A SHORT PROOF OF THE DEFORMATION PROPERTY OF BRIDGELAND STABILITY CONDITIONS

AREND BAYER

ABSTRACT. The key result in the theory of Bridgeland stability conditions is the property that they form a complex manifold. This comes from the fact that given any small deformation of the central charge, there is a unique way to correspondingly deform the stability condition.

We give a short direct proof of an effective version of this deformation property.

CONTENTS

1. INTRODUCTION

Stability conditions on triangulated categories, introduced in [Bri07], have been hugely influential, due to their connections to physics [BS15, GMN13], to mirror symmetry [Bri09a] and to representation theory [ABM15], and due to their applications within algebraic geometry, for example to Donaldson-Thomas invariants [Tod14], to the derived category itself [Huy14, BB17], or to the birational geometry of moduli spaces [ABCH13, BM14, Bay16, LZ16, MS16].

Their distinguishing property, crucial in all applications, is a strong deformation property: by [Bri07, Theorem 1.2], there is a complex manifold of stability conditions, with a map to a vector space that is a local isomorphism. We give a short proof of an effective version of this result.

Result. We refer to Section 2 for complete definitions; here we review notation and the support property. Let $\mathcal{D}$ be a triangulated category, and let $v\colon K(\mathcal{D}) \to \Lambda$ be a homomorphism from its K-group to a finitely generated free abelian group Λ. A pre-stability condition on $\mathcal{D}$ with respect to v is a pair $\sigma = (Z, \mathcal{P})$, where $\mathcal{P}$ is a *slicing* (see Definition 2.1) and $Z\colon \Lambda \to \mathbb{C}$ is a compatible (see Definition 2.2) group homomorphism.

Definition 1.1 ([Bri07], [KS08])**.** Let $Q\colon \Lambda_{\mathbb{R}} \to \mathbb{R}$ be a quadratic form. We say that a pre-stability condition $(Z, \mathcal{P})$ satisfies the support property with respect to Q if

(1) the kernel $\operatorname{Ker} Z \subset \Lambda_{\mathbb{R}}$ of the central charge is negative definite with respect to Q, and
(2) for any semistable object E, i.e. $E \in \mathcal{P}(\phi)$ for some $\phi \in \mathbb{R}$, we have $Q(v(E)) \geq 0$.

In this case, we call σ a stability condition. Let $\operatorname{Stab}_{\Lambda}(\mathcal{D})$ denote the topological space (see Section 2) of stability conditions, and $\mathcal{Z}\colon \operatorname{Stab}_{\Lambda}(\mathcal{D}) \to \operatorname{Hom}(\Lambda, \mathbb{C})$ the map $\mathcal{Z}(Z, \mathcal{P}) = Z$.

Theorem 1.2. *Let Q be a quadratic form on $\Lambda \otimes \mathbb{R}$, and assume that the stability condition $\sigma = (Z, \mathcal{P})$ satisfies the support property with respect to Q. Then:*

(1) There is an open neighbourhood $\sigma \in U_{\sigma} \subset \operatorname{Stab}_{\Lambda}(\mathcal{D})$ such that the restriction $\mathcal{Z}\colon U_{\sigma} \to \operatorname{Hom}(\Lambda, \mathbb{C})$ is a covering of the set of Z' such that Q is negative definite on $\operatorname{Ker} Z'$.
(2) All stability conditions in U_{σ} satisfy the support property with respect to Q.

In other words, $\operatorname{Stab}_{\Lambda}(\mathcal{D})$ is a manifold, and any path $Z_t \in \operatorname{Hom}(\Lambda, \mathbb{C})$ for $t \in [0, 1]$ with $Z_0 = Z$ and $\operatorname{Ker} Z_t$ negative definite for all $t \in [0, 1]$ lifts uniquely to a continuous path $\sigma_t = (Z_t, \mathcal{P}_t)$ in the space of stability conditions starting at $\sigma_0 = \sigma$.

Part (1) is an effective variant of [Bri07, Theorem 1.2] (which says that there is *some* neighbourhood of Z_0 in which paths can be lifted uniquely). The entire result first appeared as [BMS16, Proposition A.5] with an indirect proof based on reduction to Bridgeland's previous result.

Remarks. The support property can be a deep and interesting property in itself: a quadratic Bogomolov-Gieseker type inequality for Chern classes of semistable objects which, by Theorem 1.2, is preserved under wall-crossing.

Theorem 1.2 was crucial in [BMS16] in order to describe an entire component of the space of stability conditions on abelian threefolds, and on some Calabi-Yau threefolds. It also greatly simplifies the construction of stability conditions on surfaces (or of *tilt-stability* on higher-dimensional varieties [BMT14]). In this case, the quadratic form Q is the classical Bogomolov-Gieseker inequality, and Theorem 1.2 gives an open subset of stability conditions that otherwise has to be glued together from many small pieces (see e.g. [BM11, Section 4]).

Theorem 1.2 of [Bri07] also allows for components of the space of stability conditions modelled on a linear subspace $L \subset \operatorname{Hom}(\Lambda, \mathbb{C})$. When L is defined over $\mathbb{Q}$, we can recover that statement by replacing Λ with $\Lambda / \operatorname{Ker} L$. (See [MP14] for examples where this is not satisfied; however, to achieve well-behaved wall-crossing one has to assume that L is defined over $\mathbb{Q}$.)

Proof idea. Our proof is based on two ideas. First, we reduce to the case where the imaginary part of Z is constant; then we only have to deform stability in a fixed abelian category. Secondly, we use the elementary convex geometry of the *Harder-Narasimhan polygon*, see Section 3.

This avoids the need for *quasi-abelian categories*. It also avoids some of the more technical arguments of [Bri07, Section 7]. We still need a few arguments similar to ones in [Bri07], which we have reproduced for the convenience of the reader.

Application. Assume that $\mathcal{D}$ is a 2-Calabi-Yau category, i.e. for all $E, F \in \mathcal{D}$ we have a bifunctorial isomorphism $\operatorname{Hom}(E, F) = \operatorname{Hom}(F, E[2])^{\vee}$. Let Λ be the *numerical K-group* of $\mathcal{D}$, and assume that Λ is finitely generated. Then there is a surjection $v\colon K(\mathcal{D}) \to \Lambda$, and Λ admits a non-degenerate bilinear form $(\_, \_)$, called Mukai-pairing, with

$$\chi(E, F) = -\big(v(E), v(F)\big).$$

Let $\mathcal{P}_0(\mathcal{D}) \subset \mathrm{Hom}(\Lambda, \mathbb{C})$ be the set of central charges Z such that $\mathrm{Ker}\, Z$ is negative definite with respect to the Mukai pairing, and such that $\mathrm{Ker}\, Z$ contains no root $\delta \in \Lambda, (\delta, \delta) = -2$.

Corollary 1.3. *The restriction $\mathcal{Z}^{-1}\left(\mathcal{P}_0(\mathcal{D})\right) \xrightarrow{\mathcal{Z}} \mathcal{P}_0(\mathcal{D})$ is a covering map.*

The proof, given in Section 8, is fairly similar to the case of K3 surfaces [Bri08, Proposition 8.3]. The point of including it here is to show that in terms of the support property via quadratic forms, and equipped with Theorem 1.2, the proof becomes natural and short. This result was also proved previously for preprojective algebras of quivers in [Tho08, Bri09b, Ike14]. In each of these cases, there is in fact a connected component of $\mathrm{Stab}(\mathcal{D})$ that is a covering of a connected of $\mathcal{P}_0(X)$; such statements rely crucially on *non-emptiness* of moduli spaces of stable objects.

2. Review: definitions and basic properties

Throughout, $\mathcal{D}$ will be a triangulated category, equipped with a group homomorphism

$$v \colon K(\mathcal{D}) \to \Lambda$$

from its K-group to an abelian group $\Lambda \cong \mathbb{Z}^m$.

Definitions. We first recall the main definitions from [Bri07].

Definition 2.1. A *slicing* $\mathcal{P}$ on $\mathcal{D}$ is a collection of full subcategories $\mathcal{P}(\phi)$ for all $\phi \in \mathbb{R}$ with

(1) $\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]$ for all $\phi \in \mathbb{R}$;
(2) for $\phi_1 > \phi_2$ and $E_i \in \mathcal{P}(\phi_i), i = 1, 2$, we have $\mathrm{Hom}(E_1, E_2) = 0$; and
(3) for any $E \in \mathcal{D}$ there is a sequence of maps $0 = E_0 \xrightarrow{i_1} E_1 \to \ldots \xrightarrow{i_m} E_m = E$ and of real numbers $\phi_1 > \phi_2 > \cdots > \phi_m$ such that the cone of i_j is in $\mathcal{P}(\phi_j)$ for $j = 1, \ldots, m$.

The non-zero objects of $\mathcal{P}(\phi)$ are called *semistable of phase* ϕ; its simple objects are called *stable*. The sequence of maps in (3) is called the Harder-Narasimhan (HN) filtration of E.

Definition 2.2. A pre-stability condition on $\mathcal{D}$ is a pair $\sigma = (Z, \mathcal{P})$ where $Z \colon \Lambda \to \mathbb{C}$ is a group homomorphism and $\mathcal{P}$ a slicing, such that for all $0 \neq E \in \mathcal{P}(\phi)$, we have $Z(v(E)) \in \mathbb{R}_{>0} \cdot e^{i\pi\phi}$.

We will abuse notation and write $Z(E)$ instead of $Z(v(E))$.

Basic properties. Let $\mathrm{GL}_2^+(\mathbb{R})$ denote the group of real 2×2-matrices with positive determinant. Since $\mathrm{GL}_2^+(\mathbb{R})$ acts on S^1, its universal cover $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ acts on the universal cover $\mathbb{R} \to S^1$ given explicitly by $\phi \mapsto e^{i\pi\phi}$. For $\tilde{g} \in \widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ we will write g for the corresponding element of $\mathrm{GL}_2^+(\mathbb{R})$, and $\tilde{g}.\phi$ for the given action on $\mathbb{R}$.

Proposition 2.3. *There is a natural action of $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ on the set of pre-stability conditions given by $\tilde{g}.(Z, \mathcal{P}) = (g \circ Z, \mathcal{P}')$ where $\mathcal{P}'(\tilde{g}.\phi) = \mathcal{P}(\phi)$.*

The *heart of a bounded t-structure* is a full subcategory $\mathcal{A} \subset \mathcal{D}$ such that

$$\mathcal{S}(\phi) := \begin{cases} \mathcal{A}[\phi] & \text{if } \phi \in \mathbb{Z} \\ \emptyset & \text{if } \phi \notin \mathbb{Z} \end{cases}$$

is a slicing (see [Bri07, Lemma 3.2]). It is automatically an abelian subcategory; and stability conditions on $\mathcal{D}$ can be constructed from slope-stability in $\mathcal{A}$.

Definition 2.4. A stability function Z on an abelian category $\mathcal{A}$ is a morphism $Z \colon K(\mathcal{A}) \to \mathbb{C}$ of abelian groups such that for all $0 \neq E \in \mathcal{A}$, the complex number $Z(E)$ is in the semiclosed upper half plane

$$\mathbb{H} := \{z \in \mathbb{C} : \Im z > 0 \text{ or } z \in \mathbb{R}_{<0}\}.$$

For $0 \neq E \in \mathcal{A}$ we define its phase by $\phi(E) := \frac{1}{\pi} \arg Z(E) \in (0, 1]$. An object $E \in \mathcal{A}$ is called Z-semistable if for all subobjects $A \hookrightarrow E$, we have $\phi(A) \leq \phi(E)$.

Definition 2.5. We say that a stability function Z on an abelian category $\mathcal{A}$ satisfies the *HN property* if every object $E \in \mathcal{A}$ admits a Harder-Narasimhan (HN) filtration: a sequence $0 = E_0 \hookrightarrow E_1 \hookrightarrow E_2 \hookrightarrow \ldots \hookrightarrow E_m = E$ such that E_i/E_{i-1} is Z-semistable for $i = 1, \ldots, m$, with

$$\phi\left(E_1/E_0\right) > \phi\left(E_2/E_1\right) > \cdots > \phi\left(E_m/E_{m-1}\right).$$

Proposition 2.6 ([Bri07, Proposition 5.3])**.** *To give a pre-stability condition on $\mathcal{D}$ is equivalent to giving a heart $\mathcal{A}$ of a bounded t-structure, and a stability function Z on $\mathcal{A}$ with the HN property.*

Here we tacitly assume that Z factors via $K(\mathcal{A}) = K(\mathcal{D}) \xrightarrow{v} \Lambda$. Given $(Z, \mathcal{A})$, the slicing is determined by setting $\mathcal{P}(\phi)$ to be the Z-semistable objects $E \in \mathcal{A}$ of phase ϕ for $\phi \in (0, 1]$. Conversely, given $(Z, \mathcal{P})$, the heart $\mathcal{A} = \mathcal{P}(0, 1]$ is the smallest extension-closed subcategory of $\mathcal{D}$ containing $\mathcal{P}(\phi)$ for $\phi \in (0, 1]$. More generally, $\mathcal{P}(\phi, \phi + 1]$ is a heart for every $\phi \in \mathbb{R}$.

Definition 2.7. A stability condition σ is a pre-stability condition that satisfies the support property in the sense of Definition 1.1 with respect to some quadratic form Q on $\Lambda \otimes \mathbb{R}$.

Topology and local injectivity. There is a generalised metric, and thus a topology, on the set of slicings $\text{Slice}(\mathcal{D})$ given as follows. Given two slicings $\mathcal{P}, \mathcal{Q}$, we write $\phi^{\pm}(E)$ and $\psi^{\pm}(E)$ for the largest and smallest phase in the associated HN filtration of an object E for $\mathcal{P}$ and $\mathcal{Q}$, respectively. Then we define the distance of $\mathcal{P}$ and $\mathcal{Q}$ by

$$d(\mathcal{P}, \mathcal{Q}) := \sup\left\{\left|\phi^+(E) - \psi^+(E)\right|, \left|\phi^-(E) - \psi^-(E)\right| : E \in \mathcal{D}\right\} \in [0, +\infty].$$

We recall that this distance can be computed by considering $\mathcal{P}$-semistable objects alone:

Lemma 2.8 ([Bri07, Lemma 6.1])**.** *We have $d(\mathcal{P}, \mathcal{Q}) = d'(\mathcal{P}, \mathcal{Q})$, where the latter is defined by*

$$d'(\mathcal{P}, \mathcal{Q}) := \sup\left\{\psi^+(E) - \phi, \phi - \psi^-(E) : \phi \in \mathbb{R}, 0 \neq E \in \mathcal{P}(\phi)\right\}.$$

Proof. The inequality $d(\mathcal{P}, \mathcal{Q}) \geq d'(\mathcal{P}, \mathcal{Q})$ is immediate. For the converse, consider $E \in \mathcal{D}$, and let A_i be one of its HN factors with respect to $\mathcal{P}$. Then $\psi^+(A_i) \leq \phi(A_i) + d'(\mathcal{P}, \mathcal{Q}) \leq \phi^+(E) + d'(\mathcal{P}, \mathcal{Q})$. Hence E admits no maps from $\mathcal{Q}$-stable objects of phase bigger than $\phi^+(E) + d'(\mathcal{P}, \mathcal{Q})$, and so $\psi^+(E) \leq \phi^+(E) + d'(\mathcal{P}, \mathcal{Q})$. The analogous inequality $\psi^-(E) \geq \phi^-(E) - d'(\mathcal{P}, \mathcal{Q})$ follows similarly.

Finally, for $E \in \mathcal{D}$ we have a non-zero map $A_1 \to E$ with $A_1 \in \mathcal{P}(\phi^+(E))$. Therefore, $\psi^+(E) \geq \psi^-(A_1) \geq \phi^-(A_1) - d'(\mathcal{P}, \mathcal{Q}) = \phi^+(E) - d'(\mathcal{P}, \mathcal{Q})$, and so $|\psi^+(E) - \phi^+(E)| \leq d'(\mathcal{P}, \mathcal{Q})$. Combined with the same inequality for ψ^-, the claim follows. $\square$

The topology on $\mathrm{Stab}_\Lambda(\mathcal{D})$ is the coarsest topology such that both forgetful maps

$$\mathrm{Stab}_\Lambda(\mathcal{D}) \to \mathrm{Slice}(\mathcal{D}), \quad (Z, \mathcal{P}) \mapsto \mathcal{P}$$
$$\mathcal{Z} \colon \mathrm{Stab}_\Lambda(\mathcal{D}) \to \mathrm{Hom}(\Lambda, \mathbb{C}), \quad (Z, \mathcal{P}) \mapsto Z$$

are continuous.

The following Lemma is a variant of [Bri07, Lemma 6.4]:

Lemma 2.9. *Assume that $\sigma = (Z, \mathcal{P})$ and $\tau = (W, \mathcal{Q})$ are two pre-stability conditions such that σ satisfies the support property with respect to Q, such that*

$$(1) \qquad \frac{|W(v) - Z(v)|}{|Z(v)|} < \sin \pi\epsilon \quad \text{for all } v \in \Lambda \text{ with } Q(v) \geq 0,$$

and such that either $d(\mathcal{P}, \mathcal{Q}) < \frac{1}{4}$, or that σ, τ have the same heart $\mathcal{P}(0, 1] = \mathcal{Q}(0, 1]$. Then $d(\mathcal{P}, \mathcal{Q}) < \epsilon$.

Of course, (1) means in particular that the phases of $W(v)$ and $Z(v)$ differ by at most ϵ.

Proof. We want to apply Lemma 2.8, so let us consider an object $E \in \mathcal{P}(\phi)$. In the first case, $d(\mathcal{P}, \mathcal{Q}) < \frac{1}{4}$, we apply this assumption twice to see that every HN filtration factor of E with respect to $\mathcal{Q}$ is contained in $\mathcal{Q}(\phi - \frac{1}{4}, \phi + \frac{1}{4}) \subset \mathcal{P}(\phi - \frac{1}{2}, \phi + \frac{1}{2}] =: \mathcal{A}$. In particular, the first HN filtration factor $E_1 \to E$ of E with respect to $\mathcal{Q}$ is a subobject of E in the abelian category $\mathcal{A}$. The analogous claim is obvious in the second case for $\mathcal{A} = \mathcal{P}(0, 1]$. Therefore, every HN factor F of E_1 with respect to $\mathcal{P}$ has phase at most ϕ. By (1), it follows that $W(F)$ has phase less than $\phi + \epsilon$, and thus the same holds for the phase $\phi^+(E)$ of $W(E_1)$.

A similar argument shows $\psi^-(E) > \phi(E) - \epsilon$, thus proving the claim. $\square$

Corollary 2.10. *The map $\mathcal{Z} \colon \mathrm{Stab}_\Lambda(\mathcal{D}) \to \mathrm{Hom}(\Lambda, \mathbb{C})$ is locally injective.*

Moreover, consider a section $U \to \mathrm{Stab}_\Lambda(\mathcal{D}), Z \mapsto \sigma_Z = (Z, \mathcal{P}_Z)$ of $\mathcal{Z}$ defined on a subset $U \subset \mathrm{Hom}(\Lambda, \mathbb{C})$, such that every σ_Z satisfies the support property with respect to Q. Assume that U can be covered by open subsets V_i such that for all $Z, Z' \in V_i$ we have $d(\mathcal{P}_Z, \mathcal{P}_{Z'}) < \frac{1}{4}$. Then this section is continuous.

Proof. For the first statement, we just set $Z = W$ in the Lemma, to obtain $d(\mathcal{P}, \mathcal{Q}) = 0$.

For the second statement, we only need to show that (1) holds in a neighbourhood of $Z \in \mathrm{Hom}(\Lambda, \mathbb{C})$ when Q is negative definite on $\mathrm{Ker}\, Z$. Choose any metric on $\Lambda_\mathbb{R}$; then clearly (1) only needs to be checked for vectors with unit length. Since $Q(\_) \geq 0$ defines a compact subset of the unit sphere, on which $|Z(v)|$ is a positive continuous function, the claim follows. $\square$

Morever, Proposition 2.3 gives a *continuous* action of $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ on $\mathrm{Stab}_\Lambda(\mathcal{D})$.

3. Harder-Narasimhan filtrations via the Harder-Narasimhan polygon

Throughout this section, let $\mathcal{A}$ be an abelian category with a stability function Z.

Definition 3.1. The *Harder-Narasimhan polygon* $\mathrm{HN}^Z(E)$ of an object $E \in \mathcal{A}$ is the convex hull of the central charges $Z(A)$ of all subobjects $A \subset E$ of E.

(The trivial subobjects $A = 0$ or $A = E$ are included in the definition.) The idea to consider this convex set in the context of slope-stability goes back at least 40 years [Sha76].

Definition 3.2. We say that the Harder-Narasimhan polygon $\mathrm{HN}^Z(E)$ of an object $E \in \mathcal{A}$ is *polyhedral on the left* if the set has finitely many extremal points $0 = z_0, z_1, \ldots, z_m = Z(E)$ such that $\mathrm{HN}^Z(E)$ lies to the right of the path $z_0 z_1 z_2 \ldots z_m$; see fig. 1.

In other words, the intersection of $\mathrm{HN}^Z(E)$ with the closed half-plane to the left of the line through 0 and $Z(E)$ is the polygon with vertices $z_0, z_1, \ldots, z_m$. Our proof of Theorem 1.2 is based on the following well-known statement; we provide a proof for completeness:

Proposition 3.3. *The object E has a Harder-Narasimhan filtration with respect to Z if and only if its Harder-Narasimhan polygon $\mathrm{HN}^Z(E)$ is polyhedral on the left.*
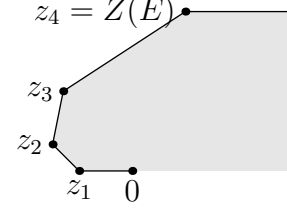

FIGURE 1. Polyhedral on the left

Assume that $\mathrm{HN}^Z(E)$ is polyhedral on the left. For each $i = 1, \ldots, m$, choose a subobject $E_i \subset E$ such that $Z(E_i) = z_i$. (This exists as z_i is extremal.)

Lemma 3.4. *This is a filtration, i.e. $E_{i-1} \subset E_i$ for $i = 1, \ldots, m$.*

Proof. Let $A := E_{i-1} \cap E_i \subset E$ be the intersection of two subsequent objects, and $B := E_{i-1} + E_i \subset E$ be their span inside E. Then there is a short exact sequence

$$A \hookrightarrow E_{i-1} \oplus E_i \twoheadrightarrow B.$$

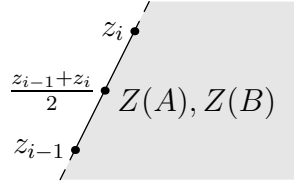Hence the midpoint of $Z(A)$ and $Z(B)$ is also the midpoint of z_{i-1} and z_i, see also figure 2.
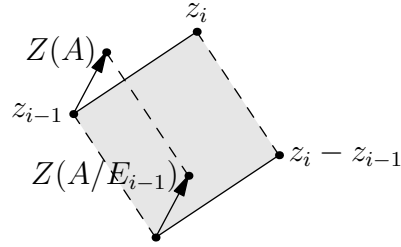

FIGURE 2. Lemma 3.4

FIGURE 3. Lemma 3.5

On the other hand, $Z(A), Z(B)$ lie in $\mathrm{HN}^Z(E)$; by convexity and the choice of z_{i-1}, z_i, they both have to lie either in the open half-plane to the right of the line $(z_{i-1} z_i)$, or on the line segment $\overline{z_{i-1} z_i}$. The former would be a contradiction to the previous paragraph, and so $Z(A) \in \overline{z_{i-1} z_i}$.

Since $A \subset E_{i-1}$, this implies $Z(A) = z_{i-1}$ and $A \cong E_{i-1}$; therefore, $E_{i-1} \subset E_i$.   □

Lemma 3.5. *The filtration quotient E_i/E_{i-1} is semistable.*

Proof. Otherwise, there is an object A with $E_{i-1} \subset A \subset E_i$ such that A/E_{i-1} has bigger phase than E_i/E_{i-1}, see fig. 3. It follows that $Z(A)$ lies to the left of the line segment $\overline{z_{i-1} z_i}$. Since $A \subset E$ and hence $Z(A) \in \mathrm{HN}^Z(E)$, this is a contradiction.   □

Proof of Proposition 3.3. The phase of E_i/E_{i-1} is determined by the argument of $z_i - z_{i-1}$; by convexity this shows $\phi(E_1/E_0) > \cdots > \phi(E_m/E_{m-1})$, and so the E_i form a HN filtration.

Conversely, assume that we are given a HN filtration $0 = E_0 \hookrightarrow E_1 \hookrightarrow \ldots \hookrightarrow E_m$ and a subobject $A \hookrightarrow E$. We have to show that $Z(A)$ lies to the right of the path $z_0 z_1 \ldots z_m$ with vertices $z_i := Z(E_i)$. By induction on m, we may assume that $Z(A \cap E_{m-1})$ lies to the right of the path $z_0 z_1 \ldots z_{m-1}$. On the other hand, $A/(A \cap E_{m-1})$ is a subobject of E_m/E_{m-1}, which is semistable; thus the central charge of $Z(A/(A \cap E_{m-1}))$ lies to the right of the line segment from 0 to $z_m - z_{m-1}$. Therefore, $Z(A) = Z(A \cap E_{m-1}) + Z(A/(A \cap E_{m-1}))$ lies to the right of the path $z_0 z_1 \ldots z_m$ as claimed. $\square$

Corollary 3.6. *Given $E \in \mathcal{A}$, assume that there are only finitely many classes $v(A)$ of subobjects $A \subset E$ with $\Re Z(A) < \max\{0, \Re Z(E)\}$. Then E admits a HN filtration.*

4. LINEAR ALGEBRA LEMMAS

Throughout Sections 4, 5 and 6, we fix a quadratic form Q on $\Lambda_{\mathbb{R}}$ with:

Assumption 4.1. *The quadratic form Q has signature $(2, \operatorname{rk}\Lambda - 2)$.*

Lemma 4.2. *Let $Z \colon \Lambda \to \mathbb{C}$ be a group homomorphism such that Q is negative definite on $K := \operatorname{Ker} Z \subset \Lambda_{\mathbb{R}}$. Let $\|\cdot\|$ be the norm on K associated to $-Q$, and let $p \colon \Lambda_{\mathbb{R}} \to \operatorname{Ker} Z$ be the orthogonal projection with respect to Q. After replacing Z by suitable $\operatorname{GL}_2^+(\mathbb{R})$-translate, we have*

$$Q(v) = |Z(v)|^2 - \|p(v)\|^2. \tag{2}$$

Proof. Let $K^{\perp}$ be the orthogonal complement of K. As $Z|_{K^{\perp}}$ is injective, Assumption 4.1 can only hold if Q is positive definite on $K^{\perp}$, and if $Z|_{K^{\perp}} \colon K^{\perp} \to \mathbb{C}$ is an isomorphism. Up to the $\operatorname{GL}_2^+(\mathbb{R})$-action, we may assume this to be an isometry. Then the the claim follows. $\square$

Remark 4.3. In [Bri08], this normalisation is used with the Mukai quadratic form replacing Q.

Consider the subset in $\operatorname{Hom}(\Lambda, \mathbb{C})$ of central charges whose kernel is negative definite with respect to Q; let $\mathcal{P}_Z(Q)$ be its connected component containing Z.

Lemma 4.4. *Assume that Q and Z satisfy equation (2).*

(1) For each $Z' \in \mathcal{P}_Z(Q)$ there exists a unique element $g \in \operatorname{GL}_2^+(\mathbb{R})$ and a linear map where $u \colon \operatorname{Ker} Z \to \mathbb{C}$ with $\|u\| < 1$ such that

$$gZ' = Z + u \circ p.$$

(2) There exist $u_{\mathbb{R}}$, $u_{i\mathbb{R}}$, Z_1 and $g_1, g_2 \in \operatorname{GL}_2^+(\mathbb{R})$ depending continuously on Z' where
   (a) $u_{\mathbb{R}} \colon \operatorname{Ker} Z \to \mathbb{R}$ and $u_{i\mathbb{R}} \colon \operatorname{Ker} Z_1 \to \mathbb{R}$ satisfy $\|u_{\mathbb{R}}\| < 1$, $\|u_{i\mathbb{R}}\| < 1$,
   (b) $Z_1 := Z + u_{\mathbb{R}} \circ p$,
   (c) $g_1 \circ Z_1$ satisfies equation (2),
   (d) $g_2 Z' = g_1 Z_1 + i u_{i\mathbb{R}} \circ p_1$ where $p_1 \colon \Lambda_{\mathbb{R}} \to \operatorname{Ker} Z_1$ is the orthogonal projection.

Proof. The restriction of Z' to the orthogonal complement $K^{\perp}$ is an isomorphism for all $Z' \in \mathcal{P}_Z(Q)$. Hence there exists a unique $g \in \operatorname{GL}_2(\mathbb{R})$ such that $gZ'|_{K^{\perp}} = Z|_{K^{\perp}}$; since $\mathcal{P}_Z(Q)$ is connected, we in fact have $g \in \operatorname{GL}_2^+(\mathbb{R})$. Let $u := Z'|_{\operatorname{Ker} Z}$; then $gZ' - Z = u \circ p$ holds both on $\operatorname{Ker} Z$ and on $K^{\perp}$, and thus on all of $\Lambda_{\mathbb{R}}$.

Next, we prove $\|u\| < 1$. Otherwise, let $v \in K$ with $\|v\| = 1$ and $|u(v)| \geq 1$, and let $\overline{v} \in K^{\perp}$ be such that $Z(\overline{v}) = u(v)$. Then $Z'(v - \overline{v}) = 0$, but $Q(v - \overline{v}) = |u(v)|^2 - \|v\|^2 \geq 0$ in

contradiction to the assumption that Q is negative definite on $\operatorname{Ker} Z'$. This completes the proof of (1).

Now let $u_{\mathbb{R}} := \Re u$, and let Z_1 be as above. If we write $\Lambda_{\mathbb{R}} = K \oplus K^{\perp}$ and identify $K^{\perp}$ with $\mathbb{C}$, then $\operatorname{Ker} Z_1$ is the graph of $-u_{\mathbb{R}}$, and its orthogonal complement is the graph of $\mathbb{C} \to K, z \mapsto \Re z \cdot u^{\vee}$, where $u^{\vee}$ corresponds to u under the identification $K \cong K^{\vee}$ induced by the symmetric form associated to $\|\cdot\|$. A straightforward computation shows that $g_1 \circ Z_1$ induces an isometry $(\operatorname{Ker} Z_1)^{\perp} \to \mathbb{C}$ for $g_1 := \begin{pmatrix} \left(1 + \|u_{\mathbb{R}}\|^2\right)^{-\frac{1}{2}} & 0 \\ 0 & 1 \end{pmatrix} \in \operatorname{GL}_2^+(\mathbb{R})$; by the proof of Lemma 4.2 this implies (2c). Moreover, $g_1 g Z' - g_1 Z_1 = g_1 \circ \Im u \circ p$ is completely imaginary. Applying part (1) to $g_1 g Z'$ and $g_1 Z_1$ then shows part (2d). $\qquad \square$

5. REAL VARIATIONS OF THE CENTRAL CHARGE

The key lemma, proved in this section and the next, treats the case where only the real part of the central charge is varying:

Lemma 5.1. *Consider a stability condition $\sigma = (Z, \mathcal{P})$, satisfying the support property with respect to Q, and Q, p as in Lemma 4.2. Given $u \colon \operatorname{Ker} Z \to \mathbb{R}$ with $\|u\| < 1$, there is a stability condition $\tau = (W, \mathcal{Q})$ with $W = Z + u \circ p$, satisfying the support property with respect to Q, and with $d(\mathcal{P}, \mathcal{Q}) < \frac{\|u\|}{2}$.*

Proof of Theorem 1.2, assuming Lemma 5.1 and Assumption 4.1. By the $\widetilde{\operatorname{GL}}_2^+(\mathbb{R})$-action, Lemma 5.1 automatically also applies to purely imaginary variations of the central charge, as in Lemma 4.4.(2d). We may assume that Q and the central charge satisfy (2); then Lemmas 4.4 and 5.1 combined with the $\widetilde{\operatorname{GL}}_2^+(\mathbb{R})$-action give a set-theoretic section $\mathcal{P}_Z(Q) \to \operatorname{Stab}_{\Lambda}(\mathcal{D})$.

Since $\operatorname{GL}_2^+(\mathbb{R})$ acts continuously on the compact set S^1, there exists an open neighbourhood $B_{\epsilon}(1)$ of $1 \in \widetilde{\operatorname{GL}}_2^+(\mathbb{R})$ such that $|\tilde{g}.\phi - \phi| < \epsilon$ for all $\tilde{g} \in B_{\epsilon}(1)$. It follows that if $(\mathcal{Q}, W) = \tilde{g}.(\mathcal{P}, Z)$, then $d(\mathcal{Q}, \mathcal{P}) < \epsilon$. Then there is an open neighborhood $U_{\epsilon}(Z)$ of Z in $\mathcal{P}_Z(Q)$ where, in the notation of Lemma 4.4, $\|u_{\mathbb{R}}\| < \epsilon$, $\|u_{i\mathbb{R}}\| < \epsilon$, and g_1, g_2 lift to $\tilde{g}_1, \tilde{g}_2 \in B_{\epsilon}(1)$. Then our lift $\tau = (Z, \mathcal{Q})$ of any Z' in this open neighbourhood satisfies $d(\mathcal{Q}, \mathcal{P}) < \frac{\epsilon}{2} + \epsilon + \frac{\epsilon}{2} + \epsilon$.

By Corollary 2.10, our section is continuous on $U_{\epsilon}(Z)$ for $\epsilon \leq \frac{1}{12}$. Since the neighbourhoods $U_{\frac{1}{12}}(Z')$ for $Z' \in \mathcal{P}_Z(\mathcal{Q})$ cover $\mathcal{P}_Z(\mathcal{Q})$, and since the lifts constructed on each such neighbourhood automatically agree on the overlaps by the local injectivity of $\mathcal{Z}$, this proves the Theorem. $\qquad \square$

Lemma 5.1 is simpler to prove since it allows (and forces) us to leave the heart $\mathcal{A} := \mathcal{P}(0, 1]$ unchanged: we will apply Proposition 2.6 and prove that $(\mathcal{A}, W)$ produces a stability condition.

Lemma 5.2. *Let Z, u be as in Lemma 5.1. Then $W = Z + u \circ p$ is a stability function on $\mathcal{A}$.*

Proof. Consider $E \in \mathcal{A}$; if $\Im Z(E) = \Im W(E) > 0$, there is nothing to prove. Otherwise, E is semistable with $Z(E) \in \mathbb{R}_{<0}$ and thus $\|p(E)\| \leq -Z(E)$. From $\|u\| < 1$ we conclude

$$W(E) = Z(E) + u \circ p(E) \leq Z(E) + \|u\| \, \|p(E)\| < Z(E) - Z(E) = 0. \qquad \square$$

We will use Proposition 3.3 and Corollary 3.6 to prove that $(\mathcal{A}, W)$ satisfies the HN property.

Let us define the *mass* $m^Z(E)$ of $E \in \mathcal{A}$ with respect to Z as the length of the boundary of $\operatorname{HN}^Z(E)$ on the left between 0 and $Z(E)$.
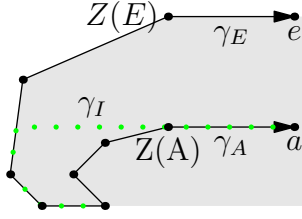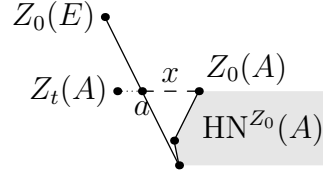
FIGURE 4. Proof of Lemma 5.5

FIGURE 5. Proof of Lemma 5.7

Lemma 5.3. *For all $E \in \mathcal{A}$ we have $\|p(E)\| \leq m^Z(E)$.*

Proof. If E is semistable, then $0 \leq Q(E) = |Z(E)|^2 - \|p(E)\|^2 = \left(m^Z(E)\right)^2 - \|p(E)\|^2$, which is exactly the claim. Otherwise, consider the HN filtration $E_0 \hookrightarrow E_1 \hookrightarrow \ldots \hookrightarrow E_m = E$. Combined with the triangle inequality, this gives

$$\|p(E)\| \leq \sum_i \|p(E_i/E_{i-1})\| \leq \sum_i |Z(E_i/E_{i-1})| = \sum_i |Z(E_i) - Z(E_{i-1})| = m^Z(E). \quad \square$$

The following Lemma needs no proof:

Lemma 5.4. *If $A \subset E$, then $\mathrm{HN}^Z(A) \subset \mathrm{HN}^Z(E)$.*

Lemma 5.5. *Given any subobject $A \subset E$, we have*

$$m^Z(A) - \Re Z(A) \leq m^Z(E) - \Re Z(E).$$

Proof. This follows from the previous Lemma, convexity and a picture, see fig. 4. Indeed, choose $x > \Re Z(A), \Re Z(E)$; let $a = x + i\Im Z(A)$ and $e = x + i\Im Z(E)$. Let γ_A be the path that follows by boundary of $\mathrm{HN}^Z(A)$ from 0 to $Z(A)$, and then continues horizontally to a; similarly γ_E follows the boundary of $\mathrm{HN}^Z(E)$ and then continues to e. Their lengths are given as

$$|\gamma_A| = m^Z(A) + x - \Re Z(A), \quad |\gamma_E| = m^Z(E) + x - \Re Z(E).$$

On the other hand, convexity and Lemma 5.4 imply $|\gamma_A| \leq |\gamma_E|$; for example, if γ_I denotes the intermediate path that follows the boundary of $\mathrm{HN}^Z(E)$ up to height $\Im Z(A)$ and then goes horizontally to a, we clearly have $|\gamma_A| \leq |\gamma_I| \leq |\gamma_E|$. $\quad \square$

Lemma 5.6. *Given $C \in \mathbb{R}$, the set of $v(A) \in \Lambda$ for subobjects $A \subset E$ with $\Re W(A) < C$ is finite.*

Proof. Given any such A, we use Lemmas 5.3 and 5.5 to obtain

$$C > \Re W(A) \geq \Re Z(A) - \|u\| \|p(A)\| > (1 - \|u\|)\Re Z(A) - \|u\| \left(m^Z(A) - \Re Z(A)\right)$$
$$\geq (1 - \|u\|)\Re Z(A) - \|u\| \left(m^Z(E) - \Re Z(E)\right).$$

Since $\|u\| < 1$, this bounds $\Re Z(A)$ from above. On the other hand, $Z(A) \in \mathrm{HN}^Z(E)$, and thus $Z(A)$ is constrained to lie in a compact region of $\mathbb{C}$. Using Lemmas 5.5 and 5.3 again, this gives an upper bound first for $m^Z(A)$ and consequently for $\|p(A)\|$. Hence $v(A)$ is contained in a compact region of $\Lambda \otimes \mathbb{R}$ depending only on E and C, and the claim follows. $\quad \square$

Therefore, Corollary 3.6 implies the existence of HN filtrations for W on $\mathcal{A}$. By Proposition 2.6, we have constructed a pre-stability condition $\tau = (\mathcal{A}, W)$; write $\mathcal{Q}$ for the associated slicing.

Lemma 5.7. *The pre-stability condition $\tau = (W, \mathcal{Q})$ satisfies $d(\mathcal{P}, \mathcal{Q}) < \frac{\|u\|}{2}$.*

Proof. We apply Lemma 2.9: by construction, by (2), and by the assumption $\|u\| < 1$ we have

$$\frac{|W(v) - Z(v)|}{|Z(v)|} = \frac{|u \circ p(v)|}{|Z(v)|} \leq \|u\| \frac{\|p(v)\|}{|Z(v)|} \leq \|u\| < \sin \pi \frac{\|u\|}{2} \text{ for } v \in \Lambda \text{ with } Q(v) \geq 0.$$

$\square$

6. THE SUPPORT PROPERTY IS PRESERVED

It remains to show that $(\mathcal{A}, W)$ satisfies the support property with respect to Q, i.e. $Q(v(E)) \geq 0$ for all W-stable $E \in \mathcal{A}$. The basic reason is that this inequality preserved by wall-crossing:

Lemma 6.1. *Let $\sigma = (Z, \mathcal{P})$ be pre-stability condition. Assume that Q is a non-degenerate quadratic form on $\Lambda_{\mathbb{R}}$ of signature $(2, \operatorname{rk} \Lambda - 2)$ such that Q is negative definite on $\operatorname{Ker} Z$. If E is strictly σ-semistable and admits a Jordan-Hölder filtration with factors $E_1, \ldots, E_m$, and if $Q(v(E_i)) \geq 0$ for $i = 1, \ldots, m$, then $Q(v(E)) \geq 0$.*

Proof. We apply Lemma 4.2; then $Q(v) \geq 0$ is equivalent to $|Z(v)| \geq \|p(v)\|$. We obtain

$$|Z(E)| = \sum_i |Z(E_i)| \geq \sum_i \|p(v(E_i))\| \geq \left\| \sum_i p(v(E_i)) \right\| = \|p(v(E))\|$$

where the first equality holds since the central charges of all E_i are aligned, the first inequality holds by assumption, and the second inequality is the triangle inequality. $\square$

The proof strategy is thus clear: we use wall-crossing for the path of stability functions $Z_t = Z + t \cdot u \circ p$ on $\mathcal{A}$, for $0 \leq t \leq 1$. If $E \in \mathcal{A}$ is Z_1-stable with $Q(v(E)) < 0$, then it must be Z_0-unstable; wall-crossing gives a $t \in [0, 1)$ such that E is strictly Z_t-semistable; by the Lemma, one of its Jordan-Hölder factors will also violate the inequality, and we proceed by induction. To conclude, we have to show that we can find such a wall, and that this process terminates.

Lemma 6.2. *Given two objects $A, E \in \mathcal{A}$, denote their phases with respect to Z_t by $\phi^t(A), \phi^t(E)$, respectively. If the set of $t \in [0, 1]$ with $\phi^t(A) \geq \phi^t(E)$ is non-empty, then it is a closed subinterval of $[0, 1]$ containing one of its endpoints.*

Proof. The condition is equivalent to $\frac{-\Re Z_t(A)}{\Im Z_t(A)} \geq \frac{-\Re Z_t(E)}{\Im Z_t(E)}$, which is a linear inequality in t. $\square$

Consider the polygon whose vertices are the extremal points of $\operatorname{HN}^{Z_0}(E)$ on the left; we will call this the *truncated HN polygon of E*, see fig. 6. Note that if $A \subset E$ is a subobject with $\phi_0(A) \geq \phi_0(E)$, then $Z_0(A)$ is contained in the truncated HN polygon of E; by Lemmas 5.5 and 5.3 there are only finitely many classes $v(A)$ of such subobjects.

FIGURE 6. The truncated HN polygon
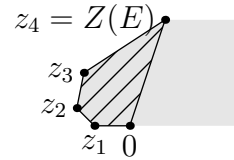
Lemma 6.3. *Every Z_1-semistable object $E \in \mathcal{A}$ satisfies $Q(E) \geq 0$.*

Proof. Otherwise, E must be Z_0-unstable. By Lemma 6.2 and the following observation, there are only finitely many classes $v(A)$ of subobjects $A \hookrightarrow E$ that destabilise E with respect Z_t for any $t \in [0, 1]$. Hence there is a wall $t_1 \in (0, 1]$ such that E is strictly semistable with respect to Z_{t_1}, and moreover E admits a Jordan-Hölder filtration with respect to Z_{t_1}. By Lemma 6.1, there are subobjects $G_1 \hookrightarrow F_1 \hookrightarrow E$ of the same phase, such that $Q(v(F_1/G_1)) < 0$.

Applying the same logic to F_1/G_1, we obtain $t_2 \in (0, t_1)$ and subobjects $G_1 \subset G_2 \subset F_2 \subset F_1 \subset E$ such that $F_2/G_1, G_2/G_1$ and F_1/G_1 all have the same phase with respect to t_2, and such that $Q(v(F_2/G_2)) < 0$. Continuing by induction, we obtain a sequence $t_1 > t_2 > t_3 > \dots$ in $(0, 1)$ and chains of subobjects $G_1 \subset G_2 \subset G_3 \subset \dots \subset E$ and $E \supset F_1 \supset F_2 \supset F_3 \supset \dots$

Lemma 6.2 gives $\phi^{t_2}(F_1) \geq \phi^{t_2}(E)$ and $\phi^{t_2}(G_1) \geq \phi^{t_2}(E)$. Since the central charge of $Z_{t_2}(F_2)$ lies on the line segment connecting $Z_{t_2}(F_1)$ and $Z_{t_2}(G_1)$, we also have $\phi^{t_2}(F_2) \geq \phi^{t_2}(E)$ (and therefore $\phi^t(F_2) \geq \phi^t(E)$ for all $t \in [0, t_2]$; similarly for G_2. Continuing this argument by induction, we see that $Z_0(F_i)$ and $Z_0(G_i)$ are all contained in the truncated HN polygon of E. Thus this process terminates. $\square$

This concludes the proof of Lemma 5.1, and thus of Theorem 1.2 under the Assumption 4.1.

7. REDUCTIONS

Finally, we will show that we can always reduce the situation to the case where Assumption 4.1 holds. By abuse of language, we call a quadratic form degenerate or non-degenerate if the associated symmetric bilinear form is degenerate or non-degenerate, respectively.

Lemma 7.1. *Assume that the quadratic form Q on $\Lambda_{\mathbb{R}}$ is degenerate. Then there exists an injective map $\Lambda_{\mathbb{R}} \hookrightarrow \overline{\Lambda}$ of real vector spaces and a non-degenerate quadratic form $\overline{Q}$ on $\overline{\Lambda}$, extending Q, such that any central charge $Z \colon \Lambda_{\mathbb{R}} \to \mathbb{C}$ with kernel negative definite with respect to Q extends to a central charge $\overline{Z} \colon \overline{\Lambda} \to \mathbb{C}$ with kernel negative definite with respect to $\overline{Q}$.*

Proof. Let $N \hookrightarrow \Lambda_{\mathbb{R}}$ be the null space of Q; we will only treat the case $\dim_{\mathbb{R}} N = 1$ (otherwise, we can iterate the construction that follows). Choose a splitting $\Lambda_{\mathbb{R}} \cong N \oplus C$; then for $n \in N, c \in C$, we have $Q(n \oplus c) = Q(c)$. Let $\overline{\Lambda_{\mathbb{R}}} := N \oplus N^{\vee} \oplus C$, let q be the canonical quadratic form on the hyperbolic plane $N \oplus N^{\vee}$, and set $\overline{Q} := q \oplus Q|_C$.

Given Z as above, the restriction $Z|_N$ is injective, and we may assume that Z maps N to the real line. Let $n \in N$ be such that $Z(n) = 1$, and let $n^{\vee} \in N^{\vee}$ be the dual vector with $(n, n^{\vee}) = 1$. We claim that for $\alpha \gg 0$, the extension of Z defined by $Z'(n^{\vee}) = \alpha$ has the desired property.

Let $K := \operatorname{Ker} Z$; then the kernel of Z' is contained in $N \oplus N^{\vee} \oplus K$, and given by vectors of the form $a \cdot n - \frac{a}{\alpha} \cdot n^{\vee} + k$ for $k \in K, a \in \mathbb{R}$. For such vectors, we have

$$Q\left(a \cdot n - \frac{a}{\alpha} \cdot n^{\vee} + k\right) = -\frac{2a^2}{\alpha} - \frac{2a}{\alpha}(n^{\vee}, k) + Q(k).$$

This is a quadratic function in a with negative constant term; its discriminant is negative if

$$\alpha > \max\left\{\frac{(n^{\vee}, k)^2}{-Q(k)} \, : \, k \in K, k \neq 0\right\}$$

(which is finite since $-Q(\cdot)$ is a positive definite form on K). $\square$

Replacing Λ by $\Lambda \oplus \mathbb{Z}$ and v by

$$K(\mathcal{D}) \xrightarrow{v} \Lambda \hookrightarrow \Lambda \oplus \mathbb{Z}$$

we can therefore restrict to the case where Q is non-degenerate: given a path Z_t of central charges in $\mathrm{Hom}(\Lambda_\mathbb{R}, \mathbb{C})$ that are negative definite with respect to Q, we can choose extensions $\overline{Z}_t$ as in the Lemma that form a continuous path in $\mathrm{Hom}(\overline{\Lambda}, \mathbb{C})$. If we can lift the latter path to a path of stability conditions $\overline{\sigma}_t = (\overline{Z}_t, \mathcal{P}_t)$ that satisfy the support property with respect to $\overline{Q}$, then $\sigma_t := (Z_t, \mathcal{P}_t)$ is a path of stability conditions satisfying the support property with respect to Q. The reduction to the case where Q has signature $(2, \mathrm{rk}\,\Lambda - 2)$ works similarly:

Lemma 7.2. *Assume that Q is non-degenerate and of signature $(p, \mathrm{rk}\,\Lambda - p)$ for $p \in \{0, 1\}$. Let $\overline{\Lambda} := \Lambda_\mathbb{R} \oplus \mathbb{R}$, and let $\overline{Q}$ be the extension given by $\overline{Q}(v, \alpha) = Q(v) + \alpha^2$ for $v \in \Lambda_\mathbb{R}$ and $\alpha \in \mathbb{R}$. Then any central charge Z on $\Lambda_\mathbb{R}$ whose kernel is negative definite with respect to Q extends to a central charge $\overline{Z}$ on $\overline{\Lambda}$ whose kernel is negative definite with respect to $\overline{Q}$.*

Proof. We claim that there exists $z \in \mathbb{C}$ such that for all $v \in \Lambda_\mathbb{R}$ with $Z(v) = z$, we have $Q(v) < -1$. Indeed, let $K \subset \Lambda_\mathbb{R}$ be the kernel of Z, and let $K^\perp$ be its orthogonal complement. Then clearly we may assume $v \in K^\perp$. Since the restriction of Z to $K^\perp$ is injective, and since $K^\perp$ either has rank one, or has signature $(1, -1)$ with respect to Q, the claim is evident.

Using the claim, we can set $\overline{Z}(v, \alpha) := Z(v) + \alpha z$. $\qquad\qquad\square$

This concludes the proof of Theorem 1.2.

8. APPLICATION

Proof of Corollary 1.3. Using the same arguments as in the previous section, we may assume that the Mukai pairing on Λ has signature $(2, \mathrm{rk}\,\Lambda - 2)$.

By Serre duality, any σ-stable object $E \in \mathcal{D}$ satisfies $\mathrm{Hom}(E, E[i]) = 0$ for $i < 0$ or $i > 3$ and $\mathrm{Hom}(E, E) = \mathbb{C} = \mathrm{Hom}(E, E[2])$; therefore, $(v(E), v(E)) \geq -2$. Moreover, Serre duality induces a non-degenerate symplectic form on $\mathrm{Ext}^1(E, E)$, and it has even dimension; thus $v(E)$ is a root, or $(v(E), v(E)) \geq 0$.

Let $\sigma = (Z, \mathcal{P})$ be a stability condition with $Z \in \mathcal{P}_0(\mathcal{D})$. As in Lemma 4.2 we may assume

$$(v, v) = |Z(v)|^2 - \|p(v)\|^2,$$

where $p\colon \Lambda_\mathbb{R} \to \mathrm{Ker}\,Z$ is the orthogonal projection onto the kernel of Z, and where $\|\cdot\|$ denotes the norm on $\mathrm{Ker}\,Z$ induced by the negative of the Mukai pairing. We claim that

$$(3) \qquad\qquad C := \inf \{|Z(\delta)| \,:\, \delta \in \Lambda, (\delta, \delta) = -2\} > 0.$$

Indeed, if $|Z(\delta)| \leq 1$, then $\|p(\delta)\| \leq \sqrt{3}$; as $|Z(\cdot)| + \|p(\cdot)\|$ is a norm on $\Lambda_\mathbb{R}$, there are only finitely many integral classes satisfying both inequalities. Since $Z(\delta) \neq 0$ by assumption, the claim follows.

Now set

$$Q(v) := (v, v) + \frac{2}{C^2} |Z(v)|^2.$$

Clearly Q is negative definite on $\mathrm{Ker}\,Z$ and depends only on Z. Moreover, $Q(\delta) \geq 0$ for all roots δ, and $Q(v) \geq 0$ for all classes with $(v, v) \geq 0$; therefore, any stability condition $\sigma' = (Z', \mathcal{P}')$ with $Z' \in \mathcal{P}_Z(Q)$ satisfies the support property with respect to Q.

Theorem 1.2 shows that the restriction of $\mathcal{Z}$ to the preimage of $\mathcal{P}_Z(Q)$ is a covering of $\mathcal{P}_Z(Q)$. Since the neighbourhoods $\mathcal{P}_Z(Q)$ cover $\mathcal{P}_0(\mathcal{D})$, this completes the proof. $\qquad\square$

References

[ABCH13] Daniele Arcara, Aaron Bertram, Izzet Coskun, and Jack Huizenga. The minimal model program for the Hilbert scheme of points on $\mathbb{P}^2$ and Bridgeland stability. *Adv. Math.*, 235:580–626, 2013, arXiv:1203.0316.

[ABM15] Rina Anno, Roman Bezrukavnikov, and Ivan Mirković. Stability conditions for Slodowy slices and real variations of stability. *Mosc. Math. J.*, 15(2):187–203, 403, 2015.

[Bay16] Arend Bayer. Wall-crossing implies Brill-Noether. Applications of stability conditions on surfaces, 2016, arXiv:1604.08261.

[BB17] Arend Bayer and Tom Bridgeland. Derived automorphism groups of K3 surfaces of Picard rank 1. *Duke Math. J.*, 166(1):75–124, 2017, arXiv:1310.8266.

[BM11] Arend Bayer and Emanuele Macrì. The space of stability conditions on the local projective plane. *Duke Math. J.*, 160(2):263–322, 2011, arXiv:0912.0043.

[BM14] Arend Bayer and Emanuele Macrì. MMP for moduli of sheaves on K3s via wall-crossing: nef and movable cones, Lagrangian fibrations. *Invent. Math.*, 198(3):505–590, 2014, arXiv:1301.6968.

[BMS16] Arend Bayer, Emanuele Macrì, and Paolo Stellari. The space of stability conditions on abelian threefolds, and on some Calabi-Yau threefolds. *Invent. math.*, pages 1–65, 2016, arXiv:1410.1585.

[BMT14] Arend Bayer, Emanuele Macrì, and Yukinobu Toda. Bridgeland stability conditions on threefolds I: Bogomolov-Gieseker type inequalities. *J. Algebraic Geom.*, 23(1):117–163, 2014, arXiv:1103.5010.

[Bri07] Tom Bridgeland. Stability conditions on triangulated categories. *Ann. of Math. (2)*, 166(2):317–345, 2007, arXiv:math/0212237.

[Bri08] Tom Bridgeland. Stability conditions on $K3$ surfaces. *Duke Math. J.*, 141(2):241–291, 2008, arXiv:math/0307164.

[Bri09a] Tom Bridgeland. Spaces of stability conditions. In *Algebraic geometry—Seattle 2005. Part 1*, volume 80 of *Proc. Sympos. Pure Math.*, pages 1–21. Amer. Math. Soc., Providence, RI, 2009, arXiv:math.AG/0611510.

[Bri09b] Tom Bridgeland. Stability conditions and Kleinian singularities. *Int. Math. Res. Not. IMRN*, 21(21):4142–4157, 2009, arXiv:math/0508257.

[BS15] Tom Bridgeland and Ivan Smith. Quadratic differentials as stability conditions. *Publ. Math. Inst. Hautes Études Sci.*, 121:155–278, 2015.

[GMN13] Davide Gaiotto, Gregory W. Moore, and Andrew Neitzke. Wall-crossing, Hitchin systems, and the WKB approximation. *Adv. Math.*, 234:239–403, 2013, arXiv:0907.3987.

[Huy14] D. Huybrechts. Introduction to stability conditions. In *Moduli spaces*, volume 411 of *London Math. Soc. Lecture Note Ser.*, pages 179–229. Cambridge Univ. Press, Cambridge, 2014, arXiv:1111.1745.

[Ike14] Akishi Ikeda. Stability conditions for preprojective algebras and root systems of Kac-Moody Lie algebras, 2014, arXiv:1402.1392.

[KS08] Maxim Kontsevich and Yan Soibelman. Stability structures, motivic Donaldson-Thomas invariants and cluster transformations, 2008, arXiv:0811.2435.

[LZ16] Chunyi Li and Xiaolei Zhao. Birational models of moduli spaces of coherent sheaves on the projective plane, 2016, arXiv:1603.05035.

[MP14] Sven Meinhardt and Holger Partsch. Quotient categories, stability conditions, and birational geometry. *Geom. Dedicata*, 173:365–392, 2014.

[MS16] Emanuele Macrì and Benjamin Schmidt. Introduction to Bridgeland stability, 2016, arXiv:1607.01262. Lecture notes for the CIMPA-ICTP-Mexico research school 2016.

[Sha76] Stephen S. Shatz. Degeneration and specialization in algebraic families of vector bundles. *Bull. Amer. Math. Soc.*, 82(4):560–562, 1976.

[Tho08] R. P. Thomas. Stability conditions and the braid group. In *Superstring theory*, volume 1 of *Adv. Lect. Math. (ALM)*, pages 209–233. Int. Press, Somerville, MA, 2008, arXiv:math/0212214.

[Tod14] Yukinobu Toda. Derived category of coherent sheaves and counting invariants, 2014, arXiv:1404.3814.

School of Mathematics and Maxwell Institute, University of Edinburgh, James Clerk Maxwell Building, Peter Guthrie Tait Road, Edinburgh, EH9 3FD, United Kingdom

E-mail address: arend.bayer@ed.ac.uk

URL: http://www.maths.ed.ac.uk/˜abayer/