Separatrix Splitting for the Extended Standard Family of Maps

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Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text. I declare that the work has not been submitted for any other degree or professional qualification except as specified.

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Abstract

This thesis presents two dimensional discrete dynamical system, the extended standard family of maps, which approximates homoclinic bifurcations of continuous dissipative systems. The main subject of study is the problem of separatrix splitting which was first discovered by Poincaré in the context of the n-body problem.

Separatrix splitting leads to chaotic behaviour of the system on exponentially small region in parameter space. To estimate the size of the region the dissipative map is extended to complex variables and approximated by differential equation on a specific domain. This approach was proposed by Lazutkin to study separatrix splitting for Chirikov's standard map.

Furthermore the complex nearly periodic function is used to estimate the width of the exponentially small region where chaos prevails and the map is related to the semistandard map. Numerical computations require solving complex differential equation and provide the constants involved in the asymptotic formula for the size of the region.

Another problem studied in this thesis is the prevalence of resonance for the dissipative standard map on a specific invariant set, which for one dimensional map corresponds to a circle. The regions in parameter space where periodic behaviour occurs on the invariant set is known as Arnold tongues. The width of Arnold tongue is studied and numerical results obtained by iterating the map and solving differential equation are related to the semistandard map.

Contents

Chapter 1

Introduction

In this thesis we study qualitative properties of the *extended standard family* of maps depending on three parameters which is motivated by its universal and extremely rich dynamics.

The extended standard map

$$\begin{aligned} x_{n+1} &= Jx_n + \omega + \epsilon \sin\theta_n \\ \theta_{n+1} &= \theta_n + x_{n+1} \end{aligned} \tag{1.1}$$

can be considered on a plane or on a cylinder $\mathbb{R} \times \mathbb{S}^1$, where $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$. The map has a constant Jacobian J and is a diffeomorphism provided $J \neq 0$. We consider the region in parameter space where ω and ϵ are close to 0 ($|\omega| \leq \epsilon$) and J is close to 1 so that the map is a perturbation of identity. If $J < 1$ the map is dissipative (area contracting) and has two hyperbolic points including one saddle, which collapse to one point when $\omega = \epsilon$. For $J = 1$ the map is area preserving and has a nonhyperbolic fixed point and a saddle point.

As after appropriate change of coordinates or parameters scaling classical maps including Chirikov's map, Hénon and whisker map [3] (p. 669) and twist map [2] (p. 349) are limiting or special cases of the extended standard map, it represents a reference system for two dimensional maps and hence plays a significant role in the field of discrete dynamical systems. The extended standard map can be also considered as a perturbation of the family of *Arnold circle map* studied in [1] and therefore it is also known as the *fattened Arnold map.*

The map exhibits a myriad of dynamically interesting behaviours; depending on the regions of the three parameters in the parameter space, it transitions from regular behaviour (existence of invariant circle) to chaotic behaviour (existence of a strange attractor).

A comprehensive study of the dissipative standard map can be found in Broer *et al* [3] (section 4). It also provides a complete classification of generic types of dynamics inherent for two dimensional maps including local bifurcations, such as period doubling and cusps, homoclinic bifurcations, in particular transversal intersection and tangencies of invariant manifolds of a saddle point, existence of globally attracting invariant circle, its destruction and eventually transition to chaos leading to existence of strange attractors.

1.1 Subject of Study

This thesis is devoted to study two types of behaviour of the extended standard map: the phenomenon of resonance zones, which originates from Arnold [1] and transversal intersection of invariant manifolds of a fixed point known as separatrix splitting which was discovered by Hénri Poincaré [31].

The problem of resonance zones first originated in the case of one dimensional map Arnold circle map [1]. It can be extended to two and more dimensions when the dynamics of the map can be restricted to an invariant circle. For a diffeomorphism f the resonance zone is associated with existence of an orbit of period p of the map f^q i.e. when the map is topologically equivalent to the rigid rotation, which is the case if and only if the rotation number is equal to $\frac{p}{q}$. Rotation number is also known as Poincaré number as it was introduced by Poincaré [30] and we provide its definition in Chapter 5. When ϵ tends to 0 the regions of resonance zone appear as a narrowing tongues, thus are called *Arnold tongues.*

Another phenomenon studied in this thesis involves separatrix splitting regions. Over one hundred twenty years ago unintentional assumption that this phenomenon is not possible led Poincaré to false conclusion on existence of the stable solution of n-body problem. He submitted his work for a competition and did win the prize despite the error made in his work. In the attempt to correct the mistake Poincaré published results that set foundations for future developments in the field of deterministic chaos.

Separatrices splitting has been studied for discrete systems including the standard map [28] and analytic Hamiltonian maps [18] and also for continuous systems: quasi-periodic high-frequency perturbation of three dimensional pendulum [10], rapidly forced system with a homoclinic orbit [24] and high-frequency time-periodic perturbation of a Hamiltonian system [14], where the semistandard map is used as a complex reference system. An asymptotic formula for the angle of separatrix splitting for the standard map can be found in [17] and the coefficients of the asymptotic expansion are calculated numerically. A thorough survey for separatrix splitting phenomenon for area preserving maps and Hamiltonian systems can be found in Gelfreich and Lazutkin [15].

Separatrices splitting leads to complex behaviour involving existence of invariant set topologically conjugate to horseshoe. In general for a diffeomorphism f of a plane with a saddle fixed point which stable and unstable manifolds intersect transversally, there is a set invariant under f^n topologically conjugate to a horseshoe, which is a source of existence of infinitely many periodic orbits [21] (section 6.6).

Extended standard map is a model system for a return map near homoclinic tangency [3] (p. 746). Consider Hamiltonian flow with a homoclinic loop. To model a return map of homoclinic tangency a perturbation of time-ϵ map of the flow is considered which gives an exponentially small splitting of separatrices. The perturbation involves small conservative perturbation leading to separatrix splitting and non conservative 'pull up' of the unstable manifold relative to the stable manifold. The stable and unstable manifolds are parametrised and the return map is constructed as a set of transformations into coordinates such that the resulting map is the extended standard map (1.1).

The general method to study dynamics of the dissipative standard map presented in this thesis is to consider ϵ close to 0, so the map (1.1) can be approximated by the flow and to study intervals in parameter space where Arnold tongues are present or homoclinic intersection appears, depending whether we fix J or let it vary.

We extend the map to a complex domain and approximate its orbits by solutions of complex ordinary differential equation. In the region where the approximation breaks due to existence of a singularity of the invariant manifold of the flow the map is approximated by the discrete system. Matching both approximations leads to exponential smallness of estimates of the interval. This approach was introduced in Lazutkin [28] for the standard map and used to measure Arnold tongues in [11] for the dissipative standard map and in Davie [9] for the sine circle map.

The method presented in [28] used to study a chaotic zone for a dissipative map can be found in Gelfreich [13]. Results of this paper include the Hénon map and are similar to those in Chapter 4, although the proof of the asymptotic formula is not complete. An accurate and direct numerical test of final formula for the chaotic zone for the Hénon map can be found in [19].

The method of extending the map to complex domain was also used to study fractal structures of fern type for the standard and semistandard map [16], where the asymptotic behaviour of the coefficients of the unstable manifold expansion (3.10) was conjectured. In [36] the conjecture was proved and relation between the constant involved in the angle of splitting and the constant involved in asymptotic formula for coefficients of the unstable manifold was given.

1.2 Structure of the Thesis

This thesis is composed of six chapters and the Appendix. In Chapter 2 we prove introductory results relating a complex map to its approximating flow. The approach is opposite to Euler approximation when we use a discrete dynamical system to solve continuous system numerically. We provide construction of the approximating flow and discuss the existence and parametrisation of the complex invariant manifolds of a fixed point for maps and flows.

In Chapter 3 we consider the semistandard map and present an alternative approach to the one used in [28] to study the width of separatrix splitting for the standard map. This approach has been proposed in [6] and involves approximation of the standard map by the semistandard map, a construction of a nearly periodic analytic function in a region of the complex plane and measure of its oscillation. The benefit of this approach comes from the fact that it can be extended to non area preserving maps. We improve the accuracy of estimates in [6] and provide numerical results which are consistent with those in Gelfreich [20].

In Chapter 4 we consider the map (1.1) with fixed parameter ϵ close to 0 and we vary J. The parameter ω is taken to ensure existence of two fixed points of the map (1.1), $\omega = a\epsilon$ where $|a| < 1$. The aim is to measure the width of the interval I_ω in J where the separatrix splitting occurs.

We extend the map to complex domain, approximate its orbit by the solution of a complex flow in a region in a complex plane and by the semistandard map outside the region and match the two approximation. The main part of the chapter is the proof of existence and construction of the nearly periodic function analytic on a specific region in the complex plane and estimation of its oscillations, see Theorem 4.7.2. To measure the exponentially small region in parameter space where the separatrix splitting occurs we estimate oscillations of nearly periodic function and give numerical results which match with [20].

Main result of Chapter 4, Theorem 4.8.1 provides the width of the interval where the homoclinic intersection appears

$$\lim_{\epsilon \to 0} e^{2\pi\sqrt{\epsilon}^{-1}\Im A_a}\sqrt{\epsilon}|I_\omega| = \Lambda|\mu_0|$$

where constants A_a and Λ depend on a, and μ_0 is related to the standard map [28].

In Chapter 5 we consider the extended standard map with fixed J, varying ϵ and ω and study the width of Arnold tongue, where there is periodic orbit, which corresponds to a rational rotation number of the map. The method to estimate the resonance zone follows the approach demonstrated in [11], whose object of study is the width of resonance zone for the Arnold circle map and the map restricted to the invariant circle. We extend the map to the complex domain and seek a periodic function analytic in a region in $\mathbb{C}$. The function is expanded as a Fourier series and its oscillation is estimated by first Fourier coefficient, which we then use to estimate the width of Arnold tongue. The approach to construct the periodic function is described in more detail in the next section, where we take a simple example as illustration.

Furthermore we extend results of [11] (Chapter 4) and provide the asymptotic behaviour of the width of resonance zone in a limiting case as $J \to 1$. We do not prove results rigorously, we use a heuristic argument followed by numerical verification. To obtain numerical results we approximate the first Fourier coefficient by the function measuring the time on the invariant manifold. Numerical results are consistent with [9], when the dissipative standard map collapses to one dimensional case and with [20], when the map folds to the standard map.

Main result of Chapter 5, Theorem 5.4.1, gives limiting width of the Arnold tongue in a weakly dissipative case and relating it to the constant present in the estimation of separatrix splitting angle for the standard map [29].

In summary in this thesis we provide estimates for oscillations of invariant manifolds in phase space and use this to get results for widths of regions in parameter space.

In the final Chapter 6 we provide topics for potential future research, which in particular could be a proof of the conjecture assumed in Chapter 4 or involve application of the methods demonstrated here to a more general class of maps, for example to the map with non constant Jacobian. We conclude this thesis with the Appendix, where for completeness we include programs used to obtain numerical results.

1.3 One Dimensional Example

We detail the method explained above by applying it to study oscillations of the complex map in the neighbourhood of the fixed point.

Consider the map

$$f(z) = z^2 + z \tag{1.2}$$

and choose $z_0 \in \mathbb{C}$ close to the origin such that $\Im z_0 > 0$. A more in depth study of this map can be found in [12] and we shall see that results presented in this section are consistent with those in [12].

An example orbit of the map (1.2) is shown in Figure 1.1. The height of the

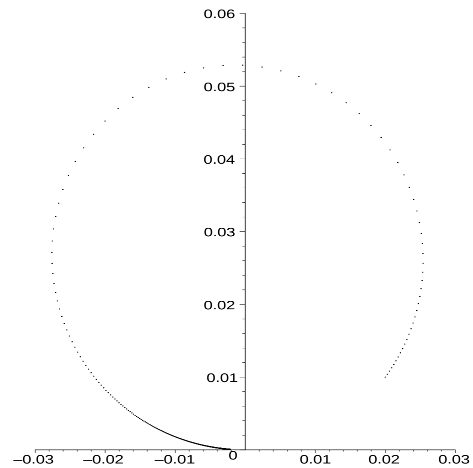

Figure 1.1: Iterates of the map (1.2)

'circle' in Figure 1.1 depends on the value of $\Im z_0$.

First we define a function

$$\varphi(z) = -z^{-1} + \ln z - \frac{1}{2}z + \frac{1}{3}z^2 \tag{1.3}$$

and note that it satisfies

$$\varphi(f(z)) - \varphi(z) - 1 = \mathcal{O}(|z|^4) \tag{1.4}$$

Denote $z_n = f^n(z_0)$, by similar argument to the one used in [11] (Section 4.3) one can show that the following limits

$$\begin{aligned} g(z_0) &= \lim_{n\to-\infty} (\varphi(z_n) - n) \\ h(z_0) &= \lim_{n\to+\infty} (\varphi(z_n) - n) \end{aligned} \tag{1.5}$$

are well defined hence one can define an analytic function

$$\sigma(w) = h\left(g^{-1}(w)\right) - w \tag{1.6}$$

in the half plane $\{\Im w > c\}$ for a positive constant c. For $w = g(z)$ we obtain

$\sigma(w) = h(z) - g(z)$. For constants $b, b' > 0$ large enough the curves $\Im g(z) = b$ and $\Im h(z) = b'$ are invariant under the iteration of the map (1.2). The curves meet if and only if there exist real v, such that $\Im\sigma(v + ib) = b' - b$. For z, such that $\Re z > 0$, the curve $\Im g(z) = b$ develops a wiggle, when approaching the origin from the left, as illustrated in Figure 1.2.

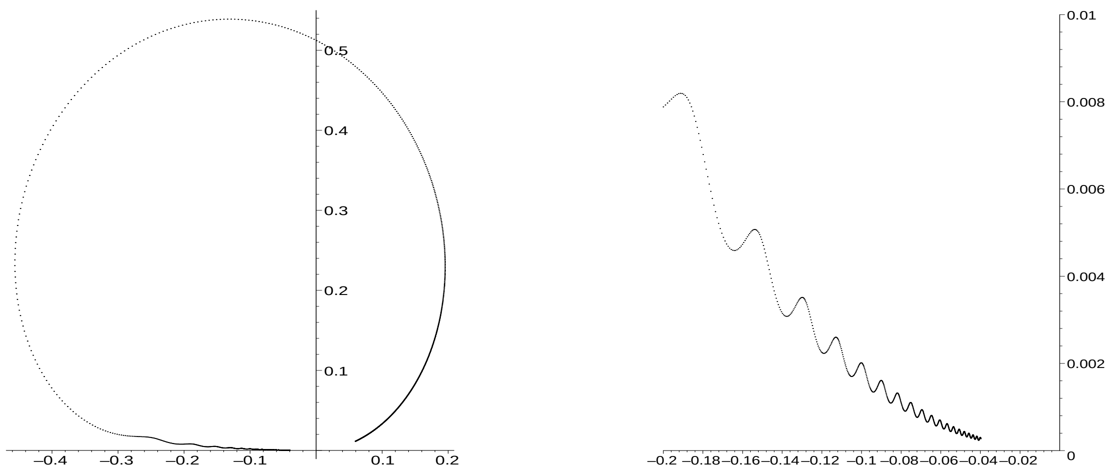

Figure 1.2: Wiggle of the curve $\Im g(z) = 3.3$ and its magnification

Similar graph can be found in [12] (Fig. 2, $\Im g(z) \in [0.05, 3]$). The aim is to measure the size of the wiggle in Figure 1.2. From definitions (1.5) it follows that

$$g(f(z)) = g(z) + 1 \tag{1.7}$$
$$g^{-1}(w + 1) = f(g^{-1}(w)) \tag{1.8}$$
$$h(f(z)) = h(z) + 1 \tag{1.9}$$

hence $\sigma(w)$ is a function of period 1 it can be expanded as a Fourier series

$$\sigma(w) = \sum_{k=1}^{\infty} \sigma_k e^{2\pi i k w}.$$

First Fourier coefficient plays a dominant role in measuring oscillation of σ and hence the wiggle in Figure 1.2. We outline the approach to measure the wiggle below.

Define

$$\begin{aligned} M(b) &= \max_{v\in\mathbb{R}} \Im\sigma(ib + v) \\ m(b) &= \min_{v\in\mathbb{R}} \Im\sigma(ib + v) \end{aligned}$$

The size of the wiggle is determined by the difference

$$M(b) - m(b) = \max_{v\in\mathbb{R}} \sum_{k=1}^{\infty} \sigma_k e^{-2\pi k b} \Im e^{2\pi i k v} - \min_{v\in\mathbb{R}} \sum_{k=1}^{\infty} \sigma_k e^{-2\pi k b} \Im e^{2\pi i k v}$$

which for a large b can be estimated by

$$W = 2|\sigma_1|e^{-2\pi b} + \mathcal{O}(e^{-4\pi b})$$

where σ_1 is the first Fourier coefficient

$$\sigma_1(\zeta) = \int_{ib}^{ib+1} \sigma(\zeta)e^{-2\pi i\zeta}d\zeta \tag{1.10}$$

which is well defined for large b.

We constructed the function, which is analytic and 1-periodic, and we explained the approach to approximate the size of the wiggle in Figure 1.2 by its first Fourier coefficient.

To compute σ_1 numerically we approximate the integral (1.10) by the sum

$$\sigma_1 = \frac{1}{N}\sum_{j=0}^{N-1} \sigma\left(\zeta_i\right) e^{-2\pi i\zeta_i}. \tag{1.11}$$

We find z satisfying

$$\varphi(z) = \zeta - n \tag{1.12}$$

and approximate

$$\sigma(\zeta_i) = \varphi\left(f^{2n}\left(z\right)\right) - \zeta_i - n + \mathcal{O}(n^{-3}) \tag{1.13}$$

where φ is defined in (1.3) and measures the time change on the orbit of the map (1.2).

In order to minimise approximation error in the estimation of W we need to choose optimal values of parameters involved in numerical computations. We choose b large enough to minimise the error in (1.10), large N to give a good approximation to the integral (1.11), sufficient number of iterations m to find z satisfying (1.12) and giving a small error in (1.4) and large n to minimise the error in (1.13). The program `sigma_1.c` which calculates the constant σ_1 for a range of values $b = \Im w$ is included in the Appendix and the results are presented in Table 1.1 with accuracy to two decimal places. Note that they are consistent with the results in [12] (Table 1), where $|\sigma_1|$ and $\arg\sigma_1$ for different values of b have been given.

The same type of phenomenon occurs for iterates of the map ze^z and following the method described above to measure the oscillation would enable to obtain the result for the width of Arnold tongues [9], which involves a constant related to the constant $E(0)$ in Chapter 5.

In subsequent chapters we study the maps with a higher degree of complexity than example (1.2), but the method summarised above is a central theme in obtaining results throughout this thesis. We define analogues of function g^{-1} satisfying (1.8) exactly and of h either increasing by 1 (satisfying (1.9)) or (almost) constant on the iterates of the map

$$h(f(z)) = h(z). \tag{1.14}$$

Table 1.1: Numerical results of sigma_1.c program

b	$\|\sigma_1\|$	$\Im\sigma_1$	$\Re\sigma_1$
3.2	22350579.22	3741603.54	-22035171.76
3.3	22350579.22	3741603.54	-22035171.76
3.4	22350579.22	3741603.54	-22035171.76
3.5	22350579.22	3741603.54	-22035171.76
3.6	22350579.22	3741603.54	-22035171.76
3.7	22350579.22	3741603.53	-22035171.76
3.8	22350579.22	3741603.53	-22035171.76
3.9	22350579.22	3741603.53	-22035171.76

In Chapter 5 we construct periodic functions, analogues of g and h satisfying (1.7) and (1.9) exactly and measure resonance region with the oscillation of periodic function analogous to σ defined in (1.6).

In chapters 3 and 4 we construct a parametrisation of the unstable manifold as an analogous to g^{-1} satisfying (1.8) exactly. We apply an analogue of function h denoted as F in Chapter 3 and as H_ν in Chapter 4 satisfying (1.14) only approximately. This gives us the construction of the nearly periodic function, an analogue of function σ defined in (1.6), and we estimate its oscillation to measure the separatrix splitting region.

As σ is periodic or nearly periodic it can be expanded as a Fourier series and its coefficients decay exponentially in $\Im w$, hence its first Fourier coefficient measures the width of the region and the estimate is exponentially small.

Similar approach can be found in the literature, in the study of one dimensional circle map [9] an analogue of function h satisfying (1.9) is defined. In the study of two dimensional maps in [11] analogous of functions h and g^{-1} are defined satisfying conditions (1.9), (1.8) exactly and we summarise those results in Chapter 5.

In the study of the standard map analogues of h and g^{-1} called analytic integrals are first conjectured in [28] and their existence is proved in [26]. In the study of the standard and the semistandard maps [6] the analogue of g^{-1} satisfying (1.8) exactly is used to parametrise the unstable manifold, the analogue of h is almost constant and satisfies (1.14) approximately. This leads to construction of the nearly periodic function, an analogue of (1.6) which motivates results in chapters 3 and 4. The problems are more complex than the simple example above, but the general idea is similar.

Chapter 2

Preliminary Results

This chapter contains auxiliary results relating a complex discrete system arising from iterations of the map to the continuous system given by solution of the ordinary differential equation of the same dimension. We use these results in Chapter 4 to study the splitting of separatrices of the dissipative map. We approximate behaviour of the dissipative map on a complex domain with the continuous system. As in subsequent chapters we study dynamics of the complex map and flow on the invariant manifolds of fixed points we discuss the existence and basic properties of stable, unstable and centre manifolds.

2.1 General Results for a Map and its Approximating Flow

In this section we study a map on a complex domain with certain properties and we relate the iterates of the map to the solution of the continuous system which gives an approximation to the map to any given order.

The results below provide construction of the continuous system approximating the map. In the real case results of this sort are fairly standard. We have not found explicit references but in their application the results below are similar to averaging [34] when we reduce dimension of the continuous dynamical system by considering its Poincaré map and approximating it by the continuous system of a lower dimension.

Another related problem to the results presented in this section explores asymptotic expansions of truncation error of discretisation and its applications, in particular to ordinary and partial differential equations, and can be found in [35]. Proposition 2.1.2 is similar to normal forms (see Theorem 3.3.1 in [21]).

We consider complex map that is a perturbation of identity and construct the flow which solutions give global approximation to the iterates of the map to a given order of the perturbation.

Proposition 2.1.1. *Suppose n is a positive integer, $K > 0$, $\mathcal{D} \subset \mathbb{C}^d$ is a domain, $\mathcal{D}'$ is a relatively compact subset of $\mathcal{D}$. Let $x \to x + \delta f(x) = \mathcal{F}(x) : \mathcal{D} \to \mathbb{C}^d$ be a map, such that $\mathcal{D} \cap \mathbb{R}^d \to \mathbb{R}^d$. Assume that f is analytic and bounded $|f(x)| < K$ on $\mathcal{D}$. Then there exist $M > 0$ and $\delta_0 > 0$ depending on K, $\mathcal{D}$ and $\mathcal{D}'$, there* *exist analytic functions $\psi_i : \mathcal{D} \to \mathbb{C}$ such that for $\delta < \delta_0$ and for $x \in \mathcal{D}'$ the flow $\dot{x} = G(x) = \sum_{i=1}^{n} \delta^i \psi_i(x)$ satisfies*

$$|H(x) - \mathcal{F}(x)| \leq M\delta^{n+1}$$

where $H(x)$ is the time-one map of the flow.

Proof. We sketch the key steps of the proof. Let $x(t) = (x_1(t), \ldots, x_d(t))$ be a solution of the flow, $x(0) = x_0$, $x_j(0) = x_{j0}$ for $j = 1, \ldots, d$. We construct functions ψ_{ji}, such that the flow

$$\dot{x}_j(t) = \sum_{i=1}^{n} \delta^i \psi_{ji}(x_1(t), \ldots, x_d(t)) \quad \text{for} \quad j = 1, \ldots, d \tag{2.1}$$

has the required properties. First we expand the solution of (2.1) in powers of δ

$$x_j(t) = x_{j0} + \sum_{k=1}^{n} \delta^k x_{jk}(t) + \mathcal{O}(\delta^{n+1})$$

for $j = 1, \ldots, d$. Differentiate both sides

$$\dot{x}_j(t) = \sum_{k=1}^{n} \delta^k \dot{x}_{jk}(t) \tag{2.2}$$

and substitute $x_j(t)$ into the differential equation (2.1)

$$\dot{x}_j(t) = \sum_{i=1}^{n} \delta^i \psi_{ji} \left(x_{10} + \sum_{k=1}^{n} \delta^k x_{1k}(t), \ldots, x_{d0} + \sum_{k=1}^{n} \delta^k x_{dk}(t) \right)$$

We expand ψ_{ji} in the Taylor series around x_0, and obtain

$$\dot{x}_j(t) = \sum_{k=1}^{n} \delta^k \sum_{|\alpha| \geq 0} \frac{D^\alpha \psi_{jk}(x_0)}{\alpha!} \left(\sum_{l=0}^{n} \delta^l x_l(t) \right)^\alpha + \mathcal{O}(\delta^{n+1}) \tag{2.3}$$

for a d dimensional multi-index α, that is $\alpha = (\alpha_1, \ldots, \alpha_d)$, $|\alpha| = \alpha_1 + \cdots + \alpha_d$, $y^\alpha = y_1^{\alpha_1} \ldots y_d^{\alpha_d}$ and $D^\alpha = (\frac{\partial}{\partial x_1})^{\alpha_1} \ldots (\frac{\partial}{\partial x_n})^{\alpha_n}$.

Now we equate equations (2.2) and (2.3) and compare terms of order δ^k. For terms of δ, δ^2 and δ^k we obtain

$$\begin{aligned}
\dot{x}_{j1}(t) &= \psi_{j1}(x_0) \\
\dot{x}_{j2}(t) &= \psi_{j2}(x_0) + \sum_{i=1}^{d} \frac{\partial \psi_{j1}(x_0)}{\partial x_i} x_{i1}(t) \\
&\vdots \\
\dot{x}_{jk}(t) &= \psi_{jk}(x_0) + \mathcal{P}
\end{aligned}$$

where $\mathcal{P}$ is a polynomial in $x_{il'}(t)$ and ψ_{jl} and its derivatives at x_0, for $l, l' < k$,

$i = 1, \dots, d$. We can integrate the system of nd ordinary differential equations $\dot{x}_{jk}(t)$, $x_{jk}(t)$ is a polynomial in t. We obtain $x_{jk}(1) = \psi_{jk}(x_0) + \mathcal{P}'$, where $\mathcal{P}'$ is a polynomial in ψ_{il} and its derivatives at x_0, for $l < k$.

Then we can solve inductively for $\psi_{jk}(x_0)$, $j = 1, \dots, d$, $k = 1, \dots, n$ to satisfy $x(1) = \mathcal{F}(x_0)$, and substitute $\psi_{jk}(x_0)$ back to the flow G. The solution of the flow (2.1) approximates the solution of flow G up to an order of δ^{n+1}.

There exist $\delta_0 > 0$ depending on n, K, $\mathcal{D}$ and $\mathcal{D}'$, such that for $\delta < \delta_0$ the solution of the flow G does not escape from a domain $\mathcal{D}'' : \overline{\mathcal{D}'} \subset \mathcal{D}'' \subset \mathcal{D}$. We apply the Cauchy inequalities [4] (p. 43) to the functions ψ_{jk} and conclude that for x in $\mathcal{D}'$ $|H(x) - \mathcal{F}(x)| \leq M\delta^{n+1}$, where M depends on n, K, $\mathcal{D}$ and $\mathcal{D}'$. $\square$

We consider the map in the neighbourhood of the fixed point and show that we can find the flow that approximates the map locally. We expand the flow at the fixed point and show that it is close to the expansion of the global flow. To simplify notation we assume that the fixed point is at the origin as the proposition below can be generalised to a flow with a fixed point elsewhere by shifting the fixed point.

Proposition 2.1.2. *Let $x \to \mathcal{F}(x)$ be d dimensional analytic map on the neighbourhood $D \in \mathbb{C}^d$ of 0. Assume that $x(0) = 0$ is a fixed point of the map and that the Jacobian of $\mathcal{F}(x)$ at the fixed point has all eigenvalues positive. Then*

i) There exists a constant L depending on D and bounds on $\mathcal{F}$ and reciprocals of eigenvalues of the Jacobian of $\mathcal{F}$ at the origin, there exists the flow $\dot{x} = g(x)$, such that $g(x) = \sum_{i_1+\dots+i_d=1}^{m} b_{i_1\dots i_d} x_1^{i_1} \dots x_d^{i_d}$ for constants $b_{i_1\dots i_d}$, $i_j \geq 0$, $i_1 + \dots + i_m = 1, \dots, m$ and the time-one map H_g of the flow satisfies

$$|H_g(x) - \mathcal{F}(x)| \leq L|x|^{m+1} \tag{2.4}$$

for $x \in D'$, where D' a relatively compact subset of D.

ii) If there exists a flow $\dot{x} = g_1(x)$, such that $g_1(x) = \sum_{i_j=1}^{m} \tilde{b}_{i_1\dots i_d} x_1^{i_1} \dots x_d^{i_d}$ and its time-one map H_{g_1} satisfies

$$|H_{g_1}(x) - \mathcal{F}(x)| < \varepsilon \tag{2.5}$$

for $x \in D'$ and $\varepsilon > 0$ then

$$|\tilde{b}_{i_1\dots i_d} - b_{i_1\dots i_d}| \leq M\varepsilon \tag{2.6}$$

for a constant M depending on D, bounds on $\mathcal{F}$ and reciprocals of eigenvalues of the Jacobian of $\mathcal{F}$ at the origin.

Proof. i) We write the map $\mathcal{F}$ in the coordinates, such that the Jacobian is a lower triangular matrix. We expand the map around the fixed point

$$\begin{aligned}\mathcal{F}(x_1, \dots, x_d) = (&a_{11}x_1 + \sum_{i_1+\dots+i_d=2}^{m} a_{1i_1\dots i_d} x_1^{i_1} \dots x_d^{i_d}, \dots, \\ &\sum_{i=1}^{d} a_{di}x_i + \sum_{i_1+\dots+i_d=2}^{m} a_{di_1\dots i_d} x_1^{i_1} \dots x_d^{i_d}) + \mathcal{O}(|x|^{m+1})\end{aligned} \tag{2.7}$$

It follows from the assumption on the Jacobian, that $a_{ii} > 0$ for $i = 1, \ldots, d$. We find constants b_{ij}, $b_{ii_1,\ldots,i_d}$ for $i, j = 1, \ldots, d$ such that the time-one map of the flow

$$\dot{x}_i = \sum_{j=1}^{i} b_{ij} x_j + \sum_{i_1+\cdots+i_d=2}^{m} b_{ii_1\ldots i_d} x_1^{i_1} \ldots x_d^{i_d} \tag{2.8}$$

satisfies (2.4). Let $u = (u_1, \ldots, u_d)$, such that $u \in D$. Denote $(u_1, \ldots, u_d) \to H(u_1, \ldots, u_d)$ as the time-one map of the flow (2.8). Consider the solution of the flow, $(x_1(t), \ldots, x_d(t))$, such that $(x_1(0), \ldots, x_d(0)) = (u_1, \ldots, u_d)$ and expand it

$$x_i(t) = \sum_{j=1}^{i} c_{ij}(t) u_j + \sum_{i_1+\cdots+i_d=2}^{m} c_{ii_1\ldots i_d}(t) u_1^{i_1} \ldots u_d^{i_d} + \text{h.o.t.} \tag{2.9}$$

Substitute the expansion to the differential equation (2.8)

$$\begin{aligned} \dot{x}_i = & \sum_{j=1}^{i} b_{ij} \left(\sum_{k=1}^{j} c_{jk}(t) u_k + \sum_{j_1+\cdots+j_d=2}^{m} c_{jj_1\ldots j_d}(t) u_1^{j_1} \ldots u_d^{j_d} \right) \\ & + \sum_{i_1+\cdots+i_d=2}^{m} b_{ii_1\ldots i_d} \prod_{k=1}^{d} \left(\sum_{j=1}^{i} c_{kj}(t) u_j + \sum_{i_1+\cdots+i_d=2}^{m} c_{ki_1\ldots i_d}(t) u_1^{i_1} \ldots u_d^{i_d} \right)^{i_k} \end{aligned} \tag{2.10}$$

Differentiate both sides of (2.9)

$$\dot{x}_i = \sum_{j=1}^{i} \dot{c}_{ij}(t) u_i + \sum_{i_1+\cdots+i_d=2}^{m} \dot{c}_{ii_1\ldots i_d}(t) u_1^{i_1} \ldots u_d^{i_d} + \text{h.o.t.} \tag{2.11}$$

Equate equations (2.10) and (2.11) and compare same order terms in powers of u_i. We obtain md differential equations linear in $c_{ii_1\ldots i_d}$ when comparing terms of the expansion of x_i of powers of $u_1^{i_1} u_d^{i_d}$

$$\begin{aligned} \dot{c}_{ii}(t) &= b_{ii} c_{ii}(t) \\ \dot{c}_{ii_1\ldots i_d}(t) &= b_{ii} c_{ii_1\ldots i_d}(t) + b_{ii_1\ldots i_d} \prod_{j=1}^{d} c_{jj}^{i_j}(t) + f_i(t) \end{aligned}$$

where $f_i(t)$ depends on the coefficients corresponding to the lower order terms in u_i and does not depend upon $b_{ii_1\ldots i_d}$. Hence we can solve equations inductively and the solution gives a relationship between constants $b_{ii_1\ldots i_d}$ and functions $c_{ii_1\ldots i_d}(t)$.

Now to match the terms of the time-one map to the terms of the expansion of $\mathcal{F}$ we require $c_{ii_1\ldots i_d}(1)$ to equate to the coefficients $a_{ii_1\ldots i_d}$. We can write $a_{ii_1\ldots i_d}$ as functions of $b_{ii_1\ldots i_d}$

$$a_{ii} = \exp b_{ii}$$

$$a_{ii_1\dots i_d} = b_{ii_1\dots i_d} \exp b_{ii} + k_i \qquad \text{for} \quad \sum_{j=1}^{d} (i_j b_{jj}) - b_{ii} = 0, \tag{2.12}$$

$$a_{ii_1\dots i_d} = \frac{\exp\left(\sum_{j=1}^{d} i_j b_{jj}\right) - \exp b_{ii}}{\sum_{j=1}^{d} (i_j b_{jj}) - b_{ii}} b_{ii_1\dots i_d} + l_i \qquad \text{for} \quad \sum_{j=1}^{d} (i_j b_{jj}) - b_{ii} \neq 0$$

where the constants k_i, l_i depend on known coefficients b_k and do not depend on $b_{ii_1\dots i_d}$. We can solve these equations for $b_{ii_1\dots i_d}$, if $a_{ii} > 0$, which we assumed to be satisfied.

For a given expansion of the map (2.7) we found the flow (2.8), such that the condition (2.4) is satisfied up to the order $m+1$. We can get the accuracy of the local approximation to any given order by considering appropriate expansions of the approximating flow.

ii) From the construction in the point *i)* we can find the map

$$\mathcal{F}_1 = \sum_{i_1+\dots+i_d=1}^{m} \tilde{a}_{i_1\dots i_d} x_1^{i_1} \dots x_d^{i_d}$$

for the flow g_1, such that $|H_{g_1} - \mathcal{F}_1| = 0$. It follows from (2.5) that $|\mathcal{F} - \mathcal{F}_1| < \varepsilon$. Thus for x in D

$$|\sum_{i_1+\dots+i_d=1}^{m} a_{i_1\dots i_d} x_1^{i_1} \dots x_d^{i_d} - \sum_{i_1+\dots+i_d=1}^{m} \tilde{a}_{i_1\dots i_d} x_1^{i_1} \dots x_d^{i_d}| < \varepsilon$$

By the Cauchy inequalities for x in D' $|a_{i_1\dots i_d} - \tilde{a}_{i_1\dots i_d}| \leq M_1\varepsilon$. Relation (2.12) implies $|b_{i_1\dots i_d} - \tilde{b}_{i_1\dots i_d}| \leq M\varepsilon$ □

Now we relate Propositions 2.1.1 and 2.1.2 to obtain the existence of the flow that approximates the map both globally (up to a given order of perturbation) and locally (to a given order of the distance from the fixed point). We state the following

Proposition 2.1.3. *Let $x \to x + \delta f(x) = \mathcal{F}(x)$ be a map $\mathbb{C}^d \supset \mathcal{D} \to \mathbb{C}^d$, such that $\mathcal{D} \cap \mathbb{R}^d \to \mathbb{R}^d$. Assume that f is analytic and bounded $|f(x)| < K$ on $\mathcal{D}$, K is independent on δ. Suppose that $x(0) = 0$ is a fixed point of the map and that the Jacobian of $\mathcal{F}(x)$ at the fixed point has all eigenvalues positive.*

Let $\dot{x} = G(x)$ be the global approximating flow as in Proposition 2.1.1, such that on a relatively compact subset $\mathcal{D}' \subset \mathcal{D}$ the time-one map of the flow satisfies

$$|H_G(x) - \mathcal{F}(x)| \leq M\delta^{n+1} \tag{2.13}$$

for $\delta < \delta_0$.

Let $\dot{x} = g(x)$ be the local approximating flow of the map around fixed point

$x(0) = 0$ *as in Proposition 2.1.2, such that the time-one map of the flow satisfies*

$$|H_g(x) - \mathcal{F}(x)| \leq L|x|^{m+1} \tag{2.14}$$

for $x \in \mathcal{D}'$. *Constants* M *and* L *depend on* K *and* $\mathcal{D}$, *in addition* L *depends on reciprocals of eigenvalues of the Jacobian of* $\mathcal{F}$ *at the origin.*

Then there exist positive constants M' *and* L' *and there exists* $\widetilde{G}$ *a perturbation of the flow* G, *such that for* $x \in \mathcal{D}'$ *the time-one map* $H_{\tilde{G}}(x)$ *satisfies*

$$|H_{\tilde{G}}(x) - \mathcal{F}(x)| \leq L'|x|^{m+1} \tag{2.15}$$

$$|H_{\tilde{G}}(x) - \mathcal{F}(x)| \leq M'\delta^{n+1} \tag{2.16}$$

for $m, n \in \mathbb{N}$ *and constants* L', M' *depending on* K *and* $\mathcal{D}$, L' *depends also on reciprocals of eigenvalues of the Jacobian of* $\mathcal{F}$ *at the origin.*

Proof. To simplify the notation we prove the proposition in one dimension. Let $G(x) = \sum_{i=1}^{m} \tilde{b}_i x^i + \text{h.o.t}$ and $g(x) = \sum_{i=1}^{m} b_i x^i + \text{h.o.t}$ be expansions around the fixed point.

Define

$$\tilde{G}(x) = G(x) + P(x) \tag{2.17}$$

where $P(x) = \sum_{i=1}^{m} \left(b_i - \tilde{b}_i\right) x^i$.

From the definition of $\tilde{G}$, expansions of $\tilde{G}$ and g agree up to order m. Since g satisfies Proposition 2.1.2 so does $\tilde{G}$ and (2.14) imply (2.15).

It follows from (2.13) that G satisfies condition (2.5) of Proposition 2.1.2 *ii)* for $\varepsilon = M\delta^{n+1}$. It follows from of the Proposition 2.1.2 that the coefficients of the expansions of the local flow g and the flow $\tilde{G}$ satisfy $|b_i - \tilde{b}_i| \leq M_1\delta^{n+1}$. It implies that $|\tilde{G}(x) - G(x)| \leq M_2\delta^{n+1}$.

For close flows, time-one maps are close as implied by (2.12), hence $|H_{\tilde{G}}(x) - \mathcal{F}(x)| \leq |H_{\tilde{G}}(x) - H_G(x)| + |H_G(x) - \mathcal{F}(x)| \leq M_3\delta^{n+1} + M\delta^{n+1} = M'\delta^{n+1}$ and (2.16) follows.

Thus for a given choice of m and n we found the flow that gives global approximation to the map of order δ^{n+1} and local approximation of order x^{m+1}. □

The above propositions can be generalised as we state below and the proofs would follow a similar argument.

Remark 2.1.1. *Propositions 2.1.2 and 2.1.3 hold for a map with two fixed points* x_1 *and* x_2 *neither placed at the origin. The bound of type (2.15) in Proposition 2.1.3 holds around both fixed points. The definition (2.17) of the flow* $\tilde{G}$ *should involve a perturbation of the global flow* G *by the polynomial* P *of order* $2m$. *Coefficients of* P *should agree with coefficients of polynomials corresponding to expansions at both fixed points up to order* m. *The resulting flow will give a global approximation to the map of order* δ^{n+1} *and local approximations of order* $|x - x_1|^{m+1}$ *and* $|x - x_2|^{m+1}$ *around fixed points* x_1, x_2 *respectively.*

Remark 2.1.2. *Consider a linear change of coordinates which transforms the map* $x_{n+1} = x_n + \delta f(x_n)$ *into the map* $y_{n+1} = y_n + c\delta h(y_n)$. *We can find the* *global approximating flow $\dot{y} = G(y)$ satisfying Proposition 2.1.1 by expanding its solution in powers of $c\delta$ and applying the transformation to go back to the original coordinates. The resulting flow is the same as the flow $\dot{x} = G_1(x)$ found directly in original coordinates by expanding its solution in powers of δ.*

We conclude this section with an observation

Remark 2.1.3. *Consider a sequence of maps f_ν converging uniformly on $\mathcal{D}$ to the map f_0 as $\nu \to 0$. If the approximating flows G_ν of the maps f_ν as constructed in Proposition 2.1.1 are expanded to the same order m then $G_\nu \to G_0$ and G_0 is the m-th order approximating flow of the map f_0.*

2.2 Parametrisation of the Manifold for the Map

We consider dynamical properties of real planar maps in the neighbourhood of the fixed point. When the fixed point is hyperbolic its qualitative properties can be deduced from the linearisation of the original system. Depending on the eigenvalues of the Jacobian matrix at the fixed point there are different types of dynamics.

For the flow when real parts of both eigenvalues are positive the fixed point is repelling, when both are negative the fixed point is attracting. When the eigenvalues are opposite signs the fixed point is a saddle and there exist uniquely defined invariant curves passing through the fixed point, which are tangent to eigenspaces of the linearised system, see *Stable Manifold Theorem for a Fixed Point* in [21] (Theorem 1.3.2). The fixed point is attracting for the stable manifold and repelling for the unstable manifold.

Similarly to continuous system a discrete system given by iterations of a diffeomorphism has a hyperbolic fixed point if the eigenvalues of the Jacobian matrix at the fixed point are not on the unit circle. If both eigenvalues are inside the unit circle then the fixed point is attracting, if both are outside of the unit circle then the fixed point is repelling. The fixed point is a saddle when the modulus of one eigenvalue is bigger than 1 and one smaller than 1. Results similar to continuous system can be obtained, see *Stable Manifold Theorem for a Fixed Point* in [21] (Theorem 1.4.2)

When at least one eigenvalue of the fixed point is on the unit circle (in case of the discrete system) or its real part is equal to 0 (in case of the continuous system), then the fixed point is nonhyperbolic and there exists, not necessarily unique, centre manifold, see *Centre Manifold Theorem* in [21] (Theorem 3.2.1). The centre manifold is not necessarily uniquely defined.

We give a brief description of some types of behaviour on the centre manifold. The fixed point can be either attracting, repelling or neutral on the centre manifold depending on the type of dynamics on the centre manifold and position of the starting point on the manifold.

- fixed point can be both attracting and repelling - example map (1.2) in Section 1.3
- the centre manifold is the whole plane - semistandard map (3.2)

- centre manifold is uniquely defined on the repelling branch and not uniquely defined on the attracting branch - system (5.23) in Chapter 5

- every point on the centre manifold is neutral - trivial example $\dot{x} = x$, $\dot{y} = 0$

Throughout this thesis a subset of points on the centre manifold that are attracted to the fixed point is called an *improper stable manifold* and a subset of points repelled from the fixed point is called an *improper unstable manifold*.

Uniqueness of solution of smooth systems guarantees that two stable (or unstable) manifolds of two distinct point cannot intersect. When the stable and unstable manifolds of two different fixed points intersect it implies rich dynamics of the system as discussed in Chapter 1. An example of such interesting behaviour is studied in Chapter 4, when the manifolds wind forming a *stochastic layer* (see [27]) and get exponentially close to each other as mentioned in Chapter 1.

For analytic systems we can extend the notion of invariant manifolds to complex plane by analytic continuation and we are interested in its parametrisation, which is an analogue of function g^{-1} satisfying (1.8). Note that complex invariant curves can be identified with two dimensional surfaces in $\mathbb{C}^2$, which intersect at the fixed point.

We provide the construction of the unstable manifold of the fixed point of the complex discrete system. A similar statement of lemma below is due to Poincaré [32] and its generalisation for invariant manifolds of n dimensional complex maps can be found in [33].

Let $\mathcal{F} : \mathbb{C}^2 \supset D \to \mathbb{C}^2$ be the analytic mapping with a saddle fixed point at the origin. Assume D a is bounded region and $|\mathcal{F}| < K$ on D. Let W_u (W_s) be the unstable (stable) global invariant manifold of the saddle point. Let λ and μ be the eigenvalues corresponding to W_u, W_s respectively.

Lemma 2.2.1. *There exists a constant a and there exists an analytic parametrisation of the unstable manifold of the fixed saddle point $\gamma_u : \mathbb{C} \supset W \to \mathbb{C}^2$ satisfying $\gamma_u(w+1) = \mathcal{F}(\gamma_u(w))$ where $W = \{w \in \mathbb{C} : \Re w < a\}$.*

Proof. Proof is by construction. We first note that since the fixed point is hyperbolic we can write $\mathcal{F}$ in eigenvectors as coordinates. From linearisation for $z = (z_1, z_2) \in \mathbb{C}^2$

$$\mathcal{F}(z_1, z_2) = (\lambda z_1, \mu z_2) + \mathcal{O}(|z|^2) \tag{2.18}$$

where eigenvalues satisfy $|\mu| < 1 < |\lambda|$ and $|z| < R$.

Take $u \in U = B(0, \min(\frac{R}{2}, d^{-1} \ln 2))$, where d is to be chosen. Define $\psi_k : \mathbb{C} \supset U \to \mathbb{C}^2$, $\psi_k(u) = \mathcal{F}^k(\lambda^{-n} u, 0)$ is analytic in U. We show by induction that for $u \in U$, $k \leq n$ the bound holds

$$|\psi_k(u)| < 2|\lambda^{-n+k} u| \tag{2.19}$$

Assume for $j \leq k$

$$|\psi_j(u)| < 2|\lambda^{-n+j} u| \tag{2.20}$$

We obtain from (2.18)

$$
\begin{aligned}
|\psi_{k+1}(u)| &= |\mathcal{F}(\psi_k(u))| \leq |\psi_k(u)|(\lambda + c|\psi_k(u)|) \\
&\leq |\psi_{k-1}(u)|(\lambda + c|\psi_{k-1}(u)|)(\lambda + c|\psi_k(u)|) \\
&\vdots \\
&\leq |\psi_0(u)| \prod_{j=0}^{k} (\lambda + c|\psi_j(u)|) \\
&= |\psi_0(u)| \lambda^{k+1} \prod_{j=0}^{k} (1 + \frac{c|\psi_j(u)|}{\lambda}) \\
&\leq \lambda^{-n+k+1} |u| \exp(\sum_{j=0}^{k} \frac{c|\psi_j(u)|}{\lambda})
\end{aligned}
$$

From induction hypothesis (2.20) we obtain

$$
|\psi_{k+1}(u)| \leq \lambda^{-n+k+1} |u| \exp(2c\lambda^{-n-1} \sum_{j=0}^{k} \lambda^j |u|)
$$

We choose a constant d independent of k and n such that $2c\lambda^{-n} \sum_{j=0}^{k} \lambda^j < d$ and hence $|\psi_{k+1}(u)| \leq 2\lambda^{-n+k+1}|u|$ provided $|u| < d^{-1} \ln 2$, which we assumed to hold. We obtain that the bound on $|\psi_k(u)| < 2|\lambda^{-n+k} u|$ holds for $k \leq n$.

Next we show that $\psi_n(u) = \mathcal{F}^n(\lambda^{-n}u, 0)$ is a uniform Cauchy sequence. Note that the difference

$$
\begin{aligned}
|\psi_{n+1}(u) - \psi_n(u)| &= |\mathcal{F}^n((\lambda^{-n}u, 0) + \mathcal{O}(\lambda^{-2(n+1)}|u|^2)) - \mathcal{F}^n(\lambda^{-n}u, 0)| \\
&= |D\mathcal{F}^n| \mathcal{O}(\lambda^{-2(n+1)}|u|^2)
\end{aligned}
\tag{2.21}
$$

where $|D\mathcal{F}|$ is the operator norm.

From (2.18) $|D\mathcal{F}| < \lambda + c|z|$ and by Chain Rule $|D\mathcal{F}^n| \leq \prod_{k=0}^{n-1}(\lambda + c|z_k|)$, which implies $|D\mathcal{F}^n| \leq \lambda^n \exp(\sum_{k=0}^{n-1} \lambda^{-1} c|z_k|)$. From the bound (2.19) on $|\psi_k(u)|$ we obtain

$$
|D\mathcal{F}^n| \leq \lambda^n \exp(\sum_{k=0}^{n-1} 2c\lambda^{-n+k-1}|u|)
$$

As the exponential term is bounded we obtain $|D\mathcal{F}^n| = \mathcal{O}(\lambda^n)$. Substitution in (2.21) implies

$$
|\psi_{n+1}(u) - \psi_n(u)| = \mathcal{O}(\lambda^{-n-2}|u|^2)
$$

which implies for $m > n$

$$
\begin{aligned}
|\psi_m(u) - \psi_n(u)| &\leq \sum_{i=1}^{m-n} |\psi_{n+i}(u) - \psi_{n+i-1}(u)| \\
&= \mathcal{O}(\lambda^{-n-2}|u|^2 \sum_{i=1}^{m-n} \lambda^{-i})
\end{aligned}
$$

The sum is bounded and we obtain that $|\psi_m(u)-\psi_n(u)| = \mathcal{O}(\lambda^{-n})$. The sequence $\psi_n(u)$ is uniformly Cauchy so there exists a limit $\psi(u) = \lim_{n\to\infty}\psi_n(u)$ and $\psi(u)$ is analytic in U. We observe that

$\mathcal{F}(\psi(u)) = \lim_{n\to\infty}\mathcal{F}^{n+1}(\lambda^{-n}u, 0) = \lim_{n\to\infty}\mathcal{F}^{n+1}(\lambda^{-(n+1)}\lambda u, 0) = \psi(\lambda u)$. Parametrisation $\psi(u)$ is unique up to the multiplication by a non zero number. The recurrence relation and derivative $\psi'(0)$ define $\psi(u)$ uniquely. Finally we define $\gamma_u(w) = \psi(\lambda^w)$ where $a = \ln(\min(\frac{R}{2}, d^{-1}\ln 2))/\ln\lambda$ and conclude that on $W = \{w \in \mathbb{C} : \Re w < a\}$, $\gamma_u : W \to \mathbb{C}^2$ is an analytic function satisfying $\gamma_u(w+1) = \mathcal{F}(\gamma_u(w))$. □

Corollary 2.2.1. *The recurrence relation $\gamma_u(w+1) = \mathcal{F}(\gamma_u(w))$ allows to define the manifold outside of the neighbourhood of the fixed point. When $\mathcal{F}$ is analytic in D then $\gamma_u(w)$ can be defined for those values of w that $\gamma_u(w-1)$ is defined and in D. When $\mathcal{F}$ is entire function then $\gamma_u(w)$ can be defined globally on whole $\mathbb{C}$.*

Note that the unstable manifold was considered for definiteness and Lemma 2.2.1 is true for the stable manifold, which can be easily deducted by taking the time in the opposite direction. A generalisation of Lemma 2.2.1 for invariant manifolds of n dimensional complex maps can be found in [33]. The proof of the general result is more advanced than the argument presented in Lemma 2.2.1, which for our purpose is sufficient. A different statement of this result is given in [32] and the proof can be found in [37].

We conclude this chapter with the statement on parametrisation of the unstable manifold for the continuous system. The proof would follow by argument similar to the one used in the proof of Lemma 2.2.1.

Lemma 2.2.2. *Suppose that $\dot{x} = f(x)$ has a hyperbolic fixed point. Then there exists a constant a and there exists an analytic parametrisation of the unstable manifold of the fixed point $\phi_u : \mathbb{C} \supset W \to \mathbb{C}^2$ satisfying $\phi_u(w+1) = T_1(\phi_u(w))$ where $W = \{w \in \mathbb{C} : \Re w < a\}$ and $T_1 : \mathbb{C}^2 \to \mathbb{C}^2$ is the time-one map of the flow f. It follows that ϕ_u is a solution i.e. it satisfies $\dot{\phi}_u(w) = f(\phi_u(w))$.*

Analogous lemma holds for the stable manifold of the fixed point.

Chapter 3

Semistandard Map

In this chapter we give an overview of the results in Davie [6], where nearly periodic function is constructed for the standard map, which motivate the approach we take in Chapter 4. We improve some estimates given in [6] to obtain good approximation for a numerical result, which concludes this chapter.

The semistandard map was also used as an approximation to the standard map in relation to study separatrix splitting angle [28]. After appropriate complex change of variables the standard map, which can be expressed as map (1.1) for $J = 1$ and $\omega = 0$, extended to complex domain can be well approximated for small ϵ by orbits of the semistandard map independent of ϵ (see [28] for more details).

As it is a special case of the extended standard map (1.1) we can relate results of subsequent chapters to the result in Lazutkin [28]. Another problem related to the semistandard map was studied in Davie [8] and involved the radius of convergence of the expansion of the invariant circle.

3.1 Introduction and Preliminaries

This section contains a summary of results from [6] that are necessary in subsequent sections of this chapter. We write the semistandard map f_0 in (φ, x) variables.

$$\begin{aligned} x_{n+1} &= x_n + ie^{-i\varphi_n} \\ \varphi_{n+1} &= \varphi_n + x_{n+1} \end{aligned} \tag{3.1}$$

After the change of variables $z = -\frac{ix}{2}$, $v = \frac{1}{\sqrt{2}}e^{-i\varphi/2}$ the map becomes

$$\begin{aligned} z_{n+1} &= z_n + v_n^2 \\ v_{n+1} &= v_n e^{z_{n+1}} \end{aligned} \tag{3.2}$$

Define $F = zz' - v^2$, $u = v^{-1}$, where $z' = z + v^2$ and denote the map (3.2) in those variables as $(u_{n+1}, F_{n+1}) = h(u_n, F_n)$

We state the following auxiliary results

Lemma 3.1.1. *If* $|v| < 1/8$ *and* $|z| < 3|v|$ *then* $|F_1 - F_0| < 128|v|^5$

Lemma 3.1.2. *There is a constant $B > 0$ such that if n is a positive integer and $|F_0| \leq 2|v_0|^2, \Re\frac{z_0}{v_0} < 0$ and $\Re u_0 < -B - n$ then $|F_k| \leq |F_0| + C_1|u_0 + k|^{-4}$ and $|u_k - u_0 - k| \leq C_2(|u_0^3 F_0| + |u_0 + k|^{-1})$ for $0 \leq k \leq n$.*

The proof of Lemma 3.1.1 and Lemma 3.1.2 can be found in [6]. Lemma 3.1.2 implies that $|F_n| \leq C_3$ and $|u_n - u_0 - n| \leq C_3$. When F_n and u_n are considered as analytic functions of F_0 and u_0 it can be shown by the Cauchy inequalities that

$$\begin{aligned} &|\partial F_n/\partial u_0| \leq C_3, & &|\partial u_n/\partial u_0| \leq 1 + C_3, \\ &|\partial F_n/\partial F_0| \leq C_3|u_0|^3, & &|\partial u_n/\partial F_0| \leq C_3|u_0|^3. \end{aligned} \tag{3.3}$$

We use F and u as coordinates, they are analogous of functions h and g^{-1} introduced in Chapter 1, with the exception that they satisfy (1.14) and (1.8) only approximately. Lemma 3.1.2 and estimates (3.3) are used to construct the invariant manifold of the map (3.2) satisfying (1.8) exactly.

Lemma 3.1.3. *For $\xi \in \{\Re\xi < -B\}$ the sequence $\zeta_n(\xi) = h^n(\xi - n, 0)$ is a Cauchy sequence.*

Proof. Let $m < n$, $\zeta_n = h^m(\alpha, \beta)$, $(\alpha, \beta) = h^{n-m}(\xi - n, 0)$. If follows from Lemma 3.1.2 that $|\alpha - (\xi - m)| < C_4 m^{-1}$ and $|\beta| < C_4 m^{-4}$. It follows from (3.3) that $|\zeta_n - \zeta_m| = |h^m(\alpha, \beta) - h^m(\xi - m, 0)| \leq C_5 m^{-1}$ so $\zeta_n(\xi)$ is a Cauchy sequence. $\square$

Note that $\zeta_n(\xi+1) = h^n(\xi+1-n, 0) = h(\zeta_{n-1}(\xi))$. As ζ_n is a Cauchy sequence we can define the limit

$$\gamma_0(\xi) = \lim_{n\to\infty} \zeta_n(\xi)$$

and we get $\gamma_0(\xi + 1) = h(\gamma_0(\xi))$, which implies that $\gamma_0(\xi)$ is an analogue to the function g^{-1} and satisfy condition (1.8) exactly. From this construction we obtain the parametrisation of the invariant manifold for the semistandard map $\gamma_0(\xi)$, which is improper unstable manifold in a sense introduced in Section 2.2 and the whole plane is on $\gamma_0(\xi)$. To distinguish from other manifolds γ defined later we use the notation γ_0 with a subscript 0 for the manifold for the semistandard map.

Take $u_0 = \xi - n$ and $(u_n, F_n) = \zeta_n(\xi)$ and note the following

Remark 3.1.1. *Lemma 3.1.2 implies that $|u_n - u_0 - n| \leq c|u_n|^{-1}$, and it follows from Lemma 3.1.3 that the sequence u_n has a limit, which implies that for $\xi \in \{\Re\xi < -B\}$ on the unstable manifold $\gamma_0(\xi)$ the following relation holds $\xi = u + \mathcal{O}(|u|^{-1})$.*

In [26] the construction of the invariant manifold is given in different coordinates hence is defined on a different region in the complex plane.

Since $\gamma_0(\xi)$ satisfies (1.8) exactly and F satisfies (1.14) approximately the analogue of function σ (1.6) is nearly periodic rather than periodic. We state proposition on nearly periodic functions, which we then use to study the behaviour of $\gamma_0(\xi)$ on a specific domain.

Proposition 3.1.1. *Suppose A, θ, K and δ are positive numbers with $\theta \geq 1$. Then we can find positive numbers Q_0, D, A and λ such that if $Q > Q_0, M \leq \theta Q$*

and g is an analytic function on an open set Ω containing

$$\Omega_0 = \{\xi = \sigma + i\tau \;\; 0 < \sigma < 2,\; -Q < \tau < -A\}$$

satisfying

i) $|g(\xi)| < K|\xi|^{-3}$*, for all* $\xi \in \Omega_0$

ii) whenever $\xi, \xi + 1 \in \Omega$ *we have* $|g(\xi + 1) - g(\xi)| < K|\xi|^{-4}$

iii) whenever $\xi \in \Omega, \Re\xi + 1 < 2M$ *and* $|g(\xi)| \leq \delta|\xi|^{-2}$ *we have* $\xi + 1 \in \Omega$*.*

Then Ω *contains the domain*

$$V = \{\xi = \sigma + i\tau : \; -Q < \tau < -A',\; 0 < \sigma < \min(\lambda e^{-\pi\tau}, M)\}$$

and there is a complex number μ_0 *with* $|\mu_0| < D$ *such that on* V *we have*

$$|g(\xi) - \mu_0 e^{-2\pi i\xi}| \leq D|\xi|^{-3}. \tag{3.4}$$

Proof of Proposition 3.1.1 can be found in [6] and is similar to the proof of Proposition 4.7.2.

Proposition 3.1.1 is applied to $g_0 = F(\gamma_0(\xi))$ with Q and M both infinite. It can be shown that g_0 satisfies assumptions of Proposition 3.1.1 if K is large enough and δ is small enough. Hence we obtain

Proposition 3.1.2. *There exists a constant* μ_0 *such that for* $\xi \in \{\xi = \sigma + i\tau : \tau < -A,\; 0 < \sigma < \lambda e^{-\pi\tau}\}$ *for suitable positive* A, λ

$$|g_0(\xi) - \mu_0 e^{-2\pi i\xi}| \leq D|\xi|^{-3} \tag{3.5}$$

The constant $2\mu_0$ was introduced in Lazutkin [28] and subsequently computed with the highest precision in Gelfreich and Simó [20]. For completeness we provide computation of μ_0, but first we refine bounds from previous section to obtain more accurate numerical results.

3.2 Further Estimates

In this section we modify results of Section 3.1 in order to improve numerical estimates of the constant μ_0. First we modify F

$$\widetilde{F}(z, v) = zz' - v^2 - \frac{1}{6}(zz')^2$$

We prove estimates analogous to Lemma 3.1.1

Lemma 3.2.1. *Under the assumptions of Lemma 3.1.1* $|\widetilde{F}_1 - \widetilde{F}_0| < c|v|^7$

Proof. First we observe that $|\widetilde{F}_1 - \widetilde{F}_0| = |v^2 f_1(\zeta) - \frac{1}{6}v^4 f_2(\zeta)|$, where $\zeta = z + v^2$, $f_1(\zeta) = (1 + e^{2\zeta})(\zeta - \frac{1}{3}\zeta^3) - e^{2\zeta} + 1$ and $f_2(\zeta) = \zeta^2(e^{4\zeta} - 1)$. Note that $f_i(0) = f_i'(0) = f_i^{(2)}(0) = 0$ for $i = \{1, 2\}$ and $f_1^{(3)}(\zeta) = \mathcal{O}(\zeta^2)$, $f_2^{(3)}(\zeta) = \mathcal{O}(1)$. From assumptions it follows that $|\zeta| < 4|v| < \frac{1}{2}$, so by Taylor expansion $|\widetilde{F}_1 - \widetilde{F}_0| < |v|^2|\zeta^3|(\mathcal{O}(|\zeta|^2) + \mathcal{O}(v^2)) = \mathcal{O}(|v|^7)$ $\square$

We define $g_1(\xi) = \widetilde{F}(\gamma_0(\xi))$ and further improve estimation (3.5). By similar argument to Proposition 3.1.1 one can show that

$$g_1(\xi) = \mu_0 e^{-2\pi i\xi} + \mu_1 e^{-4\pi i\xi} + \mathcal{O}(|\xi|^{-5}), \tag{3.6}$$

which implies $\mu_0 \approx e^{2\pi i\xi} g_1 + \mu_1 e^{-2\pi i\xi}$. To eliminate μ_1 term we integrate (3.6) and obtain

$$\mu_0 \approx \int_{\xi_0}^{\xi_0+1} e^{2\pi i\xi} g_1(\xi) d\xi$$

which for numerical computations is approximated by

$$\mu_0 \approx \frac{1}{N} \sum_{i=1}^{N} e^{2\pi i\xi_i} g_1(\xi_i). \tag{3.7}$$

We parametrise unstable manifold for the semistandard map (3.2) and use this expansion for the numerical results in the next section.

$$\begin{aligned} v(\xi) &= \frac{1}{\xi} - \frac{1}{8}\xi^{-3} + \frac{209}{3456}\xi^{-5} + \frac{8611}{138240}\xi^{-7} + \mathcal{O}(\xi^{-9}) \\ z(\xi) &= -\frac{1}{\xi} - \frac{1}{2}\xi^{-2} - \frac{1}{12}\xi^{-3} + \frac{1}{8}\xi^{-4} + \frac{193}{2160}\xi^{-5} \\ &+ \frac{59}{846}\xi^{-6} + \frac{6883}{60480}\xi^{-7} + \mathcal{O}(\xi^{-8}) \end{aligned} \tag{3.8}$$

Before we progress to numerical results we observe that the above expansion (3.8) of the unstable manifold parametrisation is consistent with the expansion given in [29]. After the change of variables $\mathrm{u} = -i\varphi, \mathrm{v} = -ix$ the semistandard map (3.1) becomes

$$\begin{aligned} \mathrm{v}_{n+1} &= \mathrm{v}_n + e^{\mathrm{u}_n} \\ \mathrm{u}_{n+1} &= \mathrm{u}_n + \mathrm{v}_{n+1} \end{aligned} \tag{3.9}$$

and the manifold expansion expressed in (v, u) variables given in [29] is

$$\begin{aligned} \mathrm{u}(w) &= -\ln\frac{w^2}{2} - \frac{1}{4}w^{-2} + \frac{91}{864}w^{-4} - \frac{319}{2880}w^{-6} + \mathcal{O}(w^{-8}) \\ \mathrm{v}(w) &= -\frac{2}{w} - w^{-2} - \frac{1}{6}w^{-3} + \frac{1}{4}w^{-4} + \frac{193}{1080}w^{-5} - \frac{59}{432}w^{-6} + \mathcal{O}(w^{-7}) \end{aligned} \tag{3.10}$$

We note that $z = \frac{1}{2}\mathrm{v}$ and $v = \frac{1}{\sqrt{2}}e^{\mathrm{u}/2}$, substitute to (3.8) and obtain v and first six terms of z, which verifies consistency of (3.1) and (3.10).

3.3 Numerical Results

In this section we explain the computation of the constant μ_0. We start iterations at $\xi \in V$, where $\Re\xi = -400$ is large negative in order to minimise the error in (3.8) and get a good approximation to the unstable manifold. Then we iterate the map (3.2) until we reach large positive $\Re\xi = 400$ to minimise the error in (3.6) and get a good approximation to $g_1(\xi)$. Finally we calculate the constant μ_0 using (3.7) by taking the average over the interval $[\xi, \xi + 1]$.

The program `mu_0.c` is included in the Appendix and the results of numerical computation are presented in Table 3.1 for a range of values $b = \Im\xi$.

Table 3.1: Numerical results of mu_0.c program

b	$\|\mu_0\|$	$\Im\mu_0$	$\Re\mu_0$
-2.6	559.41385297	559.41385297	0
-2.7	559.41385297	559.41385297	0
-2.8	559.41385297	559.41385297	0
-2.9	559.41385297	559.41385297	0
-3.0	559.41385297	559.41385297	0
-3.1	559.41385297	559.41385297	0
-3.2	559.41385297	559.41385297	0
-3.3	559.41385297	559.41385297	0
-3.4	559.41385297	559.41385297	0

The relation between z and v gives the relation between the constant μ_0 and the constant $\theta_1 = 2\mu_0$ first published in [29]. The most accurate approximation to Lazutkin's constant can be found in [20] and the constant $2\mu_0 = 1118.82770594$ matches with it up to eight decimal place.

Chapter 4

Separatrix Splitting Region for the Dissipative Standard Map

4.1 Introduction

Consider the extended standard family of maps $\mathcal{F}_{J,\omega,\epsilon}$ with three parameters

$$\begin{aligned} x_{n+1} &= Jx_n + \omega + \epsilon \sin \theta_n \\ \theta_{n+1} &= \theta_n + x_{n+1} \end{aligned} \tag{4.1}$$

where $(x_n, \theta_n) \in \mathbb{R} \times \mathbb{R}$. We study properties of this map for parameter J close to 1 and ω and ϵ close to 0. We fix the parameter $a = \frac{\omega}{\epsilon}$, $|a| < 1$ and vary J. We let $\epsilon \to 0$ and take $\omega = a\epsilon$. When $J < 1$ the map is weakly dissipative and has two hyperbolic fixed points, including one saddle. When $J = 1$ the map is area preserving and has two fixed points one nonhyperbolic, one saddle.

In particular we are interested in the problem of transversal intersection of the unstable manifold of the saddle point $(\theta_f, 0) = (\arcsin(-\frac{\omega}{\epsilon}), 0)$ and the unstable manifold of $(\theta_f + 2\pi, 0)$. We study the width of the interval of parameter J, where the manifolds intersect transversally. We also look at a limit of the width of the interval when $J \to 1$, which relates the results of this chapter to results in Chapter 3.

The approach we take to estimate the width of separatrix splitting interval is similar to Lazutkin [28] in his study of the standard map. We extend the map to the complex domain[1] and use the time-one map of a specific flow to approximate dynamics of the map in a certain region of the complex plane. Outside of the region we approximate the map with the semistandard map. Finally we combine the two approximations and relate the conclusion to the result in Chapter 3 and [28].

Main part of this chapter is the construction of the nearly periodic function analytic on a domain in $\mathbb{C}$. Bounds on oscillations of the nearly periodic function (see Theorem 4.7.2) measure the interval of splitting. Construction is motivated by results in Davie [6] in the study of the standard map.

We outline the approach in more detail. We consider parameters J, ω, ϵ,

[1] This approach has been suggested to Lazutkin by A.I.Shnirelman

such that separatrix splitting occurs. We extend the map to complex variables and prove existence of the complex approximating flow. We define a function H_ν which is constant on the solutions of the flow and 0 on the stable manifold of the saddle point of the flow. The function H_ν is defined to measure a distance from the stable manifold for the flow at the point in the region where the map (4.1) is approximated by the semistandard map. We specify its domain and prove that it is well defined in the neighbourhood of the stable manifold for the flow.

As H_ν is constant for the flow and the map (4.1) is approximated by the flow, as long as solutions of the flow are well defined, it implies that H_ν is almost constant for the map, which is an analogue of function h in Chapter 1 satisfying (1.14) and is motivated by function F in Chapter 3, which is why we construct H_ν almost constant for iterations of the map.

We verify that the unstable manifold of the map stays in the neighbourhood of the stable manifold of the flow, which follows from the fact that the manifolds of the map are close to manifolds of the flow. Then we prove that H_ν is well defined on the unstable manifold of the map and nearly periodic.

We shift the time to relate the map (4.1) to the semistandard map and to match the unstable manifolds of the maps. We parametrise the unstable manifold of the map (4.1) such that it can be regarded as the analogue of g^{-1} satisfying relation (1.8) exactly. Similarly to Chapter 3 to measure the width of the separatrix splitting region we use estimates of oscillations of the function defined analogously to (1.6), which is nearly periodic rather than periodic, because H_ν satisfies (1.14) only approximately .

We conclude this chapter with numerical results which support the relation between the width of separatrix splitting region and the angle of intersection of separatrix for the standard map.

4.2 First Order Real Approximating Flow

In this section we study real approximating flow of the map (4.1). Let $\epsilon = \nu^2$ and $x = \nu y$ then the map $\mathcal{F}_{J,\omega,\epsilon}$ can be written in the form:

$$\begin{aligned} y &\to y + \nu(a - by + \sin\theta) \\ \theta &\to \theta + \nu y + \nu^2 (a - by + \sin\theta) \end{aligned} \tag{4.2}$$

where $a = \frac{\omega}{\nu^2}$ and $b = \frac{1-J}{\nu}$.

Time-one map of the flow

$$\begin{aligned} \dot{y} &= \nu(a - by + \sin\theta) \\ \dot{\theta} &= \nu y \end{aligned} \tag{4.3}$$

gives first order approximation to $\mathcal{F}_{J,\omega,\epsilon}$.

After scaling time the time-ν map of the flow

$$\begin{aligned} \dot{y} &= a - by + \sin\theta \\ \dot{\theta} &= y \end{aligned} \tag{4.4}$$

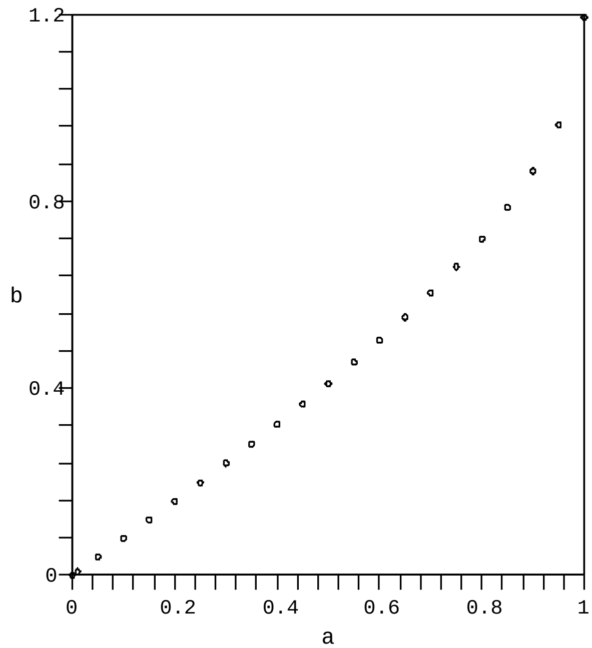

Figure 4.1: Parameters a and b giving the homoclinic orbit

approximates orbits of the map (4.2).

For $a = b = 0$ the system (4.4) becomes the pendulum equation with $(\theta, y) = (2k\pi, 0)$ a saddle and $(\theta, y) = (2(k+1)\pi, 0)$ an attracting fixed point for $k \in \mathbb{N}$. The stable and unstable manifolds of the hyperbolic fixed point create a homoclinic orbit that can be solved analytically $y = \pm 2\sin\frac{\theta}{2}$.

If $|a| < 1$ the flow (4.4) has a saddle fixed point $(\arcsin(-a) + 2k\pi, 0)$ and a nonhyperbolic fixed point $(\arcsin(-a) + 2(k+1)\pi, 0)$.

The Jacobian of the system (4.4) at the fixed point $(\theta_f, 0) = (\arcsin(-a), y)$ is equal to

$$\begin{vmatrix} -b & \cos(\theta_f) \\ 1 & 0 \end{vmatrix}$$

and the eigenvalues are $\lambda_{u/s} = \frac{1}{2}(b \pm \sqrt{b^2 - 4\cos(\theta_f)})$.

For a given parameter a, such that $|a| < 1$, there is a corresponding parameter $b = \Psi(a)$ where the stable and unstable manifolds of the saddle points of the system (4.4) coincide creating the homoclinic orbit. For $b \neq \Psi(a)$ the manifolds do not meet.

In the strip $|a| < 1$ pairs of parameters form a curve Ψ passing through the origin as shown in Figure 4.1.

We find the slope of the curve Ψ at origin. Consider the Hamiltonian of the pendulum equation

$$H_0(\theta, y) = \frac{y^2}{2} + \cos\theta - 1.$$

For $a = b = 0$, H_0 is constant on the trajectories of the pendulum equation, so for a and b small, H_0 is almost constant for the flow (4.4) and its derivative along the trajectory of (4.4) is equal to $\dot{H}_0(\theta, y) = ay - by^2$.

When we consider the homoclinic orbit for the perturbed system (4.4), then $\int \dot{H}_0(\theta, y)dt = 0$. Using this property we find a relation between the parameters a and b which give the homoclinic orbit and hence the slope of Ψ at the origin. We state it precisely in the following

Lemma 4.2.1. *The slope of the curve Ψ in the (a, b) space at the origin is equal to $\frac{\pi}{4}$.*

Proof. We integrate $\dot{H}_0(\theta, y)$ along the homoclinic trajectory of the perturbed system (4.4). We approximate the homoclinic orbit by the solution of the pendulum equation.

$$\int_{-\infty}^{\infty} \dot{H}_0(\theta, y)dt = \int_{\theta_0}^{\theta_0+2\pi} (a - by)d\theta = 2a\pi - 8b + \mathcal{O}(b^2)$$

For $b = \frac{\pi}{4}a$ the above integral is equal to 0, hence the slope of the curve Ψ in the parameter space (a, b) is equal $\Psi'(0) = \frac{\pi}{4}$. □

Homoclinic orbit of the flow (4.4) generates existence of separatrix splitting for the map (4.2). The curve close to the curve Ψ gets fattened into the exponentially narrow region and for parameters a and b inside this region the invariant stable and unstable manifolds of the saddle point of the map (4.2) intersect transversally.

To estimate the width of this region we extend the system (4.4) to a complex domain and perturb it to approximate iterates of the map (4.2).

4.3 First Order Complex Approximating Flow

We consider parameters a and b on the curve Ψ. The unstable manifold of the fixed point $(\theta_f, 0)$ and the stable manifold of the fixed point $(\theta_f + 2\pi, 0)$ coincide for the real flow. From Lemma 2.2.2 one can parametrise the unstable and stable manifolds of the fixed points of the flow (4.4). We define $\theta_m = \theta_f + \pi$ and parametrise the unstable manifold $\varphi_u(0) = (\theta_m, y_m)$ and the stable manifold $\varphi_s(0) = (\theta_m, y_m)$. We note that for real t the invariant manifolds are analytic and $\varphi_u(t) = \varphi_s(t)$. The manifolds form the invariant orbit, which we denote $\Phi(t)$.

We extend the system (4.4) to the complex domain and we denote w as the complexification of real time t. By analytic continuation the manifolds coincide, $\varphi_u(w) = \varphi_s(w)$ and the invariant manifold $\Phi(w)$ is analytic on the strip containing $\mathbb{R}$. Hence there is a maximal strip on which $\Phi(w)$ is analytic. For parameters a, b on the curve Ψ we assume that the following is true

Conjecture. *There is only one singularity of the invariant manifold $\Phi(w)$ on each boundary line of the maximal strip $\{w \in \mathbb{C} : |\Im w| < \widetilde{\tau}\}$.*

Hence the invariant manifold $\Phi(w) : \widetilde{W}_0 \to \mathbb{C}^2$ is analytic on the set

$$\widetilde{W}_0 = \{w \in \mathbb{C} : |\Im w| \leq \widetilde{\tau}\} \setminus (\{\widetilde{w}\} \cup \{\overline{\widetilde{w}}\})$$

where $\widetilde{w} = \widetilde{\sigma} + i\widetilde{\tau}$ is the singularity.

In the case when parameters $a = b = 0$ the system (4.4) can be solved and the manifolds coincide and the complex invariant manifold is analytic on the strip $\{|\Im w| < \frac{\pi}{2}\}$ and has one singularity on each boundary line of the strip at $w = \pm i\frac{\pi}{2}$ and has no other singularities inside the strip.

We believe that the conjecture is true because for $a = b = 0$ case when the system can be solved analytically the unstable manifold has one singularity on the boundary of the strip. After perturbation of parameters a and b the manifold is close to the manifold of pendulum equation hence has a singularity close to the singularity of the unperturbed system. The proof of Theorem 4.8.1 is valid for any $a \in (-1, 1)$ for which the conjecture holds.

The position of singularities in $\mathbb{C}$ is given by the integral $\widetilde{w} - w = \int_{\theta_0}^{\infty} \frac{1}{y} d\theta$, where $\theta_0 = \Phi_1(w)$, $\Phi(w) = (\Phi_1(w), \Phi_2(w))$, and y as a solution of the system (4.4) in terms of θ depends analytically on a, hence position of the singularities is analytic function of a. If the conjecture was false then because the position of singularities in $\mathbb{C}$ is analytic function of the parameter a it would be false on a set of isolated values of a. It follows that the set on which the conjecture is true is dense and open in $[-1, 1]$.

Now we study properties of function $H_0(\theta(w), y(w))$, where $(\theta(w), y(w)) = \Phi(w)$ is on the invariant manifold

Lemma 4.3.1. *There exists w_0, such that $|H_0(\theta(w_0), y(w_0))| < \frac{1}{2}e^{\Im\theta(w_0)}$.*

Proof. Assume that for all w

$$|H_0(\theta(w), y(w))| \geq \frac{1}{2}e^{\Im\theta(w)}.$$

It implies that $|y| \leq c'\sqrt{H_0(\theta, y)}$. Recall $|\dot{H}_0(\theta, y)| = |ay - by^2|$, which we can bound by $c''|H_0(\theta, y)|$. The bound $|\frac{dH_0(\theta,y)}{dt}| \leq c''|H_0(\theta, y)|$ implies exponential growth of H_0 which contradicts the existence of the singularity, which we conjectured. $\square$

As long as the solutions of the system (4.4) $y = \sqrt{2H_0(\theta, y) + 4\sin^2\frac{\theta}{2}}$ is not equal to 0 the solution of

$$\frac{dy}{d\theta} = \frac{a - by + \sin\theta}{y} \tag{4.5}$$

exists and we can consider θ as variable of the solution of (4.4). Lemma 4.3.1 implies that near $\theta_0 = \theta(w_0)$ we can express the solution of the system (4.4) in terms of θ as independent variable. Now we need to verify that we can solve (4.5) starting from θ_0. In order to do that we estimate the growth of $H_0(\theta, y)$ on the orbit of the system (4.4). Note that the derivative of $H_0(\theta, y)$ with respect to θ along a trajectory of the flow (4.4) is equal $\frac{dH_0(\theta,y)}{d\theta} = a - by$. The following lemma holds

Lemma 4.3.2. *There is a positive constant $\tilde{c}$ if $\Re\theta_0 \in (-\pi, 3\pi)$, $\Im\theta_0 > \tilde{c}$ and $|H_0(\theta_0, y_0)| < \frac{1}{2}e^{\Im\theta_0}$ then (4.5) can be solved and for (θ, y) on the solution of the system (4.4) through (θ_0, y_0) for a constant $c > 0$ depending on θ_0, for θ satisfying*

$\Re\theta \in (-\pi, 3\pi)$, $\Im\theta > \Im\theta_0$ *the bound holds*

$$|H_0(\theta, y)| < ce^{\Im\theta/2}. \tag{4.6}$$

Proof. We observe that

$$|H_0(\theta, y) - H_0(\theta_0, y_0)| = |\int_{\theta_0}^{\theta} \frac{dH_0(\phi, y(\phi))}{d\phi} d\phi|$$

holds. We consider the integral over the shortest path, such that $\Im\theta_0 \le \Im\phi \le \Im\theta$. The above implies

$$\begin{aligned} |H_0(\theta, y)| &\le |\int_{\theta_0}^{\theta} (a - b\sqrt{2H_0(\phi, y(\phi)) + 4\sin^2\frac{\phi}{2}}) d\phi| + |H_0(\theta_0, y_0)| \\ &\le \int_{\theta_0}^{\theta} (|a| + |b\sqrt{2H_0(\phi, y(\phi)) + 4\sin^2\frac{\phi}{2}}|) d\phi| + |H_0(\theta_0, y_0)| \end{aligned}$$

and we obtain

$$\begin{aligned} |H_0(\theta, y)| &\le \int_{\theta_0}^{\theta} (|a| + |2b\sin\frac{\phi}{2}|) d\phi \\ &+ \int_{\theta_0}^{\theta} |b\frac{H_0(\phi, y(\phi))}{2\sin\frac{\phi}{2}}| d\phi + |H_0(\theta_0, y_0)| \end{aligned}$$

Define $M := \max_{\Im\theta_0 \le \Im\phi \le \Im\theta} |\frac{H_0(\phi, y(\phi))}{\sin\frac{\phi}{2}}|$.

It follows that

$$M \le \max_{\Im\theta_0 \le \Im\phi \le \Im\theta} \frac{1}{|\sin\frac{\phi}{2}|} (|a(\phi - \theta_0)| + |4b(\cos\frac{\phi}{2} - \cos\frac{\theta_0}{2})| + \frac{b}{2} M|\phi - \theta_0| + |H_0(\theta_0, y_0)|)$$

We choose $\tilde{c}$, such that $\Im\theta_0 > \tilde{c}$, it follows that $\frac{b}{2}|\phi - \theta_0| < |\sin\frac{\phi}{2}|$ which implies

$$|H_0(\theta, y)| \le (|a\theta| + |8b\cos\frac{\theta}{2}| + |H_0(\theta_0, y_0)|)/(1 - \max_{\Im\theta_0 \le \Im\phi \le \Im\theta} |\frac{1}{\sin\frac{\phi}{2}}| \frac{|b|}{2} |\phi - \theta_0|)$$

From the assumption on $H_0(\theta_0, y_0)$, $\cos\frac{\theta}{2}$ is a dominant term in the numerator, i.e. absolute value of the numerator is comparable to $|\cos\frac{\theta}{2}|$, and 1 is a dominant term in the denominator. There is a constant c dependent on θ_0 such that

$$|H_0(\theta, y)| \le ce^{\Im\theta/2}$$

hence for $\Im\theta_0 < \Im\phi < \Im\theta$ $|H_0(\phi, y) + 2\sin^2\frac{\phi}{2}| \ne 0$ and the system (4.5) can be solved. $\square$

Let $W_0 = \{w \in \mathbb{C} : |\Im w| < \widetilde{\tau} - C_0\}$, where C_0 is a positive constant to be chosen, $\widetilde{\tau}$ is the imaginary part of the singularity of the invariant stable manifold $\varphi_s(w)$. Lemma 4.3.2 implies the following relation between time and θ on $\varphi_s(w)$.

Lemma 4.3.3. *For $w \in W_0$, for θ satisfying the bound (4.6) in Lemma 4.3.2 the relation between time w and θ on $\varphi_s(w)$ is*

$$w = A_a - 2e^{i\theta/2} + \mathcal{O}(e^{i\theta}) \tag{4.7}$$

Proof. From the system (4.4) $\frac{dw}{d\theta} = \sqrt{2H_0(\phi, y) + 4\sin^2\frac{\phi}{2}}^{-1}$ which implies that

$$\widetilde{w} - w = \int_\theta^\infty \sqrt{2H_0(\phi, y) + 4\sin^2\frac{\phi}{2}}^{-1} d\phi$$

where $\widetilde{w}$ is the singularity of $\varphi_s(w)$. For θ satisfying assumptions of Lemma 4.3.2 we get

$$\widetilde{w} - w = \int_\theta^\infty \frac{1}{2\sin\frac{\phi}{2}}(1 + \mathcal{O}(|\frac{e^{\Im\phi/2}}{\sin^2\frac{\phi}{2}}|))d\phi$$

Next we expand the integral in powers of $e^{i\phi/2}$ and obtain

$$w - \widetilde{w} = -2e^{i\theta/2} + \mathcal{O}(e^{i\theta})$$

The integral from θ_m to θ is equal to a constant depending on a and we denote the constant A_a. We combine the integrals and lemma holds. $\square$

Observe that for the standard map the constant $A_0 = i\frac{\pi}{2}$. Let $w_1 \in W_0$ and choose

$$\theta_1 = 2i \ln \frac{1}{C_0}; \tag{4.8}$$

denote $(\theta_1, y_1) = \varphi_s(w_1) = (\varphi_1(w_1), \varphi_2(w_1))$. For C_0 sufficiently small, $\Im\theta_1$ is large and by by Lemma 4.3.2 satisfies condition (4.6), hence $\varphi_1'(w_1) \neq 0$, which enables us to verify that θ_1 is well defined for the solution of the flow (4.4) near the stable manifold $\varphi_s(w)$.

Lemma 4.3.4. *For $d > 0$ there are constants $\delta, r > 0$ such that for (θ, y) satisfying $d((\theta, y), \varphi_s(W_0)) < \delta$ and $|(\theta, y) - (\theta_f, 0)| > d$, then for $\psi : W_0 \to \mathbb{C}^2$ a solution of the system (4.4) through (θ, y) $\exists!\, w \in W_0$, such that $|w - w_1| < r$ and $\psi_1(w) = \theta_1$.*

Proof. Take (θ, y), such that $d((\theta, y), \varphi_s(W_0)) < \delta$. There is $w_2 \in W_0$ such that $|\varphi_s(w_2) - (\theta, y)| < \delta$. Let $\psi(w)$ be the solution of the system (4.4) such that $\psi(w_2) = (\theta, y)$. From continuous dependence on initial conditions for a chosen ϵ there is δ such that $|\varphi_s(w_1) - \psi(w_1)| < \epsilon$.

Because $\varphi_1'(w_1) \neq 0$ it follows by the Inverse Function Theorem that for δ small enough there is r, there is unique w in the neighbourhood of w_1, $w \in W_0$: $|w - w_1| < r$, such that $\psi_1(w) = \theta_1$. $\square$

The above lemma and the remark below will be used in Section 4.5 to conclude that in the neighbourhood of the stable manifold the function H_ν measuring the distance from the stable manifold is well defined.

Remark 4.3.1. *If for (θ, y) close to the fixed point $d((\theta, y), \varphi_s(W_0)) < c|(\theta, y) - (\theta_f, 0)|$ then for c sufficiently small, for (θ_2, y_2) on the solution ψ through (θ, y), if $|(\theta_2, y_2) - (\theta_f, 0)| > |(\theta, y) - (\theta_f, 0)|$ then $d((\theta_2, y_2), \varphi_s(W_0)) < d((\theta, y), \varphi_s(W_0))$.*

Close to the hyperbolic fixed point (ρ distance) the solution of the flow can be approximated by its linearisation. The distance from the stable eigenspace is $|e^{\lambda_u t}\rho|$, which for a backward orbit is strictly less than ρ. For the nonlinear system the distance from the stable manifold is $|e^{\lambda_u t}\rho| + \mathcal{O}(|\rho|^2)$. Hence close to the fixed point for c sufficiently small backward orbit close to the stable manifold gets closer to it.

4.4 Existence and Properties of the Complex Approximating Flow

In this section we revert to original variables and parameters of the dissipative standard map (4.1) to approximate its iterations by the time-one map of the perturbation of the flow $G_1(\theta, x) = (G_1^1, G_1^2)$

$$\begin{aligned} \dot{\theta} &= x \\ \dot{x} &= \omega - (1-J)x + \nu^2 \sin\theta. \end{aligned} \tag{4.9}$$

Time-one map of the above flow gives the first order approximation of the map (4.1). We show that the perturbed flow approximates the map to a given order. We use auxiliary propositions from Chapter 2 relating the map to the flow.

Proposition 4.4.1. *There is a perturbation $G : D \to \mathbb{C}^2$ of the flow (4.9)*

$$G(\theta, x) = (G_1^1 + \mathcal{O}(\nu e^{\Im\theta/2})^2, G_1^2 + \mathcal{O}(\nu e^{\Im\theta/2})^3)^T,$$

such that for a relatively compact set $D' \subset D$ there is a positive constant M', that for the map $\mathcal{F}_{J,\omega,\nu}$ (4.1) the following bounds hold on D'

$$\begin{aligned} |\theta' - \theta_G| &\leq M'|\nu e^{\frac{\Im\theta}{2}}|^m \\ |\theta' - \theta_G| &\leq M'\nu^m|(\theta, \frac{x}{\nu}) - (\theta_0, 0)|^m \\ |\theta' - \theta_G| &\leq M'\nu^m|(\theta, \frac{x}{\nu}) - (\theta_0 + 2\pi, 0)|^m \\ |x' - x_G| &\leq M'|\nu e^{\frac{\Im\theta}{2}}|^{m+1} \\ |x' - x_G| &\leq M'\nu^{m+1}|(\theta, \frac{x}{\nu}) - (\theta_0, 0)|^m \\ |x' - x_G| &\leq M'\nu^{m+1}|(\theta, \frac{x}{\nu}) - (\theta_0 + 2\pi, 0)|^m \end{aligned}$$

where $D \subset \{(\theta, x) : \Im\theta < \ln(\nu^{-2}), |x| < c\nu e^{\Im\theta/2}\}$, (θ_0, x) is the hyperbolic fixed point, the iterates of the map (4.1) are denoted $(\theta', x') = \mathcal{F}_{J,\omega,\nu}(\theta, x)$ and $(\theta_G, x_G) = T_1(\theta, x)$ is the time-one map $T_1 : D \to \mathbb{C}^2$ of the flow G.

Proof. Consider a change of variables $(\phi, z) = S(\theta, x) = (ik - \theta, \frac{x}{\delta})$,where $\delta = \nu e^{\frac{k}{2}}$.

Take $(\theta, x) \in D_k$, where $D_k = \{(\theta, x) : k-1 < |\Im\theta| < k+1\}$, the bound on $\Im\theta$ gives $|\Im\phi| < 1$.

The map (4.1) becomes the map $F(\phi, z)$

$$\begin{aligned} z_{n+1} &= z_n + \delta f_1(\phi_n, z_n) \\ \phi_{n+1} &= \phi_n + \delta f_2(\phi_n, z_n) \end{aligned} \tag{4.10}$$

where $f_1(\phi_n, z_n) = ae^{-k} - be^{\frac{-k}{2}} z_n + \sin\phi_n$, $f_2(\phi_n, z_n) = z_n$. The map is real for real (ϕ_n, z_n), has two hyperbolic fixed points $(\phi_0, 0)$ and $(\phi_0 + 2\pi, 0)$, the bounds on $f_1(\phi_n, z_n), f_2(\phi_n, z_n)$ do not depend on δ hence from Proposition 2.1.1 and Remark 2.1.1 there is a δ^2 perturbation $\tilde{G}_k(\phi, z)$ of the flow (4.9) such that the time-one map T of the flow satisfies

$$|T_{\tilde{G}_k}(\phi, z) - F(\phi, z)| \leq M\delta^m, \quad k \geq 0 \tag{4.11}$$

Power m is a minimum of powers m and n in Proposition 2.1.3. Note that a δ^2 perturbation in (ϕ, z) variables corresponds to a δ^3 perturbation in x variable.

Define $G^*(\theta, x) = S^{-1}\tilde{G}_k S(\theta, x)$ on D_k. From Proposition 2.1.3 and Remark 2.1.2, $\tilde{G}_k = A^{-1}\tilde{G}_{k+1}A$ on $D_k \cap D_{k+1}$, where $A(\phi, z) = (\phi + i, e^{-1/2}z)$. It implies that the flow $G^*(\theta, x)$ is well defined globally on D and from (4.11) satisfies the bounds

$$\begin{aligned} |\theta' - \theta_{G^*}| &\leq M\delta^m \\ |x' - x_{G^*}| &\leq M\delta^{m+1} \end{aligned} \tag{4.12}$$

where (θ_{G^*}, x_{G^*}) is the time-one map of the flow G^*.

Now we perturb the global flow $G(\theta, x) = G^*(\theta, x) + P(\theta - \theta_0, \frac{x}{\nu})$, where P is a polynomial as in Remark 2.1.1. We apply Proposition 2.1.3 and Remark 2.1.1 in (ϕ, z) variables, which for $k = 0$ correspond to (θ, y) variables. Denote $F(\phi, z) = (\phi', z')$ and $T(\phi, z) = (\phi_{\tilde{G}_0}, z_{\tilde{G}_0})$, where T is the time-one map of the flow $\tilde{G}_0$. We obtain

$$\begin{aligned} |\phi' - \phi_{\tilde{G}_0}| &\leq M\nu^m \\ |\phi' - \phi_{\tilde{G}_0}| &\leq M|(\phi, z) - (\phi_0, 0)|^m \\ |\phi' - \phi_{\tilde{G}_0}| &\leq M|(\phi, z) - (\phi_0 + 2\pi, 0)|^m \\ |z' - z_{\tilde{G}_0}| &\leq M\nu^m \\ |z' - z_{\tilde{G}_0}| &\leq M|(\phi, z) - (\phi_0, 0)|^m \\ |z' - z_{\tilde{G}_0}| &\leq M|(\phi, z) - (\phi_0 + 2\pi, 0)|^m \end{aligned}$$

which imply the following bounds for the flow G for $0 < \Im\theta < 1$

$$\begin{aligned} |\theta' - \theta_G| &\leq M\nu^m \qquad (4.13)\\ |\theta' - \theta_G| &\leq M|(\theta, \frac{x}{\nu}) - (\theta_0, 0)|^m \\ |\theta' - \theta_G| &\leq M|(\theta, \frac{x}{\nu}) - (\theta_0 + 2\pi, 0)|^m \\ |x' - x_G| &\leq M\nu^{m+1} \qquad (4.14)\\ |x' - x_G| &\leq M\nu|(\theta, \frac{x}{\nu}) - (\theta_0, 0)|^m \\ |x' - x_G| &\leq M\nu|(\theta, \frac{x}{\nu}) - (\theta_0 + 2\pi, 0)|^m \end{aligned}$$

It follows from (4.13) and (4.14) by Schwarz Lemma [22] (p. 204) that

$$\begin{aligned} |\theta' - \theta_G| &\leq M\nu^m|(\theta, \frac{x}{\nu}) - (\theta_0, 0)|^m \\ |\theta' - \theta_G| &\leq M\nu^m|(\theta, \frac{x}{\nu}) - (\theta_0 + 2\pi, 0)|^m \\ |x' - x_G| &\leq M\nu^{m+1}|(\theta, \frac{x}{\nu}) - (\theta_0, 0)|^m \\ |x' - x_G| &\leq M\nu^{m+1}|(\theta, \frac{x}{\nu}) - (\theta_0 + 2\pi, 0)|^m \end{aligned}$$

For $k > 0$ we write

$$\begin{aligned} |\theta' - \theta_G| \leq |\theta' - \theta_{G^*}| + |\theta_{G^*} - \theta_G| \\ |x' - x_G| \leq |x' - x_{G^*}| + |x_{G^*} - x_G| \end{aligned} \qquad (4.15)$$

Bounds on the difference between unperturbed flow and the map iteration come from (4.12), bounds on the difference between flows come from the bound on $|P|$. We obtain

$$\begin{aligned} |\theta' - \theta_G| \leq M\delta^m + D_{ij} \\ |x' - x_G| \leq M\delta^{m+1} + \nu D_{ij} \end{aligned} \qquad (4.16)$$

where $D_{ij} = \sum_{i,j=1}^{m-1} \nu^m|\theta - \theta_0|^i|\nu^{-1}x|^j$. We bound $|\theta|^i \leq e^{\Im\theta/2}$ $|x| \leq c\nu e^{\Im\theta/2}$ and obtain $D_{ij} \leq \sum_{i=1}^{m-1} c\nu^{m-i-1}\delta^{i+1}$. If $i < m$ and $k > 0$ then $\nu < \delta$ and we obtain $D_{ij} \leq (m-1)c\delta^m$, hence

$$\begin{aligned} |\theta' - \theta_G| \leq M'\delta^m \\ |x' - x_G| \leq M'\delta^{m+1} \end{aligned} \qquad (4.17)$$

and finally on D'

$$\begin{aligned} |\theta' - \theta_G| &\leq M'|\nu e^{\frac{\Im\theta}{2}}|^m \\ |x' - x_G| &\leq M'|\nu e^{\frac{\Im\theta}{2}}|^{m+1} \end{aligned}$$

for a constant M' depending on c and M which completes the proof. $\square$

The above proposition gives existence of the high order approximating flow G for the map (4.1). To estimate the width of the separatrix splitting region we require higher order approximation than the flow (4.9). We do not give the high order approximating flow G explicitly as its construction is onerous. We perturb the first approximating flow (4.9) instead and we deduce results for the flow G from the perturbed flow.

To get a higher order approximating flow we perturb the flow (4.9) by applying the change of variable

$$v = x + \frac{1}{2}\nu^2(a - b\nu^{-1}x + \sin\theta)$$

and observe that the high order approximating flow G satisfies

$$\begin{aligned} \dot{\theta} &= v + \mathcal{O}(\delta^3) \\ \dot{v} &= \nu^2(1 + \frac{b}{2}\nu)(a - b\nu^{-1}v + \sin\theta) + \mathcal{O}(\delta^4). \end{aligned} \tag{4.18}$$

where $\delta = \nu e^{\frac{\Im\theta}{2}}$. Define the function, which is analogous to $H_0(\theta, y)$

$$H(\theta, v) = \frac{1}{2}v^2 + c(\cos\theta - 1)$$

where $c = \nu^2(1 + \frac{b}{2}\nu)$. The derivative of H with respect to θ along the solution of the system (4.18) is equal $\frac{dH}{d\theta} = c(a - b\nu^{-1}v) + \mathcal{O}(\delta^4)$. As long as $|H(\theta, v) + 2c\sin^2\frac{\theta}{2}| > 0$ and $v \sim \delta$, the solution of the system (4.18) can be parametrised by θ. To formalise this we state the result for H similar to Lemma 4.3.2, which enables us to use θ as variable of the solution of the flow (4.18).

Lemma 4.4.1. *There is a positive constant $\tilde{c}$ if $\Re\theta_0 \in (\pi, 3\pi)$, $\Im\theta_0 > \tilde{c}$ and $|H_0(\theta_0, v_0)| < \frac{1}{2}ce^{\Im\theta_0}$ then the solution of the system (4.18) can be parametrised by θ and for (θ, v) on the solution through (θ_0, v_0) for constants $c_1, c_2 > 0$ depending on θ_0, for θ satisfying $\Re\theta \in (\pi, 3\pi)$, $2\ln(C_0^*\nu)^{-1} > \Im\theta > \Im\theta_0$, for a constant $C_0^* > 0$ to be specified the bound holds*

$$|H(\theta, v)| \le c_1\nu^2 e^{\Im\theta/2} + c_2\delta^4 \tag{4.19}$$

where $\delta = \nu e^{\Im\theta/2}$.

Proof. First we note that

$$|H(\theta, v) - H(\theta_0, v_0)| = |\int_{\theta_0}^{\theta} \frac{dH(\phi, v(\phi))}{d\phi} d\phi|$$

We take the integral over the shortest path, such that $\Im\theta_0 \le \Im\phi \le \Im\theta$.

The above implies

$$
\begin{aligned}
|H(\theta, v)| &\leq |\int_{\theta_0}^{\theta} c(a - b\nu^{-1}v(\phi))d\phi| + |H(\theta_0, v_0)| + \mathcal{O}(\delta^4) \\
&\leq |c\int_{\theta_0}^{\theta}(a - b\nu^{-1}\sqrt{2H(\phi, v(\phi)) + 4c\sin^2\frac{\phi}{2}})d\phi| + |H(\theta_0, v_0)| + \mathcal{O}(\delta^4)
\end{aligned}
$$

and we obtain

$$
\begin{aligned}
&|H(\theta, v)| \leq |c\int_{\theta_0}^{\theta}(|a| + |2b\nu^{-1}\sqrt{c}\sin\frac{\phi}{2}|)d\phi| \\
&+\sqrt{c}\int_{\theta_0}^{\theta}|b\nu^{-1}\frac{H(\phi, v(\phi))}{2\sin\frac{\phi}{2}}|d\phi + |H(\theta_0, v_0)| + \mathcal{O}(\delta^4)
\end{aligned}
$$

Define $M := \max_{\Im\theta_0 \leq \Im\phi \leq \Im\theta} |\frac{H(\phi, v(\phi))}{\sin\frac{\phi}{2}}|$.

It follows that

$$
\begin{aligned}
M \quad &\leq \max_{\Im\theta_0 \leq \Im\phi \leq \Im\theta} \frac{1}{|\sin\frac{\phi}{2}|}(|ac(\phi - \theta_0) + 4b\nu^{-1}c\sqrt{c}(\cos\frac{\phi}{2} - \cos\frac{\theta_0}{2})| \\
&+\frac{b\sqrt{c}}{2\nu}M|\phi - \theta_0| + |H(\theta_0, v_0)|) + \mathcal{O}(\delta^4)
\end{aligned}
$$

We choose $\tilde{c}$ such that $\Im\theta_0 > \tilde{c}$. It follows that $\frac{b\sqrt{c}}{2\nu}|\phi - \theta_0| < |\sin\frac{\phi}{2}|$ which implies

$$
|H(\theta, v)| \leq \frac{|ac\theta + 8b\nu^{-1}c\sqrt{c}\cos\frac{\theta}{2}| + |H(\theta_0, v_0)|}{1 - \frac{b\sqrt{c}}{2\nu}\max_{\Im\theta_0 \leq \Im\phi \leq \Im\theta}|\frac{1}{\sin\frac{\phi}{2}}||\phi - \theta_0|} + \mathcal{O}(\delta^4)
$$

From the bound on $|H(\theta_0, v_0)| < \frac{1}{2}ce^{\Im\theta_0}$, which we get from the bound on H_0, $\cos\frac{\theta}{2}$ is a dominant term in the numerator and 1 is a dominant term in the denominator, which implies

$$
|H(\theta, v)| \leq c_1\nu^2 e^{\Im\theta/2} + c_2\delta^4
$$

hence for the constant C_0^* large enough $|H(\phi, v) + 2c\sin^2\frac{\phi}{2}| > 0$ and $v \sim \delta$, which implies that θ can be used a variable to parametrise solutions of the system (4.18). $\square$

The above lemma enables us to parametrise solutions of the approximating flow G with θ, which we use in the next section to verify that the stable and unstable manifolds are well defined.

4.5 Complex Invariant Manifold of the Approximating Flow

In this section we define the function that measures the distance from the stable manifold of the fixed point of the flow G and conclude that it is well defined on a suitable neighbourhood of the stable manifold.

We consider parameters (a, b), such that for the real map (4.2) stable and unstable manifolds intersect. Define sets

$$W = \{w : |\Im w| < \tau^* - C_0^*\}$$

$$W^- = \{\Re w < \sigma^* + 1\} \cap W$$
$$W^+ = \{\Re w > \sigma^* - 1\} \cap W$$
$$\Omega_0 = \{w \in \mathbb{C} : |\Im w| < \tau^* + \sigma^* - \Re w \text{ and } \sigma^* < \Re w < \tau^* + \sigma^*\} \cap W$$

where $w^* = \sigma^* + i\tau^*$ is the constant to be defined in (4.22) required to relate the dissipative map (4.1) to the semistandard map (3.1).

Consider a parametrisation of the stable manifold $\phi_s(w) = (\phi_1(w), \phi_2(w))$ of the fixed point of the approximating flow G, defined in Proposition 4.4.1 satisfying (4.18), such that $\phi_1(0) = \theta_m$.

We choose ν small, since θ_1 defined in (4.8) satisfies condition (4.6) in Lemma 4.3.2 and the flow G is a perturbation of the flow (4.4) then for ν small enough θ_1 satisfies the bound (4.19) in Lemma 4.4.1, which we are going to use to verify that the stable manifold ϕ_s is analytic.

It follows from the Conjecture on page 28 and from the above argument that

Remark 4.5.1. *The stable manifold $\phi_s(w)$ of the fixed point of the flow G satisfying Proposition 4.4.1 is an analytic function on the set $\Omega_0 \setminus K$, where $K = \{w \in \Omega_0 : |w - w^*| < 2\nu^{-1}C_0\}$.*

We relate the time w and θ on the stable manifold ϕ_s. Let A_a be the constant from Lemma 4.3.3. We state the following

Remark 4.5.2. *For $w \in W^+$ the relation between time w and θ on the stable manifold $\phi_s(w)$ of the approximating flow G, defined in Proposition 4.4.1, is*

$$w = \nu^{-1}A_a - \frac{b}{4}A_a - 2\nu^{-1}e^{i\theta/2} + \mathcal{O}(\nu^{-1}e^{i\theta}) + \mathcal{O}(\nu e^{-i\theta/2}) \tag{4.20}$$

Proof. Take the system (4.18) and apply the scaling $s = (\nu\sqrt{1 + \frac{b}{2}\nu})\, w$, $u = (\nu\sqrt{1 + \frac{b}{2}\nu})^{-1}v$. The flow becomes

$$\begin{aligned} \dot{\theta} &= u \\ \dot{u} &= a - b'u + \sin\theta \end{aligned} \tag{4.21}$$

where $b' = b\sqrt{1 + \frac{b}{2}\nu}$.

We apply Lemma 4.3.3 and get

$$s = A_a - 2e^{i\theta/2} + \mathcal{O}(e^{i\theta})$$

Then we scale back to the system (4.18) and obtain

$$w = \nu^{-1}A_a - \frac{b}{4}A_a - 2\nu^{-1}e^{i\theta/2} + \mathcal{O}(\nu^{-1}e^{i\theta}) + \mathcal{O}(\nu e^{-i\theta/2})$$

where the error $\mathcal{O}(\nu e^{-i\theta/2})$ comes from the perturbation.

Lemma 4.3.3 gives the relation on the manifold φ_s of the flow (4.4), but the flow G is a perturbation of this flow, so for ν small enough relation (4.20) holds on ϕ_s. $\square$

We define the shift in time to relate the dissipative map (4.1) to the semistandard map (3.1)

$$w^* = (\nu^{-1} - \frac{b}{4})A_a. \tag{4.22}$$

Definition of w^* comes from (4.20) and it depends on ν unlike $\widetilde{w}$. Remark 4.5.2 enables us to extend the manifold and we make it precise by stating the following

Remark 4.5.3. *The stable manifold $\phi_s(w)$ of the fixed point of the flow G satisfying Proposition 4.4.1 is an analytic function on K.*

Proof. From Lemma 4.4.1 it follows that for ν small enough the solution of the flow G, in particular the stable manifold $\phi_s(w)$, can be parametrised by θ, $v = \Psi_s(\theta)$ and it can be extended for θ, such that $\Im\theta > \Im\theta_1$, $\Re\theta \in (\pi, 3\pi)$.

From Remark 4.5.2 for a given θ we can uniquely define w on the stable manifold and from (4.20) the relation is approximately $w^* - w \approx 2\nu^{-1}e^{i\theta/2}$. It implies that $\Im\theta > \Im\theta_1$ is transformed into $|w^* - w| < 2\nu^{-1}C_0$, hence on K the stable manifold can be parametrised by w. From the fact that the image $\phi_s(\Omega_0)$ is a simply connected, bounded set by uniqueness of solution it matches with the manifold parametrised by θ hence $\phi_s(w)$ is analytic on K. $\square$

We combine Remark 4.5.1 and Remark 4.5.3 and obtain that the stable manifold is well defined on Ω_0. By similar argument we can show that the unstable manifold is well defined. We state the following

Remark 4.5.4. *There exists a constant $C_0^* > C_0$ such that the stable manifold $\phi_s(w)$ of the fixed point of the flow G satisfying Proposition 4.4.1 is an analytic function on Ω_0 and the unstable manifold $\phi_u(w)$ is an analytic function on W^-.*

Next we observe that since the flow G approximates the dissipative map the stable and unstable manifold are close.

Remark 4.5.5. *For $w \in \Omega_0$ stable and unstable manifolds of the fixed point of the flow G satisfying Proposition 4.4.1 satisfy*

$$|\phi_u(w) - \phi_s(w)| \leq |\nu e^{\Im\theta/2}|^{11} \tag{4.23}$$

Recall that we consider parameters a and b such that the manifolds of the dissipative map (4.1) intersect. Because the trajectories of the flow G approximate the orbits of the dissipative map there is a parameter $\widetilde{b}$, such that the manifolds for the flow G satisfying Proposition 4.4.1 coincide and $|b - \widetilde{b}| < \nu^{11}$, which imply (4.23).

Now we take

$$\theta_\nu = 2i \ln \frac{1}{2C_0^*\nu} \tag{4.24}$$

at which we are going to measure the distance from the stable manifold. We define the neighbourhood of the stable manifold $\phi_s(w)$

$$U = \{(\theta, v) : |(\theta, v) - \phi_s(\Omega_0)| < K|\nu \sin\frac{\theta}{2}|^5 \quad \text{and} \quad |\Im\theta| < 2\ln(C_0^*\nu)^{-1}\}$$

and verify that for solutions of the flow G in this neighbourhood the difference in function H does not change much along the solution. In particular we relate the difference in function H between two solutions at θ_m and θ_ν. To be precise we state

Proposition 4.5.1. *Take $(\theta, v), (\theta, \widetilde{v}) \in U$ on the solutions of the approximating flow G satisfying (4.18) through $(\theta_m, v_m), (\theta_m, \widetilde{v}_m)$ and $(\theta_\nu, v_\nu), (\theta_\nu, \widetilde{v}_\nu)$ respectively. If $\Im\theta_m < \Im\theta < \Im\theta_\nu$ then*

$$\begin{aligned} |H(\theta_\nu, v_\nu) - H(\theta_\nu, \widetilde{v}_\nu)| &= e^{-b\Re(A_a - \frac{b}{4}\nu A_a)} \cdot \\ \cdot |H(\theta_m, v_m) &- H(\theta_m, \widetilde{v}_m)|(1 + \mathcal{O}(\delta)) \end{aligned} \tag{4.25}$$

$$|H(\theta, v) - H(\theta, \widetilde{v})| = \mathcal{O}(|H(\theta_m, v_m) - H(\theta_m, \widetilde{v}_m)|) \tag{4.26}$$

Proof. First we consider the flow (4.4) and deduce result for the flow G satisfying (4.18). Consider two solutions $u = \frac{1}{2}x^2$ and $u_2 = u + \epsilon z$, where z measures the difference in H. Define $f(\theta, u) = \nu^2(a - b\nu^{-1}\sqrt{2u} + \sin\theta)$, from the flow (4.4) by the Chain Rule we get $f(\theta, u) = \frac{du}{d\theta}$. We consider $f(\theta, u + \epsilon z)$, from Taylor expansion we obtain $\frac{dz}{d\theta} = f_u(\theta, u)z$. Next we differentiate f with respect to u, which gives $\frac{dz}{d\theta} = -\nu b x^{-1} z$, we substitute $\frac{d\theta}{dt}$ and obtain $z = z_0 e^{-\nu b(t - t_0)}$ by solving $\frac{dz}{z} = -\nu b dt$.

As $u = \frac{1}{2}v^2 + \mathcal{O}(\nu^2)$ for ν small enough it follows from Remark 4.5.2 that $e^{-\nu bt} = e^{-b\Re(A_a - \frac{b}{4}\nu A_a)}(1 + \mathcal{O}(\delta))$. Finally we observe $|H(\theta, v) - H(\theta, \widetilde{v})| = |\frac{1}{2}v^2 - \frac{1}{2}\widetilde{v}^2|$ and (4.25) holds. A different constant applies for θ between θ_m and θ_ν, hence the difference in H at θ is of the same order as the difference in θ_m and (4.26) follows. ☐

Define

$$v_\nu = \Psi(\theta_\nu) \tag{4.27}$$

where Ψ is a parametrisation of the solution of the flow G.

Proposition 4.5.2. *For $(\theta, v) \in U$ the point (θ_ν, v_ν) on the solution of the approximating flow G satisfying (4.18) through (θ, v) is uniquely defined.*

Proof. Take $(\theta, v) \in U$. For (θ, v) close to the fixed point, for ν small enough it follows from Remark 4.3.1 that for the solution $\psi(w)$ through (θ, v) the distance to the stable manifold $\phi_s(\Omega_0)$ decreases, because the flow (4.18) is a perturbation of the flow (4.4).

For (θ, v), satisfying conditions of Lemma 4.3.4, for ν small enough there exists unique $w \in \Omega_0$, such that $\psi(w) = (\theta_1, v_1)$. It implies by Lemma 4.4.1 that ψ can be parametrised by θ; $\Psi(\theta) = v$. Hence $v_\nu = \Psi(\theta_\nu)$ is uniquely defined for $(\theta, v) \in U$. Finally we observe that from continuous dependence on initial conditions and Proposition 4.5.1 $(\theta_\nu, v_\nu) \in U$. ☐

Corollary 4.5.1. *As the consequence of Proposition 4.5.2 a function*

$$H_\nu(\theta, v) = \frac{1}{2}(v_\nu^2 - \Psi_s^2(\theta_\nu)) \tag{4.28}$$

is a well defined analytic function on U, where Ψ_s is a parametrisation of the stable manifold for the flow G satisfying Proposition 4.4.1 and (4.18).

We conclude this section with the relation between the difference in function H_ν on two different solutions in the neighbourhood of the stable manifold ϕ_s and the difference between solutions at θ_m. This relation is used in the main result of this chapter.

Corollary 4.5.2. *It follows immediately from Proposition 4.5.1 that*

$$|H_\nu(\theta_m, v) - H_\nu(\theta_m, \widetilde{v})| = e^{-b\Re(A_a - \frac{b}{4}\nu A_a)}|H(\theta_m, v) - H(\theta_m, \widetilde{v})|(1 + \mathcal{O}(\delta))$$

We have used the approximating flow to define H_ν. Now we can relate the above results to the map and define the nearly periodic function, which measures the width of separatrix splitting region.

4.6 Relating the Dissipative Map to the Approximating Flow

In this section we look at the relation between unstable manifolds of the flow G and the map and state results which enable us to define the nearly periodic function and deduce its properties from previous results on H and H_ν.

Consider the map $\mathcal{F}_{J,\omega,\epsilon}$ (4.1) in (θ, z) variables, where $z = \nu^{-1}xe^{i\theta/2}$. Denote $\gamma_u(w) = (\gamma_1(w), \gamma_2(w))$ a parametrisation of the unstable manifold of the map satisfying $\gamma_1(0) = \theta_m$.

Proposition 4.6.1. *For $s \in \Omega_0$, such that $\gamma_u(s) = (\theta, z)$ there is a constant M such that the distance between unstable manifolds is bounded*

$$|\phi_u(s) - \gamma_u(s)| \leq M(\nu e^{\Im\theta/2})^{11} \tag{4.29}$$

Proof. Take $s \in \Omega_0$ and choose $n > 0$, such that $t = s - n$. Denote $\gamma_u(s) = (\theta_n, z_n)$,$\gamma_u(t+n-i) = (\theta_{n-i}, z_{n-i})$ and $\gamma_u(t) = (\theta_0, z_0)$. Observe that the manifolds satisfy

$$
\begin{aligned}
|\phi_u(t+n)-\gamma_u(t+n)| &= |G(\phi_u(t+n-1))-\mathcal{F}(\gamma_u(t+n-1))| \\
&\leq |G(\phi_u(t+n-1))-\mathcal{F}(\phi_u(t+n-1))| \\
&\quad +|\mathcal{F}(\phi_u(t+n-1))-\mathcal{F}(\gamma_u(t+n-1))| \\
&\leq |G(\phi_u(t+n-1))-\mathcal{F}(\phi_u(t+n-1))| \\
&\quad +|D\mathcal{F}||\phi_u(t+n-1))-\gamma_u(t+n-1)| \\
&\ \ \vdots \\
\leq \textstyle\sum_{j=1}^{n-2} \Big(\Big(\prod_{i=1}^{j} &|D\mathcal{F}(\phi_u(t+n-i))|\Big) |G(\phi_u(t+n-j))-\mathcal{F}(\phi_u(t+n-j))|\Big) \\
&+\textstyle\prod_{i=0}^{n-1} |D\mathcal{F}((\phi_u(t+n-i))||\phi_u(t)-\gamma_u(t)|
\end{aligned} \tag{4.30}
$$

First we consider bounded $\Im\theta_n$, such that $e^{\Im\theta_n/2} < c_2$.
For $s \in \Omega_0$ we choose $n > 0$, such that for $t = s - n$ the manifolds $\phi_u(t), \gamma_u(t) \in B((\theta-\theta_f, 0), \nu)$, where $(\theta_f, 0)$ is the fixed point.

Define $\delta_i = \nu e^{\Im\theta_i/2}$, by argument similar to Lemma 2.2.1 we obtain the bound

$$|D\mathcal{F}(\phi_u(t+n-i)| \leq 1 + c_1\delta_{n-i}. \tag{4.31}$$

From the Proposition 4.4.1 and (4.30) it follows

$$
\begin{aligned}
&|\phi_u(s)-\gamma_u(s)| \\
&\leq \sum_{j=1}^{n-1} \exp(\sum_{i=1}^{j} c_1\nu) M'(c_2\nu)^m + \exp(\sum_{i=0}^{n} c_1\nu) M'\nu^m \\
&\leq cn\exp(nc_1\nu)\nu^m
\end{aligned}
$$

where c depends on M', c_1 and c_2.

The eigenvalue λ_u satisfies $\lambda_u - 1 \sim \mathcal{O}(\nu)$ and $\lambda_u^n \sim \mathcal{O}(\nu^{-1})$, which implies the bound on $n < a\nu^{-1}\ln\nu^{-1}$, hence we obtain

$$|\phi_u(s)-\gamma_u(s)| \leq ac\nu^{m-ac_1-2}$$

For the constants a, c_1 we choose m, such that

$$|\phi_u(s)-\gamma_u(s)| \leq ac\nu^{11}$$

which implies the bound (4.29) with a constant M dependent on M', c_1 and c_2 for bounded $\Im\theta$.

Next we consider unbounded $\Im\theta_n$. Similarly to the bounded case we take $s \in \Omega_0$ and choose n, such that for $t = s - n$ we get $\Im\theta_0 \leq c_2$. It follows from Proposition 4.4.1, (4.30) and (4.31) that

$$|\phi_u(s)-\gamma_u(s)| \leq \sum_{j=1}^{n} \left(\left(\exp(\sum_{i=1}^{j} c_1\delta_{n-i}) \right) M'\delta_{n-j}^m \right) \tag{4.32}$$

We divide θ plane in N strips $S_k = \{\theta : k-1 < \Im\theta < k+1\}$ and define

$\rho_k = \nu e^{k/2}$. The number of strips between θ_n and θ_{n-i} is equal to $2\ln\frac{\delta_n}{\delta_{n-i}}$. Lemma 4.5.2 implies that the number of steps in the strip i is of order δ_i^{-1}, i.e. there is a constant c, such that the number of steps in the strip i is bounded by $\frac{c}{\delta_i}$.

It implies that

$$\exp(\sum_{i=1}^{j} c_1\delta_{n-i}) \leq \left(\frac{\delta_n}{\delta_{n-j}}\right)^{2cc_1} \tag{4.33}$$

and we observe

$$|\phi_u(s) - \gamma_u(s)| \leq M'\delta_n^{2cc_1}\sum_{j=1}^{n}\delta_{n-j}^{m-2cc_1} \leq M'\delta_n^{2cc_1}\sum_{k=1}^{N}\rho_k^{m-2cc_1-1}$$

Since $\sum_{k=1}^{N}\rho_k^{m-2cc_1-1}$ is a geometric series we obtain

$$|\phi_u(s) - \gamma_u(s)| \leq M'C\delta_n^{2cc_1}\rho_N^{m-2cc_1-1} \leq M'C\delta_n^{m-1}$$

and for given c and c_1 we can choose m such that the manifolds satisfy

$$|\phi_u(s) - \gamma_u(s)| \leq M\delta_n^{11}$$

for M dependent on M', C, c_1 and c_2. It implies that on Ω_0

$$|\phi_u(s) - \gamma_u(s)| \leq M(\nu e^{\Im\theta/2})^{11}$$

$\square$

Define set

$$\begin{aligned}\Omega_\nu \;=\; & \{w \in \mathbb{C} : |\Im w| < \tau^* - \frac{1}{2}C_1,\ |H_\nu(\theta, x)| < |\nu\sin\frac{\theta-\theta_f}{2}|^5 \\ & \text{and } |\nu\sin\frac{\theta-\theta_f}{2}| < \frac{2}{C_1}\}\end{aligned}$$

Take (θ, x) on the unstable manifold γ_u of the map (4.1) and take $(\theta, \widetilde{x})$ on the stable manifold ϕ_s of the approximating flow G satisfying Proposition 4.4.1. Denote $(\theta', x') = \mathcal{F}_{J,\omega,\nu}(\theta, x)$ and $(\theta_1, \widetilde{x}_1) = T_1(\theta, \widetilde{x})$, where T_1 is the time-one map of the flow G. Proposition below provides estimates for the difference in time on γ_u and time on ϕ_s.

Proposition 4.6.2. *For $w \in \Omega_\nu$ if $\gamma_u(w) = (\theta, x)$ and $\phi_s(t) = (\theta, \widetilde{x})$ then $w = t + \mathcal{O}(1)$.*

Proof. Take $w \in \Omega_\nu$, consider $(\theta_1, \widetilde{x}_1)$ on ϕ_s, (θ', x') on γ_u and corresponding $(\theta', \widetilde{x}')$ on ϕ_s. From definition of Ω_ν $|x - \widetilde{x}| < |\nu\sin\frac{\theta'-\theta_f}{2}|^4$ and $|H_\nu(\theta', x')| \leq |\nu\sin\frac{\theta'-\theta_f}{2}|^5$, which by Proposition 4.5.1 implies that $|\theta' - \theta_1| \leq Ce^{\lambda_u}|\nu\sin\frac{\theta'-\theta_f}{2}|^4$.

Denote Δw the time difference between points $(\theta', \widetilde{x}')$, $(\theta_1, \widetilde{x}_1)$ on ϕ_s. Since $\Delta w = \frac{dw}{d\theta}\Delta\theta$ and close to the fixed point $\frac{dw}{d\theta} = -\nu e^{-\nu w}$ we can show that $\Delta w = Ce^{\lambda_u}|\nu\sin\frac{\theta'-\theta_f}{2}|^3$ and we obtain that $w - t = \sum_{i=1}^{n} Ce^{\lambda_u}|\nu\sin\frac{\theta'_i-\theta_f}{2}|^3$. As close to the fixed point $n = \mathcal{O}(\nu^{-2})$ we get $w = t + \mathcal{O}(1)$. Away from the fixed point the estimate is better, but since this estimate holds on Ω_ν it is sufficient for next results. $\square$

From Corollary 4.5.1 the function

$$g_\nu(w) = H_\nu(\gamma_u(w)), \tag{4.34}$$

which is a measure of the distance between the stable manifold of the flow and the unstable manifold of the map, is well defined analytic function on Ω_0.

As H_ν is defined as constant on the solutions of the flow G it is almost constant on the orbits of the map (4.1), which imply that $g_\nu(w)$ is nearly periodic and is analogous to the function h defined in Chapter 1 satisfying (1.14) approximately. In order to deduce it from properties of H_ν and other results in previous sections we define a function for $\Im w \geq 0$

$$h(w) = \begin{cases} (w - w^*)^{-1}, & \Re w \leq \frac{1}{\nu} + \sigma^* \\ \nu \exp(-\nu w), & \Re w > \frac{1}{\nu} + \sigma^*, \end{cases}$$

define $h(\overline{w}) = \overline{h(w)}$. As a consequence of Proposition 4.6.2 we obtain

Remark 4.6.1. *For $\theta = \gamma_1(w)$ on the unstable manifold the following relation holds*

$$|\nu \sin \frac{\theta - \theta_f}{2}| = \mathcal{O}(|h(w)|)$$

4.7 Unstable Manifold of the Map and Nearly Periodic Functions

The following proposition states properties of $g_\nu(w)$ and is similar to the hypothesis of Proposition 3.1.1 in Chapter 3 on nearly periodic function

Proposition 4.7.1. *For a constant $C_1 > C_0^*$ sufficiently large, there exists a constant K, such that $\Omega_\nu \supset \Omega_0$ and $g_\nu(w)$ satisfies*

i) $|g_\nu(w)| \leq K|h(w)|^{10}$ *for* $w \in \Omega_0$,

ii) If $w + 1 \in \Omega_\nu$ *then* $|g_\nu(w+1) - g_\nu(w)| \leq K|h(w)|^9$,

iii) if $w \in \Omega_\nu$ *and* $|g_\nu(w)| < |h(w)|^6$ *then* $w + 1 \in \Omega_\nu$,

iv) if $w + 1 \in \mathbb{R} \cap \Omega_\nu$ *then* $\exists\, t \in [w, w+1]$ *such that* $|g_\nu(t)| < |h(w)|^{10}$

Proof. i) Take $w \in \Omega_0$, $\phi_s(w) = (\widetilde{\theta}, \widetilde{x})$, $\gamma_u(w) = (\theta, x)$ from Proposition 4.6.1 and from Remark 4.5.5 by triangle inequality

$$|\phi_s(w) - \gamma_u(w)| \leq (\nu e^{\Im\theta/2})^{11}.$$

We bound $|H_\nu(\theta,x)-H_\nu(\widetilde{\theta},\widetilde{x})|$ by $\frac{\partial H_\nu}{\partial x}|x-\widetilde{x}|+\frac{\partial H_\nu}{\partial \theta}|\theta-\widetilde{\theta}|$. As H_ν is constant on the solution of the flow $\dot{H}_\nu=0$, which gives $\frac{\partial H_\nu}{\partial x}\frac{dx}{d\theta}=-\frac{\partial H_\nu}{\partial \theta}$ and we can bound $|H_\nu(\theta,x)-H_\nu(\widetilde{\theta},\widetilde{x})|$ by $\frac{\partial H_\nu}{\partial x}(|x-\widetilde{x}|+\frac{dx}{d\theta}|\theta-\widetilde{\theta}|)$.

Proposition 4.5.1 gives the bound on $\frac{\partial H_\nu}{\partial x}$, the bound on $\frac{dx}{d\theta}$ along the solution is of order $(\nu e^{\Im\theta/2})^{-1}$ hence from the fact that H_ν is equal to 0 on $\phi_s(w)$ we obtain

$$|H_\nu(\theta,x)|\le M|\nu e^{\Im\theta/2}|^{10}.$$

Remark 4.6.1 implies

$$|g_\nu(w)|\le K|w-w^*|^{-10}$$

and we obtain *i)*.

ii) Take $w+1\in\Omega_\nu$ and $\gamma_u(w+1)=(\theta',x')$, by triangle inequality

$$\begin{aligned}&|g_\nu(w+1)-g_\nu(w)|=|H_\nu(\gamma_u(w+1))-H_\nu(\gamma_u(w))|\\ &\le|H_\nu(\theta',x')-H_\nu(T_1(\theta,x))|+|H_\nu(T_1(\theta,x))-H_\nu(\theta,x)|.\end{aligned}$$

where T_1 is the time-one map of the flow G. As H_ν is constant on the trajectories of the flow the latter difference is 0.

As $\nu|\theta-\theta_f|=\mathcal{O}(|\nu\sin\frac{\theta-\theta_f}{2}|)$ close to the fixed point and $\nu|e^{\Im\theta/2}|=\mathcal{O}(|\nu\sin\frac{\theta-\theta_f}{2}|)$ for large values of $\Im\theta$ the estimates in Proposition 4.4.1 can be bounded by $M|\nu\sin\frac{\theta-\theta_f}{2}|^m$.

By similar argument to point *i)* we obtain

$$|g_\nu(w+1)-g_\nu(w)|\le M|\nu\sin\frac{\theta-\theta_f}{2}|^9$$

and *ii)* follows from Remark 4.6.1.

iii) The assumption $|g_\nu(w)|<|h(w)|^6$ implies $|H_\nu(\theta,x)|\le|\nu\sin\frac{\theta-\theta_f}{2}|^6$. Proposition 4.5.1 implies $|x-\widetilde{x}|\le C|\nu\sin\frac{\theta-\theta_f}{2}|^5$. Similarly to the argument in *ii)* Proposition 4.4.1 implies for $(\theta_1,x_1)=T_1(\theta,x)$

$$|(\theta',x')-(\theta_1,x_1)|\le|\nu\sin\frac{\theta-\theta_f}{2}|^{11}.$$

We also observe that

$$|(\theta_1,x_1)-(\theta',\widetilde{x}_1)|\le e^{\lambda_u}|x-\widetilde{x}|$$

which we can bound by $Ce^{\lambda_u}|\nu\sin\frac{\theta-\theta_f}{2}|^5$. From triangle inequality

$$\begin{aligned}|(\theta',x')-(\theta',\widetilde{x}')| &\le |(\theta',x')-(\theta_1,x_1)|+|(\theta_1,x_1)-(\theta',\widetilde{x}')|\\ &\le 2Ce^{\lambda_u}|\nu\sin\frac{\theta-\theta_f}{2}|^5.\end{aligned}$$

Finally it follows from Proposition 4.5.1 that

$$|H_\nu(\theta',x')|\le C'e^{\lambda_u}|\nu\sin\frac{\theta-\theta_f}{2}|^6,$$

which is less than $|\nu \sin \frac{\theta-\theta_f}{2}|^5$ and hence $w+1 \in \Omega_\nu$.

iv) Since parameters a, b are chosen such that the real manifolds of the map (4.1) meet we can choose $t \in \mathbb{R}$, such that for $t' \in \mathbb{R}$ $\gamma_u(t) = \gamma_s(t')$, for γ_s the unstable manifold of the map. Hence Proposition 4.4.1 implies that

$$|\gamma_u(t) - \phi_s(t')| \leq M' |\nu \sin \frac{\theta - \theta_f}{2}|^{11},$$

which by Proposition 4.5.1 and the fact that $H_\nu(\phi_s(t)) = 0$ imply

$$|H_\nu(\gamma_u(t))| \leq M'C' |\nu \sin \frac{\theta - \theta_f}{2}|^{11}$$

which can be bounded by $|h(w)|^{10}$ and *iv)* follows. $\square$

Next we define $\psi_0(\sigma)$ for a constant A to be specified

$$\psi_0(\sigma) = \begin{cases} \ln(\sigma - \sigma^*) + C_2, & \sigma^* + A < \sigma \leq \nu^{-1} + \sigma^* \\ (\sigma - \sigma^*)\nu - \ln \nu + C_2 - 1, & \sigma > \nu^{-1} + \sigma^* \end{cases}$$

where $C_2 = \frac{\pi}{3}A - \ln A$. Define $\psi(\sigma) = \tau^* - \frac{3}{\pi}\psi_0(\sigma)$ for $\sigma^* + A < \sigma < M$, where $M = \nu^{-1}(\frac{\pi}{3}\tau^* + \ln \nu - C_2 + 1) + \sigma^*$.

We note that for $\tau = \psi(\sigma)$

$$e^{2\pi(\tau - \tau^*)} = \begin{cases} e^{-6C_2}(\sigma - \sigma^*)^{-6}, & \sigma^* + A < \sigma \leq \nu^{-1} + \sigma^* \\ e^{-6C_2}\nu^6 e^6 e^{-6\nu(\sigma - \sigma^*)}, & \sigma > \nu^{-1} + \sigma^* \end{cases}$$

Denote $w(\sigma) = (\sigma, i\psi(\sigma))$. From definition of function $h(w)$ we observe that $e^{2\pi(\psi(\sigma) - \tau^*)}$ is of order $|h(w(\sigma))|^6$ i.e. there are positive constants c_1, c_2 independent on A, such that

$$c_1 e^{2\pi(\psi(\sigma)-\tau^*)} < e^{-6C_2}|h(w(\sigma))|^6 < c_2 e^{2\pi(\psi(\sigma)-\tau^*)}. \tag{4.35}$$

We will use this property in the proof of the main result of this thesis, which gives the approximation to oscillation of $g_\nu(w)$ defined in (4.34). It is related to the conclusion of the Proposition 3.1.1 in Chapter 3.

Proposition 4.7.2. *There exist constants $A > C_1$ and $D > 0$ such that Ω_ν contains the domain*

$$V_\nu = \{w = \sigma + i\tau : \ \Re w_0 < \sigma < M, \ |\tau| < \psi(\sigma)\}$$

where $w_0 = (\sigma^* + A, i\tau^* - A)$ and there are complex numbers b_ν and μ_ν such that $|b_\nu|$, $|\mu_\nu| < De^{-2\pi\tau^*}$ and for $w \in V_\nu$ the following holds

$$|g_\nu(w) - (b_\nu + \mu_\nu e^{-2\pi i w} + \overline{\mu}_\nu e^{2\pi i w})| \leq D|h(w)|^7 \tag{4.36}$$

Proof. First we define $V_s = \{w \in V_\nu : \Re w < s\}$ and

$$s_* = \sup\{s \leq M : \ \forall w \in V_s \text{ we have } w \in \Omega_\nu \text{ and } |g_\nu(w)| \leq \frac{1}{2}|h(w)|^6\}$$

Denote $V_* = V_{s_*}$. We begin the proof by lemma

Lemma 4.7.1. *For $w \in V_*$, $|g'_\nu(w+1) - g'_\nu(w)| < CK|h(w)|^8$ for a constant $C > 0$.*

Proof. Take $w \in V_*$ and $z \in B(w,r)$ where $r = \frac{3}{\pi}|h(w)|$.

From the definition of ∂V_* the slope of $\psi(\sigma)$ is equal

$$|\psi'(\sigma)| = \begin{cases} \frac{3}{\pi}|\sigma - \sigma^*|^{-1}, & \sigma \leq \nu^{-1} + \sigma^* \\ \frac{3}{\pi}\nu, & \sigma > \nu^{-1} + \sigma^* \end{cases}$$

which is bigger than the slope of the line through $z-1$ and w for $r = \frac{3}{\pi}|h(w)|$ which implies that $z - 1 \in V_* \cup \Omega_0$.

From definition of V_* and point *i)* we get $|g_\nu(z-1)| < \frac{1}{2}|h(z-1)|^6$. From condition *iii)* it follows that $z \in \Omega_\nu$ and from *ii)* $|g_\nu(z) - g_\nu(z-1)| < K|h(z-1)|^9$.

For A sufficiently large $\frac{3}{4}|h(z-1)|^6 < |h(z)|^6$ and $|h(z-1)| < \frac{1}{4K}$, the latter implies $K|h(z-1)|^9 < \frac{1}{4}|h(z-1)|^6$. Hence from triangle inequality $|g_\nu(z)| < |h(z)|^6$ which by *iii)* implies $z + 1 \in \Omega_\nu$ and condition *ii)* holds for $z \in B(w,r)$.

From the Cauchy inequalities

$$|g'_\nu(w+1) - g'_\nu(w)| \leq \frac{\pi}{3} \max_{B(w,|h(w)|)} |g_\nu(z+1) - g_\nu(z)|/|h(w)|$$

It implies that $|g'_\nu(w+1) - g'_\nu(w)| \leq CK|h(w)|^8$ for $C > \frac{\pi}{3}$ and lemma holds.

Define $\alpha_k(w) = \int_w^{w+1} e^{2\pi i k z} g'_\nu(z)dz$, $\beta_k(w) = \int_0^1 e^{2\pi i k s} g'_\nu(w+s)ds$. Note that $|\beta_k(w)| = |e^{-2\pi i k w}\alpha(w)|$ and $|\alpha'_k(w)| = |e^{2\pi i k w}(g'_\nu(w+1) - g'_\nu(w))|$. For convenience we use notation $a_k(\sigma) = \alpha_k(w(\sigma))$.

Lemma 4.7.1 implies that for $w \in V_*$

$$|\alpha'_k(w)| \leq CKe^{-2\pi k \Im w}|h(w)|^8. \tag{4.37}$$

We use this fact to bound $|\beta_k(w)|$. The following properties of $|\alpha'_k(w)|$ for $\Im w \geq 0$ follow from the bound (4.37) and from (4.35). Take $w_1 = w(\sigma_1)$, $w_2 = w(\sigma_2)$, $\sigma_0 \leq \sigma_1 < \sigma_2 \leq s_*$,

for $k = 1$

$$\int_{\sigma_1}^{\sigma_2} |a'_1(\sigma)|d\sigma \quad \leq CKc_2 e^{6C_2} e^{-2\pi\tau^*}|h(w_1)| \tag{4.38}$$

$$\leq CKc_1^{-1}c_2 e^{-2\pi \Im w_1}|h(w_1)|^7 \tag{4.39}$$

for $k \geq 2$

$$\begin{aligned} \int_{\sigma_1}^{\sigma_2} |a'_k(\sigma)|d\sigma &\leq CK \int_{\sigma_1}^{\sigma_2} |e^{2\pi k w(\sigma)}||h(w(\sigma))|^8 d\sigma \\ &\leq \frac{1}{k} CKc_2 e^{6C_2 k} e^{-2\pi k \tau^*}|h(w_2)|^{7-6k} \qquad (4.40) \\ &\leq \frac{1}{k} CKc_1^{-1}c_2 e^{-2\pi k \Im w_2}|h(w_2)|^7 \end{aligned}$$

Now take $w = \sigma + i\tau$, $\Im w_1 = \tau_1, \Im w_2 = \tau_2$, $0 \le \tau_1 < \tau_2 \le \psi(\sigma)$,

for $k > 0$

$$\begin{aligned} \int_{\tau_1}^{\tau_2} |\alpha_k'(w)| d\tau &\le CK \begin{cases} e^{-\tau_1} |h(w_1)|^8 \int_{\tau_1}^{\tau_2} e^{(-2\pi k+1)\tau} d\tau, & \sigma \le \frac{1}{\nu} + \sigma^* \\ |h(w_1)|^8 \int_{\tau_1}^{\tau_2} e^{-2\pi k \tau} d\tau, & \sigma > \frac{1}{\nu} + \sigma^* \end{cases} \\ &\le \frac{1}{k} CK e^{-2\pi k \tau_1} |h(w_1)|^7 \end{aligned} \tag{4.41}$$

$$\begin{aligned} \int_{\tau_1}^{\tau_2} |\alpha_{-k}'(w)| d\tau \le CK |h(w_2)|^8 \int_{\tau_1}^{\tau_2} e^{2\pi k \tau} d\tau \\ \le \frac{1}{k} CK e^{2\pi k \tau_2} |h(w_2)|^7 \end{aligned} \tag{4.42}$$

By condition *i)* and lemma similar to Lemma 4.7.1 for $k > 1$ we obtain

$$|a_k(\sigma_0)| \le CK \int_{w_0}^{w_0+1} |e^{2\pi i k s}||h(s)|^9 ds$$

As $\max_{s \in [w_0, w_0+1]} |h(s)| = |h(w_0)|$ it follows that

$$|a_k(\sigma_0)| \le CK e^{-2\pi i k \tau_0} |h(w_0)|^9 \tag{4.43}$$

Take $w = \sigma + i\tau \in V_*$ and $w(\sigma) \in \partial V_*, \tau' = \Im w(\sigma)$. The following inequality holds

$$|\alpha_k(w)| \le |a_k(\sigma_0)| + \int_{\sigma_0}^{\sigma} |a_k'(\sigma)| d\sigma + \int_{\tau}^{\tau'} |\alpha_k'(w)| d\tau \tag{4.44}$$

From (4.40), (4.41), (4.43) follows that for $k \ge 2$ and for a positive constant c dependent on c_1, c_2

$$|\alpha_k(w)| \le \frac{c}{k} CK e^{-2\pi k \tau} |h(w)|^7 \tag{4.45}$$

and hence

$$|\beta_k(w)| \le \frac{c}{k} CK |h(w)|^7 \tag{4.46}$$

Next we bound $|\alpha_1(w) - a_1(s_*)|$. We observe that

$$|\alpha_1(w) - a_1(s_*)| \le \int_{\tau}^{\tau'} |\alpha_1'(w)| d\tau + \int_{\sigma}^{s_*} |a_1'(s)| ds$$

From (4.39) and (4.41) it follows that

$$|\alpha_1(w) - a_1(s_*)| \le cCK e^{-2\pi\tau} |h(w)|^7$$

and hence

$$|\beta_1(w) - a_1(s_*) e^{-2\pi i w}| \le cCK |h(w)|^7. \tag{4.47}$$

It remains to bound $|\beta_k(w)|$ for negative k. We note that for $k > 1$

$$|\alpha_{-k}(w)| = |\alpha_{-k}(\sigma)| + \int_0^\tau |\alpha'_{-k}(w)|dt$$

Bound on $|\alpha_{-k}(\sigma)|$ comes from the fact that $\sigma \in \mathbb{R}$ and from (4.45) $|\alpha_{-k}(\sigma)| = |\alpha_k(\sigma)| \le \frac{c}{k} CK|h(\sigma)|^7$. To bound the integral we apply (4.42) to obtain $\int_0^\tau |\alpha'_{-k}(t)|dt < \frac{1}{k} CKe^{2\pi k\tau}|h(w)|^7$.

Thus for $k < -1$ $|\alpha'_k(w)| \le \frac{c}{k} CKe^{-2\pi k\tau}|h(w)|^7$.

Next we note

$$|\alpha_{-1}(w) - a_{-1}(s_*)| \le \int_0^\tau |\alpha'_{-1}(w)|dt + |\alpha_{-1}(\sigma) - a_{-1}(s_*)|$$

By similar argument that before from (4.39) and (4.42) we obtain

$$|\alpha_{-1}(w) - a_{-1}(s_*)| \le cCKe^{2\pi\tau}|h(w)|^7$$

Now consider $\Im w < 0$. Note that $\overline{\beta_k(\overline{w})} = \beta_{-k}(w)$. From the bound (4.46) on $|\beta_k(w)|$ for $\Im w \ge 0$ follows that for $\Im w \le 0$ $|\beta_{-k}(w)| < \frac{c}{k} CK|h(w)|^7$, similarly from (4.47) $|\beta_{-1}(w) - a_{-1}(s_*)e^{2\pi i w}| < cCK|h(w)|^7$.

Hence for $w \in V_*$ the following holds

$$|\beta_k(w)| < \frac{c}{k} CK|h(w)|^7 \tag{4.48}$$

for $|k| > 1$, and

$$|\beta_{\pm 1}(w) - a_{\pm 1}(s_*)e^{\mp 2\pi i w}| < cCK|h(w)|^7 \tag{4.49}$$

Define $\phi(w) = \int_0^1 g_\nu(w+s)ds$, $b_\nu = \phi(s_*)$. We bound $|\phi(w) - b_\nu|$. We can write $|\phi(w) - b_\nu| = |\int_{s_*}^w \phi'(z)dz| \le \int_{s_*}^w |g_\nu(z+1) - g_\nu(z)|dz$.

Note that

$$\int_{s_*}^w |h(z)|^8 dz = \int_0^\tau |h(z(t))|^8 dt + \int_\sigma^{s_*} |h(z(s))|^8 ds$$

after integrating the right hand side of above and applying *ii)* it follows that

$$|\phi(w) - b_\nu| \le K|h(w)|^7. \tag{4.50}$$

Now we define $h(s) = g_\nu(w+s) - s\left(g_\nu(w+1) - g_\nu(w)\right)$ for $s \in [0,1]$ and $\delta_k(w) = \int_0^1 h(s)e^{2\pi i k s}ds$. After integrating by parts we obtain that for $k \ne 0$ $\delta_k(w) = \frac{1}{2\pi i k}\beta_k(w)$. It follows that $|\delta_k(w)| = |\delta_{-k}(w)|$ and from (4.48) $|\delta_k(w)| \le \frac{c}{k^2} CK|h(w)|^7$.

We note that $h(0) = h(1) = g_\nu(w)$ and write $h(s) = \sum \delta_k(w)e^{-2\pi i k s}$. By taking $s = 0$ we get $g_\nu(w) = \sum \delta_k(w)$ which implies the following bound

$$|g_\nu(w) - (\delta_0(w) + \delta_1(w) + \delta_{-1}(w))| = |\sum_{k \ne \{0,\pm 1\}} \delta_k(w)| \le d_1|h(w)|^7$$

for a positive constant d_1 dependent on c_1, c_2, C, K. The following bounds hold from (4.49) and (4.50)

$$|\delta_0(w) - b_\nu| \leq K|h(w)|^7$$
$$|\delta_1(w) - \mu_\nu e^{-2\pi i w}| \leq cCK|h(w)|^7$$
$$|\delta_{-1}(w) - \overline{\mu}_\nu e^{2\pi i w}| \leq cCK|h(w)|^7$$

where $\mu_\nu = \frac{1}{2\pi i} a_1(s_*)$. The above imply the bound

$$|g_\nu(w) - (b_\nu + \mu_\nu e^{-2\pi i w} + \overline{\mu}_\nu e^{2\pi i w})| \leq d|h(w)|^7 \tag{4.51}$$

for a constant d dependent on c_1, c_2, C, K.

Now we show that $|\mu_\nu|$ is bounded. We observe that

$$\begin{aligned} |a_1(s_*)| &\leq |a_1(\sigma_0)| + \int_{\sigma_0}^{s_*} |a_1'(s)| ds \\ &\leq CKc_2 e^{6C_2} e^{-2\pi\tau^*} |h(w_0)| \end{aligned}$$

The latter inequality follows from (4.38) and (4.43). Hence there exists a positive constant D_1 depending on C, K, c_2 such that $|\mu_\nu| < D_1 e^{2\pi A} A^{-7} e^{-2\pi\tau^*}$.

Next we take $w + 1 \in \mathbb{R} \cap V_*$, find $x \in [w, w+1]$ and apply (4.51) at x. From point *iv)* $|g_\nu(x)| < |h(w)|^{10}$ hence we can write

$$|b_\nu| < d|h(w)|^7 + |h(w)|^{10} + 2D_1 e^{2\pi A} A^{-7} e^{-2\pi\tau^*}, \tag{4.52}$$

which implies the bound for $\Im w \geq 0$

$$\begin{aligned} |g(w)| &< d|h(w)|^7 + d|h(\Re w)|^7 + |h(\Re w)|^{10} + 4D_1 e^{2\pi A} A^{-7} e^{-2\pi(\tau^* - \Im w)} \\ &< 2d|h(w)|^7 + |h(w)|^{10} + 4D_1 e^{2\pi A} A^{-7} e^{-2\pi(\tau^* - \Im w)} \\ &< 3d|h(w)|^7 + 4D_1 e^{2\pi A} A^{-7} c_2^{-1} e^{-6C_2} |h(w)|^6 \\ &= 3d|h(w)|^7 + 4D_1 A^{-1} c_2^{-1} |h(w)|^6 \end{aligned} \tag{4.53}$$

As $|h(w)|^3 < d$ for $A > 1$ and $\Im w < \psi(\sigma)$ the inequality (4.53) holds.

We obtain $|g_\nu(w)| < \frac{1}{3}|h(w)|^6$ provided $|h(w)| < (18d)^{-1}$ and $4D_1 A^{-1} c_2^{-1} < \frac{1}{6}$ which both hold for A large enough. The bound for $\Im w < 0$ comes from the fact that $g_\nu(\overline{w}) = \overline{g_\nu(w)}$.

Next we show that $s_* = M$. Suppose $s_* < M$. By continuity there is $s' > s_*$ such that for $w \in V_{s'}$ $|g_\nu(w)| < \frac{1}{2}|h(w)|^6$, which contradicts definition of s_* and $s_* = M$ follows.

Finally bound on $|b_\nu|$ comes from applying (4.52) at $w = M$ and observing that $|h(M)|^7$ is or order $e^{-\frac{7}{3}\pi(\tau^* - A)}$ which is small compared to $e^{-2\pi(\tau^* - A)}$.

Hence there is a constant D depending on A, C, K, c_1, c_2 such that (4.36) holds, $|\mu_\nu|, |b_\nu| < De^{-2\pi\tau^*}$and the proof is complete. □

4.8 Relation with the Semistandard Map

In this section we state the result relating the manifolds of the dissipative map (4.1) and the semistandard map. We give general ideas and sketches of the proofs and omit the details. Results of this section can be proved rigorously by methods presented in previous sections.

First we observe that after the shift $\theta = \varphi + i \ln \frac{2}{\nu^2}$ the dissipative map (3.1) becomes

$$
\begin{aligned}
x_{n+1} &= Jx_n + \omega + ie^{-i\varphi_n} - \frac{i}{4}\nu^4 e^{i\varphi_n} \\
\varphi_{n+1} &= \varphi_n + x_{n+1}
\end{aligned}
\tag{4.54}
$$

When parameters $J \to 1$, $\nu, \omega \to 0$ then the map can be approximated by the semistandard map (3.1).

Second we observe that after the change of variables $(\psi, z) = (\varphi + 2k\ln(2i), \frac{x}{\varrho})$, where $\varrho = 2^{-k}$, the flow

$$
\begin{aligned}
\dot{z} &= \varrho i e^{-i\psi} \\
\dot{\psi} &= \varrho z.
\end{aligned}
\tag{4.55}
$$

is the first order approximation to the semistandard map (3.1) in (ψ, z) coordinates.

We can construct the flow G_0, a perturbation of the system (4.55), which gives a higher order approximation to the semistandard map. We expand G_0 to the same order as the approximating flow of the dissipative map. We consider the flow G_0 on the domain in $\{(\varphi, x) : \Im\varphi < 0, |x| < \sqrt{2}ce^{\Im\varphi/2}\}$ and apply Proposition 2.1.3 to the strips $\{-2(k+1)\ln 2 < \Im\varphi < -2(k-1)\ln 2\}$ in a similar manner to Proposition 4.4.1. As the argument is similar to the proof of Proposition 4.4.1 we omit the details.

Next we show the relation between the unstable manifold for the dissipative map (4.1) and the semistandard map (3.1). We recall $\gamma_0(\xi)$ a parametrisation of the unstable manifold for the semistandard map defined in variables (F, u) introduced in Section 3.1. We also denote $\gamma_\nu(w)$ a parametrisation of the unstable manifold for the dissipative map (4.1), where $w = \xi + w^*$.

We use notation $\gamma_\nu(w) = (\overline{F}_w, \overline{u}_w)$ for coordinates of the dissipative map and $(\widetilde{F}_w, \widetilde{u}_w)$ for the approximating flow G satisfying (4.18).

After appropriate shift in time unstable manifold of the semistandard map is a limit of the unstable manifold of the dissipative map as $\nu \to 0$. We state the following

Proposition 4.8.1. *For $\xi \in \mathbb{C}$, if $\nu \to 0$ then $\gamma_\nu(\xi + w^*) \to \gamma_0(\xi)$.*

Proof. Take $\xi \in \mathbb{C}, \epsilon > 0$ and choose k such that $\Re(\xi - k) < 0$. Lemma 3.1.2 implies that $|F_{w-k}| = \mathcal{O}(|\nu e^{\Im\theta/2}|^4)$.

From relation between x and v we obtain that $|\widetilde{F}_{w-k}| \le \frac{1}{2}|H| + \mathcal{O}(|\nu e^{\Im\theta/2}|^4)$. From Proposition 4.6.1 for ν small enough the bound holds for the dissipative map $|\overline{F}_{w-k}| \le \frac{1}{2}|H| + \mathcal{O}(|\nu e^{\Im\theta/2}|^4)$. For ν small enough Lemma 4.4.1 implies $|H(\theta, v)| = \mathcal{O}(|\nu e^{\Im\theta/2}|^4)$, so from relation between θ and u we obtain $|\overline{F}_{w-k}| = \mathcal{O}(|\overline{u}|^{-4})$ and it follows that

$$
\Delta F_{w-k} = |\overline{F}_{w-k} - F_{w-k}| = \mathcal{O}(|u_{w-k}|^{-4})
$$

Remark 4.5.2 and relation between θ and u imply $w - w^* = \widetilde{u} + \mathcal{O}(|\widetilde{u}|^{-1})$. From Proposition 4.6.1 for ν small enough $\xi = w - w^* = \overline{u} + \mathcal{O}(|\overline{u}|^{-1})$ and by Remark 3.1.1 it follows that

$$\Delta u_{w-k} = |\overline{u}_{w-k} - u_{w-k}| = \mathcal{O}(|u_{w-k}|^{-1})$$

Recall notation $(u_w, F_w) = h^k(u_{w-k}, F_{w-k})$ from Section 3.1 and observe that from the Cauchy estimates (3.3) if follows $|(u_w, F_w) - h^k(\overline{u}_{w-k}, \overline{F}_{w-k})| = \mathcal{O}(|u_{w-k}|^{-1})$ for orbits of the semistandard map.

For ν small enough we obtain $|h^k(\overline{u}_{w-k}, \overline{F}_{w-k}) - (\overline{u}_w, \overline{F}_w)| = \mathcal{O}(|\overline{u}_{w-k}|^{-1})$ which imply that

$$\begin{aligned} \Delta F_w &= |\overline{F}_w - F_w| = \mathcal{O}(|u_{w-k}|^{-1}) \\ \Delta u_w &= |\overline{u}_w - u_w| = \mathcal{O}(|u_{w-k}|^{-1}) \end{aligned}$$

For k large enough $|u_{w-k}|^{-1}$ is less than ϵ. Hence for $\epsilon > 0$ there is $\nu_0 > 0$, such that for $\nu < \nu_0$ we obtain $|\gamma_\nu(\xi + w^*) - \gamma_0(\xi)| < \epsilon$. □

Next consider parametrisation of the unstable manifold $\widetilde{\gamma}_0(\xi)$ for the semistandard map (3.1) in (φ, x) coordinates. Then we take (φ, x) in the neighbourhood of $\widetilde{\gamma}_0(\xi)$. We fix $\varphi_0 = i \ln \frac{1}{2A^2}$ and take (φ_0, x_0) on the solution of the semistandard map approximating flow G_0 through (φ, x). By argument similar to Proposition 4.5.2 we obtain that x_0 is uniquely defined and hence $\mathcal{H}_0(\varphi, x) = \frac{1}{2}x_0^2$ is well defined in the neighbourhood of $\widetilde{\gamma}_0(\xi)$. Next we define $\overline{g}_0(\xi) = \mathcal{H}_0(\widetilde{\gamma}_0(\xi))$ and note that after the appropriate change of variables one can show that $|F| \approx \frac{1}{2}|\mathcal{H}_0|$. Finally we define $\mathcal{H}_\nu(\theta, v) = H_\nu(\theta, v) + \frac{1}{2}\Psi_s^2(\theta_\nu)$, $\overline{g}_\nu(w) = \mathcal{H}_\nu(\gamma_\nu(w))$ and state the following

Lemma 4.8.1. *For $\xi \in V_0 = \{\Re\xi > A, \Im\xi < C_2 - \frac{3}{\pi}\ln\Re\xi\}$*

$$\lim_{\nu\to 0} \overline{g}_\nu(\xi + w^*) = \overline{g}_0(\xi) \tag{4.56}$$

Proof. Denote the change of variables $(\theta, v) = P(\varphi, x) = (\varphi + i\ln 2\nu^{-1}, x + \frac{1}{2}\nu^2(a - b\nu^{-1}x + \sin\theta))$. It follows from Remark 2.1.3 that $\mathcal{H}_\nu \circ P(\varphi, x) \to \mathcal{H}_0(\varphi, x)$, which implies (4.56). □

The limiting behaviour of the constant μ_ν is a consequence of Lemma 4.8.1 and Proposition 4.7.2 and we state the following

Proposition 4.8.2. *For the parameter a there is a constant μ*

$$\lim_{\nu\to 0} \mu_\nu e^{-2\pi i(\nu^{-1} - \frac{b}{4})A_a} \to \mu \tag{4.57}$$

where b gives existence of homoclinic orbit and A_a is the constant introduced in Remark 4.5.2.

Proof. Take $\epsilon' > 0$ and choose $\xi \in \partial V_0$, such that $\Im\xi < \frac{3}{\pi}\ln\frac{\epsilon'}{2D_1}$ for the constant D_1 to be specified below. We apply Proposition 4.7.2 to $w = \xi + w^*$ and $w + 1/2$.

It follows by (4.36) that for ν and ν'

$$|g_\nu(w)-g_{\nu'}(w')-(g_\nu(w+1/2)-g_{\nu'}(w'+1/2))-(2\mu_\nu e^{-2\pi iw}-2\mu_\nu e^{-2\pi iw'})|<D|h(w)|^7$$

Lemma 4.8.1 implies that g_ν is a Cauchy sequence, so we choose $\epsilon<\epsilon' e^{2\pi\Im\xi}$ and then we choose ν_0, such that for $\nu'<\nu<\nu_0$

$$|\mu_\nu e^{-2\pi iw}-\mu_{\nu'}e^{-2\pi iw'}|<\epsilon/2+D|h(w)|^7.$$

From (4.35) we obtain

$$|\mu_\nu e^{-2\pi iw}-\mu_{\nu'}e^{-2\pi iw'}|<\epsilon/2+D_1 e^{7\pi/3\Im\xi}$$

where D_1 depends on A, D and c_2. It follows that

$$|\mu_\nu e^{-2\pi iw^*}-\mu_{\nu'}e^{-2\pi iw'^*}|<e^{-2\pi\Im\xi}\epsilon/2+D_1 e^{\pi/3\Im\xi}$$

which for a given choice of ξ and ϵ implies

$$|\mu_\nu e^{-2\pi iw^*}-\mu_{\nu'}e^{-2\pi iw'^*}|<\epsilon'.$$

Hence $\mu_\nu e^{-2\pi iw^*}$ is a Cauchy sequence, so tends to the limit μ when $\nu\to 0$. Finally (4.57) follows from Remark 4.5.2. □

We apply Proposition 4.7.2 to g_ν as $\nu\to 0$. From Lemma 4.8.1 and Proposition 4.8.2 follows that

Remark 4.8.1. *For $\xi\in V_0$*

$$|\overline{g}_0(\xi)-\mu e^{-2\pi i\xi}|\le D|\xi|^{-7}$$

In addition by argument similar to Proposition 3.1.2 one can show that

$$|\overline{g}_0(\xi)-2\mu_0 e^{-2\pi i\xi}|\le D|\xi|^{-3}$$

where μ_0 is the constant discussed in Chapter 3. From the above we obtain that $|\mu|\approx 2|\mu_0|$.

Next we fix the parameter a, consider the flow (4.4) and take $y_u(\theta_m)$ the value on the unstable manifold of θ_0 and $y_s(\theta_m)$ the value on the stable manifold of $\theta_0+2\pi$ at θ_m. Define $f(b)=\frac{1}{2}(y_u^2(\theta_m)-y_s^2(\theta_m))$. Denote the constant $c=f'(\Psi(a))$ and the constant $\Lambda=8c^{-1}e^{(\frac{\pi}{2}\Im A_a+\Re A_a)b}$, definition of function Ψ was introduced in Section 4.2 and it is graphically presented in Figure 4.1.

We state the main result of this chapter.

Theorem 4.8.1. *For a fixed parameter a the width of the interval $|J_2-J_1|$ where the separatrix splitting occurs for the dissipative map (4.1) satisfies*

$$\lim_{\epsilon\to 0} e^{2\pi\sqrt{\epsilon}^{-1}\Im A_a}\sqrt{\epsilon}|I_\omega|=\Lambda|\mu_0| \tag{4.58}$$

Proof. We outline the idea of the proof. Define $H(b) = \frac{1}{2}(H_\nu(\theta_m, x_u) - H_\nu(\theta_m, x_s))$, for a fixed parameter a the width of $|I_a| = |b_2 - b_1|$ can be estimated as $|H(b_2) - H(b_1)|/\frac{dH(b)}{db}$.

By Proposition 4.7.2 close to the fixed point $(\theta_0 + 2\pi, 0)$, at $\theta_0 + 2\pi - \delta$ for small δ, oscillations of $g_\nu(w)$ is $4|\mu_\nu|$, and one can show that the difference in H_ν close to the fixed point is approximately the difference in H_ν at θ_m so $|H(b_2) - H(b_1)| \approx 4|\mu_\nu|$.

From Proposition 4.8.2 we obtain $4|\mu_\nu| \approx 4e^{-2\pi\Im A_a(\nu^{-1} - \frac{b}{4})}|\mu|$ and from Remark 4.8.1 the numerator can be estimated by $8e^{-2\pi\Im A_a(\nu^{-1} - \frac{b}{4})}|\mu_0|$.

By Chain Rule the denominator $\frac{dH}{db} = \frac{dH}{df}\frac{df}{db}$. Corollary 4.5.2 and relation between v and y imply $\frac{dH}{df} \approx \nu^2 e^{-b\Re A_a}$.

Finally we combine the numerator and the denominator and obtain

$$|I_a| = 8|\mu_0|e^{-2\pi\Im A_a(\nu^{-1} - \frac{b}{4})}c^{-1}\nu^{-2}e^{b\Re A_a}(1 + o(1))$$

It follows that

$$\lim_{\nu\to 0} e^{2\pi\nu^{-1}\Im A_a}\nu^2|I_a| = \Lambda|\mu_0|.$$

When we scale back from parameters a, b to J, ϵ, ω we obtain from $|I_\omega| = \nu|I_a|$ that

$$\lim_{\epsilon\to 0} e^{2\pi\sqrt{\epsilon}^{-1}\Im A_a}\sqrt{\epsilon}|I_\omega| = \Lambda|\mu_0|$$

□

In subsequent section we provide numerical results for the constants A_a, c and Λ, which support Theorem 4.8.1.

4.9 Numerical Results

In the final section of this chapter we explain the details of numerical calculations. We consider the flow (4.4) and for a given parameter a we compute parameter b leading to the homoclinic orbit, and constants A_a and c defined above.

We rewrite the system (4.4) with θ as independent variable

$$\begin{aligned} \frac{dy}{d\theta} &= \frac{a - by + \sin\theta}{y} \\ \frac{dt}{d\theta} &= \frac{1}{y} \end{aligned} \tag{4.59}$$

First to find c we consider $f_1(b) = \frac{\partial(y_u - y_s)}{\partial b}(\theta_m)$, denote $z = \frac{\partial y}{\partial b}$ and calculate $\frac{\partial^2 y}{\partial b \partial\theta}$ from (4.59). It follows that

$$\frac{dz}{d\theta} = -\frac{a + \sin\theta}{y^2}z - 1 \tag{4.60}$$

We observe that $\frac{df}{db} = f_1(b)y(\theta_m)$, which implies

$$c = (z_u(\theta_m) - z_s(\theta_m))y(\theta_m) \tag{4.61}$$

To compute b and c for a fixed a we consider (4.59) and (4.60) in real variables. We expand y at the fixed point $\theta_0 = \arcsin(-a)$ and obtain

$$y(\theta) = c_1(\theta-\theta_0) + c_2(\theta-\theta_0)^2 + c_3(\theta-\theta_0)^3 + \mathcal{O}((\theta-\theta_0)^4).$$

We substitute the expansion to the system (4.59) to find the coefficients

$$\begin{aligned} c_1 &= \pm\frac{1}{2}(b-\sqrt{b^2+4\cos\theta_0}) \\ c_2 &= -\frac{\sin\theta_0}{2(b+3c_1)} \\ c_3 &= -\frac{12c_2^2+\cos\theta_0}{6(b+4c_1)} \end{aligned}$$

The $+$ sign in c_1 corresponds to the stable manifold and the $-$ sign to the unstable manifold.

We solve the system of equations (4.59) and (4.60) using 4th order Runge-Kutta method starting from the point $\theta_0+\delta$ on the unstable manifold and from $\theta_0+2\pi-\delta$ on the unstable manifold until we reach θ_m. We use bisection method to find parameter b, where manifolds coincide and we calculate constant c using (4.61).

To find A_a we consider the system (4.59) in complex variables and we solve in imaginary direction starting from θ_m until A_a converges to a constant, which in practise is for θ of order 10^2. We calculate constant Λ as a function of b, c and A_a using the program `separatrix_splitting.c` for the range of the input parameter a. The program is included in the Appendix and the results are presented in Table 4.1.

Observations

1 For $a = 0$, $A_a = i\frac{\pi}{2}$, which corresponds to the singularity of the complex homoclinic orbit for the pendulum equation.

2 For $a = 0.01$ the ratio $b/a = 0.7854$ is approximately equal to $\frac{\pi}{4}a$, which corresponds to the slope of the curve Ψ at $a = 0$ calculated in Lemma 4.2.1.

3 For $a = 1$ parameter $b^{-2} = 0.702563563$, which corresponds to a constant given in [3] (p. 761) and we explain the relationship in more details below

4 First two columns of Table 4.1 are presented on Figure 4.1 and form a curve Ψ in the parameter space (a, b) where the homoclinic orbit Φ occurs for the system (4.4).

We conclude this chapter with relating $a = 1$ case to the result in [3]. For the parameter $a = 1$ after the change of variables $\xi = \theta + \frac{\pi}{2}$, $\eta = by$ and scaling in time $t = \frac{1}{b}s$ the flow (4.4) becomes

$$\begin{aligned} \dot{\eta} &= -\eta + 2\sin^2\frac{\xi}{2} \\ \dot{\xi} &= b'\eta \end{aligned}$$

Table 4.1: Numerical results of separatrix_splitting.c program

a	b	c	$\Re A_a$	$\Im A_a$	Λ
0.00	0.000000	7.999922	0.000000	1.570796	1.000010
0.01	0.007854	8.000042	0.003153	1.570809	1.019538
0.05	0.039285	8.002918	0.015776	1.571125	1.102088
0.10	0.078660	8.011975	0.031625	1.572114	1.215604
0.15	0.118219	8.027300	0.047620	1.573776	1.342413
0.20	0.158057	8.049251	0.063840	1.576130	1.484779
0.25	0.198278	8.078360	0.080370	1.579204	1.645489
0.30	0.238995	8.115362	0.097301	1.583038	1.828015
0.35	0.280336	8.161245	0.114740	1.587681	2.036733
0.40	0.322444	8.217323	0.132809	1.593202	2.277248
0.45	0.365489	8.285338	0.151652	1.599684	2.556844
0.50	0.409673	8.367616	0.171445	1.607237	2.885182
0.55	0.455246	8.467314	0.192409	1.616007	3.275357
0.60	0.502523	8.588798	0.214827	1.626183	3.745603
0.65	0.551915	8.738291	0.239074	1.638023	4.322149
0.70	0.603983	8.924994	0.265672	1.651892	5.044282
0.75	0.659531	9.163220	0.295381	1.668322	5.974013
0.80	0.719783	9.476826	0.329383	1.688145	7.216332
0.85	0.786799	9.909771	0.369706	1.712784	8.967765
0.90	0.864595	10.556962	0.420389	1.745055	11.659070
0.95	0.963536	11.695356	0.492120	1.792354	16.563262
1.00	1.193046	17.039464	0.702507	1.934538	40.748532

which is the flow studied in [3]. We obtain from Chain Rule $\frac{d\theta}{dt} = b'b^2y$, which implies $b' = b^{-2}$. Constant $b' = 0.702563658263$, in [3] (p. 761), agrees up to six decimal places with the value of b^{-2} in Table 4.1.

In summary the observations above verify that numerical results presented in Table 4.1 are consistent with the case $a = b = 0$, where position of the singularity can be derived analytically (*1*), they are also consistent with Lemma 4.2.1 (*2*). Finally numerical results for the case $a = 1$ match with numerical result obtained in [3].

Chapter 5

The Width of Arnold Tongues for the Dissipative Standard Map

5.1 Introduction

In this chapter we study the presence of resonance zone for the weakly dissipative map $f_{k,J,\Omega}$

$$\begin{aligned} x_{n+1} &= Jx_n - \frac{k}{2\pi}\sin 2\pi\theta_n \\ \theta_{n+1} &= \theta_n + \Omega + x_{n+1} \end{aligned} \tag{5.1}$$

where $(x_n, \theta_n) \in \mathbb{R} \times \mathbb{R}$, parameters $k, \Omega \in \mathbb{R}$ and $0 < J < 1$.

After the change of coordinates $\theta_n \to 2\pi\theta_n, x_n \to 2\pi(\Omega + x_n)$ and scaling of parameters $k = -\epsilon, \Omega = \frac{\omega}{2\pi(1-J)}$ the map (5.1) takes the form (4.1) considered in Chapter 4.

We can identify $\mathbb{R} \simeq \mathbb{R}/\mathbb{Z}$ and consider the map on a cylinder, $(x_n, \theta_n) \in \mathbb{R} \times \mathbb{S}^1$. When $k = 0$ the set $\{0\} \times \mathbb{S}^1$ is invariant. The motion on this set is determined by the parameter Ω and is a rigid rotation. For rational values of the parameter Ω the motion is periodic, for irrational values every orbit is dense on $\mathbb{S}^1$.

From the theory of normal hyperbolicity [23] it follows, that for k close to 0 there exists a set homotopic to $\mathbb{S}^1$, which is a $\mathcal{C}^r$ manifold on a cylinder $\mathbb{R} \times \mathbb{S}^1$ invariant under the iterations of the map (5.1).

Resonance zone also called Arnold tongue as alluded in Chapter 1 was first studied in relation to a family of one dimensional circle maps. For a homeomorphism f of the circle one can define a lift F, for which Poincaré [30] defined a rotation number ρ of f as

$$\rho(f) = \lim_{n\to\infty} \frac{F^n(x)}{n} \mod 1 \tag{5.2}$$

It can be shown that the rotation number is independent of the lift F and of x thus is well defined for a homeomorphism f. It can also be shown that f has a periodic orbit if and only if $\rho(f)$ is rational [2] (Proposition 1.5.1). For example for a family of circle maps

$$f(x) = x + \Omega + k \sin^2 x$$

Arnold tongues are regions in parameter space, where the rotation number is rational, which is equivalent to existence of periodic motion, hence to study the width of Arnold tongues we measure the regions in parameter space where rotation number is rational.

Both definition of rotation number and the Arnold tongues can be extended to a family of maps in more dimensions. If a family of maps has an invariant circle we can restrict the action of the map to the invariant circle and investigate regions in parameter space where periodic motion is present, which leads to a problem of Arnold tongues.

We can consider the behaviour of the map $f_{k,J,\Omega}$ restricted to the invariant circle homotopic to $\mathbb{S}^1$. It follows from one dimensional case mentioned above that the map has a periodic orbit on the invariant circle if and only if the rotation number is rational. In particular if the Poincaré number is equal to $\frac{1}{n}$ then an orbit of period n exists, which is equivalent to existence of a fixed point of the map $f^n_{k,J,\Omega}$. The region in parameter space where periodic orbit is present has been studied for example in Arnold [1] and Davie [9] for the circle map.

In subsequent section we give an overview of the results in Evans [11] regarding the width of Arnold tongues for the dissipative map (5.1). To study the width of the resonance region we fix parameter J, let $k \to 0$ and measure the width of the interval in parameter Ω, where the rotation number is equal to $\frac{1}{n}$.

As in previous chapters we extend the map to the complex domain. From Theorem 11.1 in [38] it follows that the improper unstable manifold (see page 17 for definition) is analytic on a strip containing $\mathbb{R}$, which allow us to measure the width of Arnold tongues for the dissipative standard map.

Throughout this chapter we do not provide rigorous arguments but heuristic reasoning supported by numerical results.

5.2 Preliminary Results

In this section we summarise as necessary for further investigation results presented in Chapter 4 in [11]. First we quote that for k close to 0 the length of the interval of parameter Ω, where Poincaré number of the map (5.1) is equal $\frac{1}{n}$ is

$$|I_{\frac{1}{n}}(J,k)| = \frac{\pi(1-J)}{kn^3}|\beta_2 - \beta_1| + o(n^3) \tag{5.3}$$

for some constants β_2, β_1. It gives cubic term in n, which was ascertained in [25].

We quote Theorem 4.1 in [11] which gives an estimation of the width of the interval $|\beta_2 - \beta_1|$.

Theorem 5.2.1.

$$|\beta_2 - \beta_1| \sim e^{\frac{-2\pi(1-J)}{k}} A(J) \tag{5.4}$$

as $k \to 0$, where $A(J)$ is a constant depending on J.

Proof. In order to introduce necessary notation we give a summary of the proof.

Step 1. Consider rotation number equal to 0. If $\Omega_n \in I_{\frac{1}{n}}$, then $f_{k,\Omega} \to f_{k,\Omega^*}$ as $n \to \infty$, where Ω^* is the right end point of the interval I_0, then f_{k,Ω^*} has a parabolic fixed point (x^*, θ^*). For k close to 0 there exists an invariant circle through the fixed point. Let $\theta = \theta^* + \phi$. We express the map f_{k,Ω^*} restricted to the invariant circle as

$$\phi_{n+1} = f_{k,\Omega^*}(\phi_n)$$

We are looking for a transformation Λ, such that

$$\Lambda(\phi_{n+1}) - \Lambda(\phi_n) \simeq 1$$

which is found by solving the appropriate approximating flow

$$\dot{\phi} = F(\phi)$$

where F is a polynomial with terms in ϕ. We define Λ as $\Lambda = P^{-1}$, where $\phi = P(t)$ is a solution of the above flow. Once Λ is found we are able to define mappings $g_k, h_k : (0,1) \to \mathbb{R}$

$$g_k(\phi_0) = \lim_{n \to -\infty} \Lambda(\phi_n) - n$$

$$h_k(\phi_0) = \lim_{n \to \infty} \Lambda(\phi_n) - n$$

g_k is a strictly increasing surjection, so the following function $\sigma_k : \mathbb{R} \to \mathbb{R}$

$$\sigma_k(u) = h_k \circ g_k^{-1}(u) - u \tag{5.5}$$

is well defined and 1-periodic function.

Next we quote Theorem 4.3 in [11]. It relates the interval $I_{\frac{1}{n}}$ to the map σ_k and is the key part in the proof of the Theorem 5.2.1.

Theorem 5.2.2. *The endpoints of the interval $I_{\frac{1}{n}}$ are given by*

$$\Omega^* + \frac{\pi^2}{cd}\left(\frac{1}{n^2} + \frac{\beta_i}{n^3}\right) + o(n^{-3}), \quad i = 1, 2$$

where Ω^* is the right endpoint of the interval I_0; c, d, β_1, β_2 are constants, and the length of the interval $[\beta_1, \beta_2]$ is equal to twice the length of the interval in s such that $\sigma_k(u) - s$ has a zero for the 1-periodic function $\sigma_k : \mathbb{R} \to \mathbb{R}$.

The theorem has been proved in Davie [7]. It follows, that in order to estimate the length of $[\beta_1, \beta_2]$ we need to estimate the quantity

$$\max_{u \in [0,1]} \sigma_k(u) - \min_{u \in [0,1]} \sigma_k(u)$$

Step 2. We extend the real map $f_{k,\Omega}$ (5.1) to the complex variables $(x, \theta) \in \mathbb{C} \times \mathbb{C}/\mathbb{Z}$. Parameters J, k, Ω are real constants. Consider the case, when $\Omega = \Omega^*$.

It follows from Theorem 11.1 in [38] that a parametrisation $\gamma(u)$, such that $f_{k,\Omega^*}(\gamma(u)) = \gamma(u+1)$, of the improper unstable manifold of the parabolic fixed point satisfies

$$\gamma(u) \to (x^*, \theta^*), \text{ for } \Re u \to -\infty$$

It follows from analysis in Chapter 4 in [11] that on a suitable strip containing $\mathbb{R}$ the parametrisation $\gamma(u)$ of the improper unstable manifold satisfies

$$\gamma(u) \to (x^*, \theta^* + 2\pi), \text{ for } \Re u \to \infty$$

Note that $\gamma(\mathbb{R})$ coincides with the real invariant circle excluding the fixed point, so that the image of the strip is a complex analytic invariant manifold which extends the real invariant circle. This complex invariant manifold is a subset of $\{(0,\theta) : |\Im\theta| < \frac{1}{2\pi}\ln\frac{1}{k} - C\}$, where C is a positive constant.

A complex analogue of function Λ is defined on an appropriate subset of $\mathbb{C}$. Complex analytic continuations $g_{1,k}$ and $h_{1,k}$ of g_k and h_k are well defined on the complex invariant manifold and $g_{1,k}$ is the inverse of the parametrisation $\gamma(u)$.

Step 3. After the change of variable $y = \theta - ib$, where $b = \frac{1}{2\pi}\ln\frac{1}{k}$ the map f_{k,Ω^*} (5.1) can be approximated by the map

$$\begin{aligned} x_{n+1} &= Jx_n - \frac{i}{4\pi}e^{-2\pi i y_n} \\ y_{n+1} &= y_n + x_{n+1} \end{aligned} \tag{5.6}$$

as $k \to 0$. The shift b is chosen so the exponential term above is independent of k

We determine y_{n+1} in terms of y_n as we restrict orbits $\{x_n, y_n + ib\}$ of the map (5.6) to the invariant manifold. Consider the expansion x_{n+1} in powers of $e^{-2\pi i y}$, substituting to (5.6) we obtain the following formula for the iterations of the limiting mapping on the invariant manifold

$$y_{n+1} = y_n - \frac{i}{4\pi(1-J)}e^{-2\pi i y_n} - \frac{iJ}{8\pi(1-J)^3}e^{-4\pi i y_n} \tag{5.7}$$

for $y_0 \in D_0 = \{y : |\Re y - \frac{3}{4}| < \gamma, \Im y < -C\}$ Solving approximating flow of the above recurrence equation enables to find a function $\Psi : \mathbb{C} \to \mathbb{C}$

$$\Psi(y) = 2(1-J)e^{2\pi i y} - \frac{1+J}{1-J}\pi i y \tag{5.8}$$

such that

$$\Psi(y_{n+1}) - \Psi(y_n) \simeq 1$$

Both limits $\lim_{n\to\pm\infty}\Psi(y_n) - n$ exist and are analytic functions which are related to the complex analytic continuations of real functions g_k, h_k defined in step 1.

$$\begin{aligned} G(y_0) &= \lim_{n\to-\infty}\Psi(y_n) - n \\ H(y_0) &= \lim_{n\to\infty}\Psi(y_n) - n \end{aligned} \tag{5.9}$$

The exact relationship between those limits and their counterparts $g_{1,k}$ and $h_{1,k}$ is given in [11] in Lemma 4.5.

From definition (5.5) of σ_k we can write the following equalities for the r-th Fourier coefficient of $\sigma_k(u)$

$$\begin{aligned}\sigma_r(k) &= \int_{u_0}^{u_0+1}(\sigma(u)-u)e^{2\pi i r u}du\\ &= \int_{\phi_0}^{\phi_1}(h_{1,k}(\phi)-g_{1,k}(\phi))e^{2\pi i r g_{1,k}(\phi)}g'_{1,k}(\phi)d\phi\end{aligned}$$

which implies by Lemma 4.5 in [11]

$$\lim_{k\to 0}|\sigma_r(k)|e^{\frac{2\pi r(1-J)}{k}} = e^{-\frac{3}{2}\pi^2\frac{1+J}{1-J}r}\int_{y_0}^{y_1}(H(y)-G(y))e^{2\pi i r G(y)}G'(y)dy \tag{5.10}$$

where $y_0 = G^{-1}(u_0)$ and $y_1 = G^{-1}(u_0+1)$.

As $k \to 0$, $|\sigma_{r+1}| = \mathcal{O}(e^{-\frac{2\pi(1-J)}{k}}\sigma_r(k))$ and we obtain

$$\max_{u\in[0,1]}\sigma_k(u) - \min_{u\in[0,1]}\sigma_k(u) \sim 4|\sigma_1(k)|$$

hence the length of the interval

$$|\beta_2-\beta_1| \sim e^{\frac{-2\pi(1-J)}{k}}A(J)$$

where

$$A(J) = 8e^{-\frac{3}{2}\pi^2\frac{1+J}{1-J}}B(J) \tag{5.11}$$

and

$$B(J) = \left|\int_{y_0}^{y_1}[H(y)-G(y)]e^{2\pi i G(y)}G'(y)dy\right|$$

which gives the estimation of the interval $I_{\frac{1}{n}}$ in Ω, where the map (5.1) is periodic on the invariant circle.

5.3 Change of Coordinates

In this section we consider the map (5.6) in different variables and formulate analogous results to previous section. We focus our attention on the limiting behaviour of the constant $A(J)$ as $J \to 1$ and relate it to Theorem 5.2.1 and to (5.3).

Note that after the change of variables $x = \frac{i}{\pi}z$, $y = \frac{i}{\pi}\ln(2iv)$ the map (5.6) becomes

$$\begin{aligned}z_{n+1} &= Jz_n + v_n^2\\ v_{n+1} &= v_n e^{z_{n+1}}\end{aligned} \tag{5.12}$$

which we denote as f_J.

The Jacobian of the map (5.12) at the origin is equal to

$$\begin{vmatrix} J & 0 \\ 0 & 1 \end{vmatrix}$$

so $(0,0) \in \mathbb{C} \times \mathbb{C}$ is a nonhyperbolic fixed point. We consider the map on the improper unstable manifold (see page 17 for definition).

We find the expansion of the unstable manifold at origin by substituting $z = \sum a_i v$ to z_{n+1} in (5.12) and comparing same order terms

$$z = \frac{1}{1-J}v^2 - \frac{2}{(1-J)^3}v^4 + \frac{6(1+J)}{(1-J)^5}v^6 - \frac{16(4J^2+13J+4)}{3(1-J)^7}v^8 + O(v^{10}) \quad (5.13)$$

We approximate iterations of the map f_J on the unstable manifold by substituting z to v_{n+1} in (5.12) and we obtain

$$v_{n+1} = v_n + A_1 v_n^3 + A_2 v_n^5 + A_3 v_n^7 + A_4 v_n^9 + \mathcal{O}(v_n^{11}) \quad (5.14)$$

where $A_1 = \frac{1}{1-J}$, $A_2 = \frac{1-5J}{2(1-J)^3}$, $A_3 = \frac{49J^2+22J+1}{6(1-J)^5}$, $A_4 = -\frac{729J^3+1565J^2+392J-1)}{24(1-J)^7}$.

We are seeking for the mapping $\phi(v) : \mathbb{C} \to \mathbb{C}$, such that

$$\phi(v_{n+1}) - \phi(v_n) = 1 + O(v_n^m)$$

First we consider the flow approximating the map (5.14)

$$\dot{v} = A_1 v^3 + A_2 v^5$$

we integrate

$$\int \frac{1}{A_1 v^3 + A_2 v^5} dv$$

and obtain

$$\phi_1(v) = -\frac{1-J}{2v^2} + \frac{1+J}{1-J}\ln v$$

We observe that the following holds

$$\phi_1(v_{n+1}) - \phi_1(v_n) = 1 + O(v_n^2)$$

In order to find more accurate approximation we consider

$$\phi(v) = \phi_1(v) + \alpha_1 v^2 + \alpha_2 v^4$$

and find that for $\alpha_1 = -\frac{J^2+10J+1}{3(1-J)^3}$ and $\alpha_2 = \frac{J^3+35J^2+35J+1}{4(1-J)^5}$ the following holds

$$\phi(v_{n+1}) - \phi(v_n) = 1 + O(v_n^6)$$

for

$$\phi(v) = -\frac{1-J}{2v^2} + \frac{1+J}{1-J}\ln v + \alpha_1 v^2 + \alpha_2 v^4 \quad (5.15)$$

To obtain approximation to a given order v^m we need to consider the expansion of order v^{2m+2} of the unstable manifold (5.13).

By argument similar to [11] it can be shown that both limits exist

$$\mathcal{H}(v) = \lim_{n\to-\infty} \phi(v_n) - n$$
$$\mathcal{G}(v) = \lim_{n\to+\infty} \phi(v_n) - n$$

Note that these functions play similar role to functions g and h defined in Chapter 1 and satisfy (1.7) and (1.9) exactly.

We denote $f(y) = -\frac{i}{2}e^{-\pi iy}$ and state the following lemma, which relates these limits for the map (5.12) in (z, v) variables to the corresponding limits for the map (5.6) in (x, y) variables.

Lemma 5.3.1. *For $y_0 \in D_0$ and $v_0 \in V_0 = f(D_0)$ the following approximations hold*

$$G(y_0) = \mathcal{G}(v_0) + \frac{1+J}{1-J}\ln(2i) + o(1) \tag{5.16}$$
$$H(y_0) = \mathcal{H}(v_0) + \frac{1+J}{1-J}\ln(2i) + o(1) \tag{5.17}$$

where functions $G(y)$ and $H(y)$ are defined in previous section in (5.9).

Proof. From (5.8) it follows that for $y_0 \in D_0$

$$G(y_0) = \Psi(y) - n = 2(1-J)e^{2\pi iy} - \frac{1+J}{1-J}\pi iy - n + o(1)$$

We substitute y in terms of v, simplify and obtain

$$\Psi(y) - n = \phi(v) + \frac{1+J}{1-J}\ln(2i) - n = \mathcal{G}(v_0) + \frac{1+J}{1-J}\ln(2i) + o(1)$$

for $v_0 \in V_0$, which gives relation (5.16). Argument for relation (5.17) is similar and is omitted. □

We now use Lemma 5.3.1 to reformulate (5.11) with a different constant. We relate the above results to the constant $A(J)$ in Theorem 5.2.1 and state the following

Proposition 5.3.1. *There is a constant $E(J)$, such that the width of the interval*

$$|\beta_2 - \beta_1| \sim 8e^{-2\pi\frac{1-J}{k}}e^{\frac{\pi^2}{2}\frac{1+J}{1-J}}E(J) \tag{5.18}$$

for $k \to 0$.

Proof. It follows from Lemma 5.3.1 that

$$\begin{aligned} B(J) &= \left| \int_{y_0}^{y_1} [H(y) - G(y)] e^{2\pi i G(y)} G'(y) dy \right| \\ &= \left| \int_{v_0}^{v_1} [\mathcal{H}(v) - \mathcal{G}(v)] e^{2\pi i \mathcal{G}(v)} e^{\pi^2 \frac{1+J}{1-J}} \mathcal{G}'(v) dv \right| \\ &= e^{\pi^2 \frac{1+J}{1-J}} \left| \int_{v_0}^{v_1} [\mathcal{H}(v) - \mathcal{G}(v)] e^{2\pi i \mathcal{G}(v)} \mathcal{G}'(v) dv \right| \\ &= e^{\pi^2 \frac{1+J}{1-J}} E(J) \end{aligned}$$

where

$$E(J) = \left| \int_{v_0}^{v_1} [\mathcal{H}(v) - \mathcal{G}(v)] e^{2\pi i \mathcal{G}(v)} \mathcal{G}'(v) dv \right| \tag{5.19}$$

It follows from (5.11) that

$$A(J) = 8 e^{-\frac{\pi^2}{2} \frac{1+J}{1-J}} E(J) \tag{5.20}$$

which by Theorem 5.2.1 implies (5.18). ☐

In the next section we study limiting behaviour of the constant $E(J)$ as $J \to 1$.

5.4 Relation with the Semistandard Map

In this section we relate the dissipative map (5.12) to the semistandard map and we establish the limiting behaviour of the constant $E(J)$ when $J \to 1$. We outline the main points of the argument relating the constant $E(J)$ to the constant μ_0 introduced in Chapter 3. We omit the details as the argument is similar to the proof of Theorem 4.8.1 presented in Section 4.8.

Close to the origin the dissipative map (5.12) can be approximated by the flow

$$\begin{aligned} \dot{z} &= (J-1)z + v^2 \\ \dot{v} &= vz \end{aligned} \tag{5.21}$$

We apply the scaling to eliminate the parameter J

$$\begin{aligned} Z &= (1-J)^{-1} z \\ V &= (1-J)^{-1} v \\ T &= (1-J) t \end{aligned} \tag{5.22}$$

and obtain

$$\begin{aligned} \dot{Z} &= -Z + V^2 \\ \dot{V} &= VZ \end{aligned} \tag{5.23}$$

For large values of V, Z is small relative to V^2 and solution curves of the system (5.23) can be approximated by solution curves of the flow

$$\begin{aligned} \dot{Z} &= V^2 \\ \dot{V} &= VZ \end{aligned} \tag{5.24}$$

which is the first order approximating flow of the semistandard map (3.2). The solution curves of the flow (5.24) are $Z^2 - V^2 = C$, which gives $Z = \sqrt{V^2 + C}$. This implies that the solutions of the system (5.23) can be approximated by

$$Z = V + \frac{C}{2V} + \dots \tag{5.25}$$

hence parametrised by C. Note that the error of this approximation is larger than $\frac{C}{2V}$ term as we shall see later in (5.35).

We investigate the impact of scaling (5.22) on the constant $E(J)$ involved in the estimate of the width of the Arnold tongue. After the scaling the family of dissipative maps (5.12) converges to the semistandard map (3.2). By Remark 2.1.3 one could construct the sequence of flows approximating the dissipative map and it would converge to the approximating flow of the semistandard map. Similarly to previous chapter we approximate the map using the flow. As we only give an outline of the argument we use the first order approximating flows (5.21) and (5.23).

Similarly to argument in Section 4.8 one needs to relate the manifold of the dissipative map to the manifold of the semistandard map. Denote $\gamma_J(t)$ the unstable manifold of the map (5.12) and $\gamma_0(t)$ the unstable manifold of the semistandard map. One can show by following the proof of Proposition 4.8.1 that after appropriate real shift in time s_J the manifolds γ_J and γ_0 can be matched on a specific region in $\mathbb{C}$

$$\lim_{J \to 1} \gamma_J(t + s_J) = \gamma_0(t). \tag{5.26}$$

Recall function F defined in Section 3.1. It follows from Proposition 3.1.2 that $F(\gamma_0(t)) \sim \mu_0 e^{-2\pi i t}$. Relation (5.26) implies that $F(\gamma_J(t)) \sim \mu_0 e^{-2\pi i (t - s_J)}$, where μ_0 is the constant introduced in Chapter 3. Denote $c = z^2 - v^2$, approximation to the solution of (5.21) and observe that c is approximately F hence we obtain

$$c(\gamma_J(t)) \sim \mu_0 e^{-2\pi i (t - s_J)}. \tag{5.27}$$

Since $\gamma_J(t) = \mathcal{G}^{-1}(t)$ the analogue of function σ (1.6) defined in (5.19) is

$$|\mathcal{H}(\gamma_J(t)) - t| \sim |e^{-2\pi i t}| E(J) \tag{5.28}$$

To measure oscillation of the of $|\mathcal{H}(\gamma_J(t)) - t|$ we regard it as a time shift on the orbits of the system (5.23) from the origin to infinity. To apply (5.27) we need dependence of time T on C. As time from the origin to infinity is infinite, we fix a point (V, Z) for each C, such that $|Z^2| + |V^2| = \delta$ on the orbit close to the fixed point. We define $T = f_\delta(C)$, a finite measure of time along the orbit from δ to infinity. For C corresponding to the unstable manifold of (5.23) we define the constant

$$d = \lim_{\delta \to 0} f'_\delta(C). \tag{5.29}$$

Observe that d can be approximated by $d \sim \frac{\partial T}{\partial C}$ and that $\frac{\partial t}{\partial c} = \frac{\partial t}{\partial T}\frac{\partial T}{\partial C}\frac{\partial C}{\partial c}$. From the scaling (5.22) we obtain

$$\Delta C \approx \Delta(Z^2 - V^2) = (1-J)^{-2}\Delta(z^2 - v^2) \approx (1-J)^{-2}\Delta c$$

which implies

$$\frac{\Delta t}{\Delta c} \approx d(1-J)^{-3}$$

Hence from (5.27) the time shift from δ to infinity is approximately

$$d(1-J)^{-3}\mu_0 e^{-2\pi i(t-s_J)}. \tag{5.30}$$

We combine (5.30) and (5.28) and obtain the following

Proposition 5.4.1. *For the constant d defined above the following holds*

$$\lim_{J\to 1} d^{-1}(1-J)^3 E(J) = |\mu_0| \tag{5.31}$$

From relation (5.20) and Proposition 5.4.1 follows

Corollary 5.4.1.

$$\lim_{J\to 1} \frac{1}{8d} e^{\frac{\pi^2}{2}\frac{1+J}{1-J}} A(J)(1-J)^3 = |\mu_0| \tag{5.32}$$

One can show that Theorem 5.2.1, formula (5.3) and Corollary 5.4.1 imply

Theorem 5.4.1. *The width of the interval in parameter Ω where there is n-periodic orbit of the map (5.1) on the invariant circle is*

$$\lim_{J\to 1}\lim_{k\to 0}\lim_{n\to\infty} |I_{\frac{1}{n}}(k)| \frac{kn^3}{8\pi d} e^{2\pi\frac{1-J}{k}} e^{\frac{\pi^2}{2}\frac{1+J}{1-J}} (1-J)^2 = |\mu_0| \tag{5.33}$$

The rigorous proof of the heuristic argument sketched in this section leading to the statement in the theorem above is expected to follow analogous argument to the one in Section 4.8. Proposition 5.4.1 is supported by numerical results which we discuss below.

5.5 Numerical Results

In this section we explain numerical calculations of the constants C, d and $E(J)$. First we consider the system (5.23) and find expansion of the unstable manifold at $(0,0)$.

$$Z = V^2 - 2V^4 + 12V^6 - 112V^8 + 1360V^{10} + \mathcal{O}(V^{12}) \tag{5.34}$$

Now we improve the approximation (5.25) of the solution of the system (5.23) for large values of V. We take (5.25), calculate $\dot{Z}$, substitute to (5.23) and compare same order terms to obtain next term in the expansion. By repeating the process iteratively it follows that

$$\begin{aligned}Z =&V-1+\ln(V)\left(\frac{1}{V}+\frac{2}{V^2}+\frac{3-0.5C}{V^3}\right)+\frac{C}{2V}+\frac{C+1}{V^2}\\&+\frac{2+1.5C-0.125C^2}{V^3}+(\ln(V))^2(-\frac{1}{2V^3}-\frac{8}{3V^4})+o(V^{-3})\end{aligned} \tag{5.35}$$

which gives us the approximation of the unstable manifold of the system (5.23).

In order to compute the constant C corresponding to the unstable manifold we solve the system (5.23) starting close to $(0,0)$ on the unstable manifold (5.34) until we reach large values of V, where Z is or order V and we find $C=-1.832$ iteratively from (5.35). The program `constant_C.c` is included in the Appendix.

We look at the change in time as a function of change in the orbits, $\frac{\partial T}{\partial C}$. First we find expansion of T in terms of V. For small values of V we substitute (5.34) to $\dot{V}$ in (5.23) and obtain

$$\frac{dV}{dT}=V^3-2V^5+12V^7-112V^9+1360V^{11}+\mathcal{O}(V^{13})$$

which implies

$$\frac{dT}{dV}=V^{-3}+2V^{-1}-8V+72V^3-896V^5+\mathcal{O}(V^7). \tag{5.36}$$

We integrate the above and obtain expansion of time

$$T=\phi(V)+\mathcal{O}(V^8)$$

for V small, where

$$\phi(V)=-\frac{1}{2}V^{-2}+2\ln V-4V^2+18V^4-\frac{448}{3}V^6 \tag{5.37}$$

Similarly for large values of V we substitute (5.35) to $\dot{V}$ in (5.23) and obtain approximation of time

$$T=\Phi(V)+\mathcal{O}(V^{-5})$$

where

$$\Phi(V)=-\frac{1}{V}-\frac{1}{2V^2}+\frac{3\ln V-2+1.5C}{9V^3}+\frac{4\ln V+1+2C}{4V^4} \tag{5.38}$$

As for large values of V time expansion is the explicit function of C and V is independent of C to find $\frac{\partial T}{\partial C}$ we differentiate (5.38) with respect to C and obtain

$$\frac{\partial\Phi}{\partial C}=\frac{1}{6V^3}+\frac{1}{2V^4}-\frac{0.3\ln(V)}{V^5}+\frac{0.94-0.15C}{V^5}+\mathcal{O}(V^{-6}) \tag{5.39}$$

We approximate $\frac{\partial T}{\partial C}$ by $\frac{\partial\phi}{\partial C}$ for small values of V. To compute $\frac{\partial\phi}{\partial C}$ we take a different approach. We denote $X=\frac{\partial V}{\partial C}$, $Y=\frac{\partial Z}{\partial C}$ and observe that $\frac{\partial\phi}{\partial C}=\frac{\partial\phi}{\partial V}\frac{\partial V}{\partial C}$, which by (5.36) implies

$$\frac{\partial \phi}{\partial C} = X(V^{-3} + 2V^{-1} - 8V + 72V^3 - 896V^5) \tag{5.40}$$

We compute (5.40) by solving the system

$$\begin{aligned} \dot{Z} &= -Z + V^2 \\ \dot{V} &= VZ \\ \dot{Y} &= -Y + 2VX \\ \dot{X} &= YV + XZ \end{aligned} \tag{5.41}$$

starting at (V, Z) on the unstable manifold (5.35), for $X = 0$ (as V is independent of C) and Y calculated from (5.35), until we reach V close to 0, where we calculate $\frac{d\phi}{dC}$ from (5.40).

Finally to estimate the effect of a difference in orbits, which are parametrised by C, on the change in time for the flow (5.23) we add $\frac{d\Phi}{dC}$ calculated at the starting point and $\frac{d\phi}{dC}$ calculated at the final point. We obtain the constant d defined in (5.29), such that $|d| = 2.153$ and the program `constant_d.c` is included in the Appendix.

We use the results for the flow to produce numerical results for the iteration of the map (5.12). We discuss remaining numerical computations supporting Theorem 5.4.1.

We choose point (v, z) on the invariant circle, v satisfies (5.15) and z is calculated using the expansion (5.13). We iterate the map (5.12) for a range of values $b = \Im v$ and approximate the integral (5.19) using (5.15) in a similar way to the calculation of the constant μ_0.

The limiting constant has been calculated in `E_J.c` program, which is included in the Appendix and the results for the constant $E(J)(1-J)^3$ and $D(J) = E(J)(1-J)^3 d^{-1}$ are included in Table 5.1.

We observe that $D(J)$ tends to the constant μ_0 related to the semistandard map computed in program `mu_0`, which supports (5.31).

Table 5.1: Numerical results of E_J.c program

J	b	$E(J)(1-J)^3$	$D(J)$
0.90	2.0	1152.92	535.50
0.90	3.0	1152.92	535.50
0.90	4.0	1152.97	535.52
0.92	2.0	1162.39	539.89
0.92	3.0	1162.39	539.90
0.92	4.0	1162.52	539.95
0.94	2.0	1171.94	544.33
0.94	3.0	1171.94	544.33
0.94	4.0	1171.87	544.30
0.96	2.0	1181.54	548.79
0.96	3.0	1181.55	548.79
0.96	4.0	1181.55	548.79
0.98	2.0	1191.21	553.28
0.98	3.0	1191.23	553.29
0.98	4.0	1191.24	553.29

After a slight modification the program computes the constant $E(0) = 1114.63$, which after appropriate change of coordinates is related to the constant $\Gamma = 649.97$ introduced in [9] and we note that the results of the program are consistent with relation (see [11]) $\Gamma \approx 8\pi^2 e^{-\frac{\pi^2}{2}} E(0)$.

Chapter 6

Conclusion and Further Research

In this thesis we studied the extended standard family of maps, which exhibits rich and universal dynamics and is a model system for classical families of maps. We established the width of the separatrix splitting region and provided heuristic argument for the width of Arnold tongues. We provided numerical results to support our findings.

We conclude our discussion with some directions of future research. In Chapter 4 we studied the width of the splitting region in parameter space near the curve Ψ, another possibility is to study homoclinic intersection line in the (a, b) plane, where $b = 0$.

We also conjectured existence of singularity for the flow (4.4), potential subject of further research would be to prove the conjecture or support it by numerical verification.

Results of Chapter 4 can be extended further to more general families of maps. An example extension of the extended standard map (4.1) is the inclusion of nonlinear term in x, and the map

$$\begin{aligned} x_{n+1} &= \gamma x_n - a x_n^2 + \omega + \epsilon \sin\theta_n \\ \theta_{n+1} &= \theta_n + x_{n+1} \end{aligned}$$

can be approximated by the flow studied in [5] (p. 79).

Last but not least one could rigorously prove results sketched in Chapter 5 and refine numerical calculations by using higher order approximating flow to improve approximation to the semistandard map.

Appendix

Program sigma_1.c

```
#include <math.h>
#include <stdio.h>
#include <complex.h>
/* Compute constant \sigma_1 */
main()
{ int i,j,k,m,n;
  double b,N;
  complex s,sig,sigma,S,z;
  N = 100;
  n = 500;
  b = 3.2;
  FILE *stream;
  stream = fopen("sigma1_output.txt","w+");
  for(k = 0; k < 8; k++)
    { sigma = 0;
      for(j = 0; j < N; j++)
        { s = j/N+I*b;
          z = 1/(n-s);
          for(m = 0; m <= 10; m++)
            /* Find z */
            { z = 1/(clog(z)-0.5*z+(1/3.0)*z*z-s+n);
            }
          /* Iterate the map */
          for(i = 0; i <= 2*n; i++)
            { z = z+z*z;
            }
          /* Calculate oscillation */
          sig = cexp(-2*M_PI*I*s)*(-1/z+clog(z)-0.5*z+(1/3.0)*z*z-s-n);
          sigma = sigma+sig;
        }
      S = sigma/N;
      /* Output the table */
      fprintf(stream,"%.1f & %.2f & %.2f & %.2f\\\\\n",b,cabs(S),cimag(S),creal(S));
      printf("%.1f %f %f %f \n",b,cabs(S),cimag(S),creal(S));
      b = b+0.1;
    }
}
```

Program mu_0.c

```
#include <math.h>
#include <stdio.h>
#include <complex.h>
/* Compute constant \mu_0 */
main()
{ int i,j,k,M,N;
  double Re,B,b,x;
  complex w,z,v,F,mu,mu_0;
  Re = -400;
  N = 100;
  M = 800;
  B = 2.6;
  FILE *stream;
  stream = fopen("mu_0_output.txt","w+");
  for(k = 0; k < 9; k++)
    { b = B+(0.1*k);
      mu_0 = 0;
      for(j = 0; j < N; j++)
        { x = j;
          /*Initial condition for iteration*/
          w = Re+x/N -I*b;
          /* Expansion of the unstable manifold */
          z = -1/w - 1/(2*cpow(w,2)) - 1/(12*cpow(w,3)) + 1/(8*cpow(w,4)) +
            193/(2160*cpow(w,5)) - 59/(846*cpow(w,6)) - 6883/(60480*cpow(w,7));
          v = 1/w - 1/(8*cpow(w,3)) + 209/(3456*cpow(w,5)) -
            8611/(138240*cpow(w,7));
          /* Iterate the semistandard map */
          for (i = 1; i <= M; i++)
            { z = z+v*v;
              v = v*cexp(z);
            }
          F = z*(z+v*v)-v*v-(1/6.0)*cpow(z*(z+v*v),2);
          mu = cexp(2*M_PI*I*(w+M))*F;
          /* Calculate constant mu_0 */
          mu_0 = mu_0+mu;
        }
      /* Output the table */
      fprintf(stream,"%.1f & %.8f & %.8f & %.0f\\\\\n",
      -b,cabs(mu_0)/N,cimag(mu_0/N),creal(mu_0/N));
      printf("%.1f,%.8f,%.8f,%.8f\n",
      -b,cabs(mu_0)/N,cimag(mu_0/N),creal(mu_0/N));
    }
}
```

Program separatrix_splitting.c

```
#include <stdio.h>
#include <math.h>
double a,c,L_a;
double delta = 0.01;
double y_1,y2,z1,z2,T1;
/* Calculate the difference between manifolds for a given $a$ and $b$*/
double difference(double b)
{ int i,j,n,k,l,N;
  double h1,h2,L;
  double T2,y,rT1,rT2,ry2,ry1,rz1,rz2;
  double c1,c11,c2,c21,c3,c31;
  double  k11,k12,k13,k14,k21,k22,k23,k24,l11,l12,l13,l14,l21,l22,l23,l24;
  N = 20000;
  L = 2*M_PI-2*delta;
  h1 = L/(2*N);
  h2 = -L/(2*N);
  /* Define coefficients of expansion of the unstable manifold of $\theta_0$ */
  c1 = (-0.5)*b+(0.5)*sqrt(b*b+4*cos(asin(-a)));
  c2 = ((-0.5)*(-a ))/(b+3*c1);
  c3 = ((-1/6.0)*(12*c2*c2+cos(asin(-a))))/(4*c1+b);
  /* Define coefficients of expansion of stable manifold of $\theta_0+2\pi$ */
  c11 = (-0.5)*b-(0.5)*sqrt(b*b+4*cos(asin(-a)));
  c21 = ((-0.5)*(-a ))/(b+3*c11);
  c31 = ((-1/6.0)*(12*c21*c21+cos(asin(-a))))/(4*c11+b);
  /* Initial condition near $\theta_0$ */
  T1 = asin(-a)+delta;
  y_1 = c1*(T1-asin(-a))+c2*pow((T1-asin(-a)),2)+c3*pow((T1-asin(-a)),3);
  z1 = 0;
  rT1 = 0;
  ry1 = 0;
  rz1 = 0;
  /* Initial condition near $\theta_0+2\pi$ */
  T2 = 2*M_PI+asin(-a)-delta;
  y2 = c11*(T2-asin(-a)-2*M_PI)+c21*pow((T2-asin(-a)-2*M_PI),2)
  +c31*pow((T2-asin(-a)-2*M_PI),3);
  z2 = 0;
  rT2 = 0;
  ry2 = 0;
  rz2 = 0;
  /* Integrate the real system using rk4 algorithm to compute $b$ and $c$*/
  for (i = 0;  i < N; ++i)
    { k11 = h1*((a-b*y_1+sin(T1))/y_1);
      k12 = h1*(a-b*(y_1+k11/2)+sin(T1+h1/2))/(y_1+k11/2);
      k13 = h1*(a-b*(y_1+k12/2)+sin(T1+h1/2))/(y_1+k12/2);
      k14 = h1*(a-b*(y_1+k13)+sin(T1+h1))/((y_1+k13));
      ry1 = (k11+2*(k12+k13)+k14)/6;
      l11 = h1*((-z1*(a+sin(T1))/(y_1*y_1))-1);
      l12 = h1*((-(z1+l11/2)*(a+sin(T1+h1/2)) )/(pow((y_1+k11/2),2))-1);
      l13 = h1*((-(z1+l12/2)*(a+sin(T1+h1/2)) )/(pow((y_1+k12/2),2))-1);
      l14 = h1*((-(z1+l13/2)*(a+sin(T1+h1/2)) )/(pow((y_1+k13),2))-1);
      rz1 = (l11+2*(l12+l13)+l14)/6;
      T1 = T1+h1;
      y_1 = y_1+ry1;
      z1 = z1+rz1;
      k21 = h2*((a-b*y2+sin(T2))/y2);
      k22 = h2*(a-b*(y2+k21/2)+sin(T2+h2/2))/(y2+k21/2);
      k23 = h2*(a-b*(y2+k22/2)+sin(T2+h2/2))/(y2+k22/2);
      k24 = h2*(a-b*(y2+k23)+sin(T2+h2))/((y2+k23));
      ry2 = (k21+2*(k22+k23)+k24)/6;
      l21 = h2*((-z2*(a+sin(T2))/(y2*y2))-1);
      l22 = h2*( ( -(z2+l21/2)*(a+sin(T2+h2/2)) ) /(pow((y2+k21/2),2)) -1);
      l23 = h2*((-(z2+l22/2)*(a+sin(T2+h2/2)) )/(pow((y2+k22/2),2))-1);
      l24 = h2*((-(z2+l23/2)*(a+sin(T2+h2/2)) )/(pow((y2+k23),2))-1);
      rz2 = (l21+2*(l22+l23)+l24)/6;
      T2 = T2+h2;
      y2 = y2+ry2;
      z2 = z2+rz2;
    }
```

```
  /* Output difference between manifolds at $\theta_0+\pi$ */
  return y2-y_1;
}
/* Define sign of a number */
int signum(double x)
{ int sig;
  if (x > 0) { sig = 1;}
  else if (x < 0) { sig = -1;}
  else {sig = 0;}
  return sig;
}
main ()
{ int ii;
  FILE *stream;
  stream = fopen("manifold_splitting_output.txt","w+");
  /* Compute $b$ giving a homoclinic orbit for a range of $a$
     using bisection method with accuracy epsilon */
  for (ii = 0; ii < 21; ii++)
    { a = ii/20.0;
      double b1 = a*(M_PI/4-0.1);
      double b2 = 2*b1;
      double epsilon = 1e-10;
      double r1,r2,r;
      int s = signum(r1)+signum(r2);
      r1 = difference(b1);
      r2 = difference(b2);
      do
        { r = difference(0.5*(b1+b2));
          if (signum(r1) == signum(r)) { b1 = 0.5*(b1+b2);}
          else if (signum(r2) == signum(r)) { b2 = 0.5*(b1+b2);}
          else { break;}
        }
      /* Integrate complex system */
      while(fabs(r) > epsilon);
      int j;
      double ReH,M;
      _Complex double H,Theta,Y,rY,t,rt,K1,K2,K3,K4,K11,K12,K13,K14;
      ReH = 0.001;
      H = csqrt(-1)*ReH;
      M = 100000;
      Theta = T1;
      Y = y_1;
      t = 0;
      rt = 0;
      /* Integrate the complex system using rk4 algorithm to compute $A_a$ */
      for (j = 0;  j < M; ++j)
        { K1 = H*((a-b1*Y+csin(Theta))/Y);
          K11 = H/Y;
          K2 = H*((a-b1*(Y+K1/2.0)+csin(Theta+H/2.0))/(Y+K1/2.0));
          K12 = H/(Y+K1/2);
          K3 = H*((a-b1*(Y+K2/2.0)+csin(Theta+H/2.0))/(Y+K2/2.0));
          K13 = H/(Y+K2/2);
          K4 = H*((a-b1*(Y+K3)+csin(Theta+H))/(Y+K3));
          K14 = H/(Y+K3);
          rY = (K1+2*(K2+K3)+K4)/6.0;
          rt = (K11+2*(K12+K13)+K14)/6.0;
          Theta = Theta+H;
          Y = Y+rY;
          t = t+rt;
        }
      c = (z2-z1)*y_1;
      /* Calculate constant $\Lambda$ */
      L_a = (8.0/(c))*cexp(((M_PI/2.0)*cimag(t)+creal(t))*b1);
      /* Output the table */
      fprintf(stream,"%.2f & %f & %f & %f & %f & %f\\\\\n",
      a,b1,c,creal(t),cimag(t),L_a);
      printf("a=%.2f,b=%f,c=%f,Re_t=%f,Im_t=%f,Lambda=%f\n",
      a,b1,c,creal(t),cimag(t),L_a);
    }
}
```

Program constant_C.c

```
#include <stdio.h>
#include <math.h>
main()
{ int i,j,k,n;
  double C,z,v,rz,rv;
  double h,k11,k21,k12,k22,k13,k23,k14,k24;
  n = 1000000;
  h = 0.01;
  v = 0.05;
  z = v*v-2*pow(v, 4)+12*pow(v, 6)-112*pow(v, 8)+1360*pow(v,10);
  rz = 0;
  rv = 0;
  /* Integrate the system further */
  for (k = 0; k < 3; k++)
    { rz = rz;
      rv = rv;
      v = v;
      z = z;
      /* Integrate the system using rk4 algorithm */
      for (i = 0;  i < n-1; ++i)
        { k11 = h*(-z+v*v);
          k21 = h*(z*v);
          k12 = h*(-(z+k11/2)+(v+k21/2)*(v+k21/2));
          k22 = h*((z+k11/2)*(v+k21/2));
          k13 = h*(-(z+k12/2)+(v+k22/2)*(v+k22/2));
          k23 = h*((z+k12/2)*(v+k22/2));
          k14 = h*(-(z+k13)+(v+k23)*(v+k23));
          k24 = h*((z+k13)*(v+k23));
          rz = (k11+2*(k12+k13)+k14)/6;
          rv = (k21+2*(k22+k23)+k24)/6;
          z = z+rz;
          v = v+rv;
          /* Use different step size to increase accuracy */
          if (v < 10)
            { h = 0.001;
            }
          else
            { h = 0.0000001;
            }
          C = 2*(z*v*v-pow(v,3)+v*v-v*log(v)-2*log(v)-1)/(2+v);
          for (j = 0; j < 100; j++)
            { C = 2*v*(z-v+1-log(v)/v-2*log(v)/(v*v)-(C+1)/(v*v) +0.5*(log(v)*log(v))/(pow(v,3))
                -(3-0.5*C)*log(v)/(pow(v,3))-(2+1.5*C-0.125*C*C)/(pow(v,3)));
            }
        }
      n = 100000;
      printf("%f,%f,%f,%f,%f\n",z,rz,v,rv,C);
    }
}
```

Program constant_d.c

```
#include <stdio.h>
#include <math.h>
#include <complex.h>
main()
{ int i,j,n;
  double C,h;
  complex z,v,dz,dv,rz,rv,rdz,rdv;
  complex k11,k21,k12,k22,k13,k23,k14,k24;
  complex kd11,kd21,kd12,kd22,kd13,kd23,kd14,kd24;
  complex d,d_test,dphi,dphiL,phi,phiL;
  C = -1.832;
  n = 10000000;
  h  = 0.0000001;
  v = -100;
  z = v-1+clog(v)*((1/v)+(2/(v*v))+(3-0.5*C)/(v*v*v))+(0.5*C/v)
    +((C+1)/(v*v))+(2+1.5*C-0.125*C*C)/(v*v*v)-0.5*clog(v)*clog(v)/(v*v*v);
  dv = 0;
  dz = (-0.5*clog(v))/(v*v*v)+(0.5/v)+(1/(v*v))+(1.5-0.25*C)/(v*v*v);
  dphiL = 1/(6.0*cpow(v,3))+1/(2.0*cpow(v,4))+(0.94-0.3*clog(v)-0.15*C)/cpow(v,5);
  /* Integrate the system using rk4 algorithm */
  for (i = 0;  i < n-1; ++i)
    { k11 = h*(-z+v*v);
      k21 = h*(z*v);
      k12 = h*(-(z+k11/2)+(v+k21/2)*(v+k21/2));
      k22 = h*((z+k11/2)*(v+k21/2));
      k13 = h*(-(z+k12/2)+(v+k22/2)*(v+k22/2));
      k23 = h*((z+k12/2)*(v+k22/2));
      k14 = h*(-(z+k13)+(v+k23)*(v+k23));
      k24 = h*((z+k13)*(v+k23));
      rz = (k11+2*(k12+k13)+k14)/6;
      rv = (k21+2*(k22+k23)+k24)/6;
      kd11 = h*(-dz+2*v*dv);
      kd21 = h*(dv*z+dz*v);
      kd12 = h*(-(dz+kd11/2)+2*(v+k21/2)*(dv+kd21/2));
      kd22 = h*((dv+kd11/2)*(z+k11/2)+(dz+kd11/2)*(v+k11/2));
      kd13 = h*(-(dz+kd12/2)+2*(v+k22/2)*(dv+kd22/2));
      kd23 =  h*((dv+kd12/2)*(z+k12/2)+(dz+kd12/2)*(v+k12/2));
      kd14 = h*(-(dz+kd13)+2*(v+k23)*(dv+kd23));
      kd24 =  h*((dv+kd13)*(z+k13)+(dz+kd13)*(v+k13));
      rdz = (kd11+2*(kd12+kd13)+kd14)/6;
      rdv = (kd21+2*(kd22+kd23)+kd24)/6;
      z = z+rz;
      v = v+rv;
      dz = dz+rdz;
      dv = dv+rdv;
      if (cabs(v) < 10)
        { h = 0.001;
        }
      else
        { h = 0.0000001;
        }
    }
  dphi = dv*(1/(cpow(v,3))+2/(v)-8*v+72*cpow(v,3)-2*448*cpow(v,5));
  d = dphi+dphiL;
  printf("%f,%f,%f,%f,%f,%f,%f\n",creal(v),cimag(v),creal(z),cimag(z),creal(d),cimag(d),cabs(d));
}
```

Program E_J.c

```
#include <math.h>
#include <stdio.h>
#include <complex.h>
/* Compute limiting behaviour of the constant E_J */
main()
{ int i,j,k,l,m,n;
  double d,J0,J,a1,a2,a3,a4,B,b,x,N,D,DJ;
  complex z,v,s,E,e;
  FILE *stream;
  stream = fopen("E_J_output.txt","w+");
  d = 2.153;
  J0 = 0.9;
  for(l = 0; l < 5; l++)
    { J = J0+(0.02*l);
      a1 = (J-1)/2;
      a2 = (J+1)/(1-J);
      a3 = (1+10*J+pow(J,2))/(3*pow((J-1),3));
      a4 = (pow(J,3)+35*pow(J,2)+35*J+1)/(4*pow((1-J),5));
      N = 50;
      n = 10000;
      B = 2;
      for(k = 0; k < 3; k++)
        { b = B+(k);
          E = 0;
          for(j = 0;j<N;j++)
            { x = j;
              s = x/N +I*b;
              v = (a1/(s-n));
              for(m = 0; m <= 20; m++)
                { v = csqrt((-a1)/(a2*clog(v)+a3*v*v+a4*cpow(v,4)-s+n));
                }
              z = ((v*v)/(1-J))-(2*cpow(v,4))/(pow((1-J),3))-6*(J+1)*cpow(v,6)/(pow((1-J),5))
                -16*(4*J*J+13*J+4)*cpow(v,8)/(3.0*pow((1-J),7));
              for(i = 0; i <= 2*n; i++)
                { z = J*z+v*v;
                  v = v*cexp(z);
                }
              e = cexp(-2*M_PI*I*s)*((a1/(v*v))+a2*clog(v)+a3*v*v+a4*cpow(v,4)-s-n);
              E = E+e;
            }
          D = pow((1-J),3)*cabs(E)/N;
          DJ = D/d;
          fprintf(stream,"%.2f & %.1f & %.2f & %.2f\\\\\n",J,b,D,DJ);
          printf("%.2f %.1f %.2f %.2f\n",J,b,D,DJ);
        }
    }
}
```

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