DIPLOMARBEIT

Halbeinfache Frobenius-Mannigfaltigkeiten und Quantenkohomologie

(Semisimple Frobenius Manifolds and Quantum Cohomology)

angefertigt am Mathematischen Institut

vorgelegt der

Mathematisch-Naturwissenschaftlichen Fakultät der
Rheinischen Friedrich-Wilhelms-Universität Bonn

Von

Arend Bayer

aus Tübingen

September 2003

Contents

1 Introduction: Semisimple Frobenius manifolds

In his talk at the ICM in Berlin 1998, Dubrovin proposed a surprising conjecture answering the question which Fano manifolds could have generically semisimple quantum cohomology. This diploma thesis

- explains this conjecture,
- relates it to mirror symmetry conjectures,
- gives several computations on manifolds that do have semisimple quantum cohomology, and
- proves (a part of) Dubrovin's conjecture for a large class of manifolds.

A Frobenius manifold is a complex manifold $\mathcal{M}$ equipped with a multiplication $\circ$ on the tangent bundle $\mathcal{TM}$ and a flat metric that satisfy a number of axioms. In general, one needs infinitely many numbers to describe a single Frobenius manifold. However, for those Frobenius manifolds that have a semisimple point x (i. e., $(\mathcal{T}_x\mathcal{M}, \circ)$ is isomorphic as an algebra to $\mathbb{C}^n$), there exist two independent classifications. Both classifications identify the germ of such a Frobenius manifold by a finite number of characteristic numbers. In other words, semisimple Frobenius manifolds are easier to understand.

1.1 Plan of the paper

In section 3, we recall the basic definitions and notations of Frobenius manifolds and quantum cohomology from [Man99].

Section 4 gives examples of computations of semisimple quantum cohomology. We compute the special coordinates (the classifying data in Yu. I. Manin's classification of semisimple Frobenius manifolds) of three families of Fano threefolds with minimal cohomology.

Section 5 gives a detailed definition of Dubrovin's monodromy data (that classifies semisimple Frobenius manifolds). We devote particular care to the construction of Stokes matrices; here we follow [vdPS03], partly rephrasing it in a more abstract language.

In the following section 6, we discuss exceptional systems in triangulated categories, and give the exact statements of Dubrovin's conjecture.

Section 7 is devoted to the bigger conjectural picture underlying Dubrovin's conjecture. This involves the homological mirror conjecture in its assumed form for semisimple quantum cohomology and total spaces of unfoldings of hypersurface singularities (7.6), the corresponding numerical mirror symmetry conjecture as an isomorphism of Frobenius manifolds (as in 7.4.1), and the explanation of Stokes matrices in the case of unfoldings of singularities. Parts of this section are directly inspired by the paper [HIV00] by the physicists Hori, Iqbal and Vafa.

Finally, in section 8, we prove the theorem 8.2.1. Its statement can be formulated concisely (and almost correctly) as: *If we know that Dubrovin's conjecture is true for X, then it is true for its blow-up $\hat{X}$ at a point.* The idea to the proof is similar to the case of Del Pezzo-surfaces, which was treated in [BM01].

1.2 Acknowledgements

I would like to thank Gunther Vogel for comprehensive proof-reading and my brother Tilman Bayer for useful remarks clarifying the exposition; Barbara Fantechi for suggestions related to footnote 10. Several discussions with Claus Hertling substantially helped my understanding of section 7. Finally, I am indebted to my advisor Yuri I. Manin for continued inspiring and encouraging support.

2 Deutsche Zusammenfassung

Einführung. Frobenius-Mannigfaltigkeiten wurden 1990 von Boris Dubrovin als Axiomatisierung (eines Teils) der mathematischen Struktur topologischer Feldtheorien eingeführt. Eine Frobenius-Mannigfaltigkeit ist im Wesentlichen eine flache Mannigfaltigkeit mit Metrik und einer Produkstruktur auf dem Tangentialbündel, die bestimmte Axiome erfüllen (cf. Definition 3.1.1 für Details). Das wichtigste Axiom ist eine Integrabilitätsbedingung, die die Struktur einer Frobenius-Mannigfaltigkeit erstaunlich starr macht.

Die bekannteste Beispielklasse von Frobenius-Mannigfaltigkeiten entsteht aus der Quantenkohomologie einer glatten projektiven Varietät V. In diesem Fall ist die Mannigfaltigkeit einfach der Vektorraum $H^*(V, \mathbb{C})$. Die Metrik ist gegeben durch die Poincaré-Paarung. Eine Multiplikation auf dem Tangentialbündel dieser Mannigfaltigkeit bedeutet das Folgende: an jedem Punkt $x \in H^*(V, \mathbb{C})$ ist der Tangentialraum an x natürlich identisch zu $H^*(V, \mathbb{C})$. Ein Produkt auf $\mathcal{T}_x$ ist also ein Produkt auf der Kohomologie, das aber von x als Parameter abhängt. Das Produkt ist definiert durch das Gromov-Witten-Potential, eine erzeugende Funktion aus sogenannten Gromov-Witten-Invarianten; siehe Definition 3.2.2 und Satz 3.2.3. Dass das hierdurch definierte Quantenprodukt assoziativ ist, ist eine sehr überraschende und tiefe Aussage. Man kann es verstehen als deformiertes Cup-Produkt.

Diese Diplomarbeit beschäftigt sich mit solchen Frobenius-Mannigfaltigkeiten $\mathcal{M}$, bei denen für generisches $m \in \mathcal{M}$ die Algebrastruktur auf $\mathcal{T}_m\mathcal{M}$ zerfällt, d. h. dass $\mathcal{T}_m\mathcal{M}$ als Algebra isomorph zu $\mathbb{C}^n$ mit komponentenweiser Multiplikation ist. Halbeinfache Frobenius-Mannigfaltikeiten sind aus verschiedenen Gründen einfacher zu verstehen. Ein Grund ist, dass die multiplikative Struktur auch in einer Umgebung von m sehr einfach zerfällt, siehe Proposition 3.1.4. Weiterhin sind Keime solcher Mannigfaltigkeiten vergleichsweise leicht zu klassifizieren.

Spezielle Koordinaten. Ein sehr einfach zu definierendes Klassikationsdatum sind die sog. *speziellen Koordinaten*. Sie wurden von Yuri I. Manin eingeführt und sind in gegeben Fällen auch einfach zu berechnen.

Dies wird in Kapitel an drei Familien dreidimensionaler Fano-Mannigfaltigkeiten durchgeführt. Diese Familien zeichnen sich dadurch aus, dass ihre Kohomologie nur vierdimensional ist, die minimal mögliche Dimension. Von Bondal, Kuznetsov und Orlov wurden für diese Fälle jeweils einzelne Gromov-Witten-Invarianten berechnet. Mithilfe der Relationen, die sich aus der Assoziativität der Produktstruktur der zugehörigen Frobenius-Mannigfaltigkeiten ergeben, lassen sich dann weitere Invarianten rein algebraisch berechnen.

Für die Produktstruktur berechnen wir hier explizit die Zerlegung als Algebra in $\mathbb{C}^4$ durch Angabe der vier Idempotenten. Und zwar für den Fall, dass der oben mit $x \in H^*(V)$ bezeichnete Parameter im Unterraum $H^2(V)$ liegt, bzw. in dessen erster infinitesimaler Umgebung. Daraus lassen sich die speziellen Koordinaten berechnen.

Stokes-Matrizen. Von Dubrovin stammt die Konstruktion eines flachen Zusammenhangs auf $\mathbb{P}^1 \times \mathcal{M}$ für jede Frobenius-Mannigfaltigkeit $\mathcal{M}$, der auch *erster Strukturzusammenhang* der Frobenius-Mannigfaltigkeit genannt wird; cf. Definition 5.1.1. Dieser Zusammenhang hat einen Pol zweiter Ordnung entlang $\{0\} \times \mathcal{M}$. Solche Zusammenhänge lassen sich nun wiederum klassifizieren, und das wesentliche Datum dabei sind die sogenannten Stokes-Matrizen. Zur ihrer Definition ist einiges an Theorie der Differential-Moduln über dem Differential-Ring $\mathbb{C}(\{z\})$ der Keime komplexer Funktionen (mit Derivation $\frac{\partial}{\partial z}$) nötig; eine entscheidende Technik ist dabei die Verwendung von *Funktionen mit asymptotischer Entwicklung* nahe 0 in einem Sektor in $\mathbb{C}$. Diese Theorie wird im Kapitel 5 überblicksartig vorgestellt, die Darstellung folgt (mit Änderungen) der in [vdPS03].

Gemäß Dubrovin wird erklärt, wie sich diese auf den ersten Strukturzusammenhang anwenden lässt.

Dubrovins Vermutung und halbeinfache Spiegelsymmetrie. Das darauffolgende Kapitel berichtet über eine Vermutung von Dubrovin, die besagt, bei genau welchen Varietäten V halbeinfache Quantenkohomologie zu erwarten ist. Und zwar sei dies genau für diejenigen der Fall, für die die derivierte Kategorie $D^b(V)$ der kohärenten Garben auf V in einem gewissen Sinne halbeinfach ist; genauer gesagt, falls es in $D^b(V)$ ein sogenanntes exzeptionelles System von Objekten gibt, siehe Definition 6.1.1.

Ferner vermutet Dubrovin, dass sich in diesem Fall auch die Stokes-Matrix der Frobenius-Mannigfaltigkeit zu V aus der halbeinfachen Struktur von $D^b(V)$ ablesen lässt. Das Kapitel 7 versucht zu erklären, warum dies aus dem Kontext der Spiegelsymmetrie zu V zu erwarten ist:

Zu einer solchen Varietät V ist der Spiegelpartner eine affine Varietät Y mit einer gegebenen Funktion f auf Y mit isolierten Singularitäten. Aus den Deformationen dieser Funktion f, die Entfaltungen der Singularitäten liefern, entsteht eine Frobenius-Mannigfaltigkeit. Spiegelsymmetrie würde besagen, dass diese Frobenius-Mannigfaltigkeit isomorph zu der der Quantenkohomologie von V ist; im Fall von $\mathbb{P}^n$ ist dies von Barannikov bewiesen wurden (cf. Satz 7.4.1). Wir versuchen zu zeigen (ohne vollständigen Beweis), dass die Stokes-Matrix dieser Frobenius-Mannigfaltigkeit identisch zu der rein topologisch aus der Milnor-Faserung von f definierten Seifert-Matrix ist.

Daraus, und aus einer weiteren Spiegelsymmetrievermutung analog zu Kontsevichs homologischer Spiegelsymmetrie, würde Dubrovins Behauptung über Stokes-Matrizen folgen.

Aufblasungen. Das eigentlich neue Resultat dieser Arbeit ist Satz 8.2.1.

Es sei X eine n-dimensionale projektive Varietät, und $\tilde{X}$ die Aufblasung von X an einem Punkt $x \in X$; sei $E \subset \tilde{X}$ der exzeptionelle Divisor (Faser über x) der Abbildung $\tilde{X} \to X$, mit demselben Buchstaben bezeichnen wir auch die zugehörige Kohomologieklasse in $H^2(\tilde{X})$. Es ist $H^*(\tilde{X}) \cong H^*(X) \oplus \mathbb{C} \cdot E \oplus \mathbb{C} \cdot E^2 \oplus \ldots \mathbb{C} \cdot E^{n-1}$, und damit ist auch das Cup-Produkt in $H^*(\tilde{X})$ (fast) vollständig beschrieben.

Für das deformierte Cup-Produkt der Quantenkohomologie stellt sich nun folgende Frage:

> Wenn die Frobenius-Mannigfaltigkeit der Quantenkohomologie von X halbeinfach ist, gilt dann dasselbe auch für $\tilde{X}$?

Falls nämlich die Halbeinfachheit für X gilt, müsste gemäß Dubrovins Vermutung $D^b(X)$ ein exzeptionelles System haben. Dasselbe gilt dann nach einem Resultat von Bondal auch für die Kategorie $D^b(\tilde{X})$. Folglich müsste auch $\tilde{X}$ halbeinfache Quantenkohomologie haben.

Damit ist obige Frage ein ernsthafter Test für Dubrovins Vermutung. Unter zwei technischen Zusatzvoraussetzungen beantwortet unser Satz diese Frage positiv. Der Beweis beruht wesentlich auf Resultaten von Gathmann, die die Gromov-Witten-Invarianten von $\tilde{X}$ und X in Beziehung setzen. Ein Verschwindungssatz von ihm ermöglicht eine partielle Kompaktifizierung des Parameterraums, für den das Quantenprodukt definiert ist. Entlang des neu hinzugefügten Divisors zerfällt das Quantenprodukt von $\tilde{X}$ in die direkte Summe eines halbeinfachen Anteils und eines Anteils isomorph zum Quantenprodukt von X.

3 Definitions and Notations

3.1 Frobenius Manifolds

To fix definitions and notations, we collect here the relevant definitions from [Man99].

Definition 3.1.1. *A* Frobenius manifold with flat identity *is a complex manifold $\mathcal{M}^n$ endowed with the following structures on the tangent sheaf $\mathcal{TM}$:*

- *A non-degenerate symmetric bilinear form $g\colon \mathcal{TM} \otimes \mathcal{TM} \to \mathcal{O}_{\mathcal{M}}$, called the metric.*
- *A commutative, associative multiplication*

$$\circ\colon \mathcal{TM} \otimes \mathcal{TM} \to \mathcal{TM}.$$

- *A section e of $\mathcal{TM}$ which is a unit with respect to $\circ$.*

They satisfy the following axioms:

- *The metric is multiplication invariant, i. e. for all tangent vectors X, Y, Z we have:*
$$g(X \circ Y, Z) = g(X, Y \circ Z)$$
- *The metric g is flat.*
- *If A denotes the symmetric tensor $A(X,Y,Z) := g(X \circ Y, Z)$, there is (everywhere locally) a potential Φ such that for all flat vector fields X, Y, Z we have*
$$XYZ\Phi = A(X,Y,Z).$$

Many Frobenius manifolds come together with a grading, which is expressed by the existence of an Euler field:

Definition 3.1.2. *An Euler field E with conformal weight D is a vector field on $\mathcal{M}$ such that*

$$\mathrm{Lie}_E(g) = Dg \quad \textit{and} \quad \mathrm{Lie}_E(\circ) = d_0 \cdot \circ$$

for some constant d_0.

Here Lie denotes the usual Lie derivative of tensor fields.

In the cases we consider the constant d_0 is always 1 and will therefore be omitted.

Definition 3.1.3. *A point $m \in \mathcal{M}$ is called semisimple if $(\mathcal{T}_m\mathcal{M}, \circ)$ is semisimple as an algebra. By this we mean that it is isomorphic, as a $\mathbb{C}$-algebra, to $\mathbb{C}^n$ with component-wise multiplication.*

A connected Frobenius manifold is called generically semisimple if it contains a semisimple point. (In this case, the set of semisimple points is necessarily an open and dense subset of $\mathcal{M}$.)

One can reformulate this using the *spectral cover map*: Since $(\mathcal{TM}, \circ)$ is a sheaf of rings on $\mathcal{M}$, we obtain a scheme $\mathrm{Spec}(\mathcal{TM}, \circ)$ with a natural projection

$$\mathrm{Spec}(\mathcal{TM}, \circ) \to \mathcal{M}.$$

This is called the spectral cover map. It is a finite flat morphism of degree n. Semisimple points are those where the map is unramified (or equivalently, étale) in the whole fibre. Generic semisimplicity means that the spectral cover map is generically unramified.

If we forget the metric, the structure of a Frobenius manifold with Euler field at a semisimple point is very easy to understand:

Proposition 3.1.4. *Let m be a semisimple point. There exist coordinates $u_1, \dots, u_n$, called* canonical coordinates*, in a neighbourhood U of m such that at each point $m' \in U$*

- *the vector fields $\frac{\partial}{\partial u_i}$ yield the decomposition of $(\mathcal{T}_{m'}\mathcal{M}, \circ)$ into a semisimple algebra,[1] and*
- *the eigenvalues of $E\circ$ at each point are $(u_1(m'), \dots, u_n(m'))$.*

They are unique up to reordering.

So the classification of semisimple germs of Frobenius manifolds is essentially the classifications of metrics compatible with the multiplication and Euler field given as in this proposition. A rather straightforward way to define invariants of such a germ consists of the *special coordinates*:

Definition 3.1.5. *Let $m \in \mathcal{M}$ be a semisimple point with canonical coordinates $u_1, \dots, u_n$. With $\eta_i := g(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_i})$, we call $u_i^0 = u_i(m)$, $\eta_i^0 = \eta_i(m)$ and $\eta_{ij}^0 := \frac{\partial}{\partial u_j}|_m \eta_i$ the special coordinates of the germ $(\mathcal{M}, m)$ of a Frobenius manifold.*

These easy to define invariants, together with the values of the canonical coordinates at m, actually classify semisimple germs of Frobenius manifolds. The proof uses the so-called *second structure connection*, see [Man99, II.3]. An alternative way to classify these germs is due to Dubrovin; we explain it in section 5.

3.2 Quantum Cohomology

This section will recall the definitions of Quantum Cohomology to fix notations. For more details, we refer to [Man99].

Throughout this section let V be a smooth projective variety over $\mathbb{C}$. By Δ_i we will denote cohomology classes, and β will always be an (effective) homology class in $H_2(V)$.

Definition 3.2.1. *We denote the correlator in the quantum cohomology of V by*

$$\langle \Delta_1 \dots \Delta_n \rangle_\beta.$$

[1] This means that the tangent vectors $\frac{\partial}{\partial u_i}$ are idempotents satisfying $\frac{\partial}{\partial u_i} \circ \frac{\partial}{\partial u_j} = 0$ for pairs $i \neq j$.

This is the number of appropriately counted stable maps

$$f\colon (\mathcal{C}, y_1, \ldots, y_n) \to V$$

where

- $\mathcal{C}$ is a semi-stable curve of genus zero,
- $y_1, \ldots, y_n$ are marked points on $\mathcal{C}$,
- the fundamental class of $\mathcal{C}$ is mapped to β under f,
- and $\Delta_1, \ldots, \Delta_n$ are cohomology classes representing conditions for the images of the marked points.

Such a correlator vanishes unless

$$k(\beta) := (c_1(V), \beta) = 3 - \dim V + \sum(\frac{a_i}{2} - 1) \tag{1}$$

where $a_i = |\Delta_i|$ are the degrees of the cohomology classes.

We will not say anything about the definition of the correlators, nor will we list the set of axioms they satisfy; instead, we refer to [Man99, section III.5].

From these invariants, we derive the potential of quantum cohomology:

Definition 3.2.2. *Let $\Delta_0, \Delta_1, \ldots, \Delta_r$ be a homogeneous basis of $H^*(V)$ with Δ_0 being the unity in $H^0(V)$.*

Then we define

$$\Phi(x_0, x_1, \ldots, x_r) := \sum_\beta \sum_{n=0}^{\infty} \frac{1}{n!} \langle (x_0\Delta_0 + x_1\Delta_1 + \cdots + x_n\Delta_n)^{\otimes n} \rangle_\beta,$$

where we pretend for a moment that this series converges in a non-empty domain in $H^*(V)$.

A more compact way to write this is $\Phi = \langle e^{\sum x_i \Delta_i} \rangle$.

Theorem 3.2.3. *The domain of convergence of Φ on $H^*(V)$ becomes a Frobenius manifold by*

- *letting the pairing $g(X, Y)$ be the Poincaré pairing (where we have, of course, identified the tangent space with $H^*(V)$),*
- *and defining the multiplication via the potential Φ:*

$$g(X, Y \circ Z) := XYZ\Phi$$

(which defines $Y \circ Z$ uniquely as g is non-degenerate).

While the flatness and the multiplication invariance of the metric are obvious in this construction, the associativity of the multiplication is surprising.

As an explicit formula for the multiplication we get

$$\begin{aligned}\Delta_i \circ \Delta_j &= \sum_k (\partial_i \partial_j \partial_k \Phi)\, \Delta^k \\ &= \Delta_i \cup \Delta_j + \sum_{\beta \neq 0} \sum_{k \neq 0} \langle \Delta_i \Delta_j \Delta_k e^{\sum_l x_l \Delta_l} \rangle_\beta \Delta^k,\end{aligned}$$

where Δ^k are the elements of the dual basis with respect to the Poincaré pairing.

The cohomology classes of $H^2(V)$ play a special role in quantum cohomology due to the divisor axiom (see [Man99, III.5.3]). This allows us to rewrite the above formula as follows:

$$\Delta_i \circ \Delta_j = \Delta_i \cup \Delta_j + \sum_{\beta \neq 0} \sum_{k \neq 0} \langle \Delta_i \Delta_j \Delta_k e^{\sum_{l:|\Delta_l|>2} x_l \Delta_l} \rangle_\beta \Delta^k e^{(\sum_{l:|\Delta_l|=2} x_l \Delta_l, \beta)}.$$

It is convenient to replace the coordinates x_i that have $|\Delta_i| = 2$ with $q_i = e^{x_i}$. Also we write q^β as shorthand for $\prod_{i:|\Delta_i=2|} q_i^{(\Delta_i,\beta)}$. So we finally get:

$$\Delta_i \circ \Delta_j = \Delta_i \cup \Delta_j + \sum_{\beta \neq 0} \sum_{k \neq 0} \langle \Delta_i \Delta_j \Delta_k e^{\sum_{l:|\Delta_l|>2} x_l \Delta_l} \rangle_\beta \Delta^k q^\beta. \qquad (2)$$

Now q^β can be regarded as a generic character of $H_2(V)/\text{torsion}$. Also, if we view this as a formula in the Novikov ring generated by q^β, we eliminate the possible problems of non-convergence that we have ignored so far:

Consider the polynomial ring N associated to the semi-group of effective classes $\beta \in H_2(V)/\text{torsion}$, i. e. the ring generated by the monomials q^β. As this semi-group has indecomposable zero, we can take the formal completion $\hat{N}$. This is the Novikov ring.

Instead of a Frobenius manifold in the sense of the definition in section 3.1 we then get a *formal Frobenius manifold*: Instead of the ring $\mathcal{O}_{H^*(V)}$ of functions on $H^*(V)$, all structures are defined over $A := \mathcal{O}_{H^*(V)} \otimes_N \hat{N}$. Here N is considered as a subring of $\mathcal{O}_{H^*(V)}$ via the map

$$q^\beta \mapsto \left(\sum_i x_i \Delta_i \mapsto e^{\left(\sum_{i:|\Delta_i|=2} x_i \Delta_i, \beta\right)} \right).$$

The notion of generic semisimplicity also makes sense for formal Frobenius manifolds: It means that the structure map $(H^*(V) \otimes A, \circ) \to A$ is generically semisimple, or equivalently generically unramified, or that it admits a semisimple point.

4 Semisimple Computations

4.1 Fano manifolds with minimal (p,p)-cohomology

The finite family of numbers $(u_i^0, \eta_i^0, \eta_{ij}^0)$ defined in section 3.1 essentially coincides with what was called *special coordinates* of the tame semisimple germ of a Frobenius manifold, cf. [Man99, II.7.1.1]. In this subsection, we will show how to calculate them for the $\bigoplus H^{p,p}(V)$-part of the quantum cohomology of those Fano manifolds for which $\dim H^{p,p}(V) = 1$ for all $1 \le p \le \dim V =: r$. This generalizes the computation for projective spaces done in [Man99, II.4].

We will work with a homogeneous basis $\Delta_p \in H^{p,p}(V)$ consisting of rational cohomology classes satisfying the following conditions: Δ_0 = the dual class of $[V]$, $\Delta_1 = c_1(V)/\rho$ is the ample generator of Pic$V$, and ρ is called *the index* of V. Furthermore, Δ_{r-p} is dual to Δ_p with respect to the Poincaré paring, that is $(\Delta_p, \Delta_{r-p}) = 1$, Δ_r = the dual class of a point. The dual coordinates are denoted $x_0, \ldots, x_r$. From the axioms for Gromov-Witten invariants (cf. [Man99, III.5.3, (vii)]) it follows that the non-vanishing correlators with $\beta = 0$ are the coefficients of the cubic self-intersection form

$$(x_0\Delta_0 + \cdots + x_r\Delta_r)^3.$$

We put

$$[d;\, a_1, \ldots, a_k] := \langle \Delta_{a_1} \ldots \Delta_{a_k} \rangle_{d\Delta_{r-1}}\,.$$

These symbols satisfy the following relations:

1. If $r \ge 3$, $[d;\, a_1, \ldots, a_k] \ne 0$ and $d > 0$, then necessarily
$$k > 0,\ a_i > 0 \text{ for all } i,\ \text{and } d\rho = \sum_{i=1}^{k} (a_i - 1) + 3 - r \qquad (3)$$
(see (1)).
2. $[d;\, a_1, \ldots, a_k]$ is symmetric with respect to the permutations of $a_1, \ldots, a_k$.
3. $[d;\, 1, a_2, \ldots, a_k] = d\,[d;\, a_2, \ldots, a_k]$ (divisor axiom).
4. Associativity relations, expressing the associativity of the multiplication (2).

The multiplication table in the first infinitesimal neighborhood of $H^2(V)$ (i. e. modulo J^2, where $J = (x_2, x_3, \ldots, x_r)$) involves only up to four-point correlators and looks as follows:

$$\begin{aligned}\Delta_a \circ \Delta_b &= \Delta_a \cup \Delta_b \\ &+ \sum_{d\ge1}\sum_{c\ge1} \left([d;\, a,b,c] + \sum_{f\ge2} [d;\, a,b,c,f]\, x_f \right) \Delta_{r-c} q^d\,. \qquad (4)\end{aligned}$$

Finally, the (restricted) Euler field of weight 1 is

$$E = \sum_{p=0}^{r} (1-p)\, x_p \Delta_p + \rho\, \Delta_1$$

Now if there exists a tame semisimple point in $H^2(V)$, the multiplication in the first order neighbourhood of $H^2(V)$ determines the special coordinates (see 3.1.5) and hence the full quantum cohomology of V (see [BM01, Theorem 1.8.3]). Then the eigenvalues of $E\circ$ at the generic point of H^2 are pairwise distinct and determine the canonical coordinates of this point. We have to calculate in the first infinitesimal neighborhood of H^2 and therefore we consider all the relevant quantities as consisting of two summands: restriction to H^2 and the linear (in x_a) correction term; so we write

$$u_i := u_i^{(0)} + u_i^{(1)}.$$

The remaining special coordinates are given by the following formulas.

Theorem 4.1.1. *Put*

$$e_i := \frac{\prod_{j\neq i}(E - u_j)}{\prod_{j\neq i}(u_i - u_j)} = e_i^{(0)} + e_i^{(1)}\,.$$

where the multiplication is understood in the sense of quantum cohomology with the coefficient ring extended by (u_i) and $(u_i - u_j)^{-1}$.

Then we have on H^2:

$$\eta_i = e_i^{(0)}(x_r),\;\; \eta_{ij} = e_i^{(0)} e_j\,(x_r) \tag{5}$$

where the e_i are considered as vector fields acting upon coordinates via $\Delta_a = \partial/\partial x_a$.

Proof. The elements e_i are the basic pairwise orthogonal idempotents in the quantum cohomology ring at the considered point satisfying $E \circ e_i = u_i e_i$. The metric potential η is x_r.

Here is an efficient way of computing $e_i^{(1)}$. First, compute ω_i defined by the identity in the first neighborhood:

$$e_i^{(0)} \circ e_i^{(0)} = e_i^{(0)} + \omega_i. \tag{6}$$

Then we have

$$e_i^{(1)} = -\frac{\omega_i}{2e_i^{(0)} - 1} = \omega_i \circ (1 - 2e_i^{(0)}) \tag{7}$$

where the division resp. multiplication is again made in the first neighborhood.

In fact, this follows from (6) and

$$(e_i^{(0)} + e_i^{(1)})^{\circ 2} = e_i^{(0)} + e_i^{(1)}.$$

□

4.2 Fano threefolds with minimal cohomology

4.2.1 Notation

Let V be a Fano threefold. We keep the general notation of the last section, but now consider only the case $r = 3$. Besides the index ρ, we consider the degree $\delta := (c_1(V)^3)/\rho^3$ of V.

There exist four families of Fano threefolds $V = V_\delta$ with cohomology $H^{p,p}(V,\mathbb{Z}) \cong \mathbb{Z}$ for $p = 0, \dots, 3$ and $H^{p,q}(V,\mathbb{Z}) = 0$ for $p \neq q$. Besides $V_1 = \mathbb{P}^3$ and the quadric $V_2 = Q$, they are V_5 and V_{22}, with degree as subscript; their indices are, respectively, $4, 3, 2, 1$. One can get a V_5 by considering a generic codimension three linear section of the Grassmannian of lines in $\mathbb{P}^4$ embedded in $\mathbb{P}^9$.

The nonvanishing $\beta = 0$ correlators are coefficients of the cubic self-intersection form

$$(x_0\Delta_0 + \cdots + x_3\Delta_3)^3 = \delta x_1^3 + 3x_0^2 x_3 + x_0 x_1 x_2 \,.$$

In this section, we will deal only with Q, V_5 and V_{22}, since projective spaces of any dimension were treated by various methods earlier: see [Man99, II.4] for special coordinates, [Dub98, 4.2.1] and [Guz99] for monodromy data, and [Bar01] for semiinfinite Hodge structures.

4.2.2 Tables of correlators

The following tables provide the coefficients of the multiplication table (4).

It suffices to tabulate the primitive correlators, where primitivity means that $a_i > 1$ and $a_i \leq a_{i+1}$. The symmetry and the divisor identities furnish the remaining correlators.

Manifold Q:

[1; 2,3]	[1; 2,2,2]	[2; 3,3,3]	[2; 2,2,3,3]
1	1	1	1

Manifold V_5:

[1; 3]	[1; 2,2]	[2; 3,3]	[2; 2,2,3]	[3; 3,3,3]	[2; 2,2,2,2]	[3; 2,2,3,3]	[4; 3,3,3,3]
3	1	1	1	1	1	2	3

Manifold V_{22}:

[1; 2]	[2; 3]	[2; 2,2]	[3; 2,3]	[4; 3,3]	[3; 2,2,2]	[4; 2,2,3]	[5; 2,3,3]
2	6	1	3	10	1	4	16

[6; 3,3,3]	[4; 2,2,2,2]	[5; 2,2,2,3]	[6; 2,2,3,3]	[7; 2,3,3,3]	[8; 3,3,3,3]
65	2	9	41	186	840

The tables were compiled in the following way. First, (3) furnishes the list of all primitive correlators that might be (and actually are) non-vanishing. Second, several correlators corresponding to the smallest values of n in (3) must be computed geometrically: $n = 2$ for Q, and $n = 1, 2$ for V_5, and V_{22}. These values were computed by A. Bondal, D. Kuznetsov and D. Orlov. Third, the associativity equations uniquely determine all the remaining correlators, in the spirit of the First Reconstruction Theorem of [KM94].

4.2.3 Canonical coordinates

The canonical coordinates on $H^2 \cap \{x_0 = 0\}$ expressed in terms of the flat coordinates are the roots $u_0, \ldots, u_3$ of the following characteristic equations of the operator $E\circ$:

$$\begin{aligned} Q: &\quad u^4 - 108\,q\,u &&= 0, \\ V_5: &\quad u^4 - 44\,q\,u^2 - 16\,q^2 &&= 0, \\ V_{22}: &\quad (u + 4\,q)\,(u^3 - 8\,qu^2 - 56\,q^2u - 76\,q^3) &&= 0. \end{aligned} \tag{8}$$

If $x_0 \neq 0$, one must simply add x_0 to the values above.

Question. Find "natural" functions $f(z)$ whose critical values at 0 are roots of (8) and whose unfolding space carries an appropriate flat metric.

4.2.4 Multiplication tables, idempotents, and metric coefficients

The remaining special coordinates η_i, η_{jk} were calculated using the multiplication tables in the first neighborhood of H^2 obtained by specializing (4). We calculated $e_i^{(0)}$ by determining the eigenvectors of ad E; then we used equation (7) to get $e_i^{(1)}$.

Manifold Q:

$$\begin{aligned} \Delta_1^2 &= 2\,\Delta_2 + q\,\Delta_1 x_3 + q\,\Delta_0 x_2, \\ \Delta_1\Delta_2 &= \Delta_3 + q\,\Delta_0 + q\,\Delta_2 x_3 + q\,\Delta_1 x_2, \\ \Delta_1\Delta_3 &= q\,\Delta_1 + q\,\Delta_2 x_2 + 2q^2\Delta_0 x_3, \\ \Delta_2^2 &= q\,\Delta_1 + q\,\Delta_2 x_2 + q^2\Delta_0 x_3, \\ \Delta_2\Delta_3 &= q\,\Delta_2 + q^2\Delta_0 x_2 + q^2\Delta_1 x_3, \\ \Delta_3^2 &= q^2\Delta_0 + q^2\Delta_1 x_2 + 2q^2\Delta_2 x_3. \end{aligned}$$

Let ξ_i, $i = 1, \ldots, 3$ be the three roots of $\xi^3 = 4q$. Then the respective idempotents have the following form:

$$e_0 = \frac{1}{2}\Delta_0 - \frac{1}{2q}\Delta_3 + \frac{x_2}{4}\Delta_1 + \frac{x_3}{2}\Delta_2$$

and, for $i = 1, \ldots, 3$,

$$\begin{aligned} e_i = {} & \frac{1}{6}\Delta_0 + \frac{\xi_i^2}{12q}\Delta_1 + \frac{\xi_i}{6q}\Delta_2 + \frac{1}{6q}\Delta_3 - \frac{\xi_i x_2}{36}\Delta_0 \\ & - \left(\frac{x_2}{12} + \frac{\xi_i x_3}{12}\right)\Delta_1 - \left(\frac{\xi_i^2 x_2}{18q} + \frac{x_3}{6}\right)\Delta_2 - \left(\frac{\xi_i x_2}{27q} + \frac{\xi_i^2 x_3}{12q}\right)\Delta_3. \end{aligned}$$

Probably the most direct and efficient test of our computations is to simply compute the pairwise products of these idempotents using the multiplication table above. This verifies the formulas of the e_j while checking at the same time that our multiplication table yields an associative product.

Note that $\Delta_i = \frac{\partial}{\partial x_i}$ and $\frac{\partial}{\partial x_1} q = q$ since we identify q with e^{x_1} to get a proper (non-formal) Frobenius manifold. So we get as special coordinates (where $i, j \in \{1, 2, 3\}$):

$$\eta_0 = -\frac{1}{2q}, \quad \eta_i = \frac{1}{6q}, \quad \eta_{00} = 0,$$

$$\eta_{i0} = \frac{\xi_i^2}{12q}\Delta_1\left(\frac{-1}{2q}\right) = \frac{\xi_i^2}{24q^2}, \quad \eta_{0i} = \frac{-1}{2q}\Delta_3\left(-\frac{\xi_i^2 x_3}{12q}\right) = \frac{\xi_i^2}{24q^2},$$

$$\eta_{ij} = \frac{\xi_i^2}{12q}\Delta_1\left(\frac{1}{6q}\right) - \frac{\xi_i}{6q}\Delta_2\left(\frac{\xi_j x_2}{27q}\right) - \frac{1}{6q}\Delta_3\left(\frac{\xi_j^2 x_3}{12q}\right) = -\frac{\xi_i^2}{72q^2} - \frac{\xi_i\xi_j}{162q^2} - \frac{\xi_j^2}{72q^2}.$$

The symmetry of η_{ij} is just an additional check of our computations, as we know in general that e_i are commuting vector fields.

Manifold V_5:

$$\begin{aligned}
\Delta_1^2 &= 5\Delta_2 + 3q\,\Delta_0 + 3q\,\Delta_2 x_3 + q\,\Delta_1 x_2 + 4q^2\Delta_0 x_3,\\
\Delta_1\Delta_2 &= \Delta_3 + q\,\Delta_1 + q\,\Delta_2 x_2 + 2q^2\Delta_1 x_3 + 2q^2\Delta_0 x_2,\\
\Delta_1\Delta_3 &= 3q\,\Delta_2 + 2q^2\Delta_0 + 4q^2\Delta_2 x_3 + 2q^2\Delta_1 x_2 + 3q^3\Delta_0 x_3,\\
\Delta_2^2 &= q\,\Delta_2 + q^2\Delta_0 + 2q^2\Delta_2 x_3 + q^2\Delta_1 x_2 + 2q^3\Delta_0 x_3,\\
\Delta_2\Delta_3 &= q^2\Delta_1 + 2q^2\Delta_2 x_2 + 2q^3\Delta_1 x_3 + 2q^3\Delta_0 x_2,\\
\Delta_3^2 &= 2q^2\Delta_2 + q^3\Delta_0 + 3q^3\Delta_2 x_3 + 2q^3\Delta_1 x_2 + 3q^4\Delta_0 x_3.
\end{aligned}$$

Let u_i be the roots of $u^4 - 44\,q\,u^2 - 16\,q^2$. The idempotents are given by

$$\begin{aligned}
4000q^3 e_i = \; & 1440q^3\Delta_0 - 20q^2{u_i}^2\Delta_0 + 70qu_i^3\Delta_1 - 3040q^2u_i\Delta_1\\
& -880q^2\Delta_2 + 40q{u_i}^2\Delta_2 + 4920qu_i\Delta_3 - 110{u_i}^3\Delta_3\\
& -1968u_iq^3x_2\Delta_0 + 44{u_i}^3q^2x_2\Delta_0 + 352q^4x_3\Delta_0 - 16{u_i}^2q^3x_3\Delta_0\\
& +176q^3x_2\Delta_1 - 8q^2{u_i}^2\Delta_1x_2 - 5412u_iq^3\Delta_1x_3 + 121{u_i}^3q^2\Delta_1x_3\\
& -2864u_iq^2x_2\Delta_2 + 62q{u_i}^3x_2\Delta_2 + 1056q^3\Delta_2x_3 - 48x_3{u_i}^2q^2\Delta_2\\
& -16qu_i^2x_2\Delta_3 + 6036q^2u_ix_3\Delta_3 + 352q^2x_2\Delta_3 - 138q{u_i}^3x_3\Delta_3.
\end{aligned}$$

The special coordinates η_{ii} are

$$\eta_{ii} = \frac{-964\,q + 21\,{u_i}^2}{800\,q^3} = \frac{-251 \pm 105\sqrt{5}}{400\,q^2}.$$

Now since the Galois group of $u^4 - 44\,q\,u^2 - 16\,q^2$ obviously does not act transitively on the pairs of roots, we have to distinguish two cases in determining the η_{ij}. So we fix a root u_1; the other roots are given by $u_2 = -u_1$ and $u_{3,4}^2 - 11\,q + u_1^2 = 0$. We calculated

$$\begin{aligned}
\eta_{12} = \eta_{21} &= \frac{932\,q - 21\,{u_1}^2}{800\,q^3} = \frac{47 \mp 21\sqrt{5}}{80\,q^2} \quad \text{and}\\
\eta_{13} &= \frac{-3\,u_3u_1 + 4\,q}{200q^3}.
\end{aligned}$$

All other coordinates are obtained from these via Galois permutations.

Manifold V_{22}:

$$\begin{aligned}
\Delta_1^2 &= 22\Delta_2 + 2q\,\Delta_1 + 24q^2\Delta_0 \\
&\quad +2q\,\Delta_2 x_2 + 48q^2\Delta_2 x_3 + 4q^2\Delta_1 x_2 \\
&\quad +27q^3\Delta_1 x_3 + 27q^3\Delta_0 x_2 + 160q^4\Delta_0 x_3, \\
\Delta_1\Delta_2 &= \Delta_3 + 2q\,\Delta_2 + 2q^2\Delta_1 + 9q^3\Delta_0 \\
&\quad +4q^2\Delta_2 x_2 + 27q^3\Delta_2 x_3 + 3q^3\Delta_1 x_2 \\
&\quad +16q^4\Delta_1 x_3 + 16q^4\Delta_0 x_2 + 80q^5\Delta_0 x_3, \\
\Delta_1\Delta_3 &= 24q^2\Delta_2 + 9q^3\Delta_1 + 40q^4\Delta_0 \\
&\quad +27q^3\Delta_2 x_2 + 160q^4\Delta_2 x_3 + 16q^4\Delta_1 x_2 \\
&\quad +80q^5\Delta_1 x_3 + 80q^5\Delta_0 x_2 + 390q^6\Delta_0 x_3, \\
\Delta_2^2 &= 2q^2\Delta_2 + q^3\Delta_1 + 4q^4\Delta_0 \\
&\quad +3q^3\Delta_2 x_2 + 16q^4\Delta_2 x_3 + 2q^4\Delta_1 x_2 \\
&\quad +9q^5\Delta_1 x_3 + 9q^5\Delta_0 x_2 + 41q^6\Delta_0 x_3, \\
\Delta_2\Delta_3 &= 9q^3\Delta_2 + 4q^4\Delta_1 + 16q^5\Delta_0 \\
&\quad +16q^4\Delta_2 x_2 + 80q^5\Delta_2 x_3 + 9q^5\Delta_1 x_2 \\
&\quad +41q^6\Delta_1 x_3 + 41q^6\Delta_0 x_2 + 186q^7\Delta_0 x_3, \\
\Delta_3^2 &= 40q^4\Delta_2 + 16q^5\Delta_1 + 65q^6\Delta_0 \\
&\quad +80q^5\Delta_2 x_2 + 390q^6\Delta_2 x_3 + 41q^6\Delta_1 x_2 \\
&\quad +186q^7\Delta_1 x_3 + 186q^7\Delta_0 x_2 + 840q^8\Delta_0 x_3.
\end{aligned}$$

Now let $u_i, i = 1, \ldots, 3$ be the roots of $u^3 - 8qu^2 - 56q^2u - 76q^3$. Then the respective idempotents are given by

$$\begin{aligned}
5324\,q^4\,e_i &= \left(-71742q^4 - 24552q^3u_i + 2354q^2{u_i}^2\right)\Delta_0 \\
&\quad + \left(-30272q^3 - 10186q^2u_i + 979q{u_i}^2\right)\Delta_1 \\
&\quad + \left(-118712q^2 - 38126qu_i + 3696{u_i}^2\right)\Delta_2 \\
&\quad + \left(49346q + 16192u_i - 1562\frac{{u_i}^2}{q}\right)\Delta_3 \\
&\quad + \left(-130876q^5 - 43283u_iq^4 + 4168{u_i}^2q^3\right)x_2\Delta_0 \\
&\quad + \left(-483464q^6 - 161648u_iq^5 + 15528{u_i}^2q^4\right)x_3\Delta_0 \\
&\quad + \left(-38977q^4 - 12940u_iq^3 + 1245{u_i}^2q^2\right)x_2\Delta_1 \\
&\quad + \left(-145898q^5 - 48889u_iq^4 + 4694{u_i}^2q^3\right)x_3\Delta_1 \\
&\quad + \left(-143992q^3 - 46818u_iq^2 + 4522{u_i}^2q\right)x_2\Delta_2 \\
&\quad + \left(-491334q^4 - 164832u_iq^3 + 15822{u_i}^2q^2\right)x_3\Delta_2 \\
&\quad + \left(35042q^2 + 11348u_iq - 1098{u_i}^2\right)x_2\Delta_3 \\
&\quad + \left(112272q^3 + 37824u_iq^2 - 3633{u_i}^2q\right)x_3\Delta_3.
\end{aligned}$$

The remaining idempotent is

$$\begin{aligned} e_0 &= \frac{\Delta_0}{2} + \frac{2\Delta_2}{q^2} - \frac{\Delta_3}{2q^3} + qx_2\Delta_0 + 2\,q^2x_3\Delta_0 \\ &\quad + \frac{x_2}{4}\,\Delta_1 + \frac{qx_3}{2}\Delta_1 + \frac{2\,x_2}{q}\Delta_2 + \frac{3x_3}{2}\Delta_2 - \frac{x_2}{2q^2}\Delta_3. \end{aligned}$$

From this we compute the special coordinates

$$\eta_0 = -\frac{1}{2q^3}, \qquad \eta_i = 49346q + 16192u_i - 1562\frac{{u_i}^2}{q},$$

$$\eta_{00} = -\frac{1}{q^4}, \qquad \eta_{i0} = \eta_{0i} = -\frac{-2536q^2 - 688\,u_iq + 69\,{u_i}^2}{968\,q^6},$$

$$\eta_{ii} = \frac{-3412q^2 - 260\,u_iq + 41\,{u_i}^2}{484\,q^6}$$

$$\eta_{ij} = \frac{404}{121\,q^4} - \frac{34}{121}\,\frac{u_i + u_j}{q^5} + \frac{13}{968}\,\frac{u_ju_i}{q^6}.$$

5 Dubrovin's Monodromy data

5.1 The first structure connection

When Dubrovin introduced the notion of Frobenius manifolds in [Dub96], one of the first observations he made concerned the existence of a flat meromorphic connection on $\mathbb{P}^1 \times \mathcal{M}$, called *first structure connection* in [Man99]. Its flatness is equivalent to the axioms of a Frobenius manifold with Euler field, and given this connection, the Frobenius manifold structure can be reconstructed if the unit field and the Euler field are known.

Definition 5.1.1. *The first structure connection of a Frobenius manifold with Euler field is a flat connection with singularities on the pullback $p^*\mathcal{TM}$ of the tangent bundle to $\mathbb{P}^1 \times \mathcal{M}$ via the projection $p\colon \mathbb{P}^1 \times \mathcal{M} \to \mathcal{M}$.*

If $X \in T(\mathbb{P}^1 \times \mathcal{M})$ is a horizontal tangent vector at a point $(\lambda, m) \in \mathbb{C} \times \mathcal{M}$, which we identify with the tangent vector p_*X in $\mathcal{M}$, then the connection is given by

$$\nabla_X Y = \nabla_{0,X} Y + \lambda X \circ Y,$$

where ∇_0 is the Levi-Civita connection on $\mathcal{M}$.

In λ-direction, the connection is given by

$$\nabla_{\frac{\partial}{\partial\lambda}} Y = E \circ Y - \frac{1}{\lambda}\left(\frac{D}{2}\,\mathrm{id} + [E, \cdot]\right)(Y)$$

where E is the Euler field and Y is assumed to be the pull-back of a flat section (with respect to the ordinary Levi-Civita connection on $\mathcal{M}$) of $\mathcal{TM}$.[2]

By restricting to any fibre $\{m\} \times \mathbb{P}^1$ we get a connection on $\mathbb{P}^1$ with singularities at 0 and ∞. It has a *regular singular point* (in the sense of [Del70]) at $\lambda = 0$. Such a regular singular point is characterized by its monodromy. The singularity at $\lambda = \infty$ has order 2.[3] Such singularities are classified by Stokes matrices, which we will explain in the following section.

5.2 Stokes matrices of an irregular singular point of a differential equation

Consider a differential equation of the form

$$\partial\xi = A\xi, \tag{9}$$

where A is a $n \times n$-matrix of meromorphic functions and ξ is a column vector. The singular points are by definition the singularities of A; they are called *regular singular* if A has a simple pole. Regular singular points arise very naturally as singularities of Gauß-Manin connections. However, the first structure connection of a Frobenius manifold has an irregular singularity (i. e., the matrix A

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[2] This deviates from [Man99, Definition 2.5.1] to match Dubrovin's definition, e. g. [Dub99, (2.28)]. It also matches Hertling's definition in [Her02, Definition 4.6] with $s = -\frac{1}{2}$. If D is an even integer, this is gauge equivalent to [Man99, Definition 2.5.1] by the rational gauge transformation $Y \mapsto z^{\frac{D}{2}} Y$.

[3] This corresponds to $k = 1$ in the notation of the following section.

has a second-order pole); the gauge equivalence class of this singularity will be part of Dubrovin's classification data of semisimple Frobenius manifolds. Such an irregular singularity can also arise geometrically in a twisted version of the Gauß-Manin connection; this is used to construct Frobenius manifolds from hypersurface singularities.

Let $\mathbb{C}(\{z\})$ be the field of germs of meromorphic function on the complex plane at $z = 0$, and let $\mathbb{C}((z))$ be the field of formal Laurent series. Different differential equations over $\mathbb{C}(\{z\})$ having an irregular singular point can become gauge equivalent after base change to $\mathbb{C}((z))$. So the classification of irregular singular differential equations over $\mathbb{C}(\{z\})$ can be split in two parts, which are

- classifying differential equations over $\mathbb{C}((z))$, and
- classifying differential equations over $\mathbb{C}(\{z\})$ that become isomorphic to a given differential equation over $\mathbb{C}((z))$ after base change.

The datum of Stokes matrices is one possible classification datum for the second step.

Differential modules and differential equations. As we will freely change our point of view between differential modules and that of differential equations (or rather their gauge equivalence classes), we feel obliged to spell out precisely how to pass from one to another.[4]

Definition 5.2.1. *A differential ring is a ring R (commutative, with identity) equipped with a derivation $\partial\colon R \to R$ satisfying the Leibniz rule $\partial(rr') = \partial(r)r' + r\partial(r')$.*

We will consider the differential rings $\mathbb{C}((z))$, $\mathbb{C}(\{z\})$ with $\frac{\partial}{\partial z}$ as the structure derivation.

Definition 5.2.2. *A differential module over a differential ring R is a locally free R-module M equipped with a derivation $\partial_M\colon M \to M$ satisfying the Leibniz rule $\partial_M(r.m) = \partial(r).m + r.\partial_M(m)$.*

Definition 5.2.3. *A matrix differential equation over R is an equation of the form*

$$\partial\xi = A\xi$$

where A is a $n \times n$-matrix over R. Two such equations given by matrices A and A' are called gauge equivalent if one can be obtained from the other by a gauge transformation $\xi = F\xi'$: This is the case iff $A' = F^{-1}(AF + \partial F)$ with F an invertible $n \times n$-matrix over R.

Of course, we always think of the indeterminate ξ as a column vector with entries in (some extension of) R.

[4] Some proofs will require coordinates, and hence the point of view of differential equations, yet at least in my case the word "differential module" is more likely to trigger some conceptual thinking—my apologies go to those readers whose mathematical thinking would not need this specific abstract nonsense.

We can pass from 5.2.3 to 5.2.2 by defining M to be R^n and its derivation as $\partial_M \xi := \partial\xi - A\xi$. If the ground ring is a field (or the differential module is free as an R-module), we can go the other way by choosing a basis $\xi_1, \dots, \xi_n$ of M; then A is the matrix expressing $\partial_M \xi_1, \dots, \partial_M \xi_n$ in terms of this basis. Another choice of a basis yields a gauge equivalent differential equation.

In this dictionary, fundamental matrix solutions of the differential equation correspond to trivializations of the differential module.

If, instead of a differential ring, we are given a topological space equipped with a sheaf of differential rings $\mathcal{O}$, it is immediately clear what we mean by a sheaf of differential modules over the sheaf $\mathcal{O}$. We will call this a *differential* $\mathcal{O}$*-module.*

We can get differential modules from a connection on an $\mathcal{O}$-module $\mathcal{M}$: If we choose any vector field X, we get derivations on $\mathcal{O}$ and $\mathcal{M}$ by $\partial\colon f \mapsto Xf$ and $\partial_M\colon s \mapsto \nabla_X s$. In our case of the first structure connection on $\mathbb{P}^1$, this will be a coordinate vector field $\frac{\partial}{\partial z}$.

Note that we also have to choose *one* such vector field X. In that sense, the whole theory is one-dimensional.

Differential modules over $\mathbb{C}((z))$**.** From now on, δ will denote the differential operator

$$\delta = z\frac{\partial}{\partial z}.$$

Over the field $\mathbb{C}((z))$ of formal Laurent series, the following theorem completely classifies differential modules:

Theorem 5.2.4. *[vdPS03] Let M be a finite dimensional differential module over $\mathbb{C}((z))$. Then there is a finite algebraic extension $\mathbb{C}((z^{\frac{1}{k}}))$, such that after base change M becomes a* split *differential module: M is isomorphic to the direct sum of differential modules M_i given by an equation of the form*

$$\delta\xi = (q_i + C_i)\xi$$

where q_i is a polynomial in $z^{-\frac{1}{k}}\mathbb{C}[z^{-\frac{1}{k}}]$ and $C \in \mathrm{Mat}_{n\times n}(\mathbb{C})$ is a $n \times n$-matrix (and δ is defined as above). The q_i are unique, and the matrices C_i are unique up to shifts by integers.

The fundamental short exact sequence. The whole theory of Stokes matrices is based upon a short exact sequence of sheaves on S^1. Here S^1 is identified with the set of rays in the complex plane starting in the origin, and hence an open set $U \subset S^1$ is seen as an open sector.

In the following, we will omit the proofs of all analytical lemmata and only focus on the algebraic structure of the theory. The exposition is close to [vdPS03], where all proofs can be found.

Definition 5.2.5. *The sheaf $\mathcal{A}$ of* functions with asymptotic expansion *is defined as follows: If $U \subset S^1$ is a connected open subset, then $\mathcal{A}(U)$ consists of germs at zero of functions f on the open sector U such that there exists an asymptotic expansion $\sum_{n\geq n_0} c_n z^n$ of f; this means that for every closed subset*

$W \subset U$ there exist constants $C(N, W)$ such that on an open neighbourhood of zero in W we have

$$|f(z) - \sum_{n=n_0}^{N-1} c_n z^n| < C(N, W) z^N.$$

Note that the asymptotic expansion, if it exists, is uniquely defined by f, so there is a canonical map α from $\mathcal{A}$ to the constant sheaf $\underline{\mathbb{C}((z))}$. However, the asymptotic expansion does not define f uniquely, and the map $\mathcal{A} \xrightarrow{\alpha} \underline{\mathbb{C}((z))}$ has a kernel $\mathcal{A}^0$ which consists of germs of rapidly vanishing functions. Typical sections of $\mathcal{A}^0$ are of the form $e^{az^{-k}}$ in a sector where az^{-k} has negative real part.

Note that

$$\mathcal{A}(S^1) = \mathbb{C}(\{z\}) \quad \text{and} \quad \mathcal{A}^0(S^1) = 0;$$

also, the following holds:

Lemma 5.2.6. *[vdPS03, proposition 7.22] The map of sheaves $\alpha \colon \mathcal{A} \to \underline{\mathbb{C}((z))}$ is surjective; more precisely, $\alpha(U) \colon \mathcal{A}(U) \to \underline{\mathbb{C}((z))}(U)$ is surjective iff $U \neq S^1$.*

Thus there is a short exact sequence

$$0 \to \mathcal{A}^0 \to \mathcal{A} \xrightarrow{\alpha} \underline{\mathbb{C}((z))} \to 0$$

of sheaves on S^1.

The additive Stokes phenomenon: The one-dimensional case. Let $q = a_k z^{-k} + \cdots + a_1 z^{-1}$ be an arbitrary polynomial. Consider the differential equation

$$(\delta - q)\hat{v} = w$$

where $w \in \mathbb{C}(\{z\})$ is a given germ of a meromorphic function and $\hat{v} \in \mathbb{C}((z))$ is a given formal solution. We want to know whether this formal solution can be lifted over an open sector $U \subset S^1$ to a solution $v \in \mathcal{A}(U)$ which has $\hat{v}$ as an asymptotic expansion.

To reduce this to a purely algebraic problem we need one more lemma:

Lemma 5.2.7. *[vdPS03] There is a short exact sequence*

$$0 \to \ker(\delta - q|_{\mathcal{A}^0}) \to \ker(\delta - q|_{\mathcal{A}}) \to \ker(\delta - q|_{\underline{\mathbb{C}((z))}}) \to 0$$

of sheaves on S^1. Furthermore, the map $\delta - q$ locally acts surjectively on each of the sheaves $\mathcal{A}$, $\mathcal{A}^0$ and $\underline{\mathbb{C}((z))}$.

Proof. We omit the proof of the second part of the assertion. The first part follows from the second by the 9-lemma (applied to the diagram given in the proof of the next lemma). ∎

From this, the following is easily deduced:

Proposition 5.2.8. *If $U \subsetneq S^1$ is a proper subset, then the existence of a lift of $\hat{v}$ to $v \in \mathcal{A}(U)$ solving $(\delta - q)v = w$ is controlled by $H^1(U, \ker(\delta - q|_{\mathcal{A}^0}))$.*

Any two lifts differ by an element of $H^0(U, \ker(\delta - q|_{\mathcal{A}^0}))$.

Proof. We will give a detailed proof, which only needs some diagram chasing within the following diagram, and omit similar proofs in later situations:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \ker(\delta - q)|_{\mathcal{A}^0} & \longrightarrow & \ker(\delta - q)|_{\mathcal{A}} & \xrightarrow{\alpha} & \ker(\delta - q)|_{\underline{\mathbb{C}((z))}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{A}^0 & \longrightarrow & \mathcal{A} & \xrightarrow{\alpha} & \underline{\mathbb{C}((z))} \ni \hat{v} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \delta - q} & & \downarrow{\scriptstyle \delta - q} & & \downarrow{\scriptstyle \delta - q} & & \\
0 & \longrightarrow & \mathcal{A}^0 & \longrightarrow & \mathcal{A} \ni w & \xrightarrow{\alpha} & \underline{\mathbb{C}((z))} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Let $v_1 \in \mathcal{A}(U)$ be any lift of $\hat{v}$ which exists by lemma 5.2.6. We have $\alpha(\delta - q)v_1 = (\delta - q)\hat{v} = \alpha(w)$. Hence we get an element $r \in \mathcal{A}^0(U)$ defined as $r := (\delta - q)v_1 - w$.

Now if r is mapped to zero in the canonical map $H^0(U, \mathcal{A}^0) \to H^1(U, \ker(\delta - q)|_{\mathcal{A}^0})$, we can find a preimage $r' \in H^0(U, \mathcal{A}^0)$ unique up to elements in $H^0(U, \ker(\delta - q)|_{\mathcal{A}^0})$ such that $(\delta - q)r' = r$. Then $v := v_1 + r'$ is the desired solution.

By reversing our arguments, we get the converse statement. ☐

This lemma asks us to understand the sheaf $\ker(\delta - q)|_{\mathcal{A}^0}$, the sheaf of rapidly vanishing flat sections.

Lemma 5.2.9. *Let a_k be the leading coefficient of q, i. e. $q = a_k z^{-k} + \cdots + a_1 z^{-1}$. Let $i_j \colon U_j \hookrightarrow S^1, j = 1, \ldots, k$ be the connected components of the open subset in which $\operatorname{Re}(a_k z^{-k}) > 0$, ordered counter-clockwise.*

There is an isomorphism of sheaves

$$\ker(\delta - q)|_{\mathcal{A}^0} \cong \bigoplus_j i_{j!}\underline{\mathbb{C}}_{U_j}$$

where $\underline{\mathbb{C}}_{U_j}$ is the constant sheaf $\mathbb{C}$ on U_j and $i_{j!}$ denotes the extension by zero.

Proof. All germs of holomorphic functions solving the differential equation $(\delta - q)f = 0$ are multiples of the solution

$$f(z) = e^{-\frac{a_k}{k} z^{-k} - \cdots - a_1 z^{-1}}.$$

These solutions are in $\mathcal{A}^0(U)$, i. e. are rapidly decreasing for $z \to 0$, if and only if $\operatorname{Re}(-\frac{a_k}{k} z^{-k}) < 0$ for $z \in U$. ☐

The border lines of the sectors U_j are called the Stokes rays of the differential equation.

From the cohomology of the sheaf $\underline{\mathbb{C}}_{S^1}$ and of the cokernel of $\ker(\delta-q)|_{\mathcal{A}^0} \hookrightarrow \underline{\mathbb{C}}_{S^1}$ (which is the direct sum $\bigoplus_j i_{j*}\underline{\mathbb{C}}_{V_j}$ where V_j is the closed sector in between U_j and U_{j+1}), we can immediately compute the cohomology of $\ker(\delta - q)|_{\mathcal{A}^0}$):

Corollary 5.2.10. *Let $U \subsetneq S^1$ be a closed, connected subsector. If U is contained in one of the U_j defined in lemma 5.2.9, then*

$$H^0(U, \ker(\delta - q)|_{\mathcal{A}^0}) \cong \mathbb{C},$$

otherwise it is zero.

Further, if l is the number of j such that U_j is contained in U, we have

$$H^1(U, \ker(\delta - q)|_{\mathcal{A}^0}) \cong \mathbb{C}^l.$$

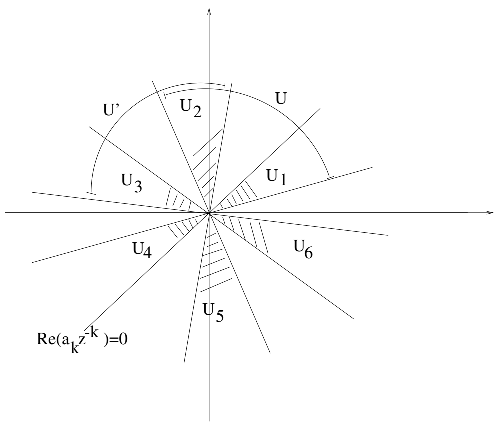

Figure 1: Stokes sectors with $k = 6$.

Now assume that U, U' are two closed connected sectors such that their intersection is non-empty, and such that both of them are neither completely contained in a sector U_j, nor do contain one of these sectors; see fig. 1. This ensures that

$$H^0(U, \ker(\delta - q)|_{\mathcal{A}^0}) = H^0(U', \ker(\delta - q)|_{\mathcal{A}^0}) =$$

$$= H^1(U, \ker(\delta - q)|_{\mathcal{A}^0}) = H^1(U, \ker(\delta - q)|_{\mathcal{A}^0}) = 0.$$

By lemma 5.2.8 we have unique lifts v, v' over U, U' of $\hat{v}$ solving the same differential equation $(\delta - q)\hat{v} = w$. However, on the intersection $U \cap U'$ they need not coincide. This is called the additive Stokes phenomenon.

So, in our language the additive Stokes phenomenon is a Čech-cocycle representation of an element in

$$H^1(U \cup U', \ker(\delta - q)|_{\mathcal{A}^0})$$

given in the covering (U, U').

Higher dimensional additive Stokes phenomena. The preceding section is easily generalized to the case of a higher-dimensional differential equation: We consider the differential equation

$$(\delta - A)\hat{v} = w \tag{10}$$

where A is a $n \times n$-matrix with coefficients in $\mathbb{C}(\{z\})$, $w \in \mathbb{C}(\{z\})^n$ is fixed and $\hat{v} \in \mathbb{C}((z))^n$ is a given solution; again we are considering the problem whether there exists a lift of $\hat{v}$ to an element $v \in \mathcal{A}(U)^n$ that has $\hat{v}$ as asymptotical expansion.

As in the one-dimensional case, we omit the proof of the

Lemma 5.2.11. *[vdPS03, Theorem 7.12] Let A be any $n\times n$-matrix with entries in $\mathbb{C}(\{z\})$.*

The operator $\delta - A$ yields surjective endomorphisms of the sheaves $\mathcal{A}^n$, $(\mathcal{A}^0)^n$ and $\underline{\mathbb{C}((z))}^n$. There is a short exact sequence

$$0 \to \ker(\delta - A)|_{(\mathcal{A}^0)^n} \to \ker(\delta - A)|_{\mathcal{A}^n} \to \ker(\delta - A)|_{\underline{\mathbb{C}((z))}^n} \to 0$$

of sheaves on S^1.

With the same proof as in the last section, we get from this:

Proposition 5.2.12. *Consider the differential equation (10), and let $U \subsetneq S^1$ be a proper open subset. Then existence and non-uniqueness of a lift $v \in \mathcal{A}(U)^n$ of $\hat{v}$ that solves the same differential equation are controlled by the cohomology H^0 and H^1 on U of the sheaf $\ker(\delta - A)|_{(\mathcal{A}^0)^n}$.*

At the moment, we can not yet compute the cohomology of this sheaf; we only note that such a lift exists locally, i. e. if U is sufficiently small.

Of course, we can rephrase the statements of this section in the language of differential modules: If M is a differential module over $\mathbb{C}(\{z\})$, we can associate to it in an obvious manner the sheaves of differential modules $\mathcal{M}^0$, $\mathcal{M}$ and $\widehat{\mathcal{M}}$ on S^1 over the sheaves of rings $\mathcal{A}^0$, $\mathcal{A}$ and $\underline{\mathbb{C}((z))}$, respectively. If $\mathcal{M}^{0f}, \mathcal{M}^f$ and $\widehat{\mathcal{M}}^f$, denote the respective subsheaves of flat sections (which we wrote as $\mathcal{M}^{0f} = \ker(\delta - A)|_{(\mathcal{A}^0)^n}$ etc.), then the short exact sequence of lemma 5.2.11 reads as

$$0 \to \mathcal{M}^{0f} \to \mathcal{M}^f \to \widehat{\mathcal{M}}^f \to 0.$$

Everything else translates in a similar way.

The key idea. Given a differential module N over $\mathbb{C}(\{z\})$, we want to understand which differential modules become isomorphic to N after base change to $\mathbb{C}((z))$.

In coordinates, if M and N are given by differential equations $\delta - A$ and $\delta - B$, a formal isomorphism of M and N means the following: *There exists a matrix $\widehat{F} \in \mathrm{GL}(n, \mathbb{C}((z)))$ that satisfies the differential equation*

$$\delta \widehat{F} = A\widehat{F} - \widehat{F}B. \tag{11}$$

(This differential equation is equivalent to the statement that $\widehat{F}$ gives a gauge equivalence between the two differential equations $(\delta - B)\xi = 0$ and $(\delta - A)\widehat{F}\xi = 0$.)

Now the essential idea of the theory of Stokes matrices is to apply the theory developed in the previous two sections to the differential equation (11). So we treat the $n \times n$-matrix $\widehat{F}$ as a column vector with n^2 entries, and we note that (11) is of the same form as (10). We immediately get from the remark after proposition 5.2.12 that, locally in S^1, we can find a matrix solution F of (11) with entries in $\mathcal{A}(U)$ that has $\hat{F}$ as asymptotic expansion.

We can rephrase this more abstractly. Let $\mathcal{M}$ and $\mathcal{N}$ be the sheaves of differential modules over the sheaf of rings $\mathcal{A}$ associated to M and N, respectively.

Lemma 5.2.13. *Locally in S^1, the differential $\mathcal{A}$-modules $\mathcal{M}$ and $\mathcal{N}$ are isomorphic (as differential $\mathcal{A}$-modules).*

This fact encourages directly the following definition and proposition:

Definition 5.2.14. *Given the differential $\mathcal{A}$-module $\mathcal{N}$, let $\mathcal{A}ut\,\mathcal{N}$ be the sheaf of $\mathcal{A}$-module automorphisms of $\mathcal{N}$. It has a subsheaf $(\mathcal{A}ut\,\mathcal{N})^f$ of flat sections which consists of the automorphisms of $\mathcal{N}$ as a differential $\mathcal{A}$-module.*

The sheaf $\mathcal{A}ut^0\,\mathcal{N}$ will denote the subsheaf of endomorphisms that have the identity as asymptotic expansion (that is, expressed in an arbitrary $\mathcal{A}$-basis, the endomorphism is represented by a matrix that has the identity matrix as asymptotic expansion). Of course $(\mathcal{A}ut^0\,\mathcal{N})^f$ denotes the corresponding subsheaf of flat sections.

Since the differential module M is obviously determined by its associated sheaf $\mathcal{M}$, one gets nearly automatically the

Proposition 5.2.15. *Let N be a differential module over $\mathbb{C}(\{z\})$. Every differential module M over $\mathbb{C}(\{z\})$ that is formally isomorphic to N is determined by an element in $H^1(S^1, (\mathcal{A}ut^0\,\mathcal{N})^f)$.*

Proof. We fix an isomorphism $F\colon \widehat{M} \xrightarrow{\sim} \widehat{N}$. We cover S^1 by open subsets U_i such that on each U_i, we have $\mathcal{M}|_{U_i} \cong \mathcal{N}|_{U_i}$ by a chosen isomorphism γ_i; we require that γ_i has $\hat{F}$ as asymptotic expansion. On the overlaps $U_i \cap U_j$, we get elements $\gamma_{ij} = \gamma_i \circ \gamma_j^{-1}$ as usual. By our choices, the automorphisms γ_{ij} have Id as asymptotic expansion: $\gamma_{ij} \in (\mathcal{A}ut^0\,\mathcal{N})^f(U_i \cap U_j)$. So the γ_{ij} yield a Čech-cocycle in $H^1(S^1, (\mathcal{A}ut^0\,\mathcal{N})^f)$, from which the sheaf $\mathcal{M}$ can be reconstructed. ☐

The Malgrange-Sibuya Theorem. The proof of the following theorem is more involved than the proofs so far (and omitted here); the result is a major step in the classification of irregular differential modules:

Theorem 5.2.16. *The natural map $H^1(S^1, \mathcal{A}ut^0\,\mathcal{N}) \to H^1(S^1, \mathcal{A}ut\,\mathcal{N})$ has image $\{1\}$.*

(Remember that sections of $\mathcal{A}ut\,\mathcal{N}$ are just the $\mathcal{A}$-module endomorphisms of the $\mathcal{A}$-module $\mathcal{N}$; so in fact we have $\mathcal{A}ut\,\mathcal{N} \cong \mathrm{GL}(n, \mathcal{A})$.)

Corollary 5.2.17. *Given a differential module N over $\mathbb{C}(\{z\})$, there is a $1:1$-correspondence between*

- *differential modules M over $\mathbb{C}(\{z\})$ equipped with an isomorphism $M \otimes_{\mathbb{C}(\{z\})} \mathbb{C}((z)) \cong N \otimes_{\mathbb{C}(\{z\})} \mathbb{C}((z))$, and*
- *elements of $H^1(S^1, (\mathcal{A}ut^0\,\mathcal{N})^f)$.*

Proof. By proposition 5.2.15, it only remains to show that to every element $\gamma \in H^1(S^1, (\mathcal{A}ut^0\,\mathcal{N})^f)$, we can find a differential module M which is sent to γ in the map described in this proposition.

By the short exact sequence

$$1 \to \mathcal{A}ut^0\,\mathcal{N} \to \mathcal{A}ut\,\mathcal{N} \to \widehat{\mathcal{A}ut\,\mathcal{N}} \to 1$$

and the Malgrange-Sibuya Theorem we can find a global section $\widehat{F} \in H^0(S^1, \widehat{\mathcal{A}ut\,\mathcal{N}}) \cong \mathrm{GL}\big(n, \mathbb{C}((z))\big)$ which is mapped to the same element in $H^1(S^1, \mathcal{A}ut^0\,\mathcal{N})$ as γ; the choice of $\widehat{F}$ is unique up to elements in $H^0(S^1, \mathcal{A}ut\,\mathcal{N})$, which will not affect the differential module we are going to construct.

If N corresponds to the matrix differential equation $\delta - B$, we would like to define M via the differential equation $\delta - A = \widehat{F}^{-1}(\delta - B)\widehat{F}$; however, it is unclear whether the matrix A defined by this equation will have convergent entries. So we will again use our fundamental short exact sequence to construct A:

We can cover S^1 by open sets U_i such that on each open set, $\widehat{F}$ can be lifted to an element $F_i \in (\mathcal{A}ut\,\mathcal{N})(U_i)$ having $\widehat{F}$ as asymptotic expansion; we can choose the F_i in such a way that $(F_i F_j^{-1})_{ij}$ is a Čech-cocycle representing γ in $H^1(S^1, (\mathcal{A}ut^0\,\mathcal{N})^f)$. We define $A_i \in \mathrm{GL}(n, \mathcal{A}(U_i))$ on U_i by the equation

$$\delta - A_i = F_i^{-1}(\delta - B)F_i.$$

By our choice, the $F_i F_j^{-1}$ define differential automorphisms of $\mathcal{N}$ on $U_i \cap U_j$, hence they commute with $\delta - B$. Thus we get

$$F_i^{-1}(\delta - B)F_i = F_j^{-1}(\delta - B)F_j,$$

which implies $A_i|_{U_i \cap U_j} = A_j|_{U_i \cap U_j}$. So the A_i glue to a well-defined global section $A \in H^0(S^1, \mathrm{GL}(n, \mathcal{A})) = \mathrm{GL}(n, \mathbb{C}(\{z\}))$ which defines the differential module M as desired. □

Stokes matrices with respect to adjacent sectors. We make the following simplifying assumption:

Assumption 5.2.18. *We assume that after base change to $\mathbb{C}((z))$, the differential module M splits: We have*

$$M \otimes_{\mathbb{C}} \mathbb{C}((z)) \cong \bigoplus_i Q_i \otimes_{\mathbb{C}} \mathbb{C}((z))$$

where each Q_i is isomorphic to the one-dimensional differential module associated to the scalar differential equation $(\delta - q_i)$, and each $q_i \in z^{-1}\mathbb{C}[z^{-1}]$ is a polynomial in z^{-1} of the same degree k for all i. Further, we assume that for all pairs $i \neq j$, the polynomial $q_i - q_j$ is of degree k as well.

This assumption contains three simplifications (compare with Theorem 5.2.4):

- In general, such a decomposition exists only after adjoining some root $\sqrt[k]{z}$ to $\mathbb{C}((z))$,
- the decomposition could contain higher dimensional irreducible modules associated to a differential equation of the form $\delta - q_i + C_i$ with some constant $n \times n$-matrix C_i, and
- the q_i and their differences $q_i - q_j$ could have different degrees.

While the first two simplifications rather just shorten our notation, the third does make a substantial difference. Without it, one needs a filtration on the sheaves $\mathcal{A}^0, \mathcal{A}, \underline{\mathbb{C}((z))}$ such that (very roughly speaking) each filtration step only notices the effects of those summands Q_i where q_i is a polynomial of fixed degree; the main ingredient is then the so-called *multisummation theorem.*

Let N be the differential module $N = \bigoplus_i Q_i$ over $\mathbb{C}(z)$. We are given an element $\hat{\gamma}$ of $\widehat{\mathrm{Hom}}(M, N)$; it corresponds to the matrix $\widehat{F}$ of equation (11). As in the additive Stokes phenomenon, we are looking for sectors U on which γ can be lifted uniquely to an element $\gamma(U) \in \mathcal{H}om(\mathcal{M}, \mathcal{N})$. Since we obviously have $\widehat{\mathrm{Hom}(M, N)} \cong \widehat{\mathrm{End}(N)}$, the following lemma will be useful:

Lemma 5.2.19. *Assume that the differential modules $L, \bar{L}$ are formally isomorphic. Then the associated sheafs $\mathcal{L}^{0f}$ and $\bar{\mathcal{L}}^{0f}$ of rapidly vanishing flat sections are isomorphic.*

We apply this to $L = \mathrm{Hom}(M, N)$ and $\bar{L} = \mathrm{End}(N)$. Using our assumptions, it is easy to describe the sheaf $(\mathcal{E}nd\, N)^{0f}$:

Lemma 5.2.20. *Let $N = \bigoplus_i Q_i$ be as above. Each Q_i is the one-dimensional differential module associated to the differential equation $\delta - q_i$ with*

$$q_i = a_{ik} z^{-k} + \cdots + a_{i1} z^{-1}.$$

For each pair $i \neq j$ let f_{ij} be the standard solution to the differential equation $(\delta + q_i - q_j) f_{ij} = 0$ which is given by

$$f_{ij} = e^{\frac{a_{ik} - a_{jk}}{k} z^{-k} + \cdots}.$$

Now let U_{ij} be the union of the k sectors where f_{ij} has zero as asymptotical expansion, i. e. where $\mathrm{Re}(a_{ik} - a_{jk}) z^{-k} < 0$.

The claim is that

$$(\mathcal{E}nd\, N)^{0f} = \bigoplus_{i \neq j} (i_{U_{ij}})_! \mathbb{C}.$$

Proof. Indeed, there are no non-zero rapidly vanishing differential endomorphisms of Q_i, and gauge transformations from the differential equation $\delta - q_i$ to $\delta - q_j$ are exactly given by (a constant multiple of) such a function f_{ij}. □

So if U neither does contain nor is contained in any of the k connected components of U_{ij} for any pair i, j, we have

$$H^0(U, (\mathcal{H}om^0(M, N))^f) = H^1(U, (\mathrm{Hom}^0(M, N))^f) = 0. \tag{12}$$

This is satisfied for almost all connected sectors U of length $\frac{\pi}{k}$.

Now suppose we have two such connected sectors U and U' with non-empty connected overlap $U \cap U'$. By proposition 5.2.12 there are unique lifts $\gamma_U, \gamma_{U'}$ of γ on these open sets. Their difference $\gamma_{U'}^{-1} \circ \gamma_U$ is an element of $(\mathcal{A}ut^0 \mathcal{N})^f(U \cap U')$.

After adjoining the solutions $f_i = e^{\frac{-a_{ik}}{k} z^{-k} + \cdots}$ of the differential equations $\delta - q_i$, $i = 1 \ldots n$ to our base field, the sheaf $\mathcal{N}$ has a canonical basis of flat sections. This yields an isomorphism $(\mathcal{A}ut \mathcal{N})^f(U \cap U') \cong \mathrm{GL}(n, \mathbb{C})$ by representing each automorphism with respect to this flat basis.

We need to identify the image of $(\mathcal{A}ut^0 \mathcal{N})^f(U \cap U')$ in $\mathrm{GL}(n, \mathbb{C})$. From lemma 5.2.20 it is clear that it consists of $n \times n$-matrices K that

- have $K_{ii} = 1$ on the diagonal, and
- whose non-diagonal entries K_{ij} vanish unless $U \cap U' \subset U_{ij}$.

Now let us consider $\gamma_{U'}^{-1} \circ \gamma_U \in (\mathcal{A}ut^0(\mathcal{N}))^f(U \cap U')$. Under the above isomorphism, this yields a matrix S of complex numbers satisfying the two conditions described above. The matrix S is the *Stokes matrix of M with respect to the sectors U and U'*.

Stokes matrices. Now we know everything to go ahead and actually construct the collection of Stokes matrices associated to a differential module M satisfying the assumption 5.2.18:

1. Choose an isomorphism $\widehat{M} \cong_\gamma \widehat{\bigoplus_{i=1}^n Q_i} = \widehat{N}$ as in 5.2.18. This includes the choice of an ordering of the irreducible modules.
2. Choose an appropriate covering $U_1, \ldots, U_m$ of S^1 by connected open sectors U_j. Each U_i must satisfy the condition necessary to ensure (12); this is automatically true if they are of size marginally larger than $\frac{\pi}{k}$ and in general position. We will assume that they are ordered counter-clockwise on S^1.

   To produce such a covering, we start with an arbitrary U_1 satisfying the condition. Having chosen $U_1, \ldots, U_{l-1}$ or $U_1', \ldots, U_{l-1}'$ respectively, we can either

   (a) choose U_l' "as close as possible" to U_{l-1}'—by this we mean that $U_l' \cap U_{l-1}'$ is only contained in U_{ij} (defined in proposition 5.2.20) for a single pair i, j (or several pairs only if the corresponding sectors U_{ij} and $U_{i'j'}$ coincide)—, or we can

   (b) choose U_l "as far away as possible" from U_{l-1}. This means that $U_l \cap U_{l-1}'$ is contained in as many sectors U_{ij} as possible, i. e. in all such sectors that contain the relevant border line of U_{l-1}.

The two different coverings obtained by starting with the same open sector $U_1 = U_1'$ are shown in figure 5.2.

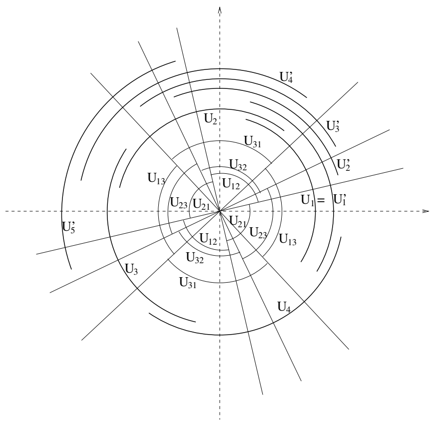

Figure 2: The two coverings (with $k = 2$ and $n = 3$): $\{U_1, U_2, U_3, U_4\}$ and $\{U_1', U_2', U_3', U_4', U_5', \dots\}$

As explained in the previous section, we get a Stokes matrix S_l associated to each pair of sectors U_l, U_{l+1} (respectively a matrix K_l from U_l', U_{l+1}'). It is the matrix representation of $\gamma_{U_l} \circ \gamma_{U_{l-1}}^{-1}$ with respect to the flat basis of $\mathcal{N}|_{U_l \cap U_{l+1}}$. Using (2a), we get $2k \cdot \binom{n}{2}$ of these matrices, compared to only $2k$ if we choose method (2b).

The procedure using (2a) is more canonical: Every choice of U_1 will produce—up to cyclic reordering—the same Stokes matrices $S_1, \dots, S_m$. Each such matrix will have only one non-zero off-diagonal entry at i, j where $U_l \cap U_{l-1} \subset U_{ij}$ (unless some of the sectors U_{ij} are identical).

We will follow Dubrovin's terminology (see [Dub99]) and call the matrices K_l obtained by method (2a) *Stokes factors*; then only the matrices S_l that we get using method (2b) will be called *Stokes matrices.* Up to reordering of the basis, the latter are alternatingly upper and lower triangular matrices. Stokes matrices and Stokes factors can easily be reconstructed from each other purely algebraically. This is due to the constraints on the matrices K_l and the relations between the two sets of matrices:

$$S_i = K_{i \cdot \binom{n}{2}} \cdot K_{i \cdot \binom{n}{2} - 1} \cdot \dots \cdot K_{(i-1) \cdot \binom{n}{2} + 1}.$$

5.3 Stokes matrices of a Frobenius manifold

As we explained in the beginning of this chapter, the Stokes matrices of a generically semisimple Frobenius manifold are the Stokes matrices at $\lambda = \infty$ of its first structure connection.

So assume that we have chosen a semisimple point $m \in \mathcal{M}$. Let $u_1, \dots, u_n$ be the canonical coordinates. We assume that they are pairwise distinct; otherwise the assumption 5.2.18 would not hold.

Now consider the first structure connection of $\mathcal{M}$ restricted to $\{m\} \times \mathbb{P}^1$. The idempotents $\frac{\partial}{\partial u_i}$ form a natural basis of $p^* \mathcal{T M}|_{\{m\} \times \mathcal{M}}$.

Let $z = \frac{1}{\lambda}$. At $z = 0$, the connection yields a differential module M over $\mathbb{C}(\{z\})$. Recalling the definition of the first structure connection, we see that its differential equation is

$$\delta v = \frac{1}{z} E \circ v - [E, v].$$

(Here $[E, v]$ denotes the Lie bracket of E with the flat extension of $v \in \mathcal{T}_m\mathcal{M}$ to a neighbourhood of m.)

Let Q_i be the one-dimensional differential module associated to the equation

$$\delta v = \frac{u_i}{z} v.$$

We see immediately that M and $\bigoplus_{i=1}^n Q_i$ are isomorphic up to $O(1)$ via the isomorphism of vector spaces $\tilde{\gamma}$ that sends $\frac{\partial}{\partial u_i}$ to $1 \in Q_i$. In fact, there exists a unique formal isomorphism

$$M \otimes \mathbb{C}((z)) \overset{\gamma}{\cong} \bigoplus_{i=1}^{n} Q_i \otimes \mathbb{C}((z)) \tag{13}$$

of differential modules that agrees with $\tilde{\gamma}$ up to higher orders of z: $\gamma = \tilde{\gamma} + O(z)$. The proof is easy but not very enlightening (cf. [Dub99, lemma 4.3]): It just constructs γ step by step as a power series.

So we have found a natural choice for step (1) in the construction of Stokes matrices as described above; using step (2a) above we only lack a choice of U_1.

There is no natural choice here. We can just note that U_1 will be a connected open sector of size approximately π. Then U_2 is a sector of similar size that has small overlaps with U_1 at both ends. Together, they cover S^1. From the two connected components of $U_1 \cap U_2$, we get two Stokes matrices S_1, S_2 of our Frobenius manifold.

These two matrices are related via

$$S_2 = S_1^T. \tag{14}$$

This follows from the good behaviour of γ with respect to the metric that M inherits from the metric of the Frobenius manifold.

Dubrovin constructs U_1 by choosing an oriented "admissible" line l; admissible means that the line does not meet any of the rays bordering a sector of any $U_{ij}, i \neq j$. The figure 5.3 shows better than any explanation how to get the covering U_1, U_2 from the line l.

Note that the remarks at the end of section 5.2 imply that one can explicitly write down how the Stokes matrix S changes if the line l is moved around. This is spelled out in [Dub99, lemma 4.8].

Note the other choice we have made: the idempotents $\frac{\partial}{\partial u_i}$ are unique only up to a factor of -1. Changing $\frac{\partial}{\partial u_i}$ to $-\frac{\partial}{\partial u_i}$ evidently changes the Stokes matrix by multiplying the i-th row and column with -1.

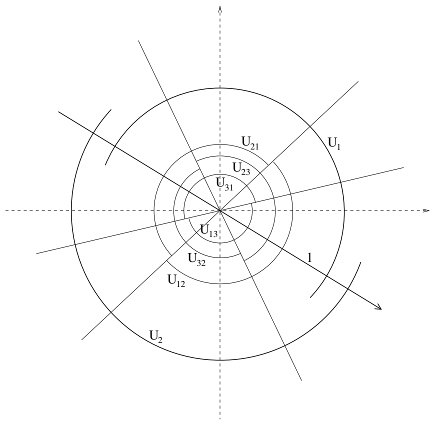

Figure 3: The admissible line l and the covering $\{U_1, U_2\}$

5.4 Dubrovin's monodromy data

Dubrovin's monodromy data is a list of invariants of the first structure connection (more precisely of a fibre $\{m\} \times \mathbb{P}^1$ of the connection) that suffice to characterize it.

The Stokes matrix $S = S_1$, together with the canonical coordinates, classifies the singularity at $\lambda = \infty$ of the first structure connection. More precisely, from this we can reconstruct the connection ∇_{U_∞} in a neighbourhood U_∞ of infinity.

Dubrovin's list begins with a vector space V that is identified with the space of flat sections of the connection (and will become the tangent space $T_m\mathcal{M}$) and a symmetric bilinear form g on V that will become the metric of the Frobenius manifold.

As mentioned before, the singularity of the connection at $\lambda = 0$ is a regular singularity. A classical invariant of such a point is the residue endomorphism $\mu \in \operatorname{End}(V)$ of the connection $\nabla_{\frac{\partial}{\partial\lambda}}$ (as defined in [Del70, II.1.16]).

As can be read off from the definitions, we have $\mu = -\frac{D}{2}\operatorname{id} - [E, \cdot]$. This endomorphism is antisymmetric with respect to the metric g, and we assume that it is diagonalizable.

Further, Dubrovin defines an endomorphism R by the following properties:

- It can be written as a finite sum $R = R_1 + R_2 + \dots$, such that the first structure connection is gauge equivalent (by a gauge transformation holomorphic on $\mathbb{C}$) to

$$\nabla_{\frac{\partial}{\partial\lambda}} v = \partial_\lambda v + (\frac{1}{\lambda}\mu + R_1 + \lambda R_2 + \lambda^2 R_3 + \dots)v.$$

- The endomorphisms R_{2k+1} are symmetric with respect to g, the R_{2k} are antisymmetric.

- Let $V = \bigoplus_r V_r$ be the isotypical decomposition of V with respect to μ, i. e. $\mu|_{V_r} = r \cdot \operatorname{id}$. Then $R_k(V_r) \subset V_{r+k}$.

The tuple (V, g, μ, R) fully describes the restriction $\nabla|_{\mathbb{C}}$ of the connection to $\mathbb{C}$.

Finally, we have to understand how $\nabla|_{\mathbb{C}}$ and ∇_{U_∞} glue to the connection ∇ on $\mathbb{P}^1$. This glueing is determined by the identification of the spaces of flat sections on any simply connected subset of $\mathbb{C} \cap U_\infty$.

Around $\lambda = \infty$, we have a canonical basis of flat sections on the open sector U_1 in the $\left(z = \frac{1}{\lambda}\right)$-plane: the lift γ_1 of the formal isomorphism γ defined in equation (13) is holomorphic on U_1. The space of flat sections of $\bigoplus_i Q_i$ (as defined in section 5.3) is identified with $\mathbb{C}^n$ via $\mathbb{C}^n \ni e_i \mapsto e^{-\frac{u_i}{z}} \cdot 1 \in Q_i$.

From $\lambda = 0$, the space of flat sections around any point is identified with V. Thus the glueing of the two connections is given by an isomorphism $C \colon \mathbb{C}^n \to V$. Dubrovin calls this the *central connection matrix*.

This completes the list of Dubrovin's monodromy data. An immediate good argument in favor of their usefulness is the following theorem:

Theorem 5.4.1. *[Dub99] With a consistent choice of the oriented line $l \subset \mathbb{C}$, the monodromy data (V, g, μ, R, S, C) does not depend on the choice of the base point $m \in \mathcal{M}$.*

It is not clear for which set of data (V, g, μ, R, S, C) a Frobenius manifold exists. One relation is obtained by comparing the monodromy around ∞ with that around 0. However, there are further implicit relations that can only be phrased by the solvability of a Riemann-Hilbert boundary value problem: Starting with arbitrary values, it is not clear whether the identification of flat sections specified by C extends to glue the sheaves $V \otimes \mathcal{O}_{\mathbb{C}}$ and $\bigoplus_i Q_i$ to a *globally free* sheaf on $\mathbb{P}^1$.

6 Exceptional systems and Dubrovin's conjecture

6.1 Exceptional systems in triangulated categories

We consider a triangulated category $\mathcal{C}$. We assume that it is defined over a ground field k, i. e. for each pair of objects $\mathcal{A}$, $\mathcal{B}$, the morphisms $\mathrm{Hom}_{\mathcal{C}}(\mathcal{A}, \mathcal{B})$ form a k-vector space. In our examples, $\mathcal{C}$ will always be the bounded derived category $D^b(X)$ of coherent sheaves on a variety X/k.

In a way, exceptional objects in triangulated categories are a substitute for simple objects in abelian categories; an exceptional system vaguely corresponds to the list of simple objects in a semisimple abelian category.

Definition 6.1.1.

- *An exceptional object in $\mathcal{C}$ is an object $\mathcal{E}$ such that the endomorphism complex of $\mathcal{E}$ is concentrated in degree zero and equal to k:*

$$\mathbf{R}\mathrm{Hom}^\bullet(\mathcal{E}, \mathcal{E}) = k[0]$$

- *An exceptional collection is a sequence $\mathcal{E}_0, \dots, \mathcal{E}_m$ of exceptional objects, such that for all $i > j$ we have no morphisms from $\mathcal{E}_i$ to $\mathcal{E}_j$:*

$$\mathbf{R}\mathrm{Hom}^\bullet(\mathcal{E}_i, \mathcal{E}_j) = 0 \quad \textit{if } i > j$$

- *An exceptional collection of objects is called a complete exceptional collection (or exceptional system), if the objects $\mathcal{E}_0, \dots, \mathcal{E}_m$ generate $\mathcal{C}$ as a triangulated category: The smallest subcategory of $\mathcal{C}$, that contains all $\mathcal{E}_i$, and is closed under isomorphisms, shifts and cones, is $\mathcal{C}$ itself.*

The basic example is the bounded derived category $D^b(\mathbb{P}^n)$ on a projective space with the series of sheaves $\mathcal{O}(i), \mathcal{O}(i+1), \dots, \mathcal{O}(i+n)$ (for any i). This was first observed (together with the special case of the Theorem 6.1.2) by Beĭlinson (cf. [Beĭ84]). Later, exceptional systems were studied by a group at the Moscow University, see e. g. the collection of papers in [Rud90].

The length of an exceptional system in $D^b(X)$ always equals the dimension of the cohomology of X, in fact an exceptional collection is complete if and only if its length is $\dim_k H^*(X)$.

To justify our vague claim about the correspondence to semisimple abelian categories we cite the following theorem, which is due to Bondal. It may be considered as a derived non-commutative analogue to the classical theorem on the structure of semisimple abelian categories:

Theorem 6.1.2. *[Bon89, Theorem 6.2] Assume that the exceptional system $\mathcal{E}_0, \dots, \mathcal{E}_m$ of the category $\mathcal{C}$ satisfies the following additional property:* For each $i < j$, the complex $\mathbf{R}\mathrm{Hom}^\bullet(\mathcal{E}_i, \mathcal{E}_j)$ is concentrated in degree zero.

Let A be the algebra of endomorphisms of $\bigoplus_i \mathcal{E}_i$. Then $\mathcal{C}$ is equivalent to the derived category $D^b(A)$ of right A-modules.

It seems very plausible that it is possible to drop this additional assumption if we replace

- the triangulated category $\mathcal{C}$ by a differential graded category and

- the endomorphism algebra A by the DG algebra of endomorphisms of $\bigoplus_i \mathcal{E}_i$.

Indeed, this has been proven (but not yet published) by B. Keller. A version with A_∞-algebras and A_∞-categories is possible as well.

Examples of varieties that do have an exceptional system include projective spaces or, more generally, flag varieties.

6.2 Dubrovin's conjecture

We are now ready to give the precise formulation of Dubrovin's conjecture.

Conjecture 6.2.1. *[Dub98] Let X be a Fano variety.*

Claim 1: The quantum cohomology of X is generically semisimple if and only if there exists an exceptional system in its derived category $D^b(X)$.

Claim 2: In this case, define the Stokes matrix S_{ij} via the Euler characteristics of the exceptional system:

$$S_{ij} := \chi(\mathcal{E}_i, \mathcal{E}_j)$$

This is an upper triangular matrix with ones on the diagonal. On the other hand, consider the Stokes matrix S_1 associated to the Frobenius manifold of the Quantum cohomology of X.

With an appropriate choice of a semisimple point and of the admissible line l producing the covering $U_1, U_2 \subset S^1$, the Stokes matrix S_1 is given by $S_1 = S_{ij}$.

Claim 3: Finally, let $C''\colon \mathbb{C}^n \to H^*(X)$ be the isomorphism that sends e_i to the Chern character $\mathbf{Ch}(\mathcal{E}_i)$. The central connection matrix C of the Frobenius manifold can be written as

$$C = C'C''$$

where C' is an endomorphism of $H^*(X)$ that commutes with multiplication by the first chern class of X. *(The precise nature of C' in general is yet unclear.)*

7 Semisimple mirror symmetries

This section will vaguely explain how Dubrovin's conjecture fits into the context of mirror symmetry of Fano varieties. This involves both the more traditional "combinatorial" mirror symmetry, which we formulate as an isomorphism of Frobenius manifolds, and the homological mirror symmetry as conjectured by Kontsevich.

The mirror partner to a Fano variety with generically semisimple quantum cohomology is not a variety, but a function $f\colon Y \to \mathbb{C}$ with isolated singularities, defined on an affine variety Y. While the construction of a Frobenius manifold from this is now rather well-understood (due to work by Barannikov, Hertling, Douai and Sabbah), the associated Fukaya category remains a bit mysterious.

7.1 The mirror construction

In the case of Calabi-Yau varieties, the mirror is constructed from deformations of another Calabi-Yau variety Y, the mirror partner. In our Fano case, mirror symmetry becomes even less symmetric.

The construction starts from a pair (Y, f), where Y is a smooth affine complex variety, and $f\colon Y \to \mathbb{C}$ is an algebraic function with isolated critical points, and whose singularities are all of type A_1. This means that the Hessian matrix $\left(\frac{\partial}{\partial y_i}\frac{\partial}{\partial y_j} f\right)_{ij}$ is non-degenerate at all critical points, so that f is a Morse-type function. We have to assume that this function behaves well at infinity. For our purposes, the following notion studied by Sabbah will be sufficient:

Definition 7.1.1. *A function f on an affine manifold Y is called* M-tame *if there is an embedding $Y \subset \mathbb{C}^N$ with:*

- *For any $r > 0$ there exists $R > 0$ such that the sphere $\{x \in \mathbb{C}^N \mid |x| = R\}$ is transversal to $f^{-1}(z)$ for $|z| \leq r$.*

Let μ be the Milnor number, i. e. the dimension of the Milnor ring $\mathcal{O}_Y/(T_Y(f))$; here $(T_Y(f))$ denotes the ideal generated by all functions that can be obtained as a partial first-order derivative of f. (This ring has finite dimension as f has isolated singularities.)

We need that there exists a deformation of f to a function $F\colon Y \times (\mathcal{M}, 0) \to \mathbb{C}$ that yields a miniversal deformation of f; here $(\mathcal{M}, 0)$ is a germ of a manifold with dimension μ, and $F|_{Y\times\{0\}} = f$: (In our case this just means that we have enough global functions on Y separating the critical points of f.)

We call the deformation F *miniversal* at $t \in \mathcal{M}$ iff the Kodaira-Spencer map

$$\mathcal{T}_t\mathcal{M} \to \mathcal{O}_{Y\times\{t\}}/(\mathcal{T}_Y(F_t)), \quad X \mapsto \tilde{X}F_t \tag{15}$$

is an isomorphism; here $\tilde{X}$ is an arbitrary lift of X to a section in $\mathcal{T}(Y \times \mathcal{M})|_{Y\times\{t\}}$, and F_t is the restriction of F to $Y \times \{t\}$.

If f has only A_1-singularities, and the values of f at the singular points are all distinct (this will give a tame semisimple point in the Frobenius manifold), it is particularly easy to write down a versal deformation F: We let $\mathcal{M} = \mathbb{C}^\mu$, with coordinates $t_1, \ldots, t_\mu$, and

$$F(y, t_1, \ldots, t_\mu) = f(y) + \sum_{i=0}^{\mu-1} t_{i+1} f(y)^i. \tag{16}$$

(This is miniversal at $t = 0$.)

7.2 The Milnor fibration

Even without the versal deformation, the geometry of the fibre at $t = 0$ carries interesting geometric information that determines part of the data of the Frobenius manifold, in particular, the Stokes matrices.

7.2.1 A homology bundle

As in the classical case of a local singularity, one has to study the geometry of the Milnor fibration. Here we will make essential use of f being M-tame.

Let $u_1, \ldots, u_\mu$ be the critical values of f, which we assume to be distinct. We choose a closed disc $\Delta \subset \mathbb{C}$ of radius r that contains all critical values. We assume that Y is $n+1$-dimensional and embedded into $\mathbb{C}^N$ as in the definition 7.1.1 of M-tameness, and choose a big ball $B = \{z \in \mathbb{C}^N \mid |z| \leq R\}$ with R corresponding to r as in the definition of M-tameness. Now let

$$\tilde{Y} = B \cap f^{-1}(\Delta).$$

Because f is M-tame, the map $f\colon \tilde{Y} \to \Delta$ is a fibration outside $\{u_1, \ldots, u_\mu\}$ in the C^∞-category (i. e. it is locally trivial): It implies condition (ii) in [AGZV88, p. 9]. The map f is called the *Milnor fibration*.

Now for each $z \in \mathbb{C}$, $z \neq 0$, let $z_0 := r \cdot \frac{z}{|z|}$ be the intersection of the ray from the origin through z with the boundary of Δ. We define a homology bundle of the Milnor fibration as follows: For $z \neq 0$, the bundle has the fibre

$$H_{n+1}(\tilde{Y}, \tilde{Y}^{(z_0)})$$

over z, where $\tilde{Y}^{(z_0)} = f^{-1}(z_0) \cap B$. Because of the local triviality of the fibration f, we get a Gauß-Manin connection, and so a flat vector bundle over $\mathbb{C}^*$.

Via the boundary map, we can relate this to the standard homology bundle:

$$H_{n+1}(\tilde{Y}, \tilde{Y}^{(z_0)}) \xrightarrow{\delta} H_n(\tilde{Y}^{z_0}). \tag{17}$$

But in general, if $\tilde{Y}$ has non-trivial homology, this map is not an isomorphism.

7.2.2 Lefschetz thimbles

Locally around z, there is a canonical trivialization of the homology bundle of the Milnor fibration. Assume that $\arg z_0$ is different from $\arg(u_i - u_j)$ for all i, j. We choose paths γ_i connecting the critical values with z_0 as in figure 4: Each path starts straight in the direction of $\arg z_0$, and then turns in to reach z_0.[5] Over each such path, we get a family of vanishing cycles that glue together to a real $n+1$-dimensional manifold Γ_i with boundary in $\tilde{Y}^{(z_0)}$, called a *Lefschetz thimble*. (For a precise definition, we refer to [AGZV88] or the construction as explicit manifolds below.) This gives canonical elements in $H_{n+1}(\tilde{Y}, \tilde{Y}^{(z_0)})$.

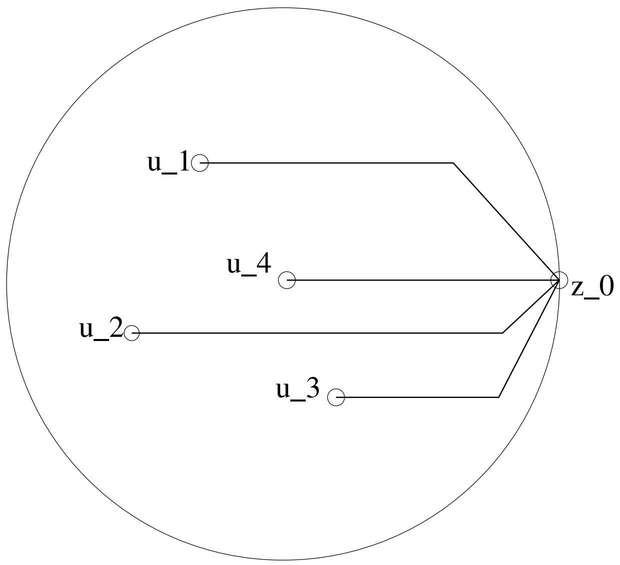

Figure 4: Paths underlying Lefschetz thimbles

We can extend this to a local trivialization by transforming the paths homotopically in $\Delta \setminus \{u_1, \dots u_n\}$ connecting the u_i with z_0' for z_0' nearby z_0. Of course, as the homotopy classeses of the paths depend on the original choice of z_0, this does not yield a global trivialization.

We will also need the Lefschetz thimbles as explicit manifolds. For this, one has to choose a Riemannian metric g on Y. We then consider the gradient flow of the real Morse functions $g_z = \mathrm{Re}(z^{-1}f(\cdot))$ (i. e. the flow generated by the vector field that is dual to the one-form dg_z via the metric g). For each critical point of g_z, we consider the unstable part of the Morse flow. This gives a $n+1$-dimensional submanifold (by general Morse theory, and as $\mathrm{Re}(z^{-1}f(\cdot))$ is locally of the form $x_1^2 + \cdots x_{n+1}^2 - y_1^2 - \cdots y_{n+1}^2$ for suitable complex coordinates $z_j = x_j + iy_j$, i. e. for which $z^{-1}f = z_1^2 + \dots z_{n+1}^2$; if the metric is given by $g = dx_1^2 + dy_1^2 + \cdots + dx_{n+1}^2 + dy_{n+1}^2$, the Lefschetz thimble is given by $y_1 = \cdots = y_{n+1} = 0$, and the vanishing cycles are the level sets of this submanifold, i. e. given by the additional equation $x_1^2 + \cdots + x_{n+1}^2 = t$).

For big discs Δ, we can view them as elements in $H_{n+1}(\tilde{Y}, \tilde{Y}^{(Z_0)}) \cong H_{n+1}(\tilde{Y}, \tilde{Y}^{(z_0)})$, where

- $\tilde{Y}^{(Z_0)}$ is defined as $B \cap f^{-1}(Z_0)$, and
- $Z_0 \subset \partial\Delta$ is a connected subset of the boundary of the disc, containing z_0,

[5]This crude description is good enough as our construction depends only on the homotopy classes of the paths γ_i in $(\Delta \setminus \{u_, \dots, u_\mu\}, z_0)$.

and big enough so that all the straight paths starting at the u_i parallel to the ray associated to z_0 intersect $\partial\Delta$ in Z_0 (see figure 5).

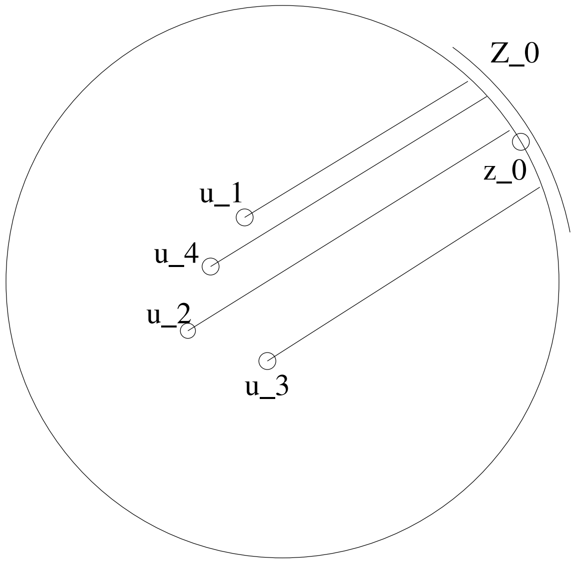

Figure 5: Paths underlying Lefschetz thimbles

The Morse theory for the function g_z shows that $(\tilde{Y}, \tilde{Y}^{(Z_0)})$ is, as a pair of spaces, homotopy equivalent to $\left(\bigcup \Gamma_i \cup Y^{(Z_0)}, Y^{(Z_0)}\right)$. In particular, we have $H_{n+1}(\tilde{Y}, \tilde{Y}^{(Z_0)}) = \mathbb{Z}^\mu$ with canonical basis $[\Gamma_1], \ldots, [\Gamma_\mu]$.

7.2.3 Seifert matrix

The canonical local trivializations that we have constructed above are not global trivializations: If we move the paths homotopically with varying z_0, we will, after z_0 has moved to some z_0', get paths that are not homotopically equivalent to the paths we would have gotten if we had started with $z_0 = z_0'$. Hence, their associated Lefschetz thimbles need not lead to the same elements in the homology $H_{n+1}(\tilde{Y}, \tilde{Y}^{(z_0)})$.

The Seifert matrix is a way to measure the difference of this local trivialization:

Definition 7.2.1. *Let z_0 be such that its argument as a complex number is different from those of $u_i - u_j$ for all i, j.*

Let $\gamma_1^+, \ldots, \gamma_n^+$ be the paths as in fig. 4, and assume that the u_i are numbered so that the γ_i^+ are ordered clock-wise at z_0.

Let $[\Gamma_1^+], \ldots, [\Gamma_n^+]$ be the homology classes in $H_{n+1}(\tilde{Y}, \tilde{Y}^{(z_0)})$ of the associated Lefschetz thimbles. Similarly, let $[\Gamma_1^-], \ldots, [\Gamma_n^-]$ be the homology classes in $H_{n+1}(\tilde{Y}, \tilde{Y}^{(-z_0)})$ of the paths starting in opposite direction.

Finally, let

$$\Phi \colon H_{n+1}(\tilde{Y}, \tilde{Y}^{(z_0)}) \to H_{n+1}(\tilde{Y}, \tilde{Y}^{(-z_0)})$$

the isomorphism induced by the flat connection by moving z_0 counter clock-wise to $-z_0$.

The Seifert matrix S_{ij} is the matrix representing the basis $\Phi([\Gamma_1^+])$, ..., $\Phi([\Gamma_n^+])$ in terms of the basis $[\Gamma_1^-], \ldots, [\Gamma_n^-]$ of $H_{n+1}(\tilde{Y}, \tilde{Y}^{(-z_0)})$.

So, similarly to the Stokes matrices, we can understand it as canonically defined Čech-cocycle. It is easy to prove that it is an upper triangular matrix with ones on the diagonal.

7.3 The twisted de Rham-complex

Given a miniversal deformation of a singularity, the manifold germ $\mathcal{M}$ is automatically equipped with a multiplication on the tangent sheaf. We simply pull back the ring structure of the Milnor ring via the Kodaira-Spencer isomorphism (15).

We get two special vector fields on $\mathcal{M}$: The Euler field E is the preimage of $F_t \in \mathcal{O}_{Y\times\{t\}}/(\mathcal{T}_Y(F_t)$ under this isomorphism, and the unit field e the preimage of the constant function 1.

So far the construction was canonical, i. e. it did not depend on choices. The multiplication satisfies an integrability condition, it is an $\mathcal{F}$-manifold with Euler field as defined in [HM99].

Constructing the flat metric of the Frobenius manifold is more involved and it depends on choices. Also, it is not known to work in complete generality. We will not try to outline the full construction of the Frobenius manifold. Instead, we will describe only as much as we need to define its Stokes matrices.

We will from here on work in the algebraic category, that is Y is an smooth affine scheme of finite type over $\mathbb{C}$, with structure sheaf $\mathcal{O}_Y$, and $f \in \mathcal{O}_Y(Y)$. As we are working with a globally defined function f instead of just a function germ, there is a problem with "singularities coming from infinity". Again, this can be seen very explicitly in the case of tame semisimplicity:

Consider the Milnor ring of the deformed function F_t as defined by (16): It is defined by the ideal generated by

$$(d_Y F_t)(y) = \left(1 + \sum_{i=1}^{\mu-1} i t_i f(y)^{i-1}\right)(d_Y f)(y).$$

We have a common zero of these functions iff $d_Y f = 0$ or $f(y)$ is a solution of

$$p(\lambda) = 1 + \sum_{i=1}^{\mu-1} i t_i \lambda^{i-1}.$$

These solutions go to infinity if the t_i go to zero.

However, since we are assuming tame simplicity at $t = 0$, we do not need the versal deformation at all to define Stokes matrices. The fibre at $t = 0$ is sufficient.

Definition 7.3.1. *The* twisted relative algebraic de Rham complex *associated to the function f is the complex of sheaves $\Omega_Y^\bullet[z, z^{-1}] \cdot e^{\frac{-f}{z}}$ on $\mathbb{C} \times Y$ with differential $d_f = z \cdot d_Y$. (Here z is the coordinate on $\mathbb{C}$.)*

A q-form in this complex is given as $\omega = \sum_i \omega_i z^i \cdot e^{\frac{-f}{z}}$*, with* ω_i *a relative q-form, given in coordinates* $y_1, \ldots, y_n$ *on* Y *as* $\omega_i = \sum g_J(y) dy_{j_1} \wedge \cdots \wedge dy_{j_q}$*. Algebraic means that* g_J *is a polynomial function on* Y*. Its differential is*

$$d_f(\omega \cdot e^{\frac{-f}{z}}) = (z\, d_Y\, \omega - df \wedge \omega) \cdot e^{\frac{-f}{z}}.$$

The definition of this complex might look a little unmotivated. By [DS02a, section 2.c], this complex is a representative for the Fourier dual of the de Rham-complex of f, in a $\mathcal{D}$-module sense.

Let $\pi\colon \mathbb{C}^* \times Y \to \mathbb{C}^*$ be the projection. The sheaves $\pi_* \left(\Omega_Y^\bullet[z, z^{-1}] \cdot e^{\frac{-f}{z}}\right)$ are equipped with a meromorphic connection:

$$\nabla_{\partial_z} \left(\omega \cdot e^{\frac{-f}{z}}\right) = \left(\frac{\partial \omega}{\partial z} + \frac{f}{z^2}\omega\right) e^{\frac{-f}{z}} \tag{18}$$

The formula

$$\nabla_{\partial_z} d_f \left(\omega \cdot e^{\frac{-f}{z}}\right) = d_f \left(-\frac{\omega}{z} \cdot e^{\frac{-f}{z}} + \nabla_{\partial_z} \left(\omega \cdot e^{\frac{-f}{z}}\right)\right)$$

shows that ∇_{∂_z} induces a connection on $H^{n+1}\left(\pi_*\Omega_Y^\bullet[z, z^{-1}] \cdot e^{\frac{-f}{z}}\right)$. One motivation behind this definition is to make the pairing in equation (19) flat.

Proposition 7.3.2. *[DS02a] The cohomology sheaf* $H^{n+1}\left(\pi_*\Omega_Y^\bullet[z, z^{-1}] \cdot e^{\frac{-f}{z}}\right)$ *of the twisted de Rham complex in degree* $n+1$ *is locally free of rank* μ*.*

Crucial for Stokes matrices is now the extension of H^{n+1} to a free sheaf at $z = 0$:

Definition 7.3.3. *The* Brieskorn lattice H_0 *is the image of the natural map*

$$\Omega^{n+1}[z] \quad \to \quad H^{n+1} = \Omega^{n+1}[z, z^{-1}] / \left((z d_Y - df \wedge)\Omega^n[z, z^{-1}]\right).$$

Proposition 7.3.4. *[DS02a] The Brieskorn lattice is a locally free sheaf of rank* μ *on* $\mathbb{C}$*, with a natural isomorphism* $H_0/zH_0 \cong \Omega^{n+1}/(df \wedge \Omega^n)$*.*

The Stokes matrix of the Frobenius manifold is the Stokes matrix of the differential module H_0 (after base change to $\mathbb{C}(\{z\})$).

7.4 Frobenius manifold mirror symmetry

To construct a Frobenius manifold, one has to extend this connection in two ways: First, one has to extend it to a connection on $\mathcal{M} \times \mathbb{C}$, where $\mathcal{M}$ is the base space of the versal deformation F above. Secondly, one has to solve a Riemann-Hilbert-Birkhoff problem, and extend it to a flat connection on a globally free sheaf $\mathcal{F}$ on $\mathcal{M} \times \mathbb{P}^1$.

Finally, one has to find a primitive form that produces an isomorphism from $\mathcal{TM}$ to the sheaf $p_*\mathcal{F}$ of fibrewise global sections of $\mathcal{F}$. Generally, neither existence nor uniqueness of such a primitive form is clear. We refer to [Her03] for an overview of the whole construction, and to [DS02a] and [DS02b] and references therein for details.

It is generally believed that this construction produces mirror partners for generically semisimple quantum cohomology of Fano varieties. This has been proven for projective spaces:

Theorem 7.4.1. *[Bar01] Let Y be the submanifold of $\mathbb{C}^{n+1}$ given by $X_0X_1\cdots X_n = 1$, and let $f = X_0 + \cdots + X_n$. We use the deformation $F\colon Y \times \mathbb{C}^{n+1} \to \mathbb{C}$ of f given by $F = f + \sum_i a_i f^i$ (where a_i are the coordinates on $\mathbb{C}^{n+1}$).*

For an appropriate choice of the metric, the Frobenius manifold associated to this deformation space is isomorphic to the Frobenius manifold of the quantum cohomology of $\mathbb{P}^n$.

7.5 Relating Stokes and Seifert matrix

Let us abbreviate the sheaves defined in sections 7.2 and 7.3 with $H_{n+1} = H_{n+1}(\tilde{Y}, \tilde{Y}^{z_0})$ and $H^{n+1} = H^{n+1}(\pi_*\Omega_Y^\bullet[z, z^{-1}] \cdot e^{\frac{-f}{z}})$. We can define a pairing between the two fibres $H^{n+1}|_z$ and $H_{n+1}|_{z'}$ for any pair of non-zero complex number z, z' with $\mathrm{Re}\left(\frac{z}{z'}\right) > 0$. It is sufficient to define it for a representative $\omega \cdot e^{\frac{-f}{z}}$ and the class of a Lefschetz thimble $[\Gamma]$.

Extend the path $\gamma\colon [0,1] \to \mathbb{C}$ as in fig. 5 to a straight path $\gamma_\infty\colon [0,\infty[\to \mathbb{C}$ going to infinity, and accordingly extend the Lefschetz thimble to Γ_∞. We set

$$\left\langle \omega \cdot e^{\frac{-f}{z}}, [\Gamma] \right\rangle = \int_{\Gamma_\infty} \omega \cdot e^{\frac{-f}{z}}. \tag{19}$$

Proposition 7.5.1. *Equation (19) gives a well-defined pairing that is flat with respect to the Gauß-Manin connection on the right-hand-side and the connection as defined in (18) on the left-hand side.*

Proof. Outside $f^{-1}(u_1, \ldots, u_\mu)$, we can write ω in the form $\omega = df \wedge \phi$, where ϕ is a $n-1$-form. For any $z' \neq u_i$, we have

$$\phi|_{f^{-1}(z')} = \mathrm{Res}_{f^{-1}(z')}(\omega).$$

This allows us to rewrite the integral as

$$\int_{\Gamma_\infty} \omega \cdot e^{\frac{-f}{z}} = \int_{\gamma_\infty} d\tau \cdot e^{\frac{-\tau}{z}} \int_{\Gamma_\tau} \phi.$$

We have written $\Gamma_\tau = \Gamma \cap f^{-1}(\tau)$ for the vanishing cycle in the fibre over τ belonging to Γ.

From the results in [Pha83], it follows that the Gauß-Manin system of functions obtained as $\int_{\Gamma_\tau} \phi$ has a regular singularity at $\tau = \infty$. In particular, these functions are of moderate growth. This is essential for all calculations.

Due to $\mathrm{Re}\left(\frac{z}{z'}\right) > 0$, the term $e^{\frac{-\tau}{z}}$ will be rapidly vanishing as τ goes to infinity. Hence the integral converges.

Now if $\omega \cdot e^{\frac{-f}{z}}$ is of the form $d_f(\omega' \cdot e^{\frac{-f}{z}}) = zd(\omega' \cdot e^{\frac{-f}{z}})$ for some algebraic $n-1$-form ω', we can by Stokes' theorem conclude for finite $t \in \mathbb{R}$:

$$\int_{\gamma|_{[0,t]}} d\tau \cdot e^{\frac{-\tau}{z}} \int_{\Gamma_\tau} \phi = ze^{\frac{-\gamma(t)}{z}} \int_{\Gamma_{\gamma(t)}} \omega'.$$

This goes to zero with $t \to \infty$.

With similar arguments, one can show that it is flat with respect to the Gauß-Manin connection on H_{n+1}. The flatness with respect to the connection in (18) is immediate. □

From proposition 7.3.4, it is clear that $z^2\nabla_{\frac{\partial}{\partial z}}$ operates on H_0/zH_0 as a diagonal matrix with entries $(u_1, \dots, u_\mu)$. Further, for $z \to 0$, all the integrals over Lefschetz thimbles become dominated by the behaviour at the critical point, and so the result is dominated by a term $e^{\frac{u_i}{z}}$. This motivates some of the following statements.

Conjecture 7.5.2.
- *The pairing (19) is non-degenerate.*
- *After base change to $\underline{\mathbb{C}((z))}$, the pair (H_0, ∇) becomes isomorphic as a differential module to the direct sum of the differential equations $\delta - \frac{u_i}{z}$ — where we have reintroduced the notation of section 5.2. Let $\hat{\omega}_i \in H_0 \otimes_{\mathbb{C}[z]} \underline{\mathbb{C}((z))}$ be the elements defining this isomorphism.*
- *Fix some $z' \neq 0$, and fix an $i, 1 \leq i \leq \mu$. Let $\mathbb{H}$ be the half plane of complex numbers z with $\operatorname{Re} \frac{z}{z'} > 0$. Let $H_0(i)$ be the subsheaf of $H_0|_{\mathbb{H}}$ defined by $\langle H_0(i), [\Gamma_j] \rangle = 0$ for all $j \neq 0$. By the preceding proposition and the first part of the conjecture, this defines a one-dimensional differential submodule of (H_0, ∇).*
- *In the isomorphism $H_0/zH_0 \cong \Omega^{n+1}/(df \wedge \Omega^n)$, the submodule $H_0(i)/zH_0(i)$ corresponds to the part concentrated at the critical point over u_i on the right hand side.*
- *Let $(\phi_i(z))_i \in H^{n+1}|_z$ be the dual basis to the basis $([\Gamma_i])_i$ of $H_{n+1}|_{z'}$ in our pairing. We define $\omega_i(z)$ by $\omega_i(z) \cdot e^{\frac{-f}{z}} = \phi_i(z) \cdot e^{\frac{-u_i}{z}}$. The conjecture is that $\omega_i(z)e^{\frac{-f}{z}}$ is an element in $H_0(i) \otimes_{(\mathbb{C}\{z\})} \mathcal{A}^6$ that has $\hat{\omega}_i$ as asymptotic expansion. (Notice that, since ϕ_i is flat, the section $\omega_i(z)e^{\frac{-f}{z}}$ is a solution to the differential equation $\delta - \frac{u_i}{z}$.)*

Since we can vary z' a little without changing the homology classes of the Γ_i, we can in fact define $H_0(i)$ and ω_i on a sector slightly larger than a half-plane. If we do the same for $-z'$ instead of z', we get analogously sections ω_i^- on a sector covering the opposite half plane. We use this sectors as the covering U_1 and U_2 needed to define the Stokes matrices of the connection.

Corollary 7.5.3. *The Stokes matrix of (H_0, ∇) with respect to (U_1, U_2) is the transpose of the Seifert matrix of f (where we chose z_0 in definition 7.2.1 as $z_0 = z'$).*

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[6] See 5.2.5 for the definition of $\mathcal{A}$.

Proof. The intersection $U_1 \cap U_2$ has two components, and we chose z arbitrary in the component that we meet when we go counter-clockwise starting from z'. The Stokes matrix is the matrix expressing the basis $(\omega_i^-(z) \cdot e^{\frac{-u_i}{z}})_i$ of $H^{n+1}|_z$ in terms of the basis $(\omega_i(z) \cdot e^{\frac{-u_i}{z}})_i$. The first is the dual basis to $\left([\Gamma_i^-]\right)_i$ of $H_{n+1}|z$, the latter to $([\Gamma_i])_i$. Here we have identified $H_{n+1}|_{z'} \cong H_{n+1}|_{-z'} \cong H_{n+1}|_z$ via the flat Gauß-Manin connection.

The Seifert matrix expresses the basis $([\Gamma_i])_i$ in terms of the basis $\left([\Gamma_i^-]\right)_i$; it is thus clear that it coincides with the Stokes matrix. □

7.6 Homological mirror symmetry and exceptional systems

In his talk at the Berkeley ICM 1994 [Kon95], Kontsevich proposed a new mirror symmetry conjecture. According to it, the bounded derived category $D^b(X)$ of coherent sheaves on X should be equivalent to the derived Fukaya category associated to the mirror partner Y.

In the case of Calabi-Yau varieties, this Fukaya category is reasonably well-defined. For our purposes, however, we will need a Fukaya category associated to the mirror $F\colon Y \times \mathcal{M} \to \mathbb{C}$. We will assume that F is the deformation of a function $f\colon Y \to \mathbb{C}$ that has only A_1-singularities.

However, apparently no one has yet succeeded in a geometric definition of this Fukaya category. Still, we can see how some aspects of the Fukaya category fit into the mirror picture with Fano varieties, as we will describe below.

The objects of the Fukaya category on a Calabi-Yau variety are (graded) Lagrangian submanifolds $\Lambda \subset Y$ equipped with a flat $U(n)$-bundle U on Λ. Graded Lagrangian submanifold means the following: Let $p\colon \mathcal{L} \to Y$ be the Lagrangian Grassmannian of the tangent bundle of Y. Let $\tilde{p}\colon \tilde{\mathcal{L}} \to Y$ be the bundle which has as fibres the universal covers of the fibres of p. A Lagrangian submanifold automatically induces a section $s\colon \Lambda \to p^{-1}(\Lambda)$. A graded Lagrangian submanifold comes with a lift $\tilde{s}$ of s to the universal cover: $\tilde{s}\colon \Lambda \to \tilde{p}^{-1}(\Lambda)$.

It is suggested to either restrict the objects to be special Lagrangian submanifolds, or to take equivalence classes of Lagrangian manifolds under Hamiltonian deformations.

The morphisms in the Fukaya category are complexes. Let (Λ_1, U_1) and (Λ_2, U_2) be two graded Lagrangian submanifolds with flat $U(n)$-bundles. We assume that the two submanifolds intersect transversely; otherwise the definition of the Hom-complexes is unclear (at least in full generality). Let x_i be the intersection points of Λ_1 and Λ_2. If we compare $\tilde{s}_1$ and $\tilde{s}_2$ at an intersection point x_i, we get a path in the Lagrangian Grassmannian of $T_{x_i}Y$. The Maslov index (or rather one of the many versions of Maslov indices) associates an integer $\mu(x_i)$ to such a path. Now $\mathrm{Hom}^\bullet(\Lambda_1, \Lambda_2)$ is a complex with objects

$$\mathrm{Hom}^k(\Lambda_1, \Lambda_2) = \bigoplus_{x_i : \mu(x_i) = k} \mathrm{Hom}(U_1|_{x_i}, U_2|_{x_i}).$$

We don't need to know the precise definition of the Maslov index[7], all we

[7] If $\gamma\colon [0,1] \to Gr_L(V)$ is a path in the Lagrangian Grassmanian of a symplectic vector

need is that $(-1)^{\mu(x_i)}$ is the local intersection multiplicity of Λ_1 and Λ_2 at x_i.[8] From this, it follows immediately that

$$\chi(\mathrm{Hom}^\bullet(\Lambda_1, \Lambda_2)) = \Lambda_1 \cdot \Lambda_2. \tag{20}$$

The following conjecture is probably believed by many people:

Conjecture 7.6.1.

The Fukaya category associated to a mirror construction as in section 7.1 has a full exceptional system $(E_1, \ldots, E_\mu)$. *The underlying manifolds of the exceptional objects can be chosen to be Lefschetz thimbles* $(\Lambda_1, \ldots, \Lambda_\mu)$ *of the function* f. *In analogy to equation (20), we have*

$$\chi(\mathrm{Hom}^\bullet(E_i, E_j)) = S_{ij}$$

for $i < j$, *with* S *being the Seifert matrix.*

This seems more a conjecture about the correct definition of the Fukaya category than anything else. We should explain the analogy to equation (20) a bit: In the case where the boundary map (17) is an isomorphism, and where n is even, the entries S_{ij} can be reconstructed via Picard-Lefschetz theory from the intersection pairing in $H_n(\tilde{Y}^{z_0})$. But in general, we are not aware of an interpretation of S_{ij} as an intersection product.

Physicists seem to have understood this part much better. In fact, the most compelling evidence for the conjecture seems that it would be the best mathematical translation of the work of Hori, Iqbal and Vafa in [HIV00]. The numbers S_{ij} are soliton numbers in their framework.

7.7 Putting it together

We can now explain claim 2 of Dubrovin's conjecture 6.2.1 in a few sentences: Let S be the Stokes matrix of a semisimple Frobenius manifold coming from the quantum cohomology of a variety X. By a combinatorial mirror symmetry statement such as 7.4.1, the underlying Frobenius manifold is isomorphic to one obtained from a mirror construction starting with a pair (Y, f) as in this section. By 7.5.3, this is the same as the Seifert matrix. By 7.6.1, it coincides with the Stokes matrix of the Fukaya category associated to (Y, f). And if Kontsevich's categorical mirror symmetry holds, this is then also the Stokes matrix of $D^b(X)$.

Needless to say, this is a very conjectural explanation.

---

space V such that $\gamma(1)$ is transversal to $\gamma(0)$, then this integer is roughly defined as follows: Let $\Gamma \subset Gr_L$ be the cycle whose support are those subspaces that do not intersect the subspace $\gamma(0)$ transversally. This is called the Maslov cycle of $\gamma(0)$. The Maslov index counts the number of intersection points of $\gamma([0,1])$ with the cycle Γ with multiplicities. The only ambiguity left is at the point $\gamma(0)$; one possible solution is to extend the path γ in a sufficiently uniform manner to a path $\gamma\colon [-\varepsilon, 1] \to Gr_L$.

[8] If we have chosen an orientation of Λ_1, then we can choose a compatible orientation of Λ_2 by comparing $\tilde{s}_1$ and $\tilde{s}_2$ at any intersection point.

8 Blow-ups

8.1 Complete exceptional system and blow-ups

The following theorem produces further examples of varieties that have an exceptional system:

Theorem 8.1.1. *[Orl92] Let Y be a smooth subvariety of the smooth projective variety X. Let $\rho\colon \tilde{X} \to X$ be the blow-up of X along Y.*

If both Y and X have an exceptional system, then the same is true for $\tilde{X}$.

We will only be interested in the case where Y is a point. In this case, the exceptional collection on $\tilde{X}$ is easy to construct: Let $\mathbb{P}^{n-1} \cong E \subset \tilde{X}$ be the exceptional divisor (n is the dimension of X). If $\mathcal{E}_0, \dots, \mathcal{E}_r$ is a given exceptional system in $D^b(X)$, then $\mathcal{O}_E(-n+1), \dots, \mathcal{O}_E(-2), \mathcal{O}_E(-1), \rho^*(\mathcal{E}_0), \dots, \rho^*(\mathcal{E}_r)$ is an exceptional system in $D^b(\tilde{X})$.

The construction in the general case is similar. Let $\rho_Y\colon \tilde{Y} \to Y$ be the exceptional divisor, which is a projective bundle. If $\mathcal{E}$ is an exceptional object in $D^b(Y)$, then $\rho_Y^*(\mathcal{E}) \otimes \mathcal{O}_{\tilde{Y}}(-i)$ is an exceptional object in $D^b(\tilde{Y})$. These objects play the role of $\mathcal{O}_E(-i)$ above.

8.2 Semisimplicity and blow-ups

So let us now assume that the variety X satisfies Dubrovin's conjecture, i. e. that it has both a complete exceptional collection and semisimple quantum cohomology. Let X_r be its blow-up at r points. By Theorem 8.1.1, this a test for Dubrovin's conjecture: We already know that X_r has an exceptional system, so we would like to show that it has semisimple quantum cohomology as well.

Under specific additional assumptions, this is done by the following theorem:

Theorem 8.2.1. *Let $X_r \to X$ be the blow-up of a smooth projective variety X at r points. Assume that X is convex (so that all the moduli spaces $\overline{M}_{g,n}(X, \beta)$ are smooth of expected dimension).*

If the quantum cohomology of X has a semisimple point in $H^2(X)$ over the Novikov ring, then the same is true for X_r.

In the case of dimension two, Del Pezzo surfaces were treated in [BM01], where the results of [GP98] on their quantum cohomology were used. The generalization presented here uses instead the results in Andreas Gathmann's paper [Gat01]. The essential idea is a variant of the idea used in [BM01]: a partial compactification of the spectral cover map where the exponentiated coordinate of an exceptional class vanishes. However, in our case, this is only possible after base change to a finite cover of the spectral cover map.

The assumption that semisimple points can already be found in H^2 (in the small quantum cohomology) is not a completely arbitrary one. It induces a Reconstruction Theorem, which states that all Gromov-Witten invariants can be reconstructed from four point correlators (see [BM01, Theorem 1.8.3] for a precise statement). As for the convexity, we refer to the discussion below.

8.2.1 Gathmann's results

The following theorem collects the results of [Gat01] that we will need later. The essential results for the proof in the next section will be part 1a and 2b.

Theorem 8.2.2. *Let $\tilde{X} \to X$ be the blow-up of a smooth projective variety X of dimension n at r points $x_1, \ldots, x_r$. Let E_i be the exceptional divisor cooresponding to x_i, and with the same letter we will denote the exceptional class in $H^2(\tilde{X})$. Let $E'_i \in H_2(\tilde{X})$ be the exceptional homology class of a line in $E_i \cong \mathbb{P}^{n-1}$.*

The following assertions relate the Gromov-Witten invariants of $\tilde{X}$ to those of X (which we will denote by $\langle \ldots \rangle_\beta^{\tilde{X}}$ and $\langle \ldots \rangle_\beta^X$, respectively):

1. (a) *If X is convex, the following holds:*
   Let $\beta \in H_2(\tilde{X})$ be any non-exceptional homology class—so β is any element of $H_2(X)$—, and let $T_1, \ldots, T_m$ be any number of non-exceptional classes in $H^*(\tilde{X})$, which we can identify with their preimages in $H^*(X)$. Then it does not matter whether we compute the following Gromov-Witten invariants with respect to $\tilde{X}$ or X:

   $$\langle T_1 \otimes \cdots \otimes T_m \rangle_\beta^{\tilde{X}} = \langle T_1 \otimes \cdots \otimes T_m \rangle_\beta^X. \tag{21}$$

   (b) *Consider the Gromov-Witten invariants $\langle T_1 \otimes \cdots \otimes T_m \rangle_\beta^{\tilde{X}}$ with β being purely exceptional, i. e. $\beta = d \cdot E'_i$.*
   If any of the cohomology classes $T_1, \ldots, T_m$ are non-exceptional, the invariant is zero. All invariants involving only exceptional cohomology classes can be computed recursively from the following:

   $$\langle E_i^{n-1} \otimes E_i^{n-1} \rangle_{E'_i}^{\tilde{X}} = 1. \tag{22}$$

   They depend only on n.

2. *If either X is convex, or part 1a is true for other reasons, the following statements hold:*

   (a) *Using the associativity relations, it is possible to compute all Gromov-Witten invariants of $\tilde{X}$ from those mentioned above in 1a and 1b.*[9]

   (b) Vanishing of mixed classes: *Write $\beta' \in H_2(\tilde{X})$ in the form $\beta' = \beta + d \cdot E'_i$ where β is the non-exceptional part with respect to E_i; assume that $\beta \neq 0$. Let $T_1, \ldots T_m$ be non-exceptional cohomology classes with respect to E_i. Let l be a non-negative integer, and let $2 \leq k_1, \ldots, k_l \leq n-1$ be integers satisfying*

   $$(k_1 - 1) + \cdots + (k_l - 1) < (d+1)(n-1).$$

   Unless we have both $d \leq 0$ and $l = 0$, this implies the vanishing of

   $$\langle T_1 \otimes \cdots \otimes T_m \otimes E_i^{k_1} \otimes \cdots \otimes E_i^{k_l} \rangle_\beta = 0.$$

[9] We will make this more precise later on.

Proof. All equations follow trivially from statements from lemma 2.2, lemma 2.4 and proposition 3.1 in [Gat01]. Note that we have made explicit the remark 2.3 of Gathmann's paper: the only part where Gathmann uses the convexity of X is in the proof of equation (21). So if it can be proven by other means, the assumption of convexity of X can be dropped. [10]

The only apparent change to Gathmann's statements is in 2b: We only assume that $\beta \neq 0$, whereas Gathmann assumes that β is not purely exceptional. But the missing cases are treated by [Gat01, lemma 2.4, (i)]. □

We want to apply Gathmann's theorems to the map $X_r \to X_{r-1}$, where X_r and X_{r-1} are the blow-up of a convex variety X at r and $r-1$ points, respectively; but we do not want to assume the convexity of X_{r-1}. So we need to prove equation (21). We will use Gathmann's algorithm to compute the Gromov-Witten invariants of X_{r-1} and X_r from those of X (i. e. we apply Gathmann's theorems to the maps $X_r \to X$ and $X_{r-1} \to X$).

We want to show that his algorithm gives the same number for Gromov-Witten invariants of X_{r-1} and the non-exceptional Gromov-Witten invariants of X_r. First, we need to introduce some of his notations.

Definition 8.2.3. *Denote by V a smooth projective variety. Let $\beta \in H_2(V)$ be an effective homology class, and let $T_1, \ldots, T_m, \mu_1, \mu_2, \mu_3, \mu_4 \in H^*(V)$ be cohomology classes. Write $\mathcal{T}$ for $\mathcal{T} = T_1 \otimes T_2 \otimes \cdots \otimes T_m$. Let $\Delta_0, \ldots, \Delta_q$ be a basis of the cohomology.*

By $\mathcal{E}_\beta(\mathcal{T}; \mu_1, \mu_2 \mid \mu_3, \mu_4)$ we denote the equation[11]

$$\begin{aligned}
0=&\langle \mathcal{T} \otimes \mu_1 \otimes \mu_2 \otimes \mu_3 \cdot \mu_4 \rangle_\beta + \langle \mathcal{T} \otimes \mu_3 \otimes \mu_4 \otimes \mu_1 \cdot \mu_2 \rangle_\beta \\
&-\langle \mathcal{T} \otimes \mu_1 \otimes \mu_3 \otimes \mu_2 \cdot \mu_4 \rangle_\beta - \langle \mathcal{T} \otimes \mu_2 \otimes \mu_4 \otimes \mu_1 \cdot \mu_3 \rangle_\beta \\
&+ \sum_{\beta_1,\beta_2} \sum_{\mathcal{T}_1,\mathcal{T}_2} \sum_{i,j} (\Delta_i, \Delta_j) \Big(\langle \mathcal{T}_1 \otimes \mu_1 \otimes \mu_2 \otimes \Delta_i \rangle_{\beta_1} \langle \mathcal{T}_2 \otimes \mu_3 \otimes \mu_4 \otimes \Delta_j \rangle_{\beta_2} \\
&\qquad - \langle \mathcal{T}_1 \otimes \mu_1 \otimes \mu_3 \otimes \Delta_i \rangle_{\beta_1} \langle \mathcal{T}_2 \otimes \mu_2 \otimes \mu_4 \otimes \Delta_j \rangle_{\beta_2} \Big)
\end{aligned}$$

Here the sums go over

- *all decompositions of β as a sum $\beta = \beta_1 + \beta_2$ of effective homology classes,*
- *decompositions of $\mathcal{T} = \mathcal{T}_1 \otimes \mathcal{T}_2$ that come from decompositions of $\{T_1, \ldots, T_m\}$ into two sets, and*
- *all $0 \leq i, j \leq q$.*

This equation is guaranteed to hold by the axioms for Gromov-Witten invariants and is necessary for associativity.

We can now describe Gathmann's recursive algorithm:

[10] The author is convinced that equation (21) is always true. However, a proof is technically highly inconvenient in the framework of the construction of the virtual fundamental classes in [Beh97]: The natural morphism $\overline{M_{0,n}}(\tilde{X}, \beta) \to \overline{M_{0,n}}(X, \beta)$ does not commute with the morphism to $\mathfrak{M}(0, n)$ (the moduli space of prestable curves), with respect to which the perfect relative obstruction theory is constructed. This makes it hard to compare the relative virtual normal cones.

If the virtual fundamental class could be constructed via an absolute obstruction theory (using the non-relative virtual normal cone of [BF97]), a proof of equation (21) might be easy.

[11] We write $\mu_i \cdot \mu_j$ for the cup product, and (Δ_i, Δ_j) for the Poincaré pairing.

Theorem 8.2.4. *Given a blow-up $\tilde{X}$ of a convex variety X at a finite number of points, we can compute all Gromov-Witten invariants of $\tilde{X}$ recursively, using the associativity relations, from those determined by 8.2.2, 1a and 1b, in the following way:*

By linearity, we can assume that all cohomology classes are of pure dimension and either non-exceptional or purely exceptional. Write $\mathcal{T} = T_1 \otimes \dots T_{m'} \otimes T_{m'+1} \otimes \dots \otimes T_m$ with $T_1, \dots, T_{m'}$ non-exceptional, and $T_{m'+1}, \dots, T_m$ exceptional. By the divisor axiom, we can assume without loss of generality that none of the exceptional cohomology classes are divisorial, and that the sum of the codimensions of $T_1, \dots, T_{m'}$ is at least $n+1$.[12] In particular, m' is at least 2. Also, we assume $\operatorname{codim} T_1 \geq \operatorname{codim} T_2 \geq \dots \geq \operatorname{codim} T_{m'}$.

Then we proceed as follows:

(A) If $m' < m$, then $T_m = E_i^k$ for some exceptional divisor class E_i. We then use

$$\mathcal{E}_\beta(\mathcal{T}'; T_1, T_2 \mid E_i, E_i^{k-1}) \quad \textit{with} \quad \mathcal{T}' = T_3 \otimes \dots \otimes T_{m-1}$$

(B) If $m' = m$, $T_1 = [pt]$ and $\operatorname{codim} T_2 \geq 2$, we choose μ, ν such that μ is divisorial, $\operatorname{codim} \nu = n-1$ and $\mu \cdot \nu = [pt]$, and we choose an exceptional divisor E_i with $E_i \cdot \beta \neq 0$ (such an E_i must exist, as otherwise the invariant would be determined by equation (21)).

We use

$$\mathcal{E}_\beta(\mathcal{T}'; \mu, \nu \mid E_i, T_2) \quad \textit{with} \quad \mathcal{T}' = T_3 \otimes \dots \otimes T_m$$

(C) In all other cases (i. e. $m' = m$, but not case (B)) we again choose an E_i as in the previous case. We use the equation

$$\mathcal{E}_{\beta + E_i'}(\mathcal{T}'; T_1, T_2 \mid E_i, E_i^{n-1}) \quad \textit{with} \quad \mathcal{T}' = T_3 \otimes \dots \otimes T_m$$

Here "using equation $\mathcal{E}_\beta(\dots)$" means

- *that there is a partial ordering on the set of pairs $(\beta, \mathcal{T})$,*[13]
- *that this ordering does not have infinite descending chains, and*
- *that at each step, the respective equation (together with the divisor axiom) determines the correlator in question uniquely from correlators that are smaller with respect to the partial ordering.*

Proposition 8.2.5. *Let $\beta \in H_2(X_{r-1})$ and $T_1, \dots, T_m \in H^*(X_{r-1})$ be (co-)homology classes in X_{r-1}, and identify them with their images in the (co-)homology of $H^*(X_r)$.*

If we compute $\langle T_1 \otimes \dots \otimes T_m \rangle_\beta^{X_{r-1}}$ and $\langle T_1 \otimes \dots \otimes T_m \rangle_\beta^{X_r}$ according to theorem 8.2.4, we can make corresponding choices at each step, and get an equivalent recursion; thus, the two values coincide. Therefore, Theorem 8.2.2 holds for the map $X_r \to X_{r-1}$.

[12] Otherwise, we can enlarge the list of the T_i by any number of divisors D with $(D, \beta) \neq 0$.

[13] Its precise definition is not relevant for us, so we will not repeat it here.

Proof. The "initial values" of the algorithm, i. e. the non-exceptional invariants of X, and the purely exceptional ones, are identical in both cases by Theorem 8.2.2, 1b and 1a.

We assume that X_r and X_{r-1} are the blow-up of X at $x_1, \dots, x_r$, and $x_1, \dots, x_{r-1}$, respectively. Having chosen a basis $\Delta_0, \dots, \Delta_{q'}$ of $H^*(X_{r-1})$, we extend this to a basis $\Delta_0, \dots, \Delta_q$ of $H^*(X_r)$ by identifying $\Delta_0, \dots, \Delta_{q'}$ with their images in $H^*(X_r)$, and setting $\Delta_{q'+k} = E_r^k$ for $1 \le k \le n-1$.

Now at each recursion step, we make the necessary choices in $H^*(X_{r-1})$, and then make the corresponding choice via the inclusion $H^*(X_{r-1}) \subset H^*(X_r)$ for X_r.

Let us assume that the claim has already been proven for all smaller classes in Gathmann's partial ordering of pairs $(\beta, \mathcal{T})$.

Consider case (A) of the algorithm. In X_r, the equation to be used reads as:

$$\begin{aligned} 0=&\langle \mathcal{T}' \otimes T_1 \otimes T_2 \otimes E_i \cdot E_i^{k-1} \rangle_\beta^{X_r} + \langle \mathcal{T}' \otimes E_i \otimes E_i^{k-1} \otimes T_1 \cdot T_2 \rangle_\beta^{X_r} + 0 + 0 \\ &+ \sum_{\mathcal{T}_{1,2}, \beta_{1,2}, i, j} (\Delta_i, \Delta_j) \big(\langle \mathcal{T}_1 \otimes T_1 \otimes T_2 \otimes \Delta_i \rangle_{\beta_1}^{X_r} \langle \mathcal{T}_2 \otimes E_i \otimes E_i^{k-1} \otimes \Delta_j \rangle_{\beta_2}^{X_r} \\ &\qquad - \langle \mathcal{T}_1 \otimes T_1 \otimes E_i \otimes \Delta_i \rangle_{\beta_1}^{X_r} \langle \mathcal{T}_2 \otimes T_2 \otimes E_i^{k-1} \otimes \Delta_j \rangle_{\beta_2}^{X_r} \big) \end{aligned}$$

(Here we have abbreviated the big sum of 8.2.3. The third and fourth summand there are zero because of $E_i \cdot T_1 = E_i \cdot T_2 = 0$.)

The first summand is the one we want to compute. The second summand is identical to $\langle \mathcal{T}' \otimes E_i \otimes E_i^{k-1} \otimes T_1 \cdot T_2 \rangle_\beta^{X_{r-1}}$ by induction hypothesis.

All summands in the big sum can, by induction hypothesis, be assumed to be identical to the corresponding terms $\langle \cdots \rangle^{X_{r-1}}$ in the corresponding equation in X_{r-1}—if they appear in that equation at all. The summands that do not appear in the corresponding equation for X_{r-1} come from

1. decompositions of $\beta = \beta_1 + \beta_2$ that are non-trivial with respect to the exceptional divisor E_r, i. e. $\beta_1 = \beta_1' + d \cdot E_r'$, $\beta_2' = \beta_2' - d \cdot E_r'$ with $d > 0$, and/or

2. the $\Delta_{i,j}$ being elements of the basis of $H^*(X_r)$ that do not appear in the basis of $H^*(X_{r-1})$, i. e. $\Delta_{i,j} = E_r^k$.

If we are in case 2, but not in case 1, both invariants $\langle \cdots \rangle_{\beta_{1,2}}^{X_r}$ vanish by 8.2.2, 2b (with $d = 0$ and $l = 1$). In the case 1, the correlator $\langle \cdots \rangle_{\beta_1}^{X_r}$ vanishes, this time by 8.2.2, 2b with $d > 0$ and $l \le 1$.

Cases (B) and (C) are completely analogous. □

8.2.2 Proof of Theorem 8.2.1

Proof.[of Theorem 8.2.1] By induction we only need to treat the case of the blow-up of a single point.

We consider the spectral cover map restricted to $H^2(X_r)$, over the Novikov ring. Let $\hat{N}_r$ and $\hat{N}_{r-1}$ be the Novikov ring of X_r and X_{r-1}, respectively. Let $\beta_1, \dots, \beta_s$ be a basis of $H_2(X_{r-1})$, and let $E' = E_r'$ be the line in the exceptional divisor of $X_r \to X_{r-1}$. Write $q_i = q^{\beta_i}$ and $Q = q^{-E'}$ for the corresponding elements in the Novikov rings. The Novikov ring $\hat{N}_r$ is generated over $\hat{N}_{r-1}$ by monomials $\prod_i q_i^{a_i} \cdot Q^{-d}$, $d \neq 0$, for which $\sum_i a_i\beta_i + dE'$ is effective.

The structure ring of small quantum cohomology is given by $H^*(X_r) \otimes \hat{N}_r$ and $H^*(X_{r-1}) \otimes \hat{N}_{r-1}$, respectively. The spectral cover map for X_r looks as follows:

$$\operatorname{Spec}\left(\hat{N}_r \otimes H^*(X_r)\right) \to \operatorname{Spec} \hat{N}_r \tag{23}$$

(The multiplication on the left-hand side is the quantum product.)

We make the base change to the cover given by adjoining $R := \sqrt[n-1]{Q}$. We want to extend the spectral cover map to the boundary where $R = 0$. To do this, we let $B = \hat{N}_{r-1}[[R]]$, and consider it as a subring of $\hat{N}_r[Q][R]/(R^{n-1} - Q)$.[14] We define M as the free B-submodule of $B \otimes H^*(X_r)$ generated by

$$\langle H^*(X_{r-1}), RE, R^2E^2, \ldots, R^{n-1}E^{n-1} = QE^{n-1}\rangle.$$

Lemma 8.2.6. • *The quantum product restricts to M, i. e. $M \circ M \subseteq M$. So after base change to the $(n-1)$-fold cover, the spectral cover map (23) extends to a map*

$$\begin{array}{ccc} \operatorname{Spec}\left(\hat{N}_r[Q][R]/(R^{n-1}-Q) \otimes H^*(X_r)\right) & \rightarrowtail & \operatorname{Spec} M \\ \downarrow & & \downarrow \\ \operatorname{Spec}\left(\hat{N}_r[Q][R]/(R^{n-1}-Q)\right) & \rightarrowtail & \operatorname{Spec} B \end{array}$$

• *The fibre at $R = 0$ of this extended spectral cover map is isomorphic to the disjoint union of $n-1$ copies of the identity map and the spectral cover map of X_{r-1}:*

$$\begin{array}{ccc} \operatorname{Spec}\left(\hat{N}_{r-1} \otimes H^*(X_{r-1})\right) \amalg \coprod_{i=1}^{n-1} \operatorname{Spec} \hat{N}_{r-1} & \rightarrowtail & \operatorname{Spec} M \\ \downarrow & & \downarrow \\ \operatorname{Spec} \hat{N}_{r-1} = \operatorname{Spec} B/(R) & \rightarrowtail & \operatorname{Spec} B \end{array}$$

We assume for a moment that this lemma holds. The map $\operatorname{Spec} M \to \operatorname{Spec} B$ is flat and finite. By the induction hypothesis, $\operatorname{Spec} \hat{N}_{r-1} \otimes H^*(X_{r-1}) \to \operatorname{Spec} \hat{N}_{r-1}$ is generically semisimple (i. e., it is unramified over an open non-empty subset of $\operatorname{Spec} \hat{N}_{r-1}$). The second part of the lemma then tells us that the map $\operatorname{Spec} M \to \operatorname{Spec} B$ is generically semisimple over the fibre of $R = 0$.

E. g. by the criterion [EGA, IV, 17.3.6] of unramifiedness, it is clear that semisimplicity is an open condition. Hence the extended spectral cover

[14] It cannot be the complete Novikov ring, as there are effective homology classes in $H_2(X_r)$ of the form $\beta = \beta_{r-1} + dE'$, $d > 0$, so that $\hat{N}_r$ contains monomials with negative powers of Q. Also note that Q itself is not an element of the Novikov ring, so we have to adjoin it first, too.

map is generically semisimple. The same must then hold for the map over $\hat{N}_r[Q][R]/(R^{n-1}–Q)$, and, as unramifiedness can be checked after a base change to a covering, the original spectral cover map (23). □

Proof.[of the lemma] Both claims follow from a close investigation of all possible products, using mainly Gathmann's vanishing result.

Let $\Delta_0, \ldots, \Delta_m$ be a graded basis of $H^*(X_{r-1})$, which is an extension of the basis $\Delta_1, \ldots, \Delta_s$ of $H^2(X_{r-1})$ that we chose above, and which contains Δ_0 as the unit and Δ_m as the dual class of a point. The matrix of the Poincaré pairing with respect to this basis has entries g^{ij}.

The following formulas compute quantum products up to higher orders of R. We denote the product taken in the quantum cohomology of X_r and X_{r-1} by $\circ_{X_r}$ and $\circ_{X_{r-1}}$, respectively. We write β as $\beta = \beta_{r-1} + d \cdot E'$ with $\beta_{r-1} \in H_2(X_{r-1})$ as the non-exceptional part. The references "no. 1a" etc. always point to the parts of Theorem 8.2.2.

We claim:

$$\begin{aligned}
\Delta_i \circ_{X_r} \Delta_j &= \sum_{k,l} \sum_{\beta} \langle \Delta_i \Delta_j \Delta_k \rangle_\beta g^{kl} \Delta_l \cdot Q^{-d} \prod_{\nu=1}^{s} q_v^{(\beta, \Delta_\nu)} \\
&\quad + (-1)^{n-1} \sum_{k=1}^{n-1} \sum_{\beta} \langle \Delta_i \Delta_j E^k \rangle_\beta E^{n-k} \cdot Q^{-d} \prod_{\nu=1}^{s} q_v^{(\beta, \Delta_\nu)} \\
&= \Delta_i \circ_{X_{r-1}} \Delta_j + O(Q) \qquad (24)
\end{aligned}$$

To show this equality we need to examine three cases according to β:

- If $d = 0$, then the contribution to the second sum vanishes by no. 2b. By no. 1a, the first sum restricted to the cases $d = 0$ computes exactly $\Delta_i \circ_{X_{r-1}} \Delta_j$.
- The term $O(Q)$ covers all summands with $d < 0$.
- If $d > 0$, all relevant Gromov-Witten invariants appearing in equation (24) vanish by no. 2b (respectively by no. 1b in the case $\beta_{r-1} = 0$).

For the next product that we have to examine (where $i, j > 0$), we get:

$$\begin{aligned}
\Delta_i \circ_{X_r} R^j E^j &= \sum_{k,l} \sum_{\beta} \langle \Delta_i E^j \Delta_k \rangle_\beta g^{kl} \Delta_l \cdot R^j Q^{-d} \prod_{\nu=1}^{s} q_v^{(\beta, \Delta_\nu)} \\
&\quad + (-1)^{n-1} \sum_{k} \sum_{\beta} \langle \Delta_i E^j E^k \rangle_\beta E^{n-k} \cdot R^j Q^{-d} \prod_{\nu=1}^{s} q_v^{(\beta, \Delta_\nu)} \\
&= O(R \cdot M) \qquad (25)
\end{aligned}$$

Again we have to treat the three different cases:

- If $d = 0$, then the contribution to the first sum vanishes by no. 2b. The vanishing result also implies that $\langle \Delta_i E^j E^k \rangle_\beta$ can only be non-zero if $j + k - 2 \geq n - 1$, i. e. $j \geq n - k + 1$. Then $R^j E^{n-k}$ lies in $R \cdot M$.

This reasoning only omitted the trivial case $\beta = 0$; here the invariant is zero since we assumed $\Delta_i \neq \Delta_0$.

- The term $O(R \cdot M)$ covers all summands with $d < 0$.
- If $d > 0$, then again all the invariants appearing in equation (25) vanish by no. 2b and no. 1b.

Further, we will show:

$$\begin{aligned} RE \circ_{X_r} R^i E^i &= \sum_{j,k} \sum_{\beta} \langle EE^i \Delta_j \rangle_\beta g^{jk} \Delta_k \cdot R^{i+1} Q^{-d} \prod_{\nu=1}^{s} q_v^{(\beta, \Delta_\nu)} \\ &\quad + (-1)^{n-1} \sum_{j} \sum_{\beta} \langle EE^i E^j \rangle_\beta E^{n-j} \cdot R^{i+1} Q^{-d} \prod_{\nu=1}^{s} q_v^{(\beta, \Delta_\nu)} \\ &= \begin{cases} R^{i+1} E^{i+1} + O(R \cdot M) & \text{if } i \leq n-2 \\ (-1)^n RE + O(R \cdot M) & \text{if } i = n-1. \end{cases} \end{aligned} \tag{26}$$

- If $d = 0$, then both terms above vanish evidently because of the divisor axiom—unless $\beta = 0$; for this case, the first sum contributes a summand $(-1)^{n-1} R^n[\mathrm{pt}] \in R \cdot M$ (for $i = n-1$ and $\Delta_j = \Delta_0$), whereas the second sum gives $R^{i+1}E^{i+1}$ (from the summand with $i + j + 1 = n$).
- As usual, the term $O(R \cdot M)$ takes care of the cases with $d < 0$.
- If $d > 0$, we have to distinguish two cases this time: If β is not purely exceptional, then we can apply no. 2b; this immediately shows that the summand of the first sum vanishes. It also shows that $\langle EE^i E^j \rangle_\beta$ is zero: We have $\nu = 3$, $a_1 = 1$ and $a_2, a_3 \leq n-1$. Hence $a_1 + a_2 + a_3 - 3 \leq 2n - 4$. On the other hand, $(n-1)(d+1) \geq (n-1)(1+1) = 2n-2$.

  It remains the case $\beta = d \cdot E'$. The first sum vanishes for this term. By the dimension axiom, the correlator in the second sum can only be non-vanishing if $k(\beta) = 3 - n + (1-1) + (i-1) + (j-1)$, or equivalently $(n-1)d = 1 - n + i + j$. This is only possible if $i = j = n-1$ and $d = 1$. The corresponding correlator is

  $$\langle EE^{n-1}E^{n-1} \rangle_{E'} = -\langle E^{n-1}E^{n-1} \rangle_{E'} = -1.$$

  This yields the summand $(-1)^n RE$ for the case $i = n-1$.

These formulae check that M is indeed a subring (with unit) of the quantum ring $\hat{N}_r \otimes H^*(X_r)$. To show the second part of our lemma, we have to investigate the fibre M/RM.

Let $Y := (-1)^n Q E^{n-1} = (-1)^n R^{n-1} E^{n-1}$. The equation (26) says that, in the ring M/RM, multiplication by Y is the identity on the span S of $RE, R^2E^2, \ldots, R^{n-1}E^{n-1}$. In particular, Y is an idempotent and gives a splitting of $M/RM \cong S \oplus K$ into the image S and kernel K of $Y\circ$. By our formulae, we see immediately that S is the image and the kernel is generated by $\Delta_1, \ldots, \Delta_m, \Delta_0 - Y$.

As Δ_0 is the unit in the quantum cohomology ring, $\Delta_0 - Y$ must be the unit in K. Then equation (24) tells us that the map

$$(K, \circ_{X_r}) \to (H^*(X_{r-1}), \circ_{X_{r-1}})$$

given by $\Delta_0 - Y \mapsto \Delta_0$, $\Delta_i \mapsto \Delta_i, i > 0$ is an isomorphism.

The image S (in which Y is the unit) is isomorphic to $B[Z]/(Z^{n-1} - (-1)^n) \cong \bigoplus_{i=1}^{n-1} B$ via the map $Z \mapsto RE$. Of course, this splits over $\mathbb{C}$ as claimed in the lemma. □

8.3 Further Questions

The main example where our theorem applies is the case of $X = \mathbb{P}^n$. For $n = 2$, this yields the semisimplicity of quantum cohomology for all Del Pezzo surfaces.

When formulating Dubrovin's conjecture in 6.2.1, we have followed him in only referring to Fano varieties. For the claims about the Stokes matrix and the central connection matrix, this makes sense, as we need a $\mathbb{C}$-valued point, and hence convergence of the potential, to even define them in the quantum cohomology case. (While a non-empty convergence domain of the quantum cohomology potential has only been proven for $\mathbb{P}^2$, one can generally hope for convergence in the Fano case.) However, our theorem suggests that the assumption of X being Fano is unnecessary for the statement about generic semicimplicity.

Of course, our theorem 8.2.1 covers only the first part of Dubrovin's conjecture. It would be very encouraging if it was possible to show his statement on Stokes matrices in a similar way. As far as I know, the only case where this part has been checked is the quantum cohomology of projective spaces; this was done by Guzzetti (cf. [Guz99]).

However, already in the case of Del Pezzo surfaces, it seems far from clear how the Stokes matrices could be computed. On the other hand, if we look again at Gathmann's theorem and his algorithm to compute the invariants of X_r (cf. 8.2.2, no. 2a), we notice that all the initial data it uses is already contained in the quantum multiplication in the special fibre $R = 0$ of our partially compactified spectral cover map. If we rephrase this statement, it says that the whole Frobenius manifold associated to the quantum cohomology of X_r is already determined by the structure at $R = 0$.

Yet our construction does not yield a Frobenius structure at the point $R = 0$. The multiplication has a pole here. If there was a formalism of divisorial Frobenius manifolds, which could contain divisors as $R = 0$ in our case, and if there was a way to extend Dubrovin's Stokes matrices to these divisorial Frobenius manifolds, this would probably be the most elegant treatment of Stokes matrices of blow-ups.

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