BRIDGELAND STABILITY CONDITIONS ON THREEFOLDS II: AN APPLICATION TO FUJITA'S CONJECTURE

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ABSTRACT. We apply a conjectured inequality on third chern classes of stable two-term complexes on threefolds to Fujita's conjecture. More precisely, the inequality is shown to imply a Reider-type theorem in dimension three which in turn implies that $K_X + 6L$ is very ample when L is ample, and that $5L$ is very ample when K_X is trivial.

CONTENTS

1. INTRODUCTION

A Bogomolov-Gieseker-type inequality on Chern classes of "tilt-stable" objects in the derived category of a threefold was conjectured in [BMT11] in the context of constructing Bridgeland stability conditions. In this paper, we show how the same inequality would allow one to extend Reider's stable-vector bundle technique ([Rei88]) from surfaces to threefolds, and in particular to obtain Fujita's conjecture in the threefold case. This follows a line of reasoning that was suggested in [AB11].

While we use the setup of tilt-stability from [BMT11], this paper is intended to be self-contained, and to be readable by birational geometers with a passing familiarity with derived categories.

Tilt-stability depends on two numerical parameters: an ample class $\omega \in \mathrm{NS}_{\mathbb{Q}}(X)$ and an arbitrary class $B \in \mathrm{NS}_{\mathbb{Q}}(X)$. It is a notion of stability on a particular abelian category, $\mathcal{B}_{\omega,B}$, of two-term complexes in $\mathrm{D}^{\mathrm{b}}(X)$, and codimension three Chern classes of stable

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objects E in this category (and not stable vector bundles) are conjectured to satisfy a Bogomolov-Gieseker inequality in Conjecture 2.3. Assuming this conjecture, we prove the following Reider-type theorem for threefolds:

Theorem 4.1. *Let X be a smooth projective threefold over $\mathbb{C}$, and let L be an ample line bundle on X such that Conjecture 2.3 holds when B and ω are scalar multiples of L. Fix a positive integer α, and assume that L satisfies the following conditions:*

- *(A) $L^3 > 49\alpha$;*
- *(B) $L^2.D \geq 7\alpha$, for all integral divisor classes D with $L^2.D > 0$ and $L.D^2 < \alpha$;*
- *(C) $L.C \geq 3\alpha$, for all curves C.*

Then $H^1(X, K_X \otimes L \otimes I_Z) = 0$ for any zero-dimensional subscheme $Z \subset X$ of length α.

Theorem 4.1 would give an effective numerical criterion for an adjoint line bundle to be globally generated ($\alpha = 1$) or very ample ($\alpha = 2$):

Corollary 1.1 (Fujita's Conjecture)**.** *Let L be an ample line bundle on a smooth projective threefold X. Assume Conjecture 2.3 holds for ω and B as above. Then:*

- *(a) $K_X \otimes L^{\otimes m}$ is globally generated for $m \geq 4$. Moreover, if $L^3 \geq 2$, then $K_X \otimes L^{\otimes 3}$ is also globally generated.*
- *(b) $K_X \otimes L^{\otimes m}$ is very ample for $m \geq 6$.*

In Proposition 4.2, we also show (assuming the conjecture) that $K_X \otimes L^5$ is very ample as long as its restriction to special degree one curves is very ample. As a consequence, $K_X \otimes L^5$ is very ample when K_X is trivial, or, more generally, when $K_X.C$ is even for all curves $C \subset X$.

Ein and Lazarsfeld proved that $K_X \otimes L^{\otimes 4}$ is globally generated [EL93]. In the case $L^3 \geq 2$, Fujita, Kawamata, and Helmke proved that $K_X \otimes L^{\otimes 3}$ is globally generated as well [Fuj93, Kaw97, Hel97]. In fact, in Proposition 4.4, we show that these results conversely give some evidence for Conjecture 2.3. Case (b) in Corollary 1.1 instead is not known in general; but also note that the strongest form of Fujita's conjecture predicts that $K_X \otimes L^{\otimes 5}$ is already very ample. For further references, we refer to [Laz04, Section 10.4]. Notice that the bounds in Theorem 4.1 are very similar to those in [Fuj93] when $\alpha = 1$ (see also [Kaw97, Hel97]) and, when $\alpha = 2$ and Z consists of two distinct points, to those in [Fuj94].

Approach. We explain our approach, which was outlined in [AB11, Section 5], but can now be made precise using the strong Bogomolov-Gieseker conjecture of [BMT11]. It is closer to Reider's original approach [Rei88] for surfaces via stability of sheaves (generalized to threefolds by extending it to derived categories), than to the Ein-Lazarsfeld-Kawamata approach mentioned above, via vanishing theorems.

Let us give first a brief recall on Reider's method for proving Fujita's Conjecture in the case of X being a surface. By Serre duality, an adjoint linear system $K_X \otimes L$ is very ample if and only if $\mathrm{Ext}^1(L \otimes I_Z, \mathcal{O}_X) = H^1(X, K_X \otimes L \otimes I_Z)^\vee = 0$, for all zero-dimensional subscheme $Z \subset X$ of length one or two. If this group was non-zero, we would get a rank 2 torsion-free sheaf E as the non-trivial extension $\mathcal{O}_X \hookrightarrow E \twoheadrightarrow L \otimes I_Z$. Reider's idea is to consider the slope-stability of E. If E is stable, then the classical Bogomolov-Gieseker inequality gives a bound on the degree L^2 of L in terms of the length of Z. If E is not stable, then the destabilizing subsheaf gives a curve of bounded degree with respect to L. Hence, if we assume that L satisfies inequalities similar to (A) and (C), we would get a contradiction.

We generalize this approach to threefolds as follows. We suppose the conclusion of Theorem 4.1 is false. Then by Serre duality,

$$0 \neq \mathrm{Ext}^2(L \otimes I_Z, \mathcal{O}_X) = \mathrm{Ext}^1(L \otimes I_Z, \mathcal{O}_X[1]).$$

For appropriate choices of ω and B, both $L \otimes I_Z$ and $\mathcal{O}_X[1]$ are objects in the abelian category $\mathcal{B}_{\omega,B}$, and thus this extension class corresponds to another object E of $\mathcal{B}_{\omega,B}$. In Section 3.1, we will show that for $\omega \to 0$, the complex E violates the inequality of Conjecture 2.3, thus it must become unstable. We show in Section 3.2 that the Chern classes of a destabilizing subobject give a contradiction to Assumptions (A) and (B) of the Theorem unless it is of the form $L \otimes I_C$, where I_C is the ideal sheaf of a curve containing Z. In Section 4, we apply our conjecture and Assumption (C) to this remaining case and deduce Theorem 4.1.

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Notation and Convention. Throughout the paper, X will be a smooth projective threefold defined over $\mathbb{C}$ and $\mathrm{D^b}(X)$ its bounded derived category of coherent sheaves. Given a line bundle L on X, we will denote by $\mathbb{D}_L \colon \mathrm{D^b}(X) \to \mathrm{D^b}(X)$ the following local dualizing functor on its derived category:

$$\mathbb{D}_L(\_) := (\_)^\vee[1] \otimes L = \mathbf{R}\mathcal{H}om(\_, L[1]).$$

We identify a line bundle L with its first Chern class $c_1(L)$, and write K_X for the canonical line bundle. While $L^{\otimes m}$ denotes the tensor powers of the line bundle, L^k denotes the intersection product of its first Chern class.

2. SETUP

In this section, we briefly recall the notion of "tilt-stability" defined in [BMT11, Section 3] and its most important properties.

Let X be a smooth projective threefold, and let $\omega, B \in \mathrm{NS}_{\mathbb{Q}}(X)$ be rational numerical divisor classes such that ω is ample. We use ω, B to define a slope function $\mu_{\omega,B}$ for coherent sheaves on X as follows: For torsion sheaves E, we set $\mu_{\omega,B}(E) = +\infty$, otherwise

$$\mu_{\omega,B}(E) = \frac{\omega^2 \operatorname{ch}_1^B(E)}{\omega^3 \operatorname{ch}_0^B(E)} = \frac{\omega^2 \operatorname{ch}_1(E)}{\omega^3 \operatorname{ch}_0^B(E)} - \frac{\omega^2 B}{\omega^3}$$

where $\operatorname{ch}^B(E) = e^{-B} \operatorname{ch}(E)$ denotes the Chern character twisted by B (explicitly, $\operatorname{ch}_0^B = \operatorname{rk}$, $\operatorname{ch}_1^B = \mathrm{c}_1 - B \operatorname{rk}$, etc.).

A coherent sheaf E is slope-(semi)stable (or $\mu_{\omega,B}$-(semi)stable) if, for all subsheaves $F \hookrightarrow E$, we have

$$\mu_{\omega,B}(F) < (\leq) \mu_{\omega,B}(E/F).$$

Due to the existence of Harder-Narasimhan filtrations (HN-filtrations, for short) with respect to slope-stability, there exists a "torsion pair" $(\mathcal{T}_{\omega,B}, \mathcal{F}_{\omega,B})$ defined as follows:

$$\mathcal{T}_{\omega,B} = \{E \in \operatorname{Coh} X : \text{ any quotient } E \twoheadrightarrow G \text{ satisfies } \mu_{\omega,B}(G) > 0\}$$
$$\mathcal{F}_{\omega,B} = \{E \in \operatorname{Coh} X : \text{ any subsheaf } F \hookrightarrow E \text{ satisfies } \mu_{\omega,B}(F) \leq 0\}$$

Equivalently, $\mathcal{T}_{\omega,B}$ and $\mathcal{F}_{\omega,B}$ are the extension-closed subcategories of $\operatorname{Coh} X$ generated by slope-stable sheaves of positive or non-positive slope, respectively.

Definition 2.1. We let $\mathcal{B}_{\omega,B} \subset \mathrm{D}^{\mathrm{b}}(X)$ be the extension-closure

$$\mathcal{B}_{\omega,B} = \langle \mathcal{T}_{\omega,B}, \mathcal{F}_{\omega,B}[1] \rangle.$$

More explicitly, $\mathcal{B}_{\omega,B}$ is the subcategory of two-term complexes $E\colon E^{-1} \xrightarrow{d} E^0$ with $H^{-1}(E) = \ker d \in \mathcal{F}_{\omega,B}$ and $H^0(E) = \operatorname{cok} d \in \mathcal{T}_{\omega,B}$. We can characterize isomorphism classes of objects in $\mathcal{B}_{\omega,B}$ by extension classes: to give an object $E \in \mathcal{B}_{\omega,B}$ is equivalent to giving $T \in \mathcal{T}_{\omega,B}$, $F \in \mathcal{F}_{\omega,B}$, and a class $\xi \in \operatorname{Ext}_X^2(T, F)$.

By the general theory of torsion pairs and tilting [HRO96], $\mathcal{B}_{\omega,B}$ is the heart of a bounded t-structure on $\mathrm{D}^{\mathrm{b}}(X)$. For the most part, we only need that $\mathcal{B}_{\omega,B}$ is an abelian category: Exact sequences in $\mathcal{B}_{\omega,B}$ are given by exact triangles in $\mathrm{D}^{\mathrm{b}}(X)$. For any such exact sequence

$$0 \to E \to F \to G \to 0$$

in $\mathcal{B}_{\omega,B}$, we have a long exact sequence in $\operatorname{Coh} X$:

$$\begin{aligned} 0 \to H^{-1}(E) \to H^{-1}(F) \to H^{-1}(G) \to \\ \to H^0(E) \to H^0(F) \to H^0(G) \to 0. \end{aligned}$$

Using the classical Bogomolov-Gieseker inequality and Hodge Index theorem, we defined the following slope function on $\mathcal{B}_{\omega,B}$: We set $\nu_{\omega,B}(E) = +\infty$ when $\omega^2 \operatorname{ch}_1^B(E) = 0$, and otherwise

$$\nu_{\omega,B}(E) = \frac{\omega \operatorname{ch}_2^B(E) - \frac{1}{6}\omega^3 \operatorname{ch}_0^B(E)}{\omega^2 \operatorname{ch}_1^B(E)}. \tag{1}$$

We showed that this is a slope function, in the sense that it satisfies the weak see-saw property for short exact sequences in $\mathcal{B}_{\omega,B}$: for any subobject $F \hookrightarrow E$, we have $\nu_{\omega,B}(F) \leq \nu_{\omega,B}(E) \leq \nu_{\omega,B}(E/F)$ or $\nu_{\omega,B}(F) \geq \nu_{\omega,B}(E) \geq \nu_{\omega,B}(E/F)$.

Definition 2.2. An object $E \in \mathcal{B}_{\omega,B}$ is "tilt-(semi)stable" if, for all non-trivial subobjects $F \hookrightarrow E$, we have

$$\nu_{\omega,B}(F) < (\leq) \nu_{\omega,B}(E/F).$$

Motivated by the case of torsion sheaves ([BMT11, Proposition 7.1.1]), by projectively flat vector bundles ([BMT11, Proposition 7.4.2]), and the case of $X = \mathbb{P}^3$ ([BMT11, Theorem 8.2.1] and [Mac12]), we stated the following conjecture:

Conjecture 2.3 ([BMT11, Conjecture 1.3.1])**.** *For any $\nu_{\omega,B}$-semistable object $E \in \mathcal{B}_{\omega,B}$ satisfying $\nu_{\omega,B}(E) = 0$, we have the following inequality*

$$\operatorname{ch}_3^B(E) \leq \frac{\omega^2}{18} \operatorname{ch}_1^B(E). \tag{2}$$

Conjecture 2.3 is analogous to the classical Bogomolov-Gieseker inequality, which can be formulated as follows: For any $\mu_{\omega,B}$-semistable sheaf E satisfying $\mu_{\omega,B}(E) = 0$, we have $\omega \operatorname{ch}_2^B(E) \leq 0$.

The original motivation for Conjecture 2.3 is to construct examples of Bridgeland stability conditions on $\mathrm{D}^{\mathrm{b}}(X)$. While any linear inequality of the form (2) would be sufficient to this end, the constant $\frac{1}{18}$ in equation (2) is chosen so that, if ω and B are proportional to the first Chern class of an ample line bundle L, the inequality is an equality for tensor power $L^{\otimes n}$ of L. More generally, it is an equality when E is a slope-stable vector bundles E whose discriminant $\Delta = (\operatorname{ch}_1^B)^2 - 2\operatorname{ch}_0^B \operatorname{ch}_2^B$ satisfies $\omega\Delta(E) = 0$, and for which $\operatorname{ch}_1^B(E)$ is proportional to L. Such vector bundles have a projectively flat connection, and are examples of tilt-stable objects:

Proposition 2.4 ([BMT11, Proposition 7.4.1])**.** *Let L be an ample line bundle, and assume that both ω and B are proportional to L. Then any slope-stable vector bundle E, with $\omega\Delta(E) = 0$ and for which $\operatorname{ch}_1^B(E)$ is proportional to L, is also tilt-stable with respect to $\nu_{\omega,B}$.*

The proof is essentially the same as for line bundles $L^{\otimes n}$ in [AB11, Proposition 3.6].

By assuming Conjecture 2.3, we can also show conversely: if an object in $\mathcal{B}_{\omega,B}$ is tilt-stable and the inequality in Conjecture 2.3 is an equality, then it must have trivial discriminant. We first recall that, based on Bridgeland's deformation theorem in [Bri07], we also showed the existence of a continuous family of stability conditions depending on *real* classes ω, B:

Proposition 2.5 ([BMT11, Corollary 3.3.3])**.** *Let $U \subset \mathrm{NS}_{\mathbb{R}}(X) \times \mathrm{NS}_{\mathbb{R}}(X)$ be the subset of pairs of real classes (ω, B) for which ω is ample. There exists a notion of "tilt-stability" for every $(\omega, B) \in U$. For every object E, the set of (ω, B) for which E is $\nu_{\omega,B}$-stable defines an open subset of U.*

By using Proposition 2.5, we can then prove the following.

Proposition 2.6. *Let L be an ample line bundle, and assume that both ω and B are proportional to L. Assume also that Conjecture 2.3 holds for such B and ω. Let $E \in \mathcal{B}_{\omega,B}$ be a $\nu_{\omega,B}$-stable object, with $\mathrm{ch}_0(E) \neq 0$ and $\mathrm{ch}_1(E)$ proportional to L, and satisfying:*

$$\frac{\omega^3}{6}\,\mathrm{ch}_0(E) = \omega\,\mathrm{ch}_2^B(E) \qquad \textit{and} \qquad \mathrm{ch}_3^B(E) = \frac{\omega^2}{18}\,\mathrm{ch}_1^B(E).$$

Then $\omega.\Delta(E) = 0$.

Proof. Write $d = L^3$, $B = b_0 L$, $\omega = T_0 L$ and $\mathrm{ch}_0(E) = r$. The idea for the proof is that, since stability is an open property, we can deform $b = b_0$ and $T = T_0$, as a function $T = T(b)$ of b, slightly such that E is still $\nu_{T(b)L,bL}$-stable with $\nu_{T(b)L,bL}(E) = 0$; then we apply Conjecture 2.3 for the pairs $\omega = T(b)L$, $B = bL$ depending on b.

Evidently, $\nu_{TL,bL}(E) = 0$ is equivalent to

$$T^2 = \frac{6}{rd} L.\,\mathrm{ch}_2^{bL}(E)$$

Since $T_0 > 0$, and since the equation is satisfied for $T = T_0$ and $b = b_0$, the equation defines a function $T = T(b)$ for b nearby b_0.

It is immediate to check from the definition that the chain rule

$$\frac{\partial}{\partial b}\,\mathrm{ch}_i^{bL}(E) = -L\,\mathrm{ch}_{i-1}^{bL}(E) \tag{3}$$

holds for $i = 1, \dots, 3$.

Consider

$$f(b) = \mathrm{ch}_3^{bL}(E) - \frac{(T(b)L)^2}{18}.\,\mathrm{ch}_1^{bL}(E) = \mathrm{ch}_3^{bL}(E) - \frac{1}{3rd} L.\,\mathrm{ch}_2^{bL}(E) \cdot L^2.\,\mathrm{ch}_1^{bL}(E)$$

as a function of b in some neighborhood of $b_0 \in \mathbb{R}$. By Proposition 2.5 and Conjecture 2.3, we have $f(b) \leq 0$ for b close to b_0, and by assumption $f(b_0) = 0$; therefore $f'(b_0) = 0$. Using equation (3), we obtain

$$\begin{aligned} f'(b) &= -L.\,\mathrm{ch}_2^{bL}(E) + \frac{1}{3rd}\bigl((L^2.\,\mathrm{ch}_1^{bL})^2 + L.\,\mathrm{ch}_2^{bL}(E) \cdot rd\bigr) \\ &= \frac{1}{3r}\bigl(L.(\mathrm{ch}_1^{bL}(E))^2 - 2L.\,\mathrm{ch}_2^{bL}(E) r\bigr) = \frac{1}{3r} L.\Delta(E). \end{aligned}$$

(Note that we used $(L^2.\operatorname{ch}_1^{bL})^2 = L^3 \cdot L.(\operatorname{ch}_1^{bL})^2$, which holds because $\operatorname{ch}_1^{bL}(E)$ is proportional to L.) This proves the claim. □

Finally, based on an alternate construction of tilt-stability, we also showed that it behaves well with respect to the dualizing functor $\mathbb{D}_L(\_) = \mathbf{R}\mathcal{H}om(\_, L[1])$ for every line bundle L. For this purpose, we fix $B = \frac{L}{2}$:

Proposition 2.7. *Let $F \in \mathcal{B}_{\omega,\frac{L}{2}}$ be an object with $\nu_{\omega,\frac{L}{2}}(A) < +\infty$ for every subobject $A \subset F$. Then there is an exact triangle $\widetilde{F} \to \mathbb{D}_L(F) \to T_0[-1]$ where T_0 is a zero-dimensional torsion sheaf and $\widetilde{F}$ an object of $\mathcal{B}_{\omega,\frac{L}{2}}$ with $\nu_{\omega,\frac{L}{2}}(\widetilde{F}) = -\nu_{\omega,\frac{L}{2}}(F)$. The object $\widetilde{F}$ is $\nu_{\omega,\frac{L}{2}}$-semistable if and only if F is $\nu_{\omega,\frac{L}{2}}$-semistable.*

Proof. Since $\mathbb{D}_L(\_)$ can be written as the composition $\_ \otimes L \circ \mathbb{D}(\_)$, this follows from [BMT11, Proposition 5.1.3] and the fact that tensoring with L corresponds to replacing B with $B - L$. □

3. Reduction to curves

In this section, we use Assumptions (A) and (B) of Theorem 4.1 to show that the non-vanishing of $H^1(X, K_X \otimes L \otimes I_Z)$ implies the existence of special low-degree curves on X. The approach, explained in the introduction, involves studying the tilt-stability of a certain object E in the category $\mathcal{B}$ constructed in the previous section.

3.1. Bogomolov-Gieseker inequalities and stability.

We will use Conjecture 2.3 in the case where L is an ample line bundle on X, $\omega = TL$ for some $T > 0$, and $B = \frac{L}{2}$. The abelian category $\mathcal{B} := \mathcal{B}_{TL,\frac{L}{2}}$ is independent of T.

To simplify notation, we will rescale the slope function: set $t = \frac{T^2}{6}$ and write ν_t for

$$\nu_t(\_) = T \cdot \nu_{TL,\frac{L}{2}}(\_) = \frac{L.\operatorname{ch}_2^{L/2}(\_) - td\operatorname{ch}_0^{L/2}(\_)}{L^2.\operatorname{ch}_1^{L/2}(\_)}, \tag{4}$$

where $d := L^3$. Then the inequality of Conjecture 2.3 states that, for every ν_t-stable object E, we have

$$\operatorname{ch}_3^{L/2}(E) \le \frac{t}{3} L^2.\operatorname{ch}_1^{L/2}(E) \quad \text{if} \quad L.\operatorname{ch}_2^{L/2}(E) = dt\operatorname{ch}_0^{L/2}(E). \tag{5}$$

Let $Z \subset X$ be a zero-dimensional subscheme of length α. Following [AB11], observe that if $H^1(X, K_X \otimes L \otimes I_Z) \neq 0$, then by Serre duality, we also have $\operatorname{Ext}^2(L \otimes I_Z, \mathcal{O}_X) \neq 0$. Any non-zero element $\xi \in \operatorname{Ext}^2(L \otimes I_Z, \mathcal{O}_X)$ gives a non-trivial exact triangle in $\mathrm{D}^{\mathrm{b}}(X)$

$$\mathcal{O}_X[1] \to E = E_\xi \to L \otimes I_Z \xrightarrow{\xi} \mathcal{O}_X[2]. \tag{6}$$

We will show that E is ν_t-semistable for $t = \frac{1}{8}$; its Chern classes invalidate the inequality of Conjecture 2.3 for $t \ll 1$, and thus it must become unstable for $t < t_0$ and some

$t_0 \in (0, \frac{1}{8}]$; finally, we will show that the the Chern classes of its destabilizing factor would give special curves or divisors on X.

Proposition 3.1. *Assume that $H^1(X, K_X \otimes L \otimes I_Z) \neq 0$, and let E be an extension as given by equation* (6)*.*

(a) $E \in \mathcal{B}$ and

$$\mathrm{ch}^{L/2}(E) = \left(0, L, 0, \frac{d}{24} - \alpha\right).$$

(b) If $t > \frac{1}{8}$, then (6) *destabilizes E with respect to ν_t.*
(c) If $t = \frac{1}{8}$, then E is ν_t-semistable.
(d) Assume Conjecture 2.3 and Assumption (A) of Theorem 4.1. Then E is not ν_t-semistable for $0 < t \ll 1$,

Proof. First of all, we have

$$\mathrm{ch}^{L/2}(\mathcal{O}_X) = \left(1, -\frac{L}{2}, \frac{L^2}{8}, -\frac{L^3}{48}\right),$$
$$\mathrm{ch}^{L/2}(L \otimes I_Z) = \left(1, \frac{L}{2}, \frac{L^2}{8}, \frac{L^3}{48} - \alpha\right).$$

As $\mathcal{O}_X$ and $L \otimes I_Z$ are slope-stable, with $\mu_{\omega,L/2}(\mathcal{O}_X) < 0$ and $\mu_{\omega,L/2}(L \otimes I_Z) > 0$, we have $\mathcal{O}_X \in \mathcal{F}$ and $L \otimes I_Z \in \mathcal{T}$. By the definition of $\mathcal{B}$, it follows that $\mathcal{O}_X[1]$, $L \otimes I_Z$ and E are all objects of $\mathcal{B}$; in particular, we have proved (a).

Moreover, we have

$$\nu_t(\mathcal{O}_X[1]) = 2\left(t - \frac{1}{8}\right), \quad \nu_t(E) = 0 \tag{7}$$

which immediately implies (b), since (6) is an exact sequence in $\mathcal{B}$.

To prove (c), simply observe that, by Proposition 2.4, both $\mathcal{O}_X[1]$ and L are ν_t-stable for all $t > 0$. Moreover, since $\nu_t(L \otimes I_Z) = \nu_t(L)$, any destabilizing subobject $A \hookrightarrow L \otimes I_Z$ would also destabilize L via the composition $A \hookrightarrow L \otimes I_Z \hookrightarrow L$ (which is an inclusion in $\mathcal{B}$); thus $L \otimes I_Z$ is also ν_t-stable. For $t = \frac{1}{8}$, we have $\nu_t(\mathcal{O}_X[1]) = \nu_t(L \otimes I_Z) = 0$, and thus the extension (6) shows that E is ν_t-semistable at $t = \frac{1}{8}$.

Finally, if E was ν_t-semistable for all $t \in (0, \frac{1}{8}]$, then by our conjectural inequality (5) we would get

$$\frac{d}{24} - \alpha \leq \frac{t}{3} d \tag{8}$$

for all such t. Hence $d \leq 24\alpha$, in contradiction to Assumption (A). $\square$

Notice that the previous proposition would answer Question 4 in [AB11]. Also observe that in part (d), instead of Assumption (A), already assuming $d > 24\alpha$ would have been enough. Similarly, instead of Conjecture 2.3, any linear inequality between ch_3^B and ch_1^B would have been sufficient.

In the following proposition, we will show that our situation is self-dual with respect to the local dualizing functor $\mathbb{D}_L(\_) = \mathbf{R}\mathcal{H}om(\_, L[1])$. As a preliminary, let us first note that we may make the following assumption:

(*) $H^1(X, K_X \otimes L \otimes I_{Z'}) = 0$ for all subschemes $Z' \subsetneq Z$, and $H^1(X, K_X \otimes L \otimes I_Z) \cong \mathbb{C}$.

Indeed, in order to show $H^1(X, L \otimes I_Z \otimes K_X) = 0$, we can proceed by induction on the length of Z (the case $\alpha = 0$ is, of course, given by Kodaira vanishing).

Proposition 3.2. *If Assumption (*) holds, and E is given by the unique non-trivial extension of the form* (6)*, then $E \cong \mathbb{D}_L(E)$.*

Proof. Due to Assumption (*), it is sufficient to show that $\mathbb{D}_L(E)$ is again a non-trivial extension of the form (6). Applying the octahedral axiom to the composition $\mathcal{O}_Z[-1] \to L \otimes I_Z \to \mathcal{O}_X[2]$, and using the two exact triangles (6) and $O_Z[-1] \to L \otimes I_Z \to L$, we obtain an exact triangle $F \to E \to L$, where F itself fits into an exact triangle

$$\mathcal{O}_X[1] \to F \to \mathcal{O}_Z[-1]. \tag{9}$$

We claim that $\mathrm{Hom}(k(x)[-1], F) = 0$ for all skyscraper sheaves of points $x \in X$. Using the long exact sequence for $\mathrm{Hom}(k(x), \_)$ applied to (9), we see that this is equivalent to the non-vanishing of the composition

$$k(x)[-1] \to \mathcal{O}_Z[-1] \to L \otimes I_Z \xrightarrow{\xi} \mathcal{O}_X[2] \tag{10}$$

for every inclusion $k(x) \hookrightarrow \mathcal{O}_Z$. Given such an inclusion, let $Z' \subset Z$ be the subscheme given by $\mathcal{O}_{Z'} \cong \mathcal{O}_Z/k(x)$. If the composition (10) vanishes, then ξ factors via $L \otimes I_Z \hookrightarrow L \otimes I_{Z'}$. This contradicts our assumption $\mathrm{Ext}^2(L \otimes I_{Z'}, \mathcal{O}_X) = H^1(X, L \otimes I_{Z'} \otimes K_X)^\vee = 0$.

Now we apply $\mathbb{D}_L$ to the exact triangle $\mathcal{O}_X[1] \to F \to \mathcal{O}_Z[-1]$. As $\mathbb{D}_L(\mathcal{O}_X[1]) = L$ and $\mathbb{D}_L(\mathcal{O}_Z[-1]) = \mathcal{O}_Z[-1]$, dualizing (9) gives an exact triangle $\mathcal{O}_Z[-1] \to \mathbb{D}_L(F) \to L \to \mathcal{O}_Z$. Since $\mathrm{Hom}(\mathbb{D}_L(F), k(x)[-1]) = \mathrm{Hom}(k(x)[-1], F) = 0$ for all $x \in X$, the map $L \to \mathcal{O}_Z$ must be surjective, and hence $\mathbb{D}_L(F) \cong L \otimes I_Z$. Consequently, applying $\mathbb{D}_L$ to the exact triangle $F \to E \to L$ shows that $\mathbb{D}_L(E)$ is indeed a non-trivial extension of the form (6). □

3.2. Chern classes of destabilizing subobjects.

By Proposition 3.1 and Proposition 2.5, Conjecture 2.3 implies the existence of $t_0 \in (0, \frac{1}{8}]$ with the following properties:

- E is ν_{t_0}-semistable.
- There exists an exact sequence in $\mathcal{B}$

$$0 \to A \to E \to F \to 0, \tag{11}$$

  with $\nu_t(A) > 0$ if $t < t_0$, and $\nu_{t_0}(A) = 0$.

In the remainder of this section, we will prove the following statement:

Proposition 3.3. *Assume that X, L, α satisfy Assumptions (A) and (B) of Theorem 4.1 and Assumption (*) of the previous section. Then in any destabilizing sequence* (11)*, the object A is of the form $L \otimes I_C$, for some purely one-dimensional subscheme $C \subset X$ containing Z.*

We will first prove this for subobjects satsfying $L^2.\operatorname{ch}_1^{L/2}(A) \leq L^2.\operatorname{ch}_1^{L/2}(F)$, or, equivalently,

$$L^2.\operatorname{ch}_1^{L/2}(A) \leq \frac{1}{2} L^2.\operatorname{ch}_1^{L/2}(E) = \frac{d}{2}. \tag{12}$$

(We will later use the derived duality $\mathbb{D}_L(\_)$ to reduce to this case.)

Lemma 3.4. *Any subobject A satisfying* (12) *is a sheaf with* $\operatorname{rk}(A) = \operatorname{rk}(H^0(A)) > 0$.

Proof. Consider the long exact cohomology sequence for $A \hookrightarrow E \twoheadrightarrow F$. If $H^{-1}(A) \neq 0$, then $H^{-1}(A) \hookrightarrow \mathcal{O}_X$ is isomorphic to an ideal sheaf of some subscheme Y of X. Since $\mathcal{O}_Y \hookrightarrow H^{-1}(F)$ and $H^{-1}(F)$ is torsion-free, we must have $H^{-1}(A) \cong \mathcal{O}_X$. Then $H^0(A)$ is also torsion-free, and (12) implies

$$L^2.\operatorname{ch}_1^{L/2}(H^0(A)) = L^2.\operatorname{ch}_1^{L/2}(A) - L^2.\operatorname{ch}_1^{L/2}(\mathcal{O}_X[1]) \leq \frac{d}{2} - \frac{d}{2} = 0.$$

On the other hand, by construction of $\mathcal{B}$, every HN-filtration factor U of $H^0(A)$ satisfies $L^2.\operatorname{ch}_1^{L/2}(U) > 0$; thus $H^0(A) = 0$ and $A = \mathcal{O}_X[1]$. This contradiction proves $H^{-1}(A) = 0$.

Finally, note that if $A = H^0(A)$ is a torsion-sheaf, then $\nu_t(A)$ is independent of t, again a contradiction. □

Lemma 3.5. *Either A is torsion-free, or its torsion-part A_t satisfies*

$$L^2.\operatorname{ch}_1(A_t) - 2L.\operatorname{ch}_2(A_t) \geq 0 \quad \textit{and} \quad L^2.\operatorname{ch}_1(A_t) > 0.$$

Proof. The sheaf A_t is a subobject of E in $\mathcal{B}$ with $\operatorname{rk} = 0$. Hence $L.\operatorname{ch}_2^{L/2}(A_t) \leq 0$, otherwise it would destabilize E at $t = \frac{1}{8}$. Expanding $\operatorname{ch}_2^{L/2}$ gives the first inequality. To show the second inequality, we just observe that there are no non-trivial morphisms from sheaves supported in dimension ≤ 1 to E. □

Lemma 3.6. *In the HN-filtration of A with respect to slope-stability, there exists a factor U of rank r such that $\Gamma := L - \frac{\operatorname{ch}_1(U)}{r}$ satisfies the following inequalities:*

$$L^2.\Gamma \leq L.\Gamma^2 + 6\alpha \tag{I}$$

$$\frac{d}{2}\left(1 - \frac{1}{r}\right) \leq L^2.\Gamma < \frac{d}{2}. \tag{II}$$

The case $r = 1$ and $L^2.\Gamma = 0$ only occurs when A is a torsion-free sheaf of rank one and $H^{-1}(F) = \mathcal{O}_X$.

If A was a line bundle, the above definition of Γ would be just as Reider's original argument for surfaces: in this case, Γ is the support of the cokernel of $A \hookrightarrow H^0(E) \cong L \otimes I_Z$.

Proof. From $\nu_{t_0}(A) = 0$ we obtain

$$t_0 = \frac{L.\operatorname{ch}_2^{L/2}(A)}{\operatorname{rk}(A)d}. \tag{13}$$

Applying the conjectured inequality (5) to E, and plugging in t_0 gives

$$\frac{d}{24} - \alpha = \operatorname{ch}_3^{L/2}(E) \leq \frac{L^2.\operatorname{ch}_1^{L/2}(E)}{3} t_0 = \frac{d}{3}\frac{L.\operatorname{ch}_2^{L/2}(A)}{\operatorname{rk}(A)d} = \frac{1}{3}\frac{L.\operatorname{ch}_2^{L/2}(A)}{\operatorname{rk}(A)}.$$

We want to bound $L.\operatorname{ch}_2^{L/2}(A)$. First we expand $\operatorname{ch}_2^{L/2}(A)$:

$$\operatorname{ch}_2^{L/2}(A) = \operatorname{ch}_2(A) - \frac{L.\operatorname{ch}_1(A)}{2} + \operatorname{rk}(A)\frac{L^2}{8}.$$

Substituting, we deduce

$$\frac{L^2.\operatorname{ch}_1(A)}{\operatorname{rk}(A)} - 2\frac{L.\operatorname{ch}_2(A)}{\operatorname{rk}(A)} \leq 6\alpha. \tag{14}$$

Let A_{tf} denote the torsion-free part of A, and consider its HN-filtration. Among the HN factors, we choose a torsion-free sheaf U for which the function

$$\eta(\_) := \frac{L^2.\operatorname{ch}_1(\_) - 2L.\operatorname{ch}_2(\_)}{\operatorname{rk}(\_)}$$

is minimal. Notice that η satisfies the see-saw property: for an exact sequence of torsion-free sheaves

$$0 \to M \to N \to P \to 0,$$

we have $\eta(N) \geq \min\{\eta(M), \eta(P)\}$. Hence we get a chain of inequalities leading to

$$\eta(U) \leq \eta(A_{tf}) \leq \eta(A) \leq 6\alpha \tag{15}$$

where we used Lemma 3.5 for the second inequality.

To abbreviate, we now write $D := \operatorname{ch}_1(U)$ and $r := \operatorname{rk}(U)$. Since U is $\mu_{\omega,L/2}$-semistable, we can combine the classical Bogomolov-Gieseker inequality with (15) to obtain

$$L^2.\frac{D}{r} = \frac{2L.\operatorname{ch}_2(U)}{r} + \eta(U) \leq L.\frac{D^2}{r^2} + 6\alpha.$$

Substituting $D = rL - r\Gamma$ yields the inequality (I).

To prove the chain of inequalities (II), we observe on the one hand that $L^2. \operatorname{ch}_1^{L/2}(U) > 0$ by the definition of $\mathcal{T}_{\omega,B} = \mathcal{B} \cap \operatorname{Coh} X$. On the other hand, U is a subquotient of A in $\mathcal{T}_{\omega,B}$; combined with inequality (12) we obtain

$$0 < L^2. \operatorname{ch}_1^{L/2}(U) \leq L^2. \operatorname{ch}_1^{L/2}(A) \leq \frac{d}{2}.$$

Plugging in $\operatorname{ch}_1^{L/2}(U) = -\frac{r}{2}L + D = \frac{r}{2}L - r\Gamma$ shows the inequality (II).

Finally, note that in the case $r = 1$ and $L^2.\Gamma = 0$ the chain of inequalities leading to the first part of (II) must be equalities; in particular $L^2. \operatorname{ch}_1^{L/2}(U) = L^2. \operatorname{ch}_1^{L/2}(A)$. This shows that A_{tf} cannot have any other HN-filtration factors besides U, i.e., $U = A_{tf}$. Additionally it implies that $\operatorname{ch}_1^{L/2}(A_t) = 0$, in contradiction to Lemma 3.5; hence $A_t = 0$ and $A = U$ is a torsion-free rank one sheaf.

As $L \otimes I_Z$ is torsion-free, if the image of $H^{-1}(F) \to A$ is non-trival, then the map is surjective, and the inclusion $A \hookrightarrow E$ factors via $A \hookrightarrow \mathcal{O}_X[1] \hookrightarrow E$, in contradiction to the stability of $\mathcal{O}_X[1]$ for all t and $\nu_{t_0-\varepsilon}(A) > 0 > \nu_{t_0-\varepsilon}(\mathcal{O}_X[1])$. Thus $H^{-1}(F) = \mathcal{O}_X$. □

Proof. (Proposition 3.3) We combine (I) and (II) with the Hodge Index Theorem (just as in [AB11, Corollary 3.9]) to obtain

$$\left(L.\Gamma^2\right) d \leq \left(L^2.\Gamma\right)^2 \leq \frac{d}{2} \left(L.\Gamma^2 + 6\alpha\right),$$

and so $L.\Gamma^2 \leq 6\alpha$.

In the case $r > 1$, we use (I) and (II) again to get

$$\frac{d}{4} \leq L^2.\Gamma \leq L.\Gamma^2 + 6\alpha \leq 12\alpha,$$

and so $d \leq 48\alpha$ in contradiction to Assumption (A).

Reider's original argument in [Rei88] deals with the case $r = 1$: In case $L^2.\Gamma \neq 0$, then $L^2.\Gamma \geq 1$. Let $\kappa := L.\Gamma^2 \leq 6\alpha$. Again combining the Hodge Index Theorem with (I), we obtain

$$\left(L.\Gamma^2\right) d \leq \left(L.\Gamma^2 + 6\alpha\right)^2,$$

and so

$$d \leq 12\alpha + \frac{\kappa^2 + 36\alpha^2}{\kappa}.$$

The RHS is strictly decreasing function for $\kappa \in (0, 6\alpha]$ and equals 49α for $\kappa = \alpha$; thus Assumption (A) implies $\kappa < \alpha$. On the other hand, Γ is integral, and hence Assumption (B) implies $L^2.\Gamma \geq 7\alpha$, in contradiction to (I).

Finally, if $L^2.\Gamma = 0$; then, according to Lemma 3.6, we have $H^{-1}(F) \cong \mathcal{O}_X$. Hence A is a subsheaf of $L \otimes I_Z$ with $\operatorname{ch}_1(A) = \operatorname{ch}_1(L)$; this is only possible if $A \cong L \otimes I_W$, for some closed subscheme $W \subset X$ with $\dim(W) \leq 1$. If W is zero-dimensional, then $\operatorname{ch}_2^{L/2}(A) = \frac{1}{2}L^2$ and equation (13) gives $t_0 = \frac{1}{2}$, in contradiction to $t_0 \in (0, \frac{1}{8}]$. Hence W

is one-dimensional, and we have shown that any subobject A with $\mathrm{ch}_1^{L/2}(A) \leq \frac{d}{2}$ is of the form $A \cong L \otimes I_W$. In particular $\mathrm{ch}_1^{L/2}(A) = \frac{d}{2}$ in this case, so there are no subobject with $\mathrm{ch}_1^{L/2}(A) < \frac{d}{2}$.

Now assume $\mathrm{ch}_1^{L/2}(A) > \frac{d}{2}$. We can apply Proposition 3.2 and Proposition 2.7 to the short exact sequence (11) obtain a short exact sequence in $\mathcal{B}$

$$0 \to \widetilde{F} \xrightarrow{u} E \to E/\widetilde{F} \to 0$$

which is again destabilizing. Indeed, since $\mathcal{B}$ is the heart of a bounded t-structure, there exists a cohomology functor $H^*_{\mathcal{B}}(\_)$. Applied to the exact triangle

$$\mathbb{D}_L(F) \to \mathbb{D}_L(E) = E \to \mathbb{D}_L(A),$$

it induces a long exact sequence in $\mathcal{B}$

$$0 \to \widetilde{F} = H^0_{\mathcal{B}}(\mathbb{D}_L(F)) \xrightarrow{u} E \to \widetilde{A} \to T_0 = H^1_{\mathcal{B}}(\mathbb{D}_L(F)) \to 0. \tag{16}$$

As $\mathbb{D}_L$ preserves $L^2.\,\mathrm{ch}_1^{L/2}(\_)$, we have that $\widetilde{F}$ is a destabilizing subobject with $\mathrm{ch}_1^{L/2}(F) = \mathrm{ch}_1^{L/2}(E) - \mathrm{ch}_1^{L/2}(A) < \frac{d}{2}$, which does not exist.

Finally, note that the long exact sequence (16) also implies that $\mathbb{D}_L(A) = \widetilde{A} \in \mathcal{B}$. This gives the vanishing of $0 = \mathrm{Hom}(\mathbb{D}_L(A), k(x)[-1]) = \mathrm{Hom}(k(x)[-1], A)$. This is equivalent to the claim that W is a purely one-dimensional scheme, as any subsheaf $k(x) \hookrightarrow \mathcal{O}_W$ gives an extension of $k(x)$ by $L \otimes I_W$. This finishes the proof of Proposition 3.3. □

4. A REIDER-TYPE THEOREM

In this section we prove our main theorem:

Theorem 4.1. *Let L be an ample line bundle on a smooth projective threefold X, and assume Conjecture 2.3 holds for B and ω proportional to L. Fix a positive integer α, and assume that L satisfies the following conditions:*

(A) $L^3 > 49\alpha$;
(B) $L^2.D \geq 7\alpha$, for all integral divisor classes D with $L^2.D > 0$ and $L.D^2 < \alpha$;
(C) $L.C \geq 3\alpha$, for all curves C.

Then $H^1(X, K_X \otimes L \otimes I_Z) = 0$, for any zero-dimensional subscheme $Z \subset X$ of length α.

Proof. As explained in Section 3.1, we may proceed by induction on the length of Z and may use Assumption (*). Let $t_0 \in (0, \frac{1}{8}]$ be as in Section 3.2 and let $t = t_0 - \epsilon$. Truncating the Harder-Narasimhan filtration of E with respect to ν_t-stability gives a short exact sequence

$$0 \to A \to E \to F \to 0$$

with $\nu_t(A) > 0$, such that any subobject $A' \hookrightarrow E$ with $\nu_t(A') > 0$ factors via $A' \hookrightarrow A$. By Proposition 3.3, A is of the form $L \otimes I_C$ for some purely one-dimensional subscheme $C \subset X$; it also implies that A is stable, as any destabilizing subobject A' of A would again be of the form $A' \cong L \otimes I_{C'}$, so that the quotient A/A' would be a torsion sheaf with $\nu_t(A/A') = +\infty$.

Let $\widetilde{F}$ be the object obtained by dualizing F and applying Proposition 2.7. The map $\mathbb{D}_L(F) \to \mathbb{D}_L(E) \cong E$ induces a map $\widetilde{F} \to E$ which is an injection in $\mathcal{B}$. Since

$$\operatorname{ch}_i^{L/2}(\widetilde{F}) = \operatorname{ch}_i^{L/2}(\mathbb{D}_L(F)) \tag{17}$$

for $i \leq 2$, we have $\nu_t(\widetilde{F}) = -\nu_t(F) > 0$; thus the map factorizes as $\widetilde{F} \hookrightarrow A \hookrightarrow E$. By Proposition 3.3, the object $\widetilde{F}$ is of the form $L \otimes I_{C'}$ for some purely one-dimensional subscheme $C' \subset X$. Equation (17) also implies $\operatorname{ch}_i^{L/2}(\widetilde{F}) = \operatorname{ch}_i^{L/2}(A)$ for $i \leq 2$; thus the (non-trivial) map $L \otimes I_{C'} \to L \otimes I_C$ has zero-dimensional cokernel. It follows that

$$\operatorname{ch}_3^{L/2}(F) = \operatorname{ch}_3^{L/2}(\mathbb{D}_L(F)) \leq \operatorname{ch}_3^{L/2}(\widetilde{F}) \leq \operatorname{ch}_3^{L/2}(A).$$

This implies that

$$2\operatorname{ch}_3^{L/2}(A) \geq \operatorname{ch}_3^{L/2}(A) + \operatorname{ch}_3^{L/2}(F) = \operatorname{ch}_3^{L/2}(E) = \frac{d}{24} - \alpha, \tag{18}$$

and the difference of the two sides is a non-negative integer.

On the other hand, as A is stable, by Conjecture 2.3, by (13) and (18), and by expanding $\operatorname{ch}^{L/2}$ we have

$$\frac{d}{48} - \frac{\alpha}{2} \leq \operatorname{ch}_3^{L/2}(A) \leq \frac{t_0}{3} L^2 . \operatorname{ch}_1^{L/2}(A) = \frac{1}{6} L . \operatorname{ch}_2^{L/2}(A) = \frac{d}{48} - \frac{L.C}{6}. \tag{19}$$

We now use Assumption (C): $L.C \geq 3\alpha$. This contradicts (19), unless $L.C = 3\alpha$ and

$$\frac{d}{48} - \frac{\alpha}{2} = \operatorname{ch}_3^{L/2}(A) = \frac{t_0}{3} L^2 . \operatorname{ch}_1^{L/2}(A).$$

Since $(TL).\Delta(A) = 3\alpha T \neq 0$, this in turn contradicts Proposition 2.6. □

We also obtain the following result characterizing the only possible counter-examples to Fujita's very ampleness conjecture in case $L = M^5$:

Proposition 4.2. *Assume that Conjecture 2.3 holds for X, $\omega = tL$ and $B = \frac{L}{2}$ and $L \cong M^5$ for an ample line bundle M. Then either $K_X \otimes L$ is very ample, or there exists a curve C of degree $M.C = 1$ and arithmetic genus $g_a(C) = \frac{5}{2} + \frac{1}{2} K_X.C$ such that $K_X \otimes L|_C$ is a line bundle of degree $2g_a(C)$ on C which is not very ample.*

Proof. Assume that $K_X \otimes L$ is not very ample. We follow the logic and the notation of the proof of Theorem 4.1, with $\alpha = 2$. As before, let $A = L \otimes I_C$ be the destabilizing subobject of E for $t = t_0 - \epsilon$; here C is a purely one-dimensional subscheme of X. By the proof of Theorem 4.1, we have $L.C < 6$ and thus necessarily $M.C = 1$ and $L.C = 5$. In particular, C is reduced and irreducible. We claim that $\mathrm{ch}_3^{L/2}(A) = \frac{d}{48} - 1$. Indeed, setting $\alpha = 2$ in (19) gives

$$\frac{d}{48} - 1 \leq \mathrm{ch}_3^{L/2}(A) \leq \frac{d}{48} - \frac{5}{6}. \tag{20}$$

On the other hand, if $\mathrm{ch}_3^{L/2}(A) \neq \frac{d}{48} - 1$, then, by (18), $\mathrm{ch}_3^{L/2}(A) \geq \frac{d}{48} - \frac{1}{2}$, a contradiction to the inequality (20).

From the claim, we obtain

$$\mathrm{ch}_3(L \otimes \mathcal{O}_C) = \mathrm{ch}_3(L) - \mathrm{ch}_3(A) = \frac{7}{2}$$

and thus

$$\mathrm{ch}_3(\mathcal{O}_C) = \mathrm{ch}_3(L \otimes \mathcal{O}_C) - L.C = -\frac{3}{2}$$

By Hirzebruch-Riemann-Roch, we get

$$1 - g_a(C) = \mathrm{ch}_3(\mathcal{O}_C) - \frac{1}{2} K_X.C.$$

Plugging in the previous equation and solving for $K_X.C$ shows that $K_X \otimes L|_C$ is a line bundle of degree $2g_a(C)$ on C. The explicit expression for $g_a(C)$ follows immediately.

Finally, the cohomology sheaves of the quotient $F \cong E/A$ are $H^{-1}(F) \cong \mathcal{O}_X$ and $H^0(F) \cong L \otimes \mathcal{O}_C(-Z)$ (where $\mathcal{O}_C(-Z)$ denotes the ideal sheaf of $Z \subset C$). If F were decomposable, $\widetilde{F}$ would be a decomposable destabilizing subobject of E, which cannot exist. Hence

$$0 \neq \mathrm{Ext}^2(L \otimes \mathcal{O}_C(-Z), \mathcal{O}_X) = H^1(C, K_X \otimes L|_C(-Z))^\vee.$$

On the other hand, $K_X \otimes L|_C$ is a line bundle of degree $2g_a(C)$ on an irreducible Cohen-Macaulay curve, and thus $H^1(K_X \otimes L|_C) = 0$. Hence $K_X \otimes L|_C$ is not very ample. □

Remark 4.3. Notice that Proposition 4.2 implies Fujita's conjecture when K_X is numerically trivial (or, more generally, when $K_X.C$ is even for all integral curve classes C).

In case the curve $C \subset X$ of Proposition 4.2 is l.c.i, one can be even more precise. Let ω_C be the dualzing sheaf (which agrees with the dualizing complex, as $\mathcal{O}_C$ is pure and thus C Cohen-Macaulay). The sheaf $K_X \otimes L(-Z)|_C$ is torsion-free of rank one and degree $2g_a(C) - 2$ with $H^1(K_X \otimes L(-Z)|_C) \neq 0$, and thus Serre duality implies $K_X \otimes L(-Z)|_C \cong \omega_C$. If N is the normal bundle, adjunction gives $\Lambda^2 N \cong L(-Z)$. In particular, the normal bundle has degree 3. Since $M.C = 1$, bend-and-break implies that such a curve cannot be rational.

In conclusion, we show how to reverse part of the argument in this section when Z has length one. Indeed, in such a case we can use Ein-Lazarsfeld theorem (or better, its variant by Kawamata and Helmke) to show that Conjecture 2.3 holds true for this particular case, coherently with our result:

Proposition 4.4. *Let L be an ample line bundle on a smooth projective threefold X. Assume that L satisfies the following conditions:*

(a) $L^3 \geq 28$*;*
(b) $L^2.D \geq 9$*, for all integral effective divisor classes* D*.*

Assume also that there exists $x \in X$ such that $H^1(X, K_X \otimes L \otimes I_x) \neq 0$. Then Conjecture 2.3 holds for all objects $E \in \mathcal{B}$ given as non-trivial extensions

$$\mathcal{O}_X[1] \to E \to L \otimes I_x \to \mathcal{O}_X[2].$$

Proof. The argument is very similar to [Kaw97], Proposition 2.7 and Theorem 3.1, Step 1. We freely use the notation from [Laz04, Sections 9 & 10]. By [Kaw97, Lemma 2.1], given a rational number t satisfying $3/\sqrt[3]{L^3} < t < 1$, there exists a $\mathbb{Q}$-divisor D numerically equivalent to tL such that $\mathrm{ord}_x D = 3$. Let $c \leq 1$ the log-canonical threshold of D at x.

By [Kaw97, Theorem 3.1] (also [Hel97]) and our assumptions, the LC-locus $\mathrm{LC}(cD; x)$ (i.e., the zero-locus of the multiplier ideal $\mathcal{J}(c \cdot D)$ passing through x) must be a curve C satisfying $1 \leq L.C \leq 2$. We can now apply Nadel's vanishing theorem to cD to deduce that $H^1(X, K_X \otimes L \otimes I_C) = 0$, and so that the restriction map $H^0(X, K_X \otimes L) \twoheadrightarrow H^0(X, K_X \otimes L|_C)$ is surjective.

Consider the composition $u\colon L \otimes I_C \to L \otimes I_x \to \mathcal{O}_x[2]$. Then, $u \neq 0$ if and only if x is a base point of $K_X \otimes L$ which is not a base point of $K_X \otimes L|_C$. The surjectivity of the restriction map implies that $u = 0$. Hence, we get an inclusion $L \otimes I_C \hookrightarrow E$ in $\mathcal{B}$ which destabilizes E, if (2) is not satisfied. □

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