XVI.—On the Faraday-Tube Theory of Electro-Magnetism. By the late William Gordon Brown. Communicated by Dr C. G. Knott, F.R.S., General Secretary, along with a Biographical Note of the Author.

(Read January 9, 1922.)

1. The method of describing a field of force by means of lines or tubes of induction, which originated with Faraday, was given a quantitative form by Sir J. J. Thomson,* and further discussed by N. Campbell in his book *Modern Electrical Theory*. Since Maxwell himself looked on his work as a mathematical theory of Faraday's lines of force, one is tempted to examine the original physical theory for hints as to the modification of the Maxwellian theory to suit certain modern requirements.

What is attempted in the present paper is a reconstruction of the quantitative theory of Faraday tubes on a dynamical basis from the minimum of hypotheses: partly to enable the electro-magnetic consequences of altering the Principle of Action to be estimated, and partly to suggest plausible directions for modification of the electro-magnetic relations themselves. It will incidentally be shown that the stress which may be supposed to act in the electro-magnetic field requires certain modifications if the theory of lines of force is adopted.

2. The first assumption required is as follows:—A tube of induction, or Faraday tube, may be defined as a continuous line having certain physical properties. Any tube may either be a closed curve, or its ends be connected to a positive and a negative electric particle respectively; the positive direction will then be from the positive to the negative particle. It would be superfluous at present to specify any further properties of the electric particles.

The tubes at any point may be divided into sets distinguished by each set having a common direction and a common velocity of translation.

In what follows the vectorial notation of Heaviside is employed,† and

* *Recent Researches*, chap. i; *Electricity and Matter*, chap. i.

† [Heaviside's vector notation is a modification of Hamilton's quaternion notation, the main difference being that the quaternion product of two vectors **AB** is not used in Hamilton's sense but is used to mean the scalar of the complete product—that is, Heaviside's **AB** is equivalent to Hamilton's $-S\mathbf{AB}$, and may be defined geometrically as equal to AB cos θ, where A, B are the lengths of **A**, **B**, and θ the angle between them. As in other non-associative vector algebras, the square of a vector is equal to the square of its length; in quaternions $\mathbf{A}^2 = -A^2$. The notation introduced by Gordon Brown in equations (9), (10), etc., has been suggested by others but generally discarded. Burali-Forti and Marcolongo, however, make it a feature of their system of vector analysis. As a notation it is misleading; as an operator it is inferior to the quaternion ∇.—C. G. K.]

electrical quantities are measured in rational units. Let the density of the tubes of the mth set and their direction, at any point, be represented by the magnitude and direction of the vector $\mathbf{d}_m$; then the number of tubes of that set passing through unit area normal to the unit vector $\mathbf{N}$ will be $\mathbf{N}\mathbf{d}_m$.

Let

$$\mathbf{D} = \Sigma \mathbf{d}_m \, , \qquad (1)$$

the summation including all the sets present at the point; then the total of all sets passing through the same unit area is

$$\Sigma \mathbf{N}\mathbf{d}_m = \mathbf{N}\mathbf{D} \, ,$$

where tubes passing through the area in the direction of $\mathbf{N}$ are reckoned positive, and the algebraic total is intended. Thus $\mathbf{D}$ represents vectorially the total flux of tubes; it is to be identified with the $\mathbf{D}$ of Heaviside, and, except for the question of units, with the (f, g, h) of Maxwell.

Let $\mathbf{q}_m$ be the (vector) velocity of the tubes of the mth set at the point in question, and let

$$\mathbf{H} = \Sigma V \mathbf{q}_m \mathbf{d}_m \qquad (2)$$

The quantity thus defined will be shown to have the properties of magnetic force.

This completes the geometrical and kinematic specification of the properties of the tubes. It is not difficult to see that if we define the charge of an electric particle as the number of tubes *leaving* it, in the sense that the direction of the tubes at a positive particle is outwards, then the density of electric charge will be given by

$$\rho = \operatorname{div} \mathbf{D} \qquad (3)$$

If we take the curl of (2) and expand the right member fully, interpreting the terms kinematically, we obtain the equation

$$\begin{aligned} \operatorname{curl} \mathbf{H} &= \dot{\mathbf{D}} + \Sigma \mathbf{q}_m \operatorname{div} \mathbf{d}_m \\ &= \dot{\mathbf{D}} + \mathbf{u}\rho \, , \end{aligned} \qquad (4)$$

where $\dot{\mathbf{D}}$ is the time rate of change of $\mathbf{D}$ *at a fixed point*, and $\mathbf{u}$ is the mean velocity of translation of the electric particles calculated so as to make $\mathbf{u}\rho$ the convection current.

3. The second assumption made is dynamical. Let us write

$$\mathbf{E} = \frac{\mathbf{D}}{K} \qquad (5)$$

$$\mathbf{B} = \mu \mathbf{H} \, , \qquad (6)$$

where μ and K are constants, and $\mathbf{E}$ and $\mathbf{B}$ are new vectors, the electric intensity and magnetic induction.

Then we assume that the volume densities of kinetic and potential energy are given by

$$U = \tfrac{1}{2}\mathbf{ED} \qquad (7)$$

$$T = \tfrac{1}{2}\mathbf{HB} \qquad (8)$$

The meaning attached to the above quantities is that if we write

$$L = \iiint (T - U)\,dv\,,$$

where the volume integral is extended throughout all space, then L may be used as the Lagrangian function in equations of motion of the usual form. For the sake of brevity, vectorial general coordinates will be employed. In order to preserve the form of the equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0\,,$$

it is sufficient to write, in the case of a vector coordinate $\mathbf{r}$ (equivalent to the three scalar coordinates x, y, z),

$$\left.\begin{aligned}\frac{\partial}{\partial \mathbf{r}} &= \nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\\ \frac{\partial}{\partial \dot{\mathbf{r}}} &= \quad\;\; i\frac{\partial}{\partial \dot{x}} + j\frac{\partial}{\partial \dot{y}} + k\frac{\partial}{\partial \dot{z}}\end{aligned}\right\} \qquad (9)$$

This notation in vectorial analysis is of course not generally applicable but is convenient for the purposes of the present paper. The general results of differentiation which will be required are

$$\frac{\partial}{\partial \mathbf{s}}\mathbf{as} = \mathbf{a} \qquad (10)$$

$$\frac{\partial}{\partial \mathbf{s}}\mathbf{s}\psi\mathbf{s} = 2\psi\mathbf{s}\,, \qquad (11)$$

where $\mathbf{s}$ is any vector variable, $\mathbf{a}$ is a constant vector, and ψ is a constant self-conjugate linear and vector operator.

4. To define the general coordinates, let all tubes at a given moment be divided into small unit lengths; and let $\mathbf{r}$ be the vector from a fixed origin to the centre of one such unit segment, which forms part of a tube of the mth set, then the Lagrangian equation corresponding to $\mathbf{r}$ will be

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{r}}} - \frac{\partial L}{\partial \mathbf{r}} = 0\,. \qquad (12)$$

Now, when a unit length of a tube of the mth set is added to, or removed from, an element of volume, the increase or decrease of the whole Lagrangian function due to this element will be

$$\begin{aligned}\delta L &= \frac{\partial L}{\partial \mathbf{d}_m}\delta\mathbf{d}_m\\ &= -\,\delta\mathbf{d}_m(\mathbf{E} + V\mathbf{q}_m\mathbf{B})\,, \qquad (13)\end{aligned}$$

for

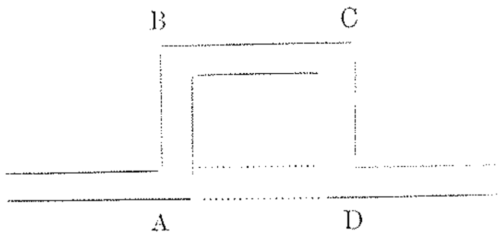
$$\frac{\partial}{\partial \mathbf{d}_m}(\mathrm{T}-\mathrm{U}) = \frac{\partial}{\partial \mathbf{d}_m}[\tfrac{1}{2}\mu(\Sigma \mathrm{V}\mathbf{q}_m\mathbf{d}_m)^2 - \tfrac{1}{2}\mathbf{D}^2/\mathrm{K}]$$

$$= \frac{\partial}{\partial \mathbf{d}_m}[\Sigma\Sigma\tfrac{1}{2}\mu(-\mathbf{d}_m \mathrm{V}\mathbf{q}_m \mathrm{V}\mathbf{q}_s\mathbf{d}_s) - \tfrac{1}{2}(\Sigma\mathbf{d}_m)^2/(\mathrm{K})]$$

$$= -(\mathbf{E}+\mathrm{V}\mathbf{q}_m\mathbf{B}), \quad \ldots \quad (14)$$

where the summations include all values of the suffixes n, s, the differentiation of terms such as $(-\mathbf{d}_m \mathrm{V}\mathbf{q}_m \mathrm{V}\mathbf{q}_m\mathbf{d}_m)$ being performed by means of (11), since $(-\mathrm{V}\mathbf{q}_m\mathrm{V}\mathbf{q}_m)$ is a self-conjugate operator; and that of cross-products, such as :—

$(-\mathbf{d}_m \mathrm{V}\mathbf{q}_m \mathrm{V}\mathbf{q}_s\mathbf{d}_s)$ by means of (10), writing $\mathbf{a} = -\mathrm{V}\mathbf{q}_m\mathrm{V}\mathbf{q}_s\mathbf{d}_s$.



Thus, if in the figure the unit segment is removed from the position AD (at which (14) has the value $-(\mathbf{E}+\mathrm{V}\mathbf{q}_m\mathbf{B})$ to the parallel position BC (at which (14) has the value $-(1+\delta\mathbf{r}\nabla.)$ $(\mathbf{E}+\mathrm{V}\mathbf{q}_m\mathbf{B})$, $\mathrm{AB}=\delta\mathbf{r}$), then the total increase in L is given by

$$\delta\mathrm{L} = -\delta\mathbf{r}\nabla \,.\, \delta\mathbf{d}_m(\mathbf{E}+\mathrm{V}\mathbf{q}_m\mathbf{B})\,.$$

It will now be convenient to suppose (as we may without loss of generality) that the mth set consists of but one tube, so that $\delta\mathbf{d}_m = \mathbf{d}_m$ and is in fact a unit vector.

Then

$$\delta_1\mathrm{L} = -\delta\mathbf{r}\nabla \,.\, \mathbf{d}_m(\mathbf{E}+\mathrm{V}\mathbf{q}_m\mathbf{B})\,, \quad \ldots \quad (15)$$

and in applying the axial differentiator $\delta\mathbf{r}\nabla$ we must remember that neither $\mathbf{d}_m$ nor $\mathbf{q}_m$ as they occur explicitly are to be considered variable.

But to preserve the continuity of the tube we require to introduce the segments AB, CD, as shown in the figure, so that, again applying (13), we have the change of L due to this cause.

$$\delta_2\mathrm{L} = \mathbf{d}_m\nabla \,.\, \delta\mathbf{r}(\mathbf{E}+\mathrm{V}\mathbf{q}_m\mathbf{B})\,, \quad \ldots \quad (16)$$

in which $\mathbf{q}_m$ is variable (but not $\mathbf{d}_m$).

Hence

$$\delta\mathrm{L} = \delta_1\mathrm{L} + \delta_2\mathrm{L} = \delta\mathbf{r}[\mathbf{d}_m\nabla \,.\, (\mathbf{E}+\mathrm{V}\mathbf{q}_m\mathbf{B}) - \nabla \,.\, \mathbf{d}_m(\mathbf{E}+\mathrm{V}\mathbf{q}_m\mathbf{B})]\,,$$

$\mathbf{q}_m$ varying in the first term only, and $\mathbf{d}_m$ not at all, and finally

$$\frac{\partial \mathrm{L}}{\partial \mathbf{r}} = \mathbf{d}_m \nabla . (\mathbf{E} + \mathrm{V}\mathbf{q}_m\mathbf{B}) - \nabla . \mathbf{d}_m(\mathbf{E} + \mathrm{V}\mathbf{q}_m\mathbf{B}) \quad . \quad . \quad . \quad (17)$$

with the same convention.

In calculating the momentum term $\frac{\partial \mathrm{L}}{\partial \dot{\mathbf{r}}}$ we have $\dot{\mathbf{r}} = \mathbf{q}_m$. Then by the method employed above in calculating (14), since T is symmetrical in $\mathbf{q}_m$ and $\mathbf{d}_m$,

$$\frac{\partial \mathrm{T}}{\partial \mathbf{q}_m} = \mathrm{V}\mathbf{d}_m\mathbf{B} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

This will be the value of $\frac{\partial \mathrm{L}}{\partial \dot{\mathbf{r}}}$ when $\mathbf{d}_m$ is a unit length of tube, but in performing the complete differentiation to time in (7) we must remember that any length of tube will in general be continually varying in direction and magnitude. It is clear that

$$\frac{d}{dt}\mathbf{d}_m = \mathbf{d}_m \nabla . \mathbf{q}_m , \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

since the rate of change of a segment of a straight line, as AD in the figure, will be the relative velocity of its ends (vectorially); while, of course, if $\mathbf{q}_m$ expresses the velocity of any point of the tube, as A, the velocity at D will be $(1 + \overline{\mathrm{AD}}\nabla .)\mathbf{q}_m$, where $\overline{\mathrm{AD}}$ is the vector element.

Thus

$$\frac{d}{dt}\frac{\partial \mathrm{L}}{\partial \dot{\mathbf{r}}} = \frac{d}{dt}\mathrm{V}\mathbf{d}_m\mathbf{B}$$
$$= \mathrm{V}(\mathbf{d}_m\nabla . \mathbf{q}_m)\mathbf{B} + \mathrm{V}\mathbf{d}_m\dot{\mathbf{B}} + \mathrm{V}\mathbf{d}_m(\mathbf{q}_m\nabla . \mathbf{B}) , . \quad . \quad . \quad (20)$$

where $\dot{\mathbf{B}}$ is the rate of change of $\mathbf{B}$ at a fixed point coincident with the moving centre of the segment, $\mathbf{q}_m\nabla . \mathbf{B}$ being of course the term in the rate of change due to motion of the segment with velocity $\mathbf{q}_m$.

Equation (12) is therefore by (17) and (20),

$$\mathrm{V}(\mathbf{d}_m\nabla . \mathbf{q}_m)\mathbf{B} + \mathrm{V}\mathbf{d}_m\dot{\mathbf{B}} - \mathrm{V}\mathbf{d}_m(\mathbf{q}_m\nabla . \mathbf{B})$$
$$- \mathbf{d}_m\nabla . (\mathbf{E} + \mathrm{V}\mathbf{q}_m\mathbf{B}) + \nabla . \mathbf{d}_m(\mathbf{E} + \mathrm{V}\mathbf{q}_m\mathbf{B}) = 0 , . \quad . \quad . \quad (21)$$

$\mathbf{d}_m$ and $\mathbf{q}_m$ being constant in the last term, and ∇ operating forwards only.

In carrying out the simplifying transformations we may drop for the moment the suffix m.

From the last two terms we have, in part,

$$- \mathbf{d}\nabla . \mathbf{E} + \nabla . \mathbf{d}\mathbf{E} = + \mathrm{V}\mathbf{d}\mathrm{V}\nabla\mathbf{E}$$
$$= + \mathrm{V}\mathbf{d} \operatorname{curl} \mathbf{E} \quad . \quad . \quad . \quad . \quad . \quad (22)$$

From the remainder we find

$$\begin{aligned}
&V\mathbf{d}\dot{\mathbf{B}} + V(\mathbf{d}\nabla \,.\, \mathbf{q})\mathbf{B} + V\mathbf{d}(\mathbf{q}\nabla \,.\, \mathbf{B}) - \mathbf{d}\nabla \,.\, V\mathbf{q}\mathbf{B} + \nabla_1 \,.\, \mathbf{d}V\mathbf{q}\mathbf{B}_1 \\
&= V\mathbf{d}\dot{\mathbf{B}} + V(\mathbf{d}\nabla \,.\, \mathbf{q})\mathbf{B} + V \,.\, \mathbf{d}(\mathbf{q}\nabla \,.\, \mathbf{B}) \\
&\qquad - V(\mathbf{d}\nabla \,.\, \mathbf{q})\dot{\mathbf{B}} - V \,.\, \mathbf{q}(\mathbf{d}\nabla \,.\, \mathbf{B}) + \nabla_1 \,.\, \mathbf{d}V\mathbf{q}\mathbf{B}_1 \\
&= V\mathbf{d}\dot{\mathbf{B}} + V \,.\, \mathbf{d}(\mathbf{q}\nabla \,.\, \mathbf{B}) - V \,.\, \mathbf{q}(\mathbf{d}\nabla \,.\, \mathbf{B}) \\
&\qquad - V \,.\, \mathbf{d}(\mathbf{q}\nabla \,.\, \mathbf{B}) + V \,.\, \mathbf{q}(\mathbf{d}\nabla \,.\, \mathbf{B}) + V\mathbf{d}\mathbf{q} \,.\, \nabla\mathbf{B} \\
&= V\mathbf{d}\dot{\mathbf{B}} + V\mathbf{d}\mathbf{q} \operatorname{div} \mathbf{B}, \qquad (23)
\end{aligned}$$

where the suffix restricts the action of ∇ to the vector carrying the same suffix.

Equation (21) then reduces to

$$V\mathbf{d}_m(\operatorname{curl} \mathbf{E} + \dot{\mathbf{B}} + \operatorname{div} \mathbf{B}) \,. \qquad (24)$$

Now $\mathbf{d}_m$ will have different values according to the different directions of the various sets of tubes; hence (unless all the tubes are parallel) we may write

$$\operatorname{curl} \mathbf{E} + \dot{\mathbf{B}} + \mathbf{q}_m \operatorname{div} \mathbf{B} = 0 \,. \qquad (25)$$

From this, since $\mathbf{q}_m$ is the velocity of any set of tubes, unless all the sets have a common velocity, we must have

$$\operatorname{div} \mathbf{B} = 0 \,, \qquad (26)$$

and thus

$$-\operatorname{curl} \mathbf{E} = \dot{\mathbf{B}} \,. \qquad (27)$$

We have now shown that the first four laws of the ordinary theory of electro-magnetism are consequences of the assumptions which have been made. It may be observed that whereas, in the proof of the first two laws (3) and (4), no departure of importance is made from the method of *Recent Researches*, the proof just given of the laws (26) and (27) is quite different from that adopted in that work. This is rendered necessary by the purpose of the present paper, which is not to deduce the properties of the tubes from the known laws of electro-magnetism, but to show that, given the tubes with the (essential) properties assigned to them by Sir J. J. Thomson, the laws of electro-magnetism follow.

5. It remains to discuss the forces acting on the electric particles. Referring to the figure on p. 228, let B be a particle at the end of the tube B, C, D. Then the change in L due to the displacement of the end of the tube from B to A (introducing a new segment BA), is by (13)

$$\delta L = \delta \mathbf{r}(\mathbf{E} + V\mathbf{q}_m\mathbf{B}) \,, \qquad (28)$$

since

$$\delta\mathbf{d}_m = \overline{AB} = -\delta\mathbf{r} \,,$$

B being the positive end of the tube, and thus equivalent to a positive unit of electricity. Hence the force acting per unit charge moving with velocity $\mathbf{q}$ is

$$\mathbf{F} = \mathbf{E} + \mathrm{V}\mathbf{q}\mathbf{B}\,, \qquad (29)$$

the Fifth Law of Electro-magnetism.

6. The definite dynamical assumptions of this theory enable us to examine very thoroughly such questions as the stress in the field and the mechanism of radiation.

Heaviside* has given a general discussion of the problem of stresses from which it is not difficult to deduce the following general result:—

Let ψ_0 be the operator of Maxwell's stress,

$$\psi_0 = \mathbf{E}\,.\,\mathbf{D} + \mathbf{H}\,.\,\mathbf{B} - \tfrac{1}{2}(\mathbf{ED} + \mathbf{HB})\,, \qquad (30)$$

where any vector operand forms with $\mathbf{D}$ and $\mathbf{B}$ scalar products in the first and second terms. When this operand is a unit vector $\mathbf{N}$, $\psi_0\mathbf{N}$ is the stress on the plane perpendicular to $\mathbf{N}$.

Let ψ be the stress derived from ψ_0 by putting for $\mathbf{E}$, $\mathbf{E} + \mathrm{V}\mathbf{q}\mathbf{B}$, and for $\mathbf{H}$, $\mathbf{H} - \mathrm{V}\mathbf{q}\mathbf{D}$, namely

$$\begin{aligned} \psi &= \psi_0 + \mathrm{V}\mathbf{qB}\,.\,\mathbf{D} - \mathrm{V}\mathbf{qD}\,.\,\mathbf{B} - \tfrac{1}{2}(\mathrm{V}\mathbf{qB})\mathbf{D} + \tfrac{1}{2}(\mathrm{V}\mathbf{qD})\mathbf{B} \\ &= \psi_0 + \mathrm{V}\mathbf{qB}\,.\,\mathbf{D} + \mathrm{V}\mathbf{Dq}\,.\,\mathbf{B} - \mathbf{D}\mathrm{V}\mathbf{qB} \\ &= \psi_0 + \mathrm{V}\mathbf{DB}\,.\,\mathbf{q} \qquad (31) \end{aligned}$$

by mere vector transformation.

Then if $\mathbf{N}$ is unit normal to a surface moving with a velocity $\mathbf{q}$ at any point, $\psi\mathbf{N}$ is the flux of momentum through the surface in the direction opposite to the positive direction of $\mathbf{N}$, per unit surface per unit time.

To see that this is true we have only to apply the theorem of divergence; in the first place we note that since

$$\frac{\partial \mathrm{T}}{\partial \mathrm{q}_m} = \mathrm{V}\mathbf{d}_m\mathbf{B}\,, \qquad (18)$$

summing for all value of m we have $\mathrm{V}\mathbf{DB}$ equal to the momentum per unit volume. But

$$\psi_0\nabla = \frac{\partial}{\partial t}\mathrm{V}\mathbf{DB}\,, \qquad (32)$$

a result easily deduced (Heaviside, *loc. cit.*) from the circuital laws, and usually expressed in words by stating that Maxwell's stress gives rise to a translational force per unit volume equal to the rate of change at a fixed point of the momentum per unit volume (the absence of electrification being assumed). We are thus entitled to say that $\psi_0\mathbf{N}$ is the flux of momentum per unit area of a fixed surface. Now it is clear that $\mathrm{V}\mathbf{DB}\,.\,\mathbf{qN}$

* *Electrical Papers*, vol. ii, pp. 521 *et seq.*; *also Phil. Trans.*, A, 1892.

is the flux per unit area due to the motion of the surface with velocity $\mathbf{q}$. Hence ψ is the general operator giving the flux of momentum. The equation of rate of change of momentum per unit volume at a point whose velocity is $\mathbf{q}$ is

$$\psi\nabla = \frac{\partial}{\partial t}\mathrm{V}\mathbf{DB} + \nabla\mathbf{q}\,.\,\mathrm{V}\mathbf{DB}$$

$$= \frac{\partial}{\partial t}\mathrm{V}\mathbf{DB} + \mathbf{q}\nabla\,.\,\mathbf{DB} + \mathrm{V}\mathbf{DB}\,.\,\mathrm{div}\,\mathbf{q}, \qquad (33)$$

the first two terms giving the rate of change of density of momentum at the moving point, and the last term the rate of change due to expansion at the rate div $\mathbf{q}$.

This flux of momentum ψ is partly due to convection, and partly to be ascribed to a stress. It is interesting to note that if all the tubes were of one set, we could determine the stress by simply putting $\mathbf{q}$ equal to this velocity. We should then have $\mathbf{H} = \mathrm{V}\mathbf{qD}$, and the stress would be

$$\phi = (\mathbf{E} + \mathrm{V}\mathbf{qB})\,.\,\mathbf{D} - \tfrac{1}{2}(\mathbf{E} + \mathrm{V}\mathbf{qB})\mathbf{D}$$

$$= \mathbf{F}\,.\,\mathbf{D} - \tfrac{1}{2}\mathbf{FD}$$

$$= \mathbf{F}\,.\,\mathbf{D} + \tfrac{1}{2}(\mathbf{HB} - \mathbf{ED}) \qquad (34)$$

In general the stress operator will be obtained by subtracting from ψ_0 the operator $-\Sigma(\mathrm{V}\mathbf{d}_m\mathbf{B}\,.\,\mathbf{q}_m)$ which gives the convective flux of momentum relative to a fixed point; thus the stress is

$$\phi = \psi_0 + \Sigma(\mathrm{V}\mathbf{d}_m\mathbf{B}\,.\,\mathbf{q}_m) \qquad (35)'$$

$$= \mathbf{E}\,.\,\mathbf{D} + \mathbf{H}\,.\,\mathbf{B} - \tfrac{1}{2}\mathbf{ED} - \tfrac{1}{2}\mathbf{HB} + \Sigma\mathrm{V}\mathbf{q}_m\mathbf{B}\,.\,\mathbf{d}_m - \Sigma\mathrm{V}\mathbf{q}_m\mathbf{d}_m\,.\,\mathbf{B}$$
$$+ \Sigma(\mathrm{V}\mathbf{q}_m\mathbf{d}_m)\mathbf{B}$$

$$= \mathbf{E}\,.\,\mathbf{D} + \mathbf{H}\,.\,\mathbf{B} - \tfrac{1}{2}\mathbf{ED} - \tfrac{1}{2}\mathbf{HB} + \Sigma\mathrm{V}\mathbf{q}_m\mathbf{B}\,.\,\mathbf{d}_m - \mathbf{H}\,.\,\mathbf{B} + \mathbf{HB}$$

$$= \Sigma\{(\mathbf{E} + \mathrm{V}\mathbf{q}_m\mathbf{B})\,.\,\mathbf{d}_m\} - \tfrac{1}{2}\mathbf{ED} + \tfrac{1}{2}\mathbf{HB} \qquad (35)$$

From (35)′ we see that the stress coincides with Maxwell's stress when there is no convection of momentum relative to the (so-called) fixed reference frame; and from (35) that it consists in general of a quasi-tension equal to $\mathbf{E} + \mathrm{V}\mathbf{q}_m\mathbf{B}$ per tube of the mth set together with a hydrostatic pressure $\tfrac{1}{2}(\mathbf{ED} - \mathbf{HB})$. The torque per unit volume is seen to be

$$\phi - \phi' = \mathbf{S} = -\Sigma\mathrm{V}(\mathbf{E} + \mathrm{V}\mathbf{q}_m\mathbf{B})\mathbf{d}_m$$

$$= +\Sigma\mathrm{V}\mathbf{d}_m\mathrm{V}\mathbf{q}_m\mathbf{B}$$

$$= +\Sigma\mathrm{V}\mathbf{q}_m\mathrm{V}\mathbf{d}_m\mathbf{B} - \Sigma\mathrm{V}(\mathrm{V}\mathbf{q}_m\mathbf{d}_m)\mathbf{B}$$

$$= +\Sigma\mathrm{V}\mathbf{q}_m\mathrm{V}\mathbf{d}_m\mathbf{B}, \qquad (36)$$

the last expression being the rate of change of moment of momentum about a fixed point due to component of velocity perpendicular to the momentum, familiar in the hydrodynamics of the motion of bodies in a fluid.

7. The flux of energy also consists of two parts: the convective flux due to the motion of the tubes, and the flux due to the activity of the stress. To find the convective flux we require to localise the energy in a manner rather difficult to justify. The whole energy per unit volume may be written

$$\tfrac{1}{2}\mathbf{NB}+\tfrac{1}{2}\mathbf{ED}$$
$$=\tfrac{1}{2}\Sigma\mathbf{d}_m(\mathbf{E}-\mathrm{V}\mathbf{q}_m\mathbf{B}) \qquad (37)$$

Then we may suppose the part $\mathbf{d}_m(\mathbf{E}-\mathrm{V}\mathbf{q}_m\mathbf{B})$ of the energy to be moving with velocity $\mathbf{q}_m$, and so on. The total convection of energy will therefore be

$$\tfrac{1}{2}\Sigma\mathbf{d}_m(\mathbf{E}-\mathrm{V}\mathbf{q}_m\mathbf{B})\,.\,\mathbf{q}_m \qquad (38)$$

To find the stress-activity flux from (35), consider first the term $(\mathbf{E}+\mathrm{V}\mathbf{q}_m\mathbf{B})\,.\,\mathbf{d}_m$; the appropriate velocity is clearly $\mathbf{q}_m$, and the flux (by Heaviside's method)

$$-\mathbf{q}_m(\mathbf{E}+\mathrm{V}\mathbf{q}_m\mathbf{B})\,.\,\mathbf{d}_m=-\mathbf{q}_m\mathbf{E}\,.\,\mathbf{d}_m\,.$$

Again, we may write the second term

$$-\tfrac{1}{2}\mathbf{ED}+\tfrac{1}{2}\mathbf{HB}=-\tfrac{1}{2}\{(\Sigma\mathbf{d}_m)\mathbf{E}-(\Sigma\mathrm{V}\mathbf{q}_m\mathbf{d}_m)\mathbf{B}\}$$
$$=-\tfrac{1}{2}\Sigma\mathbf{d}_m(\mathbf{E}+\mathrm{V}\mathbf{q}_m\mathbf{B}),$$

and it seems permissible to write the activity flux due to the term $-\tfrac{1}{2}\mathbf{d}_m(\mathbf{E}+\mathrm{V}\mathbf{q}_m\mathbf{B})$ as $+\tfrac{1}{2}\mathbf{q}_m\,.\,\mathbf{d}_m(\mathbf{E}+\mathrm{V}\mathbf{q}_m\mathbf{B})$. Hence the total activity flux will be

$$-\Sigma\{\mathbf{q}_m\mathbf{E}\,.\,\mathbf{d}_m-\tfrac{1}{2}\mathbf{d}_m(\mathbf{E}+\mathrm{V}\mathbf{q}_m\mathbf{B})\}, \qquad (39)$$

and the whole flux, adding (38) and (39),

$$\begin{aligned}\mathbf{W}&=\tfrac{1}{2}\Sigma\mathbf{d}_m(\mathbf{E}-\mathrm{V}\mathbf{q}_m\mathbf{B})\,.\,\mathbf{q}_m-\Sigma\mathbf{q}_m\mathbf{E}\,.\,\mathbf{d}_m+\tfrac{1}{2}\Sigma\mathbf{d}_m(\mathbf{E}+\mathrm{V}\mathbf{q}_m\mathbf{B})\mathbf{q}_m\\&=\Sigma(\mathbf{d}_m\mathbf{E}\,.\,\mathbf{q}_m-\mathbf{q}_m\mathbf{E}\,.\,\mathbf{d}_m)\\&=\mathrm{V}\mathbf{E}\Sigma\mathrm{V}\mathbf{q}_m\mathbf{d}_m\\&=\mathrm{V}\mathbf{EH}\,. \qquad (40)\end{aligned}$$

8. Since we have shown that this theory leads to the ordinary equations of the electro-magnetic field, it is unnecessary to give a separate proof of the uniform propagation of disturbances with velocity $1/\sqrt{\mu\mathrm{K}}$. It is perhaps as well, however, to examine shortly the mechanism of propagation, particularly since the mental picture of electro-magnetic radiation afforded by the theory is in many respects very satisfactory.

N. Campbell gives a short discussion of the question, and shows that a tube at rest may be compared to a flexible cord of linear density $\mu\mathrm{D}$ under a tension D/K; the square of the velocity of propagation of transverse disturbances being then $1/\mu\mathrm{K}$ by the elementary dynamics of cords. To extend this result to the case of a tube having a general velocity v perpendicular to its own direction, we have only to remember that, by equation (35) above, the stress to which the restoring force is due will now be the quasi-tension $\mathbf{E}+\mathrm{V}\mathbf{q}\mathbf{B}$, where $\mathbf{q}$ is the velocity of the tubes, of which we shall suppose that only one set need be taken into account; and with this last assumption we may drop the suffix m and so write

$$\mathbf{B}=\mu\mathrm{V}\mathbf{q}\mathbf{D},\ \mathbf{E}=\frac{\mathbf{d}}{\mathrm{K}}.$$

The $\mathbf{d}$ component of $\mathbf{E}+\mathrm{V}\mathbf{q}\mathbf{D}$ is the only effective part of the stress, and its magnitude is given by

$$(\mathbf{E}+\mathrm{V}\mathbf{q}\mathbf{B})\mathbf{d}_1=\left(\frac{\mathbf{d}}{K}+\mu\mathrm{V}\mathbf{q}\mathrm{V}\mathbf{q}\mathbf{d}\right)\mathbf{d}_1,$$

where $\mathbf{d}_1$ is the unit vector parallel to $\mathbf{d}$, or $\mathbf{d}=d\mathbf{d}_1$. This equals

$$\begin{aligned}&\frac{d}{\mathrm{K}}(1+\mu\mathrm{K}\mathbf{d}_1\mathrm{V}\mathbf{q}\mathrm{V}\mathbf{q}\mathbf{d})\\ &=\frac{d}{\mathrm{K}}\{1-\mu\mathrm{K}(\mathrm{V}\mathbf{d}_1\mathbf{q})^2\}\\ &=\frac{d}{\mathrm{K}}\left\{1-\frac{v^2}{c^2}\right\}, \qquad (41)\end{aligned}$$

where $c^2\mu\mathrm{K}=1$.

The linear density will remain μd, so that the velocity of propagation along the tube will be $\sqrt{c^2-v^2}$. Since the tube itself is in motion with velocity v in a perpendicular direction, the propagation of the disturbance in space will be with velocity c in a direction making an angle $\sin^{-1}\frac{v}{c}$ with the tube. When $v=c$ the disturbance will not be propagated at all along the tube, which will lie in the wave-front; and the traction $(\mathbf{E}+\mu\mathrm{V}\mathbf{q}\mathrm{V}\mathbf{q}\mathbf{D})$ will vanish.

9. To take into account a general velocity of the tube in the direction of its length, let us restrict ourselves to plane-polarised radiation. We shall take the x-axis in the direction of propagation, and the y-axis in that of the disturbance. Since we are dealing only with transverse vibrations, the velocity of the tubes in the direction of the ray will be constant from point to point along a tube. Let u be this x-component of velocity. Also let (x, y) be the coordinates of a point on some particular tube at time t, so that y is a function of x and t. Then the whole y-component of velocity of the point will be

$$v=\frac{dy}{dt}=\frac{\partial y}{\partial t}+u\frac{\partial y}{\partial x} \qquad (42)$$

It is obvious that the shearing motion perpendicular to the x-axis of the tubes in their vibration will not affect the number of tubes per unit area passing through a plane normal to the x-axis. Thus the quantity

d_x, the x-component of electric displacement, will be constant at a point on the tube, or

$$\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}\right)d_x=0.$$

Also, if d_y is the y-component, we shall have

$$\frac{d_y}{d_x}=\frac{\partial y}{\partial x}.$$

And thus

$$\begin{aligned} d_y &= \frac{\partial y}{\partial x}d_x \\ d^2 &= d_x^2+d_y^2 \\ &= d_x^2\left\{1+\left(\frac{\partial y}{\partial x}\right)^2\right\} \qquad (43) \end{aligned}$$

The momentum per unit length along the tube is

$$\begin{aligned} \mathrm{V}\mathbf{d}_1\mathbf{B} &= \mu\mathrm{V}\mathbf{d}_1\mathrm{V}\mathbf{q}\mathbf{d} \\ &= \mu(\mathbf{q}\,.\,d-\mathbf{d}\,.\,\mathbf{q}\mathbf{d}_1). \end{aligned}$$

Multiply this by $\dfrac{d}{d_x}$ to find the value appropriate to unit length along the x-axis, and, taking the y-component, we have

$$\begin{aligned} &\mu\frac{d}{d_x}\left[\left(\frac{\partial y}{\partial t}+u\frac{\partial y}{\partial x}\right)d-d_x\frac{\partial y}{\partial x}\frac{1}{d}\left\{ud_x+\left(\frac{\partial y}{\partial t}+u\frac{\partial y}{\partial x}\right)d_x\frac{\partial y}{\partial x}\right\}\right] \\ &=\mu d_x\left[\left(\frac{\partial y}{\partial t}+u\frac{\partial y}{\partial x}\right)\frac{d^2}{d_x^2}-\frac{\partial y}{\partial x}\left\{u+\left(\frac{\partial y}{\partial t}+u\frac{\partial y}{\partial x}\right)\frac{\partial y}{\partial x}\right\}\right] \\ &=\mu d_x\left[\frac{\partial y}{\partial t}\left\{1+\left(\frac{\partial y}{\partial x}\right)^2-\left(\frac{\partial y}{\partial x}\right)^2\right\}+u\frac{\partial y}{\partial x}\left\{1+\left(\frac{\partial y}{\partial x}\right)^2-1-\left(\frac{\partial y}{\partial x}\right)^2\right\}\right] \\ &=\mu d_x\frac{\partial y}{\partial t} \qquad (44) \end{aligned}$$

Hence the rate of change of momentum in the y-direction per unit length along the x-axis is

$$\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}\right)\mu d_x\frac{\partial y}{\partial t}=\mu d_x\left[\frac{\partial^2 y}{\partial t^2}+u\frac{\partial^2 y}{\partial x\partial t}\right]. \qquad (45)$$

The force to be equated to this arises from the quasi-tension

$$\begin{aligned} \mathbf{E}+\mathrm{V}\mathbf{q}\mathbf{B} &= \frac{\mathbf{d}}{\mathrm{K}}+\mu\mathrm{V}\mathbf{q}\mathrm{V}\mathbf{q}\mathbf{D} \\ &= \frac{\mathbf{d}}{\mathrm{K}}+\mu\mathbf{q}\,.\,\mathbf{q}\mathbf{d}-\mu\mathbf{d}\,.\,\mathbf{q}^2, \end{aligned}$$

of which the y-component is

$$\begin{aligned} &\frac{1}{\mathrm{K}}d_x\frac{\partial y}{\partial x}+\mu\left(\frac{\partial y}{\partial t}+u\frac{\partial y}{\partial x}\right)\left\{ud_x+\left(\frac{\partial y}{\partial t}+u\frac{\partial y}{\partial x}\right)d_x\frac{\partial y}{\partial x}\right\}-\mu d_x\frac{\partial y}{\partial x}\left\{u^2+\left(\frac{\partial y}{\partial t}+u\frac{\partial y}{\partial x}\right)^2\right\} \\ &=\frac{1}{\mathrm{K}}d_x\frac{\partial y}{\partial x}+\mu\left(\frac{\partial y}{\partial t}+u\frac{\partial y}{\partial x}\right)ud_x-\mu d_x\frac{\partial y}{\partial x}u^2 \\ &=\frac{1}{\mathrm{K}}d_x\frac{\partial y}{\partial x}+\mu d_x u\frac{\partial y}{\partial t} \qquad (46) \end{aligned}$$

Differentiating with respect to x we have the force per unit length

$$d_x\left[\frac{1}{K}\frac{\partial^2 y}{\partial x^2}+\mu u\frac{\partial^2 y}{\partial x\partial t}\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

Equating therefore expressions (45) and (47) and dividing by d_x, we have

$$\frac{1}{K}\frac{\partial^2 y}{\partial x^2}+\mu u\frac{\partial^2 y}{\partial x\partial t}=\mu u\frac{\partial^2 y}{\partial x\partial t}+\mu\frac{\partial^2 y}{\partial t^2}$$

or

$$\frac{\partial^2 y}{\partial t^2}=c^2\frac{\partial^2 y}{\partial x^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

exhibiting the uniform propagation with velocity c independent of the general motion of the tube.

The relation between the electro-magnetic disturbance and the displacement y of the tube is easily seen to be given by

$$E_y=\frac{d_y}{K}=\frac{1}{K}d_x\frac{\partial y}{\partial x} \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

$$H_z=-\frac{\partial y}{\partial t}d_x$$

$$=\frac{1}{\mu c}\mathbf{E} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

But while in plane-polarised radiation the displacement of the tube from its normal position is thus perpendicular to the plane of polarisation, in circularly polarised rays it is easy to see that the reverse is the case.

10. The intention in presenting the theory of Faraday tubes in the present form was to suggest possibilities of modification which might explain various phenomena of which no entirely satisfactory electrical explanation has been given so far.

In making attempts of this kind we may, for instance, take advantage in various ways of the fact that the electric displacement has been considered as a mean value taken over a small, but not infinitesimal, area. From this point of view the Maxwellian theory is microscopical, and a more microscopical theory may be what is required in various regions of modern physics.

Again, the present theory rests on the localisation of electric and magnetic energy as functions of **D** and **H**, and on the derivation from these of equations of motion. Hence it would be comparatively simple to estimate the effects either of a modified distribution of energy, or of substituting any different hypothesis for the principle of action.

Lastly, quite a variety of hypotheses are possible as to the exact nature of the electric particles.

11. It will be observed that in describing the properties of the tubes

of force we have so far assumed that two oppositely directed tubes at the same point exactly cancel each other in their effects, if they are moving with the same velocity. Now, just as the electrical theory of matter explains all the phenomena of neutral bodies as due to the existence of the equal mixture of positive and negative electricity, which on the two-fluid theory was supposed to have no recognisable physical properties, so on the lines of force theory we may perhaps speculate with advantage on the possibility of explaining by means of properties of equal mixtures of oppositely directed tubes the phenomenon of gravitation, which seems for many reasons to be on a different level from the ordinary electrical phenomena. Let us consider the potential energy of such a mixture of tubes. So long as we choose an element of area large enough to include many tubes, the density of energy $\frac{1}{2}$ED must always vanish; but as we take smaller and smaller elements of area, there will be an increasing probability of the number of tubes passing through it in one direction being not quite equal to the number passing through it in the opposite direction: in other words, what to ordinary microscopic electrical measurements is a uniform absence of electric displacement may consist of alternate regions of opposite displacement so small that only the mean field of a considerable number of regions is measured. Such a field would have positive potential energy; but since the more closely the tubes are packed, the smaller is the element of area we can take without considering this effect, it seems reasonable to suppose that the effect will become smaller the more numerous are the tubes of either sign. Not improbably a mathematical form might be given to this hypothesis which would explain and locate the energy of gravitation. Let $de_1, -de_1$; $de_2, -de_2$, be pairs of opposite charges; r_1, r_2 the (small) distances apart of the components of each pair; and R the distance between the pairs. Then if the hypothesis could be so formulated that the potential energy of the system would include a term of the form

$$\frac{-\gamma de_1{}^2 de_1{}^2}{r_1 r_2 \mathrm{R}},$$

where γ is a positive constant, the law of gravitation would be completely satisfied, and gravitational mass would be identified exactly with electro-magnetic mass; for

$$\frac{de_1{}^2}{r_1}$$

is proportional to the element of electro-magnetic mass due to two elements of charge $de_1, -de_1$.

This last question is of some interest in the theory of atomic structure; a number of writers have laid stress on the importance of mutual electromagnetic mass, and in particular Harkins and E. D. Wilson * have used this phenomenon to explain the departure of atomic *weights* from whole numbers. It appears, however, that such an explanation could alone be valid if mutual mass were ponderable.

12. The theory of Faraday tubes might possibly be employed with advantage in other investigations connected with atom theory. Sir J. J. Thomson † has made several suggestions of this nature; his conception of the electron as possibly simply the end of a single Faraday tube would of course have very important consequences if adhered to in any theory of atomic structure.

Again, if we suppose that electrons and positive nuclei have the property of excluding the tubes of other electrons and nuclei, the attractions between particles of opposite sign would become a repulsion at very small distances. Or we may suppose that some or all of the tubes of an electron in an atom simply end at a nucleus, instead of spreading equally outwards in all directions; and different states of an atom, with different periods of vibration, might arise according to the number of tubes so connected. Suggestions have also been made as to the application of the theory in connection with a possible discrete structure in radiation.‡

Conclusion.

13. It has been shown that the general equation of the Maxwell-Lorentz-Heaviside theory of electro-magnetism can be derived as macroscopic consequences of a simple dynamical theory of Faraday tubes.

This theory also gives explicit and non-contradictory expression to the ideas of electro-magnetic stress, momentum, and flux of energy, and an electro-mechanical picture of radiation explaining the law of uniform propagation in spite of the motion of the source.

A number of suggestions are made as to applications to the theory of gravitation and other problems.

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* *Phil. Mag.*, Nov. 1915, p. 723.
† *Phil. Mag.* (6), xxvi, p. 792.
‡ Jeans, "Report on Quantum Theories," *Proc. Lond. Phys. Soc.*, 1915.