [BM96, Definition 7.1, (5)] by careful diagram computation:

Since none of the squares of the cube necessarily yield cartesian squares of moduli spaces, we need to consider the products $P_{\text{back}} = \overline{M}(\tau^s) \times_{\overline{M}(\sigma^s)}$

$\overline{M}(V, \sigma)$, $P_{\text{front}} = \overline{M}((\tau^1)^s) \times_{\overline{M}((\sigma^1)^s)} \overline{M}(V, \sigma^1)$ and $P_{\text{diag}} = \overline{M}(\tau^s) \times_{\overline{M}(\sigma^s)} \overline{M}(V, \sigma^1)$. Let h_{back} and h_{front} be the induced map from the corresponding corner of the cube to P_{back} and P_{front} , respectively, and $h_{\text{d} \rightarrow \text{b}}: P_{\text{diag}} \rightarrow P_{\text{back}}$, $h_{\text{f} \rightarrow \text{b}}: P_{\text{front}} \rightarrow P_{\text{back}}$ and $h_{\text{f} \rightarrow \text{d}}: P_{\text{front}} \rightarrow P_{\text{diag}}$ the maps induced by the commutative cube. We obtain

$$\begin{aligned}
\overline{M}(\Phi)^! J(V, \sigma) &= \overline{M}(\Phi)^! \overline{M}(W(\sigma))_* J(V, \sigma^1) && \text{(by definition)} \\
&= h_{\text{d} \rightarrow \text{b}*} \overline{M}(\Phi)^! J(V, \sigma^1) && \text{(push-forward)} \\
&= h_{\text{d} \rightarrow \text{b}*} h_{\text{f} \rightarrow \text{d}*} \overline{M}(\Phi^1)^! J(V, \sigma^1) && \text{(lemma 7.3.4)} \\
&= h_{\text{f} \rightarrow \text{b}*} h_{\text{front}*} \sum_i J(V, \tau_i^1) && (*) \\
&= h_{\text{back}*} \sum_i W(\tau_i)_* J(V, \tau_i^1) \\
&= h_{\text{back}*} \sum_i J(V, \tau_i), && \text{(by definition)}
\end{aligned}$$

where (*) holds according to [BM96, Definition 7.1, (5)]. So it remains to prove the two claims above.

The definition of Φ_i^1 is obvious and necessarily unique, as the graphs τ_i and τ_i^1 , as well as σ_i and σ_i^1 , are identical as marked graphs after forgetting the weighting. Commutativity of the top square is equivalent to the claim that the combinatorial morphism $w(\tau_i): \tau_i \rightarrow \tau_i^1$ is the stable pull-back (see p. 44) of $w(\sigma): \sigma \rightarrow \sigma^1$ along Φ_i^1 , which is equally obvious.

For the bottom square involving Φ^1 , we need to review the construction of cartesian isogenies. Consider any tail $f \in S_{\sigma^s}$; it corresponds to a long tail in σ consisting of edges $\{f_1, \bar{f}_1\}, \dots, \{f_n, \bar{f}_n\}$, of vertices v_1, \dots, v_n and of tails f_{n+1}, \dots, f_{n+k} attached to v_n . Its preimage $\Phi^F(f) \in S_{\tau^s}$ corresponds to an identical long tail $\{\Phi_i^F(f_1), \Phi_i^F(\bar{f}_1)\}, \dots$ etc. in τ_i . After adjusting the weights to one, we again see identical long tails as part of σ^1 respectively τ_i^1 ; these will have identical stabilization in $(\sigma^1)^s$ resp. $(\tau^1)^s$. This shows that Φ^1 is uniquely determined on the stabilization of this long tail. The same discussion applies to any edge of σ^s corresponding to a long edge in σ^s . Finally, any part of τ^s contracted by Φ will appear identically in τ_i , and thus in τ_i^1 and $(\tau^1)^s$. Hence Φ^1 will necessarily contract it, too.

We have thus constructed Φ^1 so that the front square is a cartesian isogeny diagram. At the same time, the above discussion shows that the stable pull-back of $\sigma^s \rightarrow (\sigma^1)^s$ along Φ^1 will recover $\tau^s \rightarrow (\tau^1)^s$, i.e. the bottom square is indeed commutative.

- (5) This holds by definition.
- (6) This follows from the definition and the fact that reduction morphisms are compatible with composition (Proposition 2.2.1).

(3) By properties (4a) and (4b), we can consider only graphs having a single vertex. Further, we may assume that the combinatorial morphism a combines exactly two tails $f_1, f_2 \in S_\tau$ to a single tail $f = a_F(f_1) = a_F(f_2) \in S_\sigma$.

Let ρ be the V -graph obtained from σ^1 by adding second vertex of class and genus zero, having two tails f'_1, f'_2 of weight 1 and one edge whose second flag connects it to the original vertex and replaces the tail f ; geometrically, we replace the tail f with a tripod.

The morphism $\rho \rightarrow \sigma^1$ induced by the combinatorial morphism $\sigma^1 \rightarrow \rho$ gives an isomorphism of moduli spaces $\overline{M}(\rho) \rightarrow \overline{M}(\sigma^1)$, which respects the virtual fundamental classes by properties (1), (4a) and (4b).

There is a morphism $\Psi: \rho \rightarrow \tau^1$ contracting the edge in ρ and sending f'_i to f_i . Thus we have the following commutative diagram:

$$\begin{array}{ccc} \overline{M}(\rho) \cong \overline{M}(\sigma^1) & \xrightarrow{\overline{M}(\Psi)} & \overline{M}(\tau^1) \\ \downarrow W(\sigma) & & \downarrow W(\tau) \\ \overline{M}(\sigma) & \xrightarrow{\overline{M}(\Phi)} & \overline{M}(\tau) \end{array}$$

A discussion similar to the one in the proof of (4c) shows that this is a cartesian square. Let $\Xi: \tau^1 \rightarrow \sigma^1$ be the morphism obtained by forgetting the tail f_1 and mapping f_2 to f . Then $\overline{M}(\Psi)$ is a section of $\overline{M}(\Xi)$, so $\overline{M}(\Psi)^! [\overline{M}(\tau^1)]^{\text{virt}} = \overline{M}(\Psi)^! \overline{M}(\Xi)^* [\overline{M}(\sigma^1)]^{\text{virt}} = [\overline{M}(\sigma^1)]^{\text{virt}}$. The desired equality follows by push-forward and the vanishing of excess intersection.

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Summary

Semisimple Quantum Cohomology, deformations of stability conditions and the derived category

by Arend Bayer

The introduction discusses various motivations for the following chapters of the thesis, and their relation to questions around mirror symmetry.

The main theorem of chapter 2 says that if the quantum cohomology of a smooth projective variety V yields a generically semisimple product, then the same holds true for its blow-up at any number of points (theorem 3.1.1). This is a positive test for a conjecture by Dubrovin, which claims that quantum cohomology of V is generically semisimple if and only if its bounded derived category $D^b(V)$ has a complete exceptional collection.

Chapter 3 generalizes Bridgeland's notion of stability condition on a triangulated category. The generalization, a *polynomial stability condition* (definition 2.1.4), allows the *central charge* to have values in polynomials $\mathbb{C}[N]$ instead of complex numbers \mathbb{C} . We show that polynomial stability conditions have the same deformation properties as Bridgeland's stability conditions (theorem 3.2.5). In section 4, it is shown that every projective variety has a canonical family of polynomial stability conditions.

In chapter 4, we define and study the notion of a weighted stable map (definition 2.1.2), depending on a set of weights. We show the existence of moduli spaces of weighted stable maps as proper Deligne-Mumford stacks of finite type (theorem 2.1.4), and study in detail their birational behaviour under changes of weights (section 4). We introduce a category of weighted marked graphs in section 6, and show that it keeps track of the boundary components of the moduli spaces, and natural morphisms between them. We introduce weighted Gromov-Witten invariants by defining virtual fundamental classes, and prove that these satisfy all properties to be expected (sections 5 and 7). In particular, we show that Gromov-Witten invariants without gravitational descendants do not depend on the choice of weights.

Zusammenfassung

Halbeinfach Quanten-Kohomologie, Deformation von Stabilitätsbedingungen und die Derivierte Kategorie

von Arend Bayer

Die Einleitung erläutert verschiedene Ausgangspunkte für die nachfolgenden Kapitel, und ihre Verbindungen zu Fragen rund um Spiegelsymmetrie.

Hauptaussage von Kapitel 2 ist Satz 3.1.1: wenn das Produkt der Quantenkohomologie einer glatten projektiven Varietät V generisch halbeinfach ist, dann gilt das gleiche für die Aufblasung von V an beliebig vielen Punkten. Dies ist ein erfolgreicher Test für eine Vermutung von Dubrovin, die besagt, dass die Quantenkohomologie von V genau dann generisch halbeinfach ist, wenn die beschränkte derivierte Kategorie $D^b(V)$ ein vollständiges exzeptionelles System besitzt.

Kapitel 3 verallgemeinert Bridgelands Begriff einer Stabilitätsbedingung in einer triangulierten Kategorie. Diese Verallgemeinerung, eine *polynomiale Stabilitätsbedingung* (Definition 2.1.4), lässt eine zentrale Ladung mit Werten in Polynomen $\mathbb{C}[N]$ statt komplexen Zahlen \mathbb{C} zu. Es wird gezeigt, dass polynomiale Stabilitätsbedingungen dieselben Deformationseigenschaften wie Bridgelands Stabilitätsbedingungen haben (Satz 3.2.5). Abschnitt 4 zeigt, dass es für jede projektive Varietät V eine kanonische Familie von polynomialen Stabilitätsbedingungen in $D^b(V)$ gibt.

Kapitel 4 führt den Begriff einer gewichteten stabilen Abbildung ein (Definition 2.1.2), in Abhängigkeit einer Menge von Gewichten. Satz 2.1.4 zeigt die Existenz der Modulräume gewichteter stabiler Abbildung als eigentliche Deligne-Mumford-Stacks endlichen Typs, und Abschnitt 4 beschäftigt sich im Detail mit dem birational Verhalten der Modulräume bei Änderungen der Gewichte. In Abschnitt 6 führen wir eine Kategorie gewichteter markierter Graphen ein, und zeigen, dass sie natürlicherweise Randkomponenten der Modulräume und die natürliche Morphismen zwischen ihnen Buch indiziert. Gewichtete Gromov-Witten-Invarianten werden durch die Definition von virtuellen Fundamentalklassen eingeführt, und wir zeigen, dass diese alle zu erwartenden Eigenschaften erfüllen (Abschnitte 5 und 7). Insbesondere zeigen wir, dass Gromov-Witten-Invarianten ohne Kopplung an Gravitation nicht von der Wahl der Gewichte abhängen.