Chapter 1
Using the Kelly Criterion for Investing

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Abstract This chapter describes the use of the Kelly capital growth model. This model, dubbed *Fortune's Formula* by Thorp and used in the title by Poundstone (Fortune's Formula: The Untold Story of the Scientific System That Beat the Casinos and Wall Street, 2005), has many attractive features such as the maximization of asymptotic long-run wealth; see Kelly (Bell System Technical Journal 35:917–926, 1956), Breiman (Proceedings of the 4th Berkely Symposium on Mathematical Statistics and Probability 1:63–68, 1961), Algoet and Cover (Annals of Probability 16(2):876–898, 1988) and Thorp (Handbook of Asset and Liability Management, 2006). Moreover, it minimizes the expected time to reach asymptotically large goals (Breiman, Proceedings of the 4th Berkeley Symposium on Mathematical Statistics and Probability 1:63–68, 1961) and the strategy is myopic (Hakansson, Journal of Business 44:324–334, 1971). While the strategy to maximize the expected logarithm of expected final wealth computed via a nonlinear program has a number of good short- and medium-term qualities (see MacLean, Thorp, and Ziemba, The Kelly Capital Growth Investment Critria, 2010b), it is actually very risky short term since its Arrow–Pratt risk aversion index is the reciprocal of wealth and that is essentially zero for non-bankrupt investors. The chapter traces the development and use of this strategy from the log utility formulation in 1738 by Bernoulli (Econometrica 22:23–36, 1954) to current use in financial markets, sports betting, and other applications. Fractional Kelly wagers that blend the E log maximizing strategy with cash tempers the risk and yield smoother wealth paths but with generally less final wealth. Great sensitivity to parameter estimates, especially the means, makes the strategy dangerous to those whose estimates are in error and leads them to poor betting and possible bankruptcy. Still, many investors with repeated investment periods and considerable wealth, such as Warren Buffett and George Soros, use strategies that approximate full Kelly which tends to place most of one's wealth in a few assets and lead to many monthly losses but large final wealth most of the time. A simulation study is presented that shows the possibility of huge gains most of the time, possible losses no matter how good the investments appear to be, and possible extreme losses from overbetting when bad scenarios occur. The study and discussion shows that

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Samuelson's objections to E log strategies are well understood. In practice, careful risk control or financial engineering is important to deal with short-term volatility and the design of good wealth paths with limited drawdowns. Properly implemented, the strategy used by many billionaires has much to commend it, especially with many repeated investments.

Keywords Kelly investment criterion · Long-range investing · Logarithmic utility functions · Fractional Kelly strategies

1.1 Introduction

The Kelly capital growth strategy is defined as *allocate your current wealth to risky assets so that the expected logarithm of wealth is maximized period by period*. So it is a one-period static calculation that can have transaction costs and other market imperfections considered. Log utility dates to Daniel Bernoulli in 1738 who postulated that marginal utility was monotone increasing but declined with wealth and, specifically, is equal to the reciprocal of wealth, w, which yields the utility of wealth $u(w) = \log w$. Prior to this it was assumed that decisions were made on an expected value or linear utility basis. This idea ushered in declining marginal utility or *risk aversion* or *concavity* which is crucial in investment decision making. In his chapter, in Latin, he also discussed the St. Petersburg paradox and how it might be analyzed using $\log w$.

The St. Petersburg paradox actually originates from Daniel's cousin, Nicolas Bernoulli, a professor at the University of Basel where Daniel was also a professor of mathematics. In 1708, Nicolas submitted five important problems to Professor Pierre Montmort. This problem was how much to pay for the following gamble:

> A fair coin with $\frac{1}{2}$ probability of heads is repeatedly tossed until heads occurs, ending the game. The investor pays c dollars and receives in return 2^{k-1} with probability 2^{-k} for $k = 1, 2, \ldots$ should a head occur. Thus, after each succeeding loss, assuming a head does not appear, the bet is doubled to 2, 4, 8, ... etc. Clearly the expected value is $\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots$ or infinity with linear utility.

Bell and Cover (1980) argue that the St. Petersburg gamble is attractive at any price c, but the investor wants less of it as $c \to \infty$. The proportion of the investor's wealth invested in the St. Petersburg gamble is always positive but decreases with the cost c as c increases. The rest of the wealth is in cash.

Bernoulli offers two solutions since he felt that this gamble is worth a lot less than infinity. In the first solution, he arbitrarily sets a limit to the utility of very large payoffs. Specifically, any amount over 10 million is assumed to be equal to 2^{24}. Under that bounded utility assumption, the expected value is

$$\frac{1}{2}(1) + \frac{1}{4}(2) + \frac{1}{8}(4) + \cdots + \left(\frac{1}{2}\right)^{24}(2^{24}) + \left(\frac{1}{2}\right)^{25}(2^{24}) + \left(\frac{1}{2}\right)^{26}(2^{24}) + \ldots = 12 + \text{ the original } 1 \ = 13.$$

When utility is log the expected value is

$$\frac{1}{2}\log 1 + \frac{1}{4}\log 2 + \cdots + \frac{1}{8}\log 4 + \cdots = \log 2 = 0.69315.$$

Use of a concave utility function does not eliminate the paradox.

For example, the utility function $U(x) = x/\log(x + A)$, where $A > 2$ is a constant, is strictly concave, strictly increasing, and infinitely differentiable yet the expected value for the St. Petersburg gamble is $+\infty$.

As Menger (1967) pointed out in 1934, the log, the square root, and many other, but not all, concave utility functions eliminate the original St. Petersburg paradox but it does not solve one where the payoffs grow faster than 2^n. So if log is the utility function, one creates a new paradox by having the payoffs increase at least as fast as log reduces them so one still has an infinite sum for the expected utility. With exponentially growing payoffs one has

$$\frac{1}{2}\log(e^1) + \frac{1}{4}\log(e^2) + \cdots = \infty.$$

The super St. Petersburg paradox, in which even $E \log X = \infty$ is examined in Cover and Thomas (2006: p. 181, 182) where a satisfactory resolution is reached by looking at relative growth rates of wealth. Another solution to such paradoxes is to have bounded utility. To solve the St. Petersburg paradox with exponentially growing payoffs, or any other growth rate, a second solution, in addition to that of bounding the utility function above, is simply to choose a utility function which, though unbounded, grows "sufficiently more" slowly than the inverse of the payoff function, e.g., like the log of the inverse function to the payoff function. The key is whether the valuation using a utility function is finite or not; if finite, the specific value does not matter since utilities are equivalent to within a positive linear transformation ($V = aU + b, a > 0$). So for any utility giving a finite result there is an equivalent one that will give you any specified finite value as a result. Only the behavior of $U(x)$ as $x \to \infty$ matters and strict monotonicity is necessary for a paradox. For example, $U(x) = x,\ x \leqslant A$, will not produce a paradox. But the continuous concave utility function

$$U(x) = \frac{x}{2} + \frac{A}{2}, \quad x > A$$

will have a paradox. Samuelson (1977) provides an extensive survey of the paradox; see also Menger (1967) and Aase (2001).

Kelly (1956) is given credit for the idea of using log utility in gambling and repeated investment problems and it is known as the Kelly criterion. Kelly's analyses use Bernoulli trials. Not only does he show that log is the utility function which maximizes the long-run growth rate, but that this utility function is myopic in the sense that period by period maximization based only on current capital is optimal. Working at Bell Labs, Kelly was strongly influenced by information theorist Claude Shannon.

Kelly defined the long-run growth rate of the investor's fortune using

$$G = \lim_{N \to \infty} \log \frac{W_N}{W_0},$$

where W_0 is the initial wealth and W_N is the wealth after N trials in sequence. With Bernoulli trials, one wins = $+1$ with probability p and losses –1 with probability $q = 1 - p$. The wealth with M wins and $L = N - M$ losses is

$$W_N = (1 + f)^M (1 - f)^{N-M} W_0,$$

where f is the fraction of wealth wagered on each of the N trials. Substituting this into G yields

$$G = \lim_{N \to \infty} \left(\frac{M}{N} \log(1 + f) + \left(\frac{N - M}{N} \right) \log(1 - f) \right) = p \log(1+f) + q \log(1-f) = E \log W$$

by the strong law of large numbers.

Maximizing G is equivalent to maximizing the expected value of the log of each period's wealth. The optimal wager for this is

$$f^* = p - q, \qquad p \geqslant q > 0,$$

which is the expected gain per trial or the edge. If there is no edge, the bet is zero.

If the payoff is $+B$ for a win and -1 for a loss, then the edge is $Bp - q$, the odds are B, and

$$f^* = \frac{Bp - q}{B} = \frac{\text{edge}}{\text{odds}}.$$

Latané (1978) introduced log utility as an investment criterion to the finance world independent of Kelly's work. Focussing, like Kelly, on simple intuitive versions of the expected log criteria he suggested that it had superior long-run properties. Hakansson and Ziemba (1995) survey economic analyses and applications.

Kelly bets can be very large and quite risky short term. For example, if $p = 0.99$ and $q = 0.01$ then $f^* = 0.98$ or 98% of one's current wealth. A real example of this is by Mohnish and Pabrai (2007) who won the bidding for the 2008 lunch with Warren Buffett paying more than \$600,000. He had the following investment in Stewart Enterprises as discussed by Thorp (2008). Over a 24-month period, with probability 0.80 the investment at least doubles, with 0.19 probability the investment breaks even, and with 0.01 probability all the investment is lost. The optimal Kelly bet is 97.5% of wealth and half Kelly is 38.75%. Pabrai invested 10%. While this seems rather low, other investment opportunities, miscalculation of probabilities, risk tolerance, possible short-run losses, bad scenario Black Swan events, price pressures, buying in and exiting suggest that a bet a lot lower than 97.5% is appropriate.

Risk aversion is generally measured by the Arrow–Pratt risk aversion index, namely

$$R_{A(w)} = \frac{-u''(w)}{u'(w)}$$

for absolute wagers and $R_A = wR_{A(w)}$ for proportional wagers.

For log, $R_A = 1/w$ which is close to zero for non-bankrupt investors, so we will argue that log is the most risky utility function one should ever consider. Positive power utility functions like $w^{1/2}$ lead to overbetting and are growth-security dominated. That means that growth and security both decrease.

Breiman (1961), following his earlier intuitive paper Breiman (1960), established the basic mathematical properties of the expected log criterion in a rigorous fashion. He proves three basic asymptotic results in a general discrete time setting with intertemporally independent assets.

Suppose in each period, N, there are K investment opportunities with returns per unit invested $X_{N_1}, \ldots, X_{N_K}$. Let $\Lambda = (\Lambda_1, \ldots, \Lambda_K)$ be the fraction of wealth invested in each asset. The wealth at the end of period N is

$$W_N = \left(\sum_{i=1}^{K} \Lambda_i X_{N_i} \right) W_{N-1}.$$

Property 1 In each time period, two portfolio managers have the same family of investment opportunities, X, and one uses a Λ^* which maximizes $E \log W_N$ whereas the other uses an *essentially different* strategy, Λ, so they differ infinitely often, that is,

$$E \log W_N(\Lambda^*) - E \log W_N(\Lambda) \to \infty.$$

Then

$$\lim_{N\to\infty} \frac{W_N(\Lambda^*)}{W_N(\Lambda)} \to \infty.$$

So the wealth exceeds that with any other strategy by more and more as the horizon becomes more distant.

This generalizes the Kelly Bernoulli trial setting to intertemporally independent and stationary returns.

Property 2 The expected time to reach a preassigned goal A is asymptotically least as A increases with a strategy maximizing $E \log W_N$.

Property 3 Assuming a fixed opportunity set, there is a fixed fraction strategy that maximizes $E \log W_N$, which is independent of N.

1.2 Risk Aversion

We can break risk aversion, both absolute and relative, into categories of investors as Ziemba (2010) has done in his response to letters he received from Professor Paul A Samuelson (2006, 2007, 2008) (Table 1.1).

Ziemba named Ida after Ida May Fuller who paid \$24.75 into US social security and received her first social security check numbered 00-000-001 on January 31, 1940, the actual first such check. She lived in Ludlow, Vermont, to the age of 100 and collected \$22,889. Such are the benefits and risks of this system; see Bertoccchi, Schwartz, and Ziemba (2010) for more on this. Victor is named for the hedge fund trader Victor Niederhoffer who seems to alternate between very high returns and blowing up; see Ziemba and Ziemba (2007) for some but not all of his episodes. The other three investors are the overbetting Tom who is growth-security dominated in the sense of MacLean, Ziemba, and Blazenko (1992), our E log investor Dick and Harriet, approximately half Kelly, who Samuelson says fits the data well. We agree that in practice, half Kelly is a toned down version of full Kelly that provides a lot more security to compensate for its loss in long-term growth. Figure 1.1 shows this behavior in the context of Blackjack where Thorp first used Kelly strategies.

The edge for a successful card counter varies from about – 5 to +10% depending upon the favorability of the deck. By wagering more in favorable situations and less or nothing when the deck is unfavorable, an average weighted edge is about 2%. An approximation to provide insight into the long-run behavior of a player's fortune is to assume that the game is a Bernoulli trial with a probability of success $= 0.51$ and probability of loss $1 = 0.49$.

Figure 1.1 shows the relative growth rate $f \ln(1+p) + (1-f)\ln(1-p)$ versus the fraction of the investor's wealth wagered, f. This is maximized by the Kelly log bet $f^* = p - q = 0.02$. The growth rate is lower for smaller and for larger bets than the Kelly bet. Superimposed on this graph is also the probability that the investor doubles or quadruples the initial wealth before losing half of this initial wealth. Since the growth rate and the security are both decreasing for $f > f^*$, it follows that it is never advisable to wager more than f^*.

Observe that the E log investor maximizes long-run growth and that the investor who wagers exactly twice this amount has a growth rate of zero plus the risk-free rate of interest. The fractional Kelly strategies are on the left and correspond to

Table 1.1 Samuelson's three investors plus Ziemba's two tail investors

	Victor	Tom	Dick	Harriet	Ida
	w linear	$w^{1/2}$ positive power	$\log w$ geometric mean optimizer	$-\frac{1}{w}$ half Kelly	$-\frac{N}{w}$, $N \to \infty$ finite risk averse
Absolute $R_A - \frac{u'}{u'w}$	0	$\frac{1}{2w}$	$\frac{1}{2}$	$\frac{2}{w}$	∞
Relative $R_A - \frac{wv''(w)}{u'(w)}$	0	$\frac{1}{2}$	1	2	∞

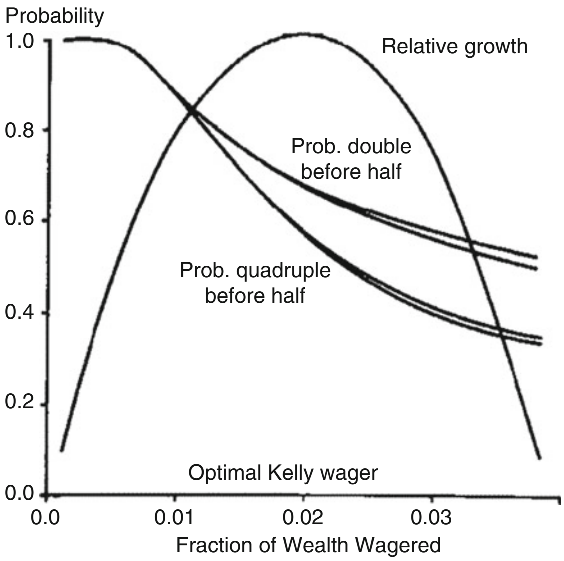


Fig. 1.1 Probability of doubling and quadrupling before halving and relative growth rates versus fraction of wealth wagered for Blackjack (2% advantage, $p = 0.51$ and $q = 0.49$)
Source: MacLean, Ziemba, and Blazenko (1992)

various negative power utility functions αw^{α} for $\alpha < 0$ such as 1/2 Kelly, $\alpha = -1$, and 1/4 Kelly, $\alpha = -3$. These values come from the handy formula for the fractional Kelly

$$f = \frac{1}{1-\alpha} = \frac{1}{R_R},$$

which is exactly correct for lognormal assets and approximately correct otherwise; see MacLean, Ziemba, and Li (2005) for proof. Thorp (2008) shows that this approximation can be very poor.

1.3 Understanding the Behavior of *E* log Strategies

There are many possible investment situations and E log Kelly wagering is useful for some of them. Good uses of the strategy are in situations with many repeated bets that approximate an infinite sequence as in the Breiman, etc., theory. See the papers in MacLean, Thorp, and Ziemba (2010b) for such extensions; MacLean, Thorp, and Ziemba (2010a) for good and bad Kelly and fractional Kelly properties; and MacLean, Thorp, Zhao, and Ziemba (2011) for simulations of typical behavior. Luenberger (1993) looks at long-run asymptotic behavior. Futures and options trading, sports betting, including horseracing, are good examples. The policies tend to non-diversify, plunge on a small number of the best assets, have a lot of volatility,

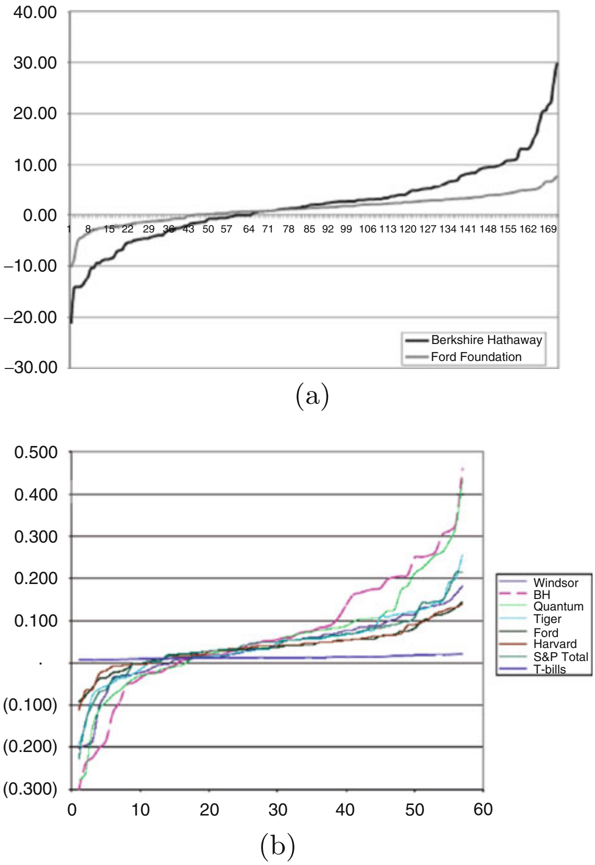


Fig. 1.2 Monthly returns for some funds ranked worst to best. **(a)** Berkshire Hathaway versus Ford Foundation, monthly returns distribution, January 1977 to April 2000. **(b)** Return distributions of all the funds, quarterly returns distribution, December 1985 to March 2000
Source: Ziemba (2005)

and produce more total wealth in the end than other strategies. Notable investors who use such strategies are Warren Buffett of Berkshire Hathaway, George Soros of the Quantum funds, and John Maynard Keynes who ran the King's College Cambridge endowment from 1927 to 1945. Figure 1.2a, b shows the best and worst months for the Buffett and Soros funds. Observe that Buffett and Soros are asymptotically equivalent in both the left and right tails. Figure 1.3 shows their wealth graphs. These correspond to typical Kelly behavior. Some Kelly advocates with a gambling background have produced nice smooth graphs such as those of Hong Kong racing guru Bill Benter, famed hedge fund traders Ed Thorp and Jim Simons; see Fig. 1.4a–d for the various wealth graphs.

According to Ziemba (2005), Keynes was approximately an 80% Kelly bettor with a utility function of $-w^{-0.25}$. In Ziemba (2005) it is argued that Buffett and

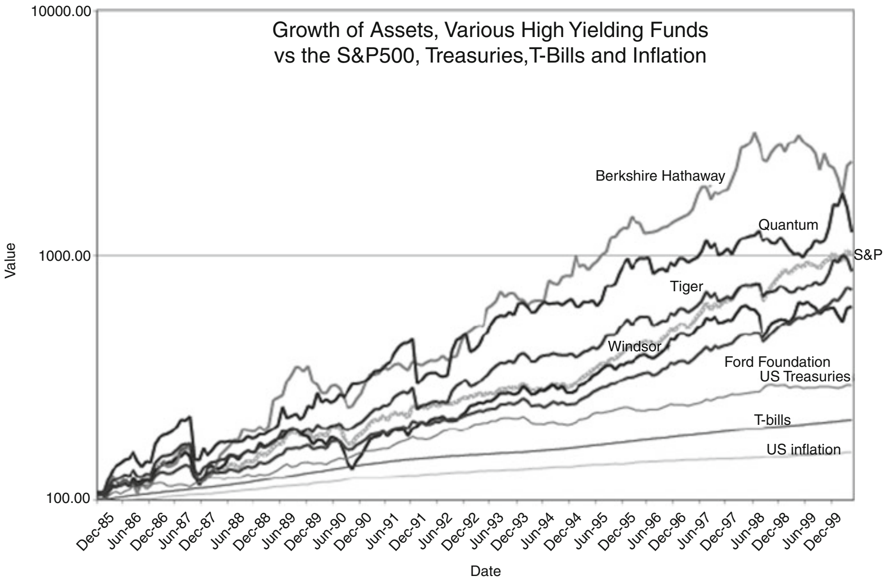


Fig. 1.3 The wealth levels from December 1985 to April 2000 for the Windsor Fund of George Neff, the Ford Foundation, the Tiger Fund of Julian Robertson, the Quantum Fund of George Soros, and Berkshire Hathaway, the fund run by Warren Buffett, as well as the S&P500 total return index *Source*: Ziemba (2005)

Soros are full Kelly bettors. They focus on long run wealth gains, not worrying about short term monthly losses. They tend to have few positions and try not to lose on any of them and not focusing on diversification. Table 1.2 supports this showing their top 10 equity holdings on September 30, 2008. Soros is even more of a plunger with more than half his equity portfolio in just one position.

The basic optimization of an E log strategy is to maximize the expected utility of a logarithmic utility function of final wealth subject to its constraints. Figure 1.5 shows a model formulation where transaction costs are present. Here in this horseracing example q_i is the probability that horse i wins a given race. The probability of an ijk finish is approximated using the Harville (1973) formulas as shown under the assumption that the probability the j wins a race that does not contain i equals $\frac{q_i}{1-q_i}$, etc. In practice these q_i are modified because in reality favorites who do not win do not come second or third as often as these formulas indicate. See Hausch, Lo, and Ziemba (1994, 2008) for these discounted Harville formulas and other approaches to this problem.

The final wealth W inside $E\,\log(W)$ is the amount not bet plus the winnings from place and show bets to come first or second, or first, second, or third, respectively, namely, the p_i and s_i where the P_i and S_i are the bets of other people. So the expression computes the payoffs after our bets are made assuming we bet last.

There are a number of practical details in current racetrack betting. First, there is rebate so when you bet $B = \sum p_i$ and $\sum s_i$ you receive back a percent, say ΔB

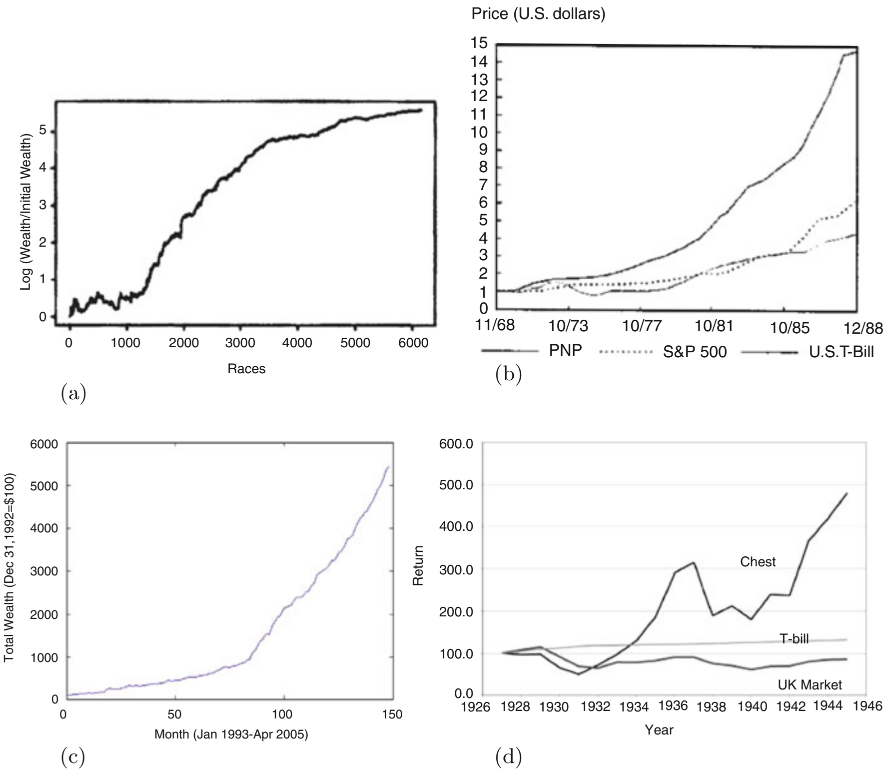


Fig. 1.4 The records of Bill Benter, Edward O. Thorp, Jim Simons, and John Maynard Keynes. **(a)** Benter's record in the Hong Kong Racing Syndicate; **(b)** Thorp's record in Princeton-Newport; **(c)** Jim Simon's record in Renaissance Medallion; and **(d)** John Maynard Keynes' record at King's College Cambridge Endowment
Source: Ziemba (2005) and Ziemba and Ziemba (2007)

where Δ varies depending on the bet, track, etc. The net effect is that the track take instead of being $1 - Q$ =0.13 to 0.30 is actually about 0.1–0.12. So professional bettors have lower transaction costs. Second, this model is just an approximation since about half the money bet does not get recorded in the pools until the race is running because of delays in reporting off track betting. Hence the probabilities must be estimated. Nevertheless, the old 1981 system modified in Hausch and Ziemba (1985) and discussed in the trade books Ziemba and Hausch (1986, 1987) does still seem to work and produce profits when rebate is included. Figure 1.6 shows a 2004 application performed by John Swetye and William Ziemba. A 5000 dollar initial wealth was churned into \$1.5 million of bets. The system lost 7% but gained 2% after an average 9% rebate so 2%(1.5 million) = \$30,000 profit for full Kelly. Observe that half and one-third Kelly have slightly smoother wealth paths but less final wealth.

Table 1.2 Top 10 equity holdings of Soros Fund Management and Berkshire Hathaway, September 30, 2008

Company	Current value × 1000	Shares	% portfolio
Soros fund management			
Petroleo Brasileiro SA	$ 1,673,048	43,854,474	50.53
Potash Corp Sask Inc.	378,020	3,341,027	11.58
Wal Mart Stores Inc.	195,320	3,791,890	5.95
Hess Corp	115,001	2,085,988	4.49
ConocoPhillips	96,855	1,707,900	3.28
Research in Motion Ltd.	85,840	1,610,810	2.88
Arch Coal Inc.	75,851	2,877,486	2.48
iShares TR	67,236	1,300,000	2.11
Powershares QQQ Trust	93,100	2,000,000	2.04
Schlumberger Ltd.	33,801	545,000	1.12
Berkshire Hathaway			
ConocoPhillips	$ 4,413,390	7,795,580	8.17
Procter & Gamble Co.	4,789,440	80,252,000	8.00
Kraft Foods Inc.	3,633,985	120,012,700	5.62
Wells Fargo & Co.	1,819,970	66,132,620	3.55
Wesco Finl Corp.	1,927,643	5,703,087	2.91
US Bancorp	1,1366,385	49,461,826	2.55
Johnson & Johnson	1,468,689	24,588,800	2.44
Moody's	1,121,760	48,000,000	2.34
Wal Mart Stores, Inc.	1,026,334	19,944,300	1.71
Anheuser Busch Cos, Inc.	725,201	13,845,000	1.29

Source: SEC Filings.

$$\underset{\{p_l\}\,\{s_l\}}{\text{Maximize}} \sum_{i=1}^{n} \sum_{\substack{j=1\\ j\neq i}}^{n} \sum_{\substack{k=1\\ k\neq i,j}}^{n} \frac{q_i q_j q_k}{(1-q_i)(1-q_i-q_j)} \log \left[\begin{array}{l} \dfrac{Q\left(P+\sum_{l=1}^{n} p_l\right)-(p_i+p_j+P_{ij})}{2} \\ \quad \times \left[\dfrac{p_i}{p_i+P_i}+\dfrac{p_j}{p_j+P_j}\right] \\ + \dfrac{Q\left(S+\sum_{l=1}^{n} s_l\right)-(s_i+s_j+s_k+S_{ijk})}{3} \\ \quad \times \left[\dfrac{s_i}{s_i+S_i}+\dfrac{s_j}{s_j+S_j}+\dfrac{s_k}{s_k+S_k}\right] \\ + \omega_0 - \sum\limits_{\substack{l=1\\ l\neq i,j,k}}^{n} s_l - \sum\limits_{\substack{l=1\\ l\neq i,j}}^{n} p_l \end{array} \right]$$

$$\text{s.t.} \quad \sum_{l=1}^{n} (p_l+s_l) \le \omega_0, \qquad p_l \ge 0, \qquad s_l \ge 0, \qquad l=1,\ldots,n,$$

Fig. 1.5 *E* log transaction cost model for place and show wagering
Source: Hausch, Ziemba, and Rubinstein (1981)

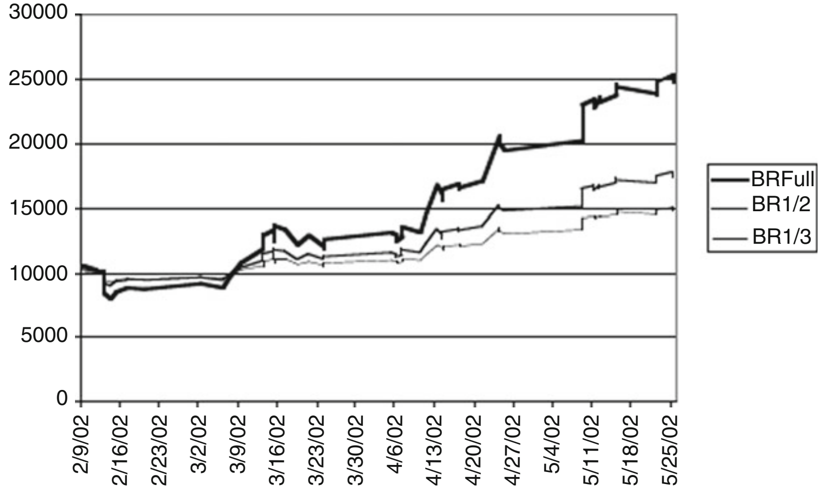


Fig. 1.6 Results of Dr. Z place and show betting with real money in 2004

1.4 A Simulated Example – Equity Versus Cash

In our experiment based on a similar example in Bicksler and Thorp (1973), there are two assets: US equities and US T-bills.[1] According to Siegel (2002), during 1926–2001 US equities returned 10.2% with a yearly standard deviation of 20.3%, and the mean return was 3.9% for short-term government T-bills with zero standard deviation. We assume the choice is between these two assets in each period. The Kelly strategy is to invest a proportion of wealth $x = 1.5288$ in equities and sell short the T-bill at $1 - x = -0.5228$ of current wealth. With the short selling and levered strategies, there is a chance of substantial losses. For the simulations, the proportion λ of wealth invested in equities[2] and the corresponding Kelly fraction f are

λ	0.4	0.8	1.2	1.6	2.0	2.4
f	0.26	0.52	0.78	1.05	1.31	1.57

[1] This example was modified from one in MacLean, Thorp, Zhao, and Ziemba (2011).
[2] The formula relating λ and f for this example is as follows. For the problem

$$\mathrm{Max}_x \{E(\ln(1 + r + x(R - r)\},$$

where R is assumed to be Gaussian with mean μ_R and standard deviation σ_R, and $r =$ the risk-free rate. The solution is given by Merton (1990) as

$$x = \frac{\mu_R - r}{\sigma_R}.$$

Since $\mu_R = 0.102$, $\sigma_R = 0.203$, $r = 0.039$, the Kelly strategy is $x = 1.5288$.

Table 1.3 Final wealth statistics by Kelly fraction for the equity versus cash example

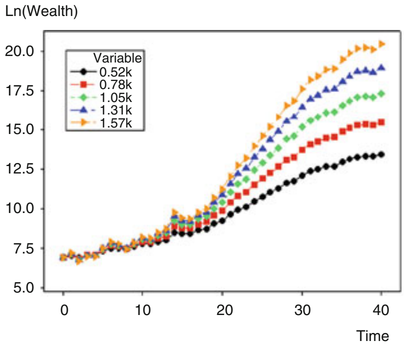
Fraction						
Statistic	0.26 k	0.52 k	0.78 k	1.05 k	1.31 k	1.57 k
Max	65,842.09	673,058.45	5,283,234.28	33,314,627.67	174,061,071.4	769,753,090
Mean	12,110.34	30,937.03	76,573.69	182,645.07	416,382.80	895,952.14
Min	2367.92	701.28	– 4969.78	– 133,456.35	– 6,862,762.81	– 102,513,723.8
St. Dev.	6147.30	35,980.17	174,683.09	815,091.13	3,634,459.82	15,004,915.61
Skewness	1.54	4.88	13.01	25.92	38.22	45.45
Kurtosis	4.90	51.85	305.66	950.96	1755.18	2303.38
$>5 \times 10$	3000	3000	2998	2970	2713	2184
10^2	3000	3000	2998	2955	2671	2129
$>5 \times 10^2$	3000	3000	2986	2866	2520	1960
$>10^3$	3000	2996	2954	2779	2409	1875
$>10^4$	1698	2276	2273	2112	1794	1375
$>10^5$	0	132	575	838	877	751
$>10^6$	0	0	9	116	216	270

Bicksler and Thorp used 10 and 20 yearly decision periods, and 50 simulated scenarios. MacLean et al. used 40 yearly decision periods, with 3000 scenarios.

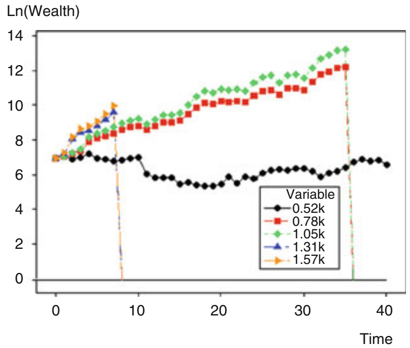
The results from the simulations appear in Table 1.3 and Figs. 1.7, 1.8 and 1.9. The striking aspects of the statistics in Table 1.3 are the sizable gains and losses. In his lectures, Ziemba always says when in doubt bet less – that is certainly borne out in these simulations. For the most aggressive strategy (1.57 k), it is possible to lose 10,000 times the initial wealth. This assumes that the shortselling is permissible through the decision period at the horizon $T = 40$.

The highest and lowest final wealth trajectories are presented in Fig. 1.7. In the worst case, the trajectory is terminated to indicate the timing of vanishing wealth. There is quick bankruptcy for the aggressive overbet strategies.

The substantial downside is further illustrated in the distribution of final wealth plot in Fig. 1.8. The normal probability plots are almost linear on the upside



(a)



(b)

Fig. 1.7 Trajectories with final wealth extremes for the equity versus cash example II: **(a)** maximum trajectories and **(b)** minimum trajectories

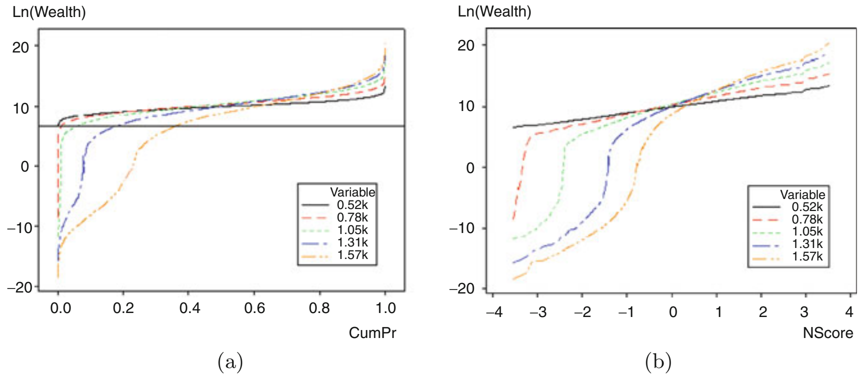


Fig. 1.8 Final ln(wealth) distributions by fraction for the equity versus cash example: **(a)** inverse cumulative and **(b)** normal plot

(log normality), but the downside is much more extreme than log-normal for all strategies except for 0.52 k. Even the full Kelly is very risky in this example largely because the basic position is levered. The inverse cumulative distribution shows a high probability of large losses with the most aggressive strategies. In constructing these plots the negative growth was incorporated with the formula, growth $= \left[\text{sign } W_T\right] \ln(|W_T|)$.

The mean–standard deviation trade-off in Fig. 1.9 provides more evidence concerning the riskiness of the high proportion strategies. When the fraction exceeds the full Kelly, the drop-off in growth rate is sharp, and that is matched by a sharp increase in the standard deviation.

The results of this experiment lead to the following conclusions:

1. The statistics describing the end of the horizon ($T = 40$) wealth are monotone in the fraction of wealth invested in the Kelly portfolio. Specifically (i) the

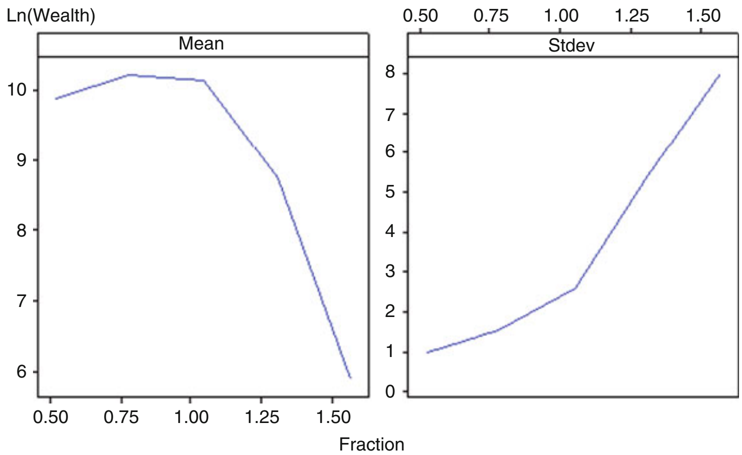


Fig. 1.9 Mean–standard deviation trade-off in the equity versus cash example

maximum terminal wealth and the mean terminal wealth increase in the Kelly fraction and (ii) the minimum wealth decreases as the fraction increases and the standard deviation grows as the fraction increases. The growth and decay are pronounced and it is possible to have extremely large losses. The fraction of the Kelly optimal growth strategy exceeds 1 in the most levered strategies and this is very risky. There is a trade-off between return and risk, but the mean for the levered strategies is growing far less than the standard deviation. The disadvantage of leveraged investment is illustrated with the cumulative distributions in Fig. 1.8. The log normality of final wealth does not hold for the levered strategies.

2. The maximum and minimum final wealth trajectories show the return – risk of levered strategies. The worst and best scenarios are not the same for all Kelly fractions. The worst scenario for the most levered strategy shows a rapid decline in wealth. The mean–standard deviation trade-off confirms the extreme riskiness of the aggressive strategies.

1.5 Final Comments

The Kelly optimal capital growth investment strategy is an attractive approach to wealth creation. In addition to maximizing the asymptotic rate of long-term growth of capital, it avoids bankruptcy and overwhelms any essentially different investment strategy in the long run. See MacLean, Thorp, and Ziemba (2010a) for a discussion of the good and bad properties of these strategies. However, automatic use of the Kelly strategy in any investment situation is risky and can be very dangerous. It requires some adaptation to the investment environment: rates of return, volatilities, correlation of alternative assets, estimation error, risk aversion preferences, and planning horizon are all important aspects of the investment process. Chopra and Ziemba (1993) show that in typical investment modeling, errors in the means average about 20 times in importance in objective value than errors in co-variances with errors in variances about double the co-variance errors. This is dangerous enough but they also show that the relative importance of the errors is risk aversion dependent with the errors compounding more and more for lower risk aversion investors and for the extreme log investors with essentially zero risk aversion the errors are worth about 100:3:1. So log investors must estimate means well if they are to survive. This is compounded even more by the observation that when times move suddenly from normal to bad the correlations/co-variances approach 1 and it is hard to predict the transition from good times to bad. Poundstone's (2005) book, while a very good read with lots of useful discussions, does not explain these important investment aspects and the use of Kelly strategies by advisory firms such as *Morningstar* and *Motley Fools* is flawed; see, for example, Fuller (2006) and Lee (2006). The experiments in Bicksler and Thorp (1973), Ziemba and Hausch (1986), and MacLean, Thorp, Zhao, and Ziemba (2011) and that described here represent some of the diversity in the investment environment. By considering the Kelly and its variants we get a concrete look at the plusses and minuses of the capital growth model. We can conclude that

- The wealth accumulated from the full Kelly strategy does not stochastically dominate fractional Kelly wealth. The downside is often much more favorable with a fraction less than 1.
- There is a trade-off of risk and return with the fraction invested in the Kelly portfolio. In cases of large uncertainty, from either intrinsic volatility or estimation error, security is gained by reducing the Kelly investment fraction.
- The full Kelly strategy can be highly levered. While the use of borrowing can be effective in generating large returns on investment, increased leveraging beyond the full Kelly is not warranted as it is growth-security dominated. The returns from over-levered investment are offset by a growing probability of bankruptcy.
- The Kelly strategy is not merely a long-term approach. Proper use in the short and medium run can achieve wealth goals while protecting against drawdowns. MacLean, Sanegre, Zhao, and Ziemba (2004) and MacLean, Zhao, and Ziemba (2009) discuss a strategy to reduce the Kelly fraction to stay above a prespecified wealth path with high probability and to be penalized for being below the path.

The great economist Paul Samuelson was a long-time critic of the Kelly strategy which maximizes the expected logarithm of final wealth; see, for example, Samuelson (1969, 1971, 1979) and Merton and Samuelson (1974). His criticisms are well dealt with in the simulation example in this chapter and we see no disagreement with his various analytic points:

1. The Kelly strategy maximizes the asymptotic long-run growth of the investor's wealth, and we agree;
2. The Kelly strategy maximizes expected utility of only logarithmic utility and not necessarily any other utility function, and we agree;
3. The Kelly strategy always leads to more wealth than any essentially different strategy; this we know from the simulation in this chapter is not true since it is possible to have a large number of very good investments and still lose most of one's fortune.

Samuelson seemed to imply that Kelly proponents thought that the Kelly strategy maximizes for other utility functions but this was neither argued nor implied.

It is true that the expected value of wealth is higher with the Kelly strategy but bad outcomes are very possible.

We close this chapter with the main conclusions of the simulation studies

1. that the great superiority of full Kelly and close to full Kelly strategies over longer horizons with very large gains a large fraction of the time;
2. that the short-term performance of Kelly and high fractional Kelly strategies is very risky;
3. that there is a consistent trade-off of growth versus security as a function of the bet size determined by the various strategies; and

4. that no matter how favorable the investment opportunities are or how long the finite horizon is, a sequence of bad scenarios can lead to very poor final wealth outcomes, with a loss of most of the investor's initial capital.

Hence, in practice, financial engineering is important to deal with the short-term volatility and long-run situations with a sequence of bad scenarios. But properly used, the strategy has much to commend it, especially in trading with many repeated investments.

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