STABILITY CONDITIONS IN FAMILIES AND FAMILIES OF HYPERKÄHLER VARIETIES

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In this talk, I described a new construction of families of Hyperkähler varieties associated to families of cubic fourfolds, obtained in work in progress with the co-authors listed above. Our construction is based on crucial technical progress in the theory of Bridgeland stability conditions on derived categories of algebraic varieties. More specifically, we develop a notion of a "family of stability conditions" on a family of varieties, as well as a version of that for families with *Kuznetsov subcategories* of the derived categories of the fibers; both come with a notion of *relative moduli spaces* of stable objects. Our construction allows us to prove analogues of the powerful results for moduli spaces of stable sheaves on K3 surfaces, due to Mukai, Huybrechts, O'Grady, Yoshioka and others, in the setting of cubic fourfolds.

0.1. **Setting: The Kuznetsov category of a cubic fourfold, and stability conditions.** Let $X \subset \mathbb{P}^5$ be a smooth cubic fourfold. By its *Kuznetsov category* we denote the triangulated subcategory

$$\mathcal{K}\mathrm{u}(X) := \mathcal{O}_X^{\perp} \cap \mathcal{O}_X(1)^{\perp} \cap \mathcal{O}_X(2)^{\perp} \subset \mathrm{D^b}(X) = \mathrm{D^b}(\mathrm{Coh}\, X)$$

of its bounded derived category of coherent sheaves. This category shares many properties with the derived category of K3 surfaces; its foundations were developed in [Kuz10, AT14, Huy17]:

(1) $\mathcal{K}\mathrm{u}(X)$ is a CY 2-category: $\mathrm{Hom}(E, F) = \mathrm{Hom}(F, E[2])^\vee$.
(2) Topological K-theory of $\mathcal{K}\mathrm{u}(X)$, along with the faithful functor $\mathcal{K}\mathrm{u}(X) \to \mathrm{D^b}(X)$ and the Hodge structure on $H^4(X)$ equips $\mathcal{K}\mathrm{u}(X)$ with an extended *Mukai lattice*, which by some abuse of notation we will denote $\widetilde{H}(\mathcal{K}\mathrm{u}(X), \mathbb{Z})$: as a lattice, it is isomorphic to $H^*(K3)$; it carries a weight two Hodge structure with $h^{2,0} = 1$; and it admits a Mukai vector $v\colon K(\mathcal{K}\mathrm{u}(X)) \to \widetilde{H}(\mathcal{K}\mathrm{u}(X), \mathbb{Z})$ satisfying $(v(E), v(F)) = -\chi(E, F)$.

Often, $\mathcal{K}\mathrm{u}(X)$ is equivalent to the derived category of a K3 surface, see Corollary 0.4.

By $\widetilde{H}_{\mathrm{Hodge}}(\mathcal{K}\mathrm{u}(X), \mathbb{Z})$ we will denote the sublattice of integral (1,1)-classes.

The recent preprint [BLMS17] gives a contruction of a component $\mathrm{Stab}^\dagger(\mathcal{K}\mathrm{u}(X))$ of the space of *Bridgeland stability conditions* on $\mathcal{K}\mathrm{u}(X)$. A stability condition consists of the datum of a subcategory $\mathcal{P}(\phi)$ of semistable objects of phase ϕ for all $\phi \in \mathbb{R}$, and a central charge, i.e. group homomorphism $Z\colon \widetilde{H}_{\mathrm{Hodge}}(\mathcal{K}\mathrm{u}(X)) \to \mathbb{C}$, such that there is a notion of Harder-Narasimhan filtrations for all objects in $\mathcal{K}\mathrm{u}(X)$, and such that the central charge $Z(E)$ of a semistable object $E \in \mathcal{P}(\phi)$ of phase ϕ is a complex number with argument $\pi\phi$.

0.2. **Main results.** Let $\mathbf{v} \in \widetilde{H}_{\mathrm{Hodge}}(\mathcal{K}\mathrm{u}(X), \mathbb{Z})$ be a primitive class, and let $\sigma \in \mathrm{Stab}^\dagger(\mathcal{K}\mathrm{u}(X))$ be a stability condition as constructed in [BLMS17]. The first result concerns the existence and non-emptiness of the moduli space $M_\sigma(\mathcal{K}\mathrm{u}(X), \mathbf{v})$ of σ-stable objects in $\mathcal{K}\mathrm{u}(X)$ of Mukai vector $\mathbf{v}$.

Theorem 0.1. *If σ is generic with respect to $\mathbf{v}$, then $M_\sigma(\mathcal{K}\mathrm{u}(X), \mathbf{v})$ is non-empty if and only if $\mathbf{v}^2 \geq -2$. It is a smooth projective irreducible holomorphic symplectic variety of dimension $\mathbf{v}^2 + 2$, deformation-equivalent to a Hilbert scheme of points on a K3 surface.*

Here *generic* means that σ is not on a wall, so that stability and semistability coincide for objects of Mukai vector $\mathbf{v}$. We can also describe $H^2(M_\sigma(\mathcal{K}u(X), \mathbf{v}))$ in terms of the Hodge structure on $\widetilde{H}(\mathcal{K}\mathrm{u}(X))$, and thus on $H^4(X)$, analogous to the corresponding result by Yoshioka for moduli of sheaves on K3s

Theorem 0.1 is proved by deformation to the case where $\mathcal{K}\mathrm{u}(X)$ is known to be equivalent to the derived category of a K3 surface. Such deformation arguments rely on the existence of *relative moduli spaces* given by Theorem 0.2 below.

Consider a family $\mathcal{X} \to S$ of smooth cubic fourfolds. Let $\mathbf{v}$ be a primitive section of the local system given by the Mukai lattices $\widetilde{H}(\mathcal{K}\mathrm{u}(\mathcal{X}_s), \mathbb{Z})$ of the fibers over $s \in S$, such that $\mathbf{v}$ is algebraic on all fibers. Assume that for $s \in S$ very general, there exists a stability condition $\sigma_s \in \mathrm{Stab}^\dagger(\mathcal{K}\mathrm{u}(\mathcal{X}_s))$ that is generic with respect to $\mathbf{v}$, and such that the associated central charge $Z\colon \widetilde{H}_{\mathrm{Hodge}}(\mathcal{K}\mathrm{u}(\mathcal{X})) \to \mathbb{C}$ is monodromy-invariant. (This is, for example, automatic when S is the moduli space of all cubic fourfolds.)

Theorem 0.2. (1) *There exists a finite cover $\widetilde{S} \to S$, an algebraic space $\widetilde{M}(\mathbf{v})$, and a proper morphism $\widetilde{M} \to \widetilde{S}$ that makes $\widetilde{M}$ a* relative moduli space *over $\widetilde{S}$: the fibers over $s \in \widetilde{S}$ are a moduli space $M_{\sigma_s}(\mathcal{K}\mathrm{u}(\mathcal{X}_s), \mathbf{v})$ of stable objects in the Kuznetsov category of the corresponding cubic.*
(2) *There exists an open subset $S^0 \subset S$, a projective variety $M^0(\mathbf{v})$, and a* projective *morphism $M^0(\mathbf{v}) \to S^0$ that makes $M^0(\mathbf{v})$ a relative moduli space over S^0.*

Note that every fiber of the morphism $\widetilde{M}(\mathbf{v}) \to \widetilde{S}$ is projective, but the morphism itself might not be.

Example 0.3. Let S be the moduli space of cubic fourfolds. For a very general cubic fourfold, $\widetilde{H}_{\mathrm{Hodge}}(\mathcal{K}\mathrm{u}(X), \mathbb{Z})$ is isomorphic to the A_2-lattice, generated by two roots $\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2$ with $(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2) = -1$. If we choose $\mathbf{v} = \boldsymbol{\lambda}_1$ in Theorem 0.2, then

$S^0 = \widetilde{S} = S$, and $M(\mathbf{v})$ is the Fano variety of lines. For $\mathbf{v} = \boldsymbol{\lambda}_1 + 2\boldsymbol{\lambda}_2$, we have $S^0 \subset S$ the complement of cubics containing a plane, $\widetilde{S} = S$, and $M^0(\mathbf{v})$ is the family of Hyperkähler eight-folds constructed in recent work [LLSvS15] of Lehn, Lehn, Sorger and van Straten. In particular, the algebraic space $\widetilde{M}(\mathbf{v})$ partially compactifies their family, at the cost of losing projectivity, over cubics containing a plane; here our moduli spaces agree with those considered by Ouchi in [Ouc17]. Finally, for $\mathbf{v} = 2\boldsymbol{\lambda}_1$, we get an algebraic construction of a 20-dimensional family of singular 10-dimensional O'Grady spaces.

0.3. **Applications.** Recall from Hassett's work on cubic fourfolds, [Has00], that there is a countable union of divisors of special cubics for which one can Hodge-theoretically associate a K3 surface to its primitive cohomology in $H^4(X)$. In our notation, a cubic is contained in one of Hassett's special divisors if and only if $\widetilde{H}_{\mathrm{Hodge}}(\mathcal{K}\mathrm{u}(X), \mathbb{Z})$ contains a hyperbolic plane.

Corollary 0.4. *Let X be a cubic fourfold. Then X has a Hodge-theoretically associated K3 if and only if there exists a smooth projective K3 surface S and an equivalence $\mathcal{K}\mathrm{u}(X) \cong \mathrm{D}^{\mathrm{b}}(S)$.*

This (literally) completes a result by Addington and Thomas, [AT14], who proved that every divisor described by Hassett contains an open subset of cubics admitting a derived equivalence as above. A version of the Corollary also holds for K3s with a Brauer twist; the corresponding Hodge-theoretic condition is the existence of a square-zero class in $\widetilde{H}_{\mathrm{Hodge}}(\mathcal{K}\mathrm{u}(X))$.

As pointed out to us by Voisin, the non-emptiness of moduli spaces also produces enough algebraic cohomology classes to reprove her result on the integral Hodge conjecture for cubic fourfolds:

Corollary 0.5 ([Voi07, Theorem 18])**.** *The integral Hodge conjecture holds for X.*

Our results also provide the full machinery of [BM14], describing the birational geometry of $M_\sigma(\mathcal{K}\mathrm{u}(X), \mathbf{v})$ in terms of wall-crossing.

0.4. **Stability conditions in families.** As hinted at in the introductory paragraph, the notion of relative moduli spaces depends on developing a notion and construction of stability conditions for a familiy of varieties $\pi\colon \mathcal{Y} \to S$.

Here we only sketch the underlying definition, in the simplest possible case where $S = C$ is a smooth curve over $\mathbb{C}$. Both the definitions and the technical setup borrows results and ideas from the work [AP06, Pol07] by Abramovich and Polishchuk on sheaves of t-structures over a base.

The first ingredient of a stability condition on $\mathrm{D}^{\mathrm{b}}(\mathcal{Y})$ over C is again a *slicing*, i.e. a list $\mathcal{P}(\phi)$ of semistable objects of phase ϕ satisfying Harder-Narasimhan filtration; we require that each $\mathcal{P}(\phi)$ is invariant under tensoring with pull-backs of line bundles from C. Second, our central charge $Z\colon \mathrm{D}^{\mathrm{b}}(Y)_{C\text{-tor}} \to \mathbb{C}$ is defined for C-torsion objects, i.e. objects whose pull-back to the generic fiber over C vanishes. We require Z is constant in families, in the sense that for any object $F \in \mathrm{D}^{\mathrm{b}}(Y)$, the complex number $Z(F|_{\pi^{-1}(c)})$ is independent of the closed point $c \in C$. Finally, we require that stability is an open property in families.

We show the existence of such stability conditions over C in the same generality that the existence of stability conditions on the fibers can be proved in the framework of [BMT14]. It comes with proper relative moduli spaces of semistable objects, generalizing work by Piyaratne and Toda, [PT15], and it extends to the notion of stability on Kuznetsov categories, as for cubic fourfolds in the setup above.

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