

DUALITY AND S-THEORY

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This is an account of a systematization of certain parts of homotopy theory by means of the suspension category (also called the S-category). One of the most important applications is that a formal analogy becomes a rigorous duality in the S-category. We begin by summarizing some of the background material and results leading to the S-theory and duality.

1. Given topological spaces X and Y and two continuous mappings f_0 and f_1 from X to Y (denoted by $f_0, f_1: X \rightarrow Y$) we say that f_0 is *homotopic* to f_1 (denoted by $f_0 \simeq f_1$) if there exists a continuous family of continuous mappings $h_t: X \rightarrow Y$ for $0 \leq t \leq 1$ such that $h_0 = f_0$ and $h_1 = f_1$. Intuitively $f_0 \simeq f_1$ if the map f_0 can be continuously deformed into the map f_1 . It is easily seen [4; 7] that the relation of homotopy thus defined is reflexive, symmetric, and transitive and, therefore, partitions the set of continuous maps from X to Y into disjoint equivalence classes called *homotopy classes*. We denote the set of homotopy classes of mappings from X to Y by $[X, Y]$, and if $f: X \rightarrow Y$, then $[f]$ will denote the homotopy class of f . It is of fundamental importance in present day topology to determine the structure of $[X, Y]$. Specifically, we would like to have a method of determining whether two given mappings are homotopic, and also we would like to determine how many elements there are in $[X, Y]$. In many cases our information is so limited that we do not even know whether $[X, Y]$ contains more than one element. (In the above when X is contractible we assume Y is arcwise connected.)

It is clear that if either X or Y is a contractible space then there is exactly one homotopy class of maps from X to Y . In particular if either X or Y is a cell of any dimension, the structure of $[X, Y]$ is completely known as this set consists of a single element.

Perhaps the most natural spaces to consider after the cells are the spheres. We denote by S^n the unit sphere in a Euclidean space of $(n+1)$ dimensions. We want to discuss the homotopy classes $S^n \rightarrow Y$ and $X \rightarrow S^n$.

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2. Let us consider first the case where we map S^n into a space Y . We choose a base point $p \in S^n$ and a base point $y \in Y$ and restrict our attention to maps $f: S^n \rightarrow Y$ such that $f(p) = y$ and to homotopies h_t such that $h_t(p) = y$ for all $0 \leq t \leq 1$. Given two maps $f, g: S^n \rightarrow Y$ with $f(p) = g(p) = y$ we define a map

$$(f + g): S^n \rightarrow Y$$

with $(f+g)(p) = y$ to be the composite

$$S^n \xrightarrow{\Lambda} S^n \vee S^n \xrightarrow{f \vee g} Y$$

where $S^n \vee S^n$ denotes the subset $S^n \times p \cup p \times S^n$ of $S^n \times S^n$, Λ pinches an equator of S^n through p to a point and maps one hemisphere onto $S^n \times p$ and the other hemisphere onto $p \times S^n$, and $f \vee g$ maps $S^n \times p$ by f and $p \times S^n$ by g [4; 7; 9; 10]. It turns out that this “addition” of maps defines a group structure on the set of homotopy classes of maps of S^n into Y which take p into y , and this group is abelian if $n > 1$. The resulting group is denoted by $\pi_n(Y, y)$ and is called the *nth homotopy group* of Y with *base point* y . For $n = 1$ the group is also known as the *fundamental group* or *Poincaré group* of Y .

The group structure helps greatly in studying the structure of the set of homotopy classes from S^n to Y ; not that the solution is known in general but that many natural mappings between sets of homotopy classes are homomorphic in terms of this group structure, and these homomorphisms have provided a solution in some cases.

For example, a continuous map $f: X \rightarrow Y$ induces homomorphisms of the integral homology groups $f_*: H_p(X) \rightarrow H_p(Y)$ for every p and $f \simeq g$ implies $f_* = g_*$ [5]. Since $H_n(S^n)$ is infinite cyclic, choice of a generator $s \in H_n(S^n)$ determines a map $\phi: \pi_n(Y, y) \rightarrow H_n(Y)$ defined by $\phi[f] = f_* s$, and this map is a homomorphism between these two groups. If Y is arcwise connected and simply connected and if $H_i(Y) = 0$ for $0 < i < n$ (where $n > 1$), then $\pi_i(Y, y) = 0$ for $0 < i < n$ and ϕ is an isomorphism of $\pi_n(Y, y)$ onto $H_n(Y)$. This result, known as the *Hurewicz isomorphism theorem*, shows that the first nontrivial homotopy group of a simply connected space can easily be determined from its homology structure. It also follows that for such a space Y two maps $f, g: S^n \rightarrow Y$ are homotopic if and only if $f_* s = g_* s$, which can be regarded as a generalization of the theorem of Brouwer to the effect that two maps of S^n into itself are homotopic if and only if they have the same degree [3].

3. Consider next the case where we map a space X into S^n . If we assume dimension $X \leq 2n - 2$, it turns out that we can make

$[X, S^n]$ into an abelian group. Given two maps $f, g: X \rightarrow S^n$ we define a map $(f, g): X \rightarrow S^n \times S^n$ by $(f, g)(x) = (f(x), g(x))$. Let $S^n \vee S^n$ denote the subset $S^n \times p \cup p \times S^n$ of $S^n \times S^n$ as above and define

$$\Omega: S^n \vee S^n \rightarrow S^n$$

by $\Omega(z, p) = \Omega(p, z) = z$. Because of the dimension restriction on X the map (f, g) is homotopic to a map $h: X \rightarrow S^n \times S^n$ such that $h(X) \subset S^n \vee S^n$. Then the composite Ωh is a map of X into S^n representing the sum of $[f]$ and $[g]$. The definition of addition in this case is due to Borsuk [2; 12], and the resulting group is called the n th cohomotopy group of X and denoted by $\pi^n(X)$.

A continuous map $f: X \rightarrow Y$ induces homomorphisms

$$f^*: H^p(Y) \rightarrow H^p(X)$$

of the integral cohomology groups for every p and $f \simeq g$ implies $f^* = g^*$. Since $H^n(S^n)$ is infinite cyclic, a generator $s^* \in H^n(S^n)$ determines a homomorphism

$$\phi^*: \pi^n(X) \rightarrow H^n(X)$$

defined by $\phi^*[f] = f^*s^*$. If X is a complex and $H^i(X) = 0$ for $i > n$, then $\pi^i(X) = 0$ for $i > n$ and ϕ^* is an isomorphism of $\pi^n(X)$ onto $H^n(X)$. This theorem is known as the Hopf classification theorem [4; 8; 22]. As in the case of the Hurewicz isomorphism theorem, it can also be regarded as a generalization of Brouwer's theorem on the degree.

The Hurewicz isomorphism theorem and the Hopf classification theorem are examples of dual theorems. That is, if we interchange homotopy group with cohomotopy group, homology group with cohomology group, the map ϕ with the map ϕ^* , and lowest nontrivial dimension for homology with highest nontrivial dimension for cohomology, then one of them becomes the other (assuming simple connectedness for the Hurewicz theorem and a suitable dimension restriction for the Hopf theorem). The duality we seek is such as to make these two theorems dual. There are, of course, other theorems dual in the same sense. For example, the second nontrivial homotopy group from below is related to the homology groups [21] in a way dual to the way that the second nontrivial cohomotopy group from above is related to the cohomology groups [12; 18] (again if suitable dimension restrictions are imposed on the first nontrivial group in each theorem).

4. When we consider maps $S^m \rightarrow S^n$ the resulting set $[S^m, S^n]$ can be regarded as a homotopy group and, if $m \leq 2n - 2$, also as a cohomotopy group (the base point condition being unnecessary for the

homotopy groups of spheres because a sphere is simple in all dimensions [4; 7; 9]). The two group structures on $[S^m, S^n]$ defined when $m \leq 2n - 2$ turn out to be the same, and the groups $[S^m, S^n]$ are particularly important as the other groups of homotopy classes of mappings between complexes are built from them. A complete knowledge of the homotopy groups of spheres would be the major step in obtaining information about the homotopy classes of maps from one complex to another.

To study the homotopy groups of spheres Freudenthal [6] introduced the *suspension homomorphism* (*Einhängung*) $S: \pi_k(S^n) \rightarrow \pi_{k+1}(S^{n+1})$ defined as follows. We regard S^{k+1} as the join of S^k with a pair of points (a, b) ; that is, S^{k+1} is the set of line segments joining a point of an equator S^k to the two poles a and b . Similarly S^{n+1} is the join of S^n with a pair of points (a', b') . Given a map $f: S^k \rightarrow S^n$ its suspension $Sf: S^{k+1} \rightarrow S^{n+1}$ is defined to map S^k into S^n (regarded as equators of S^{k+1} , S^{n+1} , respectively) as f does, to map a into a' , b into b' , and a line segment from a point x of S^k to a (or b) linearly onto the line segment from $f(x)$ to a' (or b'). The map $f \rightarrow Sf$ induces the suspension homomorphism

$$S: \pi_k(S^n) \rightarrow \pi_{k+1}(S^{n+1}).$$

This homomorphism is of great importance in the study of the homotopy groups of spheres. For $k \leq 2n - 2$ it is an isomorphism onto [6; 7; 19]. It follows that for fixed d in the sequence

$$\pi_{n+d}(S^n) \xrightarrow{S} \pi_{n+1+d}(S^{n+1}) \xrightarrow{S} \dots \xrightarrow{S} \pi_{m+d}(S^m) \xrightarrow{S} \dots$$

all the maps S except for a finite number at the beginning are isomorphisms onto. Note that S is an isomorphism onto for the same range of values for which the cohomotopy group $\pi^n(S^k)$ is defined.

5. The suspension map can be extended to more general spaces. If X denotes an arbitrary space, SX will denote the join of X with an ordered pair of points. Then by a natural generalization of the map S defined above we define a map

$$S: [X, Y] \rightarrow [SX, SY]$$

called the *suspension map*. The isomorphism theorem for the suspension homomorphism for spheres has an extension to the more general case [13; 16] and states that if $\pi_i(Y) = 0$ for $i < n$ and if X is a complex with dimension $\leq 2n - 2$, then S is a 1-1 correspondence between $[X, Y]$ and $[SX, SY]$. In particular, it follows that for the range of values for which the cohomotopy groups are defined S is an isomorphism of

$\pi^k(X)$ onto $\pi^{k+1}(SX)$. On the other hand, it is not true for homotopy groups that the map $S: \pi_k(Y) \rightarrow \pi_{k+1}(SY)$ is always an isomorphism onto.

Hence, there are two difficulties to the immediate establishment of a duality interchanging homotopy groups with cohomotopy groups. First, the cohomotopy groups are only defined when certain dimension restrictions are satisfied, and secondly, the suspension map is not always an isomorphism onto for the homotopy groups. Since the dimension restriction in order that the cohomotopy group be defined is the same as the one for which the suspension map is a 1-1 correspondence, we might expect both difficulties to disappear if we could construct a system in which the suspension map is always a 1-1 correspondence.

6. In order to obtain a system in which the suspension map is always a 1-1 correspondence we pass to the limit under the suspension map. This leads us to the *suspension category* or *S-category* which we now describe. For arbitrary spaces X, Y consider the sequence

$$[X, Y] \xrightarrow{S} [SX, SY] \xrightarrow{S} \dots \xrightarrow{S} [S^k X, S^k Y] \xrightarrow{S} \dots$$

and define $\{X, Y\} = \lim_k [S^k X, S^k Y]$. The elements of $\{X, Y\}$ are called *S-maps*, and the *S-category* is the category [5] whose objects are topological spaces and whose mappings are *S-maps*. If $f: S^k X \rightarrow S^k Y$ for some $k \geq 0$, then $\{f\}$ will denote the *S-map* determined by f . It is clear from the definition that $\{X, Y\} = \{SX, SY\}$ so if we use *S-maps* instead of the usual homotopy classes we have a system in which the suspension map is always a 1-1 correspondence. It is also clear from the theorem stated in §5 that if $\pi_i(Y) = 0$ for $i < n$ and X is a complex with dimension $X \leq 2n - 2$, then $[X, Y]$ is in 1-1 correspondence with $\{X, Y\}$. For k sufficiently large it is known [1] that $[S^k X, S^k Y]$ can be given a law of composition so that it becomes an abelian group and $S: [S^k X, S^k Y] \rightarrow [S^{k+1} X, S^{k+1} Y]$ is a homomorphism. Therefore, $\{X, Y\}$ is an abelian group for every pair of spaces X, Y . Furthermore, composition defines a bilinear pairing of $\{X, Y\}$ and $\{Y, Z\}$ to $\{X, Z\}$.

7. Having defined the *S-category* we are ready to formulate the duality for *S-maps*. The *S-homotopy* groups will be dual to the *S-cohomotopy* groups. The examples of dual theorems quoted above will be consequences of the duality because in the range of values for which they hold the set of *S-maps* is in 1-1 correspondence with the set of usual homotopy classes. It is also clear that dual theorems in

the S -category give rise to dual theorems about ordinary homotopy classes for the range of values of dimension and connectedness assumptions so that $[X, Y]$ is in 1-1 correspondence with $\{X, Y\}$.

The duality is an extension of the Alexander duality which equates the p th homology group of a subset of S^n with the $n-p-1$ st cohomology group of its complement. We summarize its main properties below [15].

Given a subpolyhedron X of a triangulation of S^n we define a polyhedron $D_n X \subset S^n - X$ as n -dual to X if $D_n X$ is an S -deformation retract of $S^n - X$; that is, if the inclusion map $D_n X \subset S^n - X$ represents an S -map which is an S -equivalence. Any subpolyhedron of an n -sphere has an n -dual because if we triangulate S^n so that the subpolyhedron X is a complete subcomplex (every simplex of S^n having all its vertices in X belongs to X) then the set of simplexes of S^n with no vertex in X is an n -dual of X because every simplex of S^n is the join of a unique simplex of X and of a unique simplex with no vertex in X . It can be shown that if $D_n X$ is an n -dual of X , then X is an n -dual of $D_n X$ and $SD_n X$ is an $(n+1)$ -dual of X (where $SD_n X$ denotes that subset of S^{n+1} obtained by regarding S^n as an equator of S^{n+1} and taking the join of $D_n X$, which lies in this equator, with the poles of S^{n+1}).

The main theorem on duality follows. Given subpolyhedra $X, Y \subset S^n$ and n -duals $D_n X, D_n Y \subset S^n$ there exists a unique map

$$D_n \{X, Y\} \rightarrow \{D_n Y, D_n X\}$$

with the properties:

- (1) If $i: X \subset Y$ and $i': D_n Y \subset D_n X$, then $D_n \{i\} = \{i'\}$.
- (2) Given $\{f\} \in \{X, Y\}$, $\{g\} \in \{Y, Z\}$ and n -duals $D_n X, D_n Y, D_n Z$, then $D_n (\{g\} \{f\}) = D_n \{f\} D_n \{g\}$.
- (3) Using X, Y as n -duals of $D_n X, D_n Y$, respectively, we have maps

$$\{X, Y\} \xrightarrow{D_n} \{D_n Y, D_n X\} \xrightarrow{D_n} \{X, Y\},$$

and this composition is the identity.

- (4) D_n is a homomorphism.
- (5) Taking $SD_n X, SD_n Y$ as $(n+1)$ -dual to X, Y , then

$$SD_n = D_{n+1}.$$

- (6) Taking $D_n X, D_n Y$ as $(n+1)$ -dual to SX, SY , then

$$D_{n+1}S = D_n.$$

- (7) Commutativity holds in the diagram

$$\begin{array}{ccc} H_p(X) & \xrightarrow{f_*} & H_p(Y) \\ \mathfrak{D}_n \downarrow & & \downarrow \mathfrak{D}_n \\ H^{n-p-1}(D_n X) & \xrightarrow{(D_n f)^*} & H^{n-p-1}(D_n Y) \end{array}$$

where \mathfrak{D}_n denotes the Alexander duality isomorphism.

The first four properties assert that D_n is an isomorphism from $\{X, Y\}$ onto $\{D_n Y, D_n X\}$ with certain functorial properties. Properties (5) and (6) assert that this isomorphism doesn't depend on the integer n and is stable under suspension in the sense that complete commutativity holds in the diagram

$$\begin{array}{ccc} \{X, Y\} & \xrightarrow{D_n} & \{D_n Y, D_n X\} \\ S \downarrow & \searrow & \downarrow S \\ \{SX, SY\} & \xrightarrow{D_{n+2}} & \{SD_n Y, SD_n X\} \end{array}$$

where each of the diagonal maps is D_{n+1} .

As an example of this duality let us take $S^p \subset S^{p+q+1}$ and as $(p+q+1)$ -dual to S^p we can take a sphere $S_1^q \subset S^{p+q+1}$. Assuming $q \leq 2p - 2$ we know from the suspension theorem that

$$\{S_1^q, S^p\} \approx [S_1^q, S^p] = \pi_q(S^p).$$

Taking $D_{p+q+1}(S^p) = S_1^q$, $D_{p+q+1}(S_1^q) = S^p$, we get an automorphism of period two,

$$D_{p+q+1}: [S_1^q, S^p] \approx \{D_{p+q+1}S^p, D_{p+q+1}S_1^q\} \approx [S_1^q, S^p].$$

Hence, the duality defines an automorphism of period two in the stable (under suspension) homotopy groups of spheres, and this automorphism is itself stable. It need not be the identity. In fact, for $p=q$ it can be shown that $D_{2p+1}: [S_1^p, S^p] \approx [S_1^p, S^p]$ is given by $D_{2p+1}[f] = -[f]$. On the other hand, if $q=p+1$ and $p \geq 3$, then $[S_1^{p+1}, S^p]$ is a group of order two so D_{2p+2} is necessarily the identity on this group. Similarly for $q=p+2$ and $p \geq 3$, D_{2p+3} is the identity.

We define the S -homotopy and S -cohomotopy groups by

$$\Sigma_p(X) = \{S^p, X\}, \quad \Sigma^p(Y) = \{Y, S^p\}.$$

Since S^p and S^{n-p-1} are n -dual, it follows from the above properties that D_n is an isomorphism onto

$$D_n: \Sigma_p(X) \approx \Sigma^{n-p-1}(D_n X).$$

Analogous to the homomorphisms defined for the ordinary homotopy

and cohomotopy groups there are homomorphisms

$$\phi: \Sigma_p(X) \rightarrow H_p(X), \quad \phi^*: \Sigma^p(Y) \rightarrow H^p(Y).$$

It is a consequence of (7) and the other properties of the duality that commutativity holds in the diagram

$$\begin{array}{ccc} \Sigma_p(X) & \xrightarrow{\phi} & H_p(X) \\ D_n \downarrow & & \downarrow \mathcal{D}_n \\ \Sigma^{n-p-1}(D_n X) & \xrightarrow{\phi^*} & H^{n-p-1}(D_n X). \end{array}$$

In the *S*-category the Hurewicz isomorphism theorem asserts that if $H_i(X) = 0$ for $i < m$ (where $H_0(X)$ is taken to be the reduced group [5]), then $\Sigma_i(X) = 0$ for $i < m$ and $\phi: \Sigma_m(X) \approx H_m(X)$. Similarly in the *S*-category the Hopf classification theorem asserts that if Y is a polyhedron with $H^i(Y) = 0$ for $i > m$, then $\Sigma^i(Y) = 0$ for $i > m$ $\phi^*: \Sigma^m(Y) \approx H^m(Y)$. Both theorems as thus stated are stable, and we note that from the commutativity of the last diagram above D_n takes one theorem into the other, which was one of the features we desired in trying to find the duality.

To define the map $D_n: \{X, Y\} \rightarrow \{D_n Y, D_n X\}$ note that (1) defines it in the case that we are considering an inclusion map (and the duals satisfy a reverse inclusion). (1) and (2) show that if $Y \subset X$ and the inclusion map $i: Y \subset X$ is an *S*-equivalence and if $i': D_n X \subset D_n Y$, then i' is an *S*-equivalence and

$$D_n(\{i\}^{-1}) = (D_n\{i\})^{-1} = \{i'\}^{-1}.$$

It can be shown directly that i' is an *S*-equivalence when i is and then the above equation defines D_n on $\{i\}^{-1}$. By using the concept of a mapping cylinder any map can be seen to be homotopic to a map which is a composition of inclusion maps and *S*-retractions by deformation (i.e. *S*-inverses of inclusion maps). The remarks above and (2) show how to define D_n for such a composition. D_n is then defined for an arbitrary map by noting that if it is defined for a homotopic map which is a composition of inclusion maps and *S*-retractions by deformation then the resulting map doesn't depend on the choice of the representation as a composition of such maps. The properties (1)–(7) are then established in straightforward fashion.

8. If we consider spaces which are not subpolyhedra of spheres, there is a duality for them if they are of the same *S*-homotopy type as subpolyhedra of spheres. We restrict attention to CW complexes [20], and we shall consider maps $\xi: X \rightarrow X'$ which are *S*-equivalences

of the CW complex X onto some subpolyhedron X' of S^n . We say that X , ξ and X^* , ξ^* are *weakly n-dual* if ξ , ξ^* are S-equivalences of X , X^* onto n -dual subpolyhedra of S^n . The duality D_n can be extended to weakly n -dual complexes in an obvious manner. The importance of this generalization lies in the fact that it enables us to dualize the concept of adjoining a cell to a complex by a given map. The basic result in this direction is that if $f: X \rightarrow Y$ is weakly n -dual to $f^*: Y^* \rightarrow X^*$ and if $Z = CX \cup Y$ is the space obtained by taking a cone over X disjoint from Y and identifying $x \in X$ with $fx \in Y$ and Z^* is similarly defined, then Z and Z^* are weakly $(n+1)$ -dual in a natural way [15].

This last result can be used to extend the duality to relative theory [17]. To any CW complex X there corresponds, for sufficiently large n , a CW complex X^* and an anti-isomorphism $\sigma \leftrightarrow \sigma^*$ of the cells of X with those of X^* such that if A is a subcomplex of X and A^* denotes the subcomplex of X^* of cells which correspond to the cells of $X - A$, then the complexes A and $X^* \cup CA^*$ are weakly n -dual in such a way that the natural maps $A \rightarrow X \rightarrow X \cup CA$ are weakly n -dual to the natural maps $X^* \cup CA^* \leftarrow X^* \leftarrow A^*$. It follows that the S -homotopy exact couple [11] of X is isomorphic to the S -cohomotopy exact couple of X^* so that theorems about these exact couples can also be dualized.

If $\phi: X \rightarrow Y$ is a cellular carrier [13; 16] and X^* , Y^* are weakly n -dual complexes as in the above, there is defined in a natural way a dual carrier $\phi^*: Y^* \rightarrow X^*$. Then the relative theorem [17] for D_n asserts that D_n maps the S -homotopy classes carried by ϕ (denoted by $\{\phi\}$) isomorphically onto $\{\phi^*\}$. This duality converts an obstruction to extending a continuous map [4] into an obstruction to compressing a continuous map into a subset [14; 16].

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