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JAMES COOK MATHEMATICAL NOTES

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ROLL A BALL

J.M. Hammersley

A spherical ball of unit radius rests on a horizontal table, and needs to be given an arbitrary prescribed reorientation. You may roll the ball along any suitable path P on the table; but the rolling must be pure in the sense that the instantaneous axis of rotation must always be horizontal and perpendicular to P and there must be no slipping. Prove that, if P has the shortest possible length T, then P is a straight segment or an arc of a circle, that $0 \leq T \leq \pi\sqrt{3}$, and that $T = \pi\sqrt{3}$ if and only if the desired reorientation is a rotation through an angle π about the vertical. My own proof of this uses the calculus of variations, and is long and complicated. Can anybody provide a short snappy proof of this simple result, preferably a proof depending only upon elementary Euclidean geometry? What can be said about the more difficult problem when the ball also has to finish at a prescribed point on the table?

HIGHER DIMENSIONAL ROTATIONS

H. Kestelman

Every rotation in two or three dimensions is a product of two reflections. The proposition below is an analogue in n dimensions.

A matrix is called involutary if its square is the identity. Prove that every real orthogonal matrix is either involutary or the product of two involutary matrices.

LINEAR RECURRENCES AND DETERMINANTS

A.P. Guinand & P.L. Manley

Many well-known sequences, such as the Bernoulli, Euler, and Fibonacci numbers, the Legendre polynomials, and the Bessel functions can be generated by linear recurrence formulae. Such formulae can be regarded as sets of linear equations and solved by determinants, but the resulting determinant formulae are seldom mentioned in the literature. So here are some examples; in each case the determinant is to be cut off when its size is n×n.

(i) The Bernoulli numbers (in even suffix notation) are generated by

$B_0 = 1$ and $B_0 + 2B_1 = 0,$

$$B_0 + 3B_1 + 3B_2 = 0,$$

$$B_0 + 4B_1 + 6B_2 + 4B_3 = 0,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$B_0 + \binom{n+1}{1}B_1 + \binom{n+1}{2}B_2 + \ldots + \binom{n+1}{n-1}B_{n-1} + \binom{n+1}{n}B_n = 0.$$

Solving these equations for the ratio of B_0 to B_n, we have

$$\frac{B_0}{\begin{vmatrix} 2 & 0 & 0\cdots\cdots 0 \\ 3 & 3 & 0\cdots\cdots 0 \\ 4 & 6 & 4\cdots\cdots 0 \\ \cdots & \cdots & \cdots\cdots 0 \\ \binom{n}{1} & \binom{n}{2} & \binom{n}{3}\cdots\cdots 0 \\ \binom{n+1}{1} & \binom{n+1}{2} & \cdots\cdots \binom{n+1}{n} \end{vmatrix}} = \frac{(-)^n B_n}{\begin{vmatrix} 1 & 2 & 0 & 0\cdots\cdots 0 \\ 1 & 3 & 3 & 0\cdots\cdots 0 \\ 1 & 4 & 6 & 4\cdots\cdots 0 \\ \cdots & \cdots & \cdots & \cdots\cdots 0 \\ \binom{n}{0} & \binom{n}{1} & \binom{n}{2} & \cdots\cdots \binom{n}{n-1} \\ \binom{n+1}{0} & \binom{n+1}{1} & \cdots & \cdots\cdots \binom{n+1}{n-1} \end{vmatrix}}$$

The first of these determinants reduces to (n+1)! and it follows that

$$B_n = \frac{(-)^n}{(n+1)!}\begin{vmatrix} 1 & 2 & 0 & 0\cdots\cdots 0 \\ 1 & 3 & 3 & 0\cdots\cdots 0 \\ 1 & 4 & 6 & 4\cdots\cdots 0 \\ \cdots & \cdots & \cdots & \cdots\cdots 0 \\ \binom{n}{0} & \binom{n}{1} & \binom{n}{2} & \cdots\cdots \binom{n}{n-1} \\ \binom{n+1}{0} & \binom{n+1}{1} & \cdots & \cdots\cdots \binom{n+1}{n-1} \end{vmatrix}$$

Similar methods lead to the following examples:

(ii) The Euler numbers, generated by $E_0 = 1$ and the recurrence

$$\binom{n}{n}E_n + \binom{n}{n-2}E_{n-2} + \binom{n}{n-4}E_{n-4} + \ldots = 0, \quad (n = 1, 2, 3, \ldots).$$

Then

$$E_n = \begin{vmatrix} 0 & 1 & 0 & 0 & 0 & 0\cdots\cdots 0 \\ 1 & 0 & 1 & 0 & 0 & 0\cdots\cdots 0 \\ 0 & 3 & 0 & 1 & 0 & 0\cdots\cdots 0 \\ 1 & 0 & 6 & 0 & 1 & 0\cdots\cdots 0 \\ 0 & 5 & 0 & 10 & 0 & 1\cdots\cdots 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}$$

where the r^{th} row consists of alternate zeros and binomial coefficients of order r.

(iii) The Fibonacci numbers, generated by $F_0 = F_1 = 1$,

$F_{n+2} = F_{n+1} + F_n$.

$$F_n = \begin{vmatrix} 1 & -1 & 0 & 0 & 0 & 0\cdots\cdots 0 \\ 1 & 1 & -1 & 0 & 0 & 0\cdots\cdots 0 \\ 0 & 1 & 1 & -1 & 0 & 0\cdots\cdots 0 \\ 0 & 0 & 1 & 1 & -1 & 0\cdots\cdots 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0\cdots & \cdots & \cdots & 1 & 1 & -1 \\ 0\cdots & \cdots & \cdots & 0 & 1 & 1 \end{vmatrix}$$

(iv) The Legendre polynomials, generated by $P_0(x) = 1$ and the recurrence $P_n(x) - (2n-1)xP_{n-1}(x) + (n-1)P_{n-2}(x) = 0$.

$$P_n(x) = \frac{1}{n!}\begin{vmatrix} x & -1 & 0 & 0 & 0\cdots\cdots & \cdots & 0 \\ -1 & 3x & -2 & 0 & 0\cdots\cdots & \cdots & 0 \\ 0 & -2 & 5x & -3 & 0\cdots\cdots & \cdots & 0 \\ 0\cdots & \cdots & \cdots & \cdots & \cdots\cdots & \cdots & 0 \\ 0\cdots & \cdots & \cdots & \cdots & -(n-2) & (2n-3)x & -(n-1) \\ 0\cdots & \cdots & \cdots & \cdots & 0 & -(n-1) & (2n-1)x \end{vmatrix}$$

(v) Bessel functions of the first kind, for which

$$J_{n-1}(z) + J_{n+1}(z) = \frac{2n}{z} J_n(z).$$

$$z^n J_n(z) = \begin{vmatrix} zJ_1(z) & -z & 0 & 0 & 0 & \cdots & & & 0 \\ -zJ_0(z) & 2 & -z & 0 & 0 & \cdots & & & 0 \\ 0 & -z & 4 & -z & 0 & \cdots & & & 0 \\ 0 & 0 & -z & 6 & -z & \cdots & & & 0 \\ \cdots & & & & & & & & 0 \\ \cdots & & & & & 0 & -z & 2n-4 & -z \\ \cdots & & & & & 0 & 0 & -z & 2n-2 \end{vmatrix}$$

UNIFORM WAVES IN DEEP WATER

Mathematically the problem is to find $\phi(x, y, t)$ with

$$\phi_{xx} + \phi_{yy} = u_x + v_y = 0$$

such that the free surface $y = \eta(x, t)$ determined by

$$\phi_t + gy + \tfrac{1}{2}(u^2 + v^2) = 0$$

satisfies the kinematic free surface condition $\partial\eta/\partial t + u\partial\eta/\partial x = v$.

The first order approximation is well known,

$$\phi = \alpha(n/k)\exp(ky)\sin(kx-nt)$$

and $\eta = \alpha\cos(kx-nt)$ where $n^2 = gk$, and α is infinitesimal.

Less well known is the fact that the same velocity potential also gives the second order approximation, the relation $n^2 = gk$ is the same, and the free surface is given by

$$\eta = \alpha\cos(kx-nt) + (\alpha^2 k/2)\cos 2(kx-nt)$$

It may be noted that Gerstner's trochoidal waves (which satisfy the free surface condition exactly but have vorticity non-zero) in the case of small amplitude give the same second order approximation to the free surface.

INEQUALITY FOR POLYNOMIALS

J.B. Parker

Let f be any real polynomial and N any positive integer, then

$$\sum_{r=0}^{2N} \int_{-\infty}^{\infty} (-1)^r (r!)^{-1} (d^r f/dx^r)^2 \exp - x^2/2 \, dx \geq 0 \quad .$$

Proof. Define the polynomials $V_j(x)$ by

$$\exp(tx - t^2/2) = \sum_{j=0}^{\infty} t^j (j!)^{-\frac{1}{2}} V_j(x)$$

so that for instance $V_0(x) = 1$, $V_1(x) = x$, $V_2(x) = (x^2 - 1)/\sqrt{2}$, ...
Apart from the $\sqrt{(j!)}$ factor these are the Tchebycheff-Hermite polynomials. Differentiating both sides of the generating equation s times with respect to x gives:

$$t^s \exp(tx - t^2/2) = \sum_0^{\infty} t^j (j!)^{-\frac{1}{2}} (d/dx)^s V_j(x) \quad . \qquad (1)$$

For any non-negative integers j, k and s, consider

$$(j!\,k!)^{-\frac{1}{2}} \int_{-\infty}^{\infty} V_j^{(s)}(x)\, V_k^{(s)}(x) \exp - \frac{x^2}{2}\, dx$$

(where $f^{(s)}(x)$ denotes $(d/dx)^s f(x)$ in the usual way).
It is the coefficient of $t^j u^k$ in

$$\int_{-\infty}^{\infty} \sum_{m=0}^{\infty} \frac{t^m V_m^{(s)}(x)}{\sqrt{m!}} \sum_{n=0}^{\infty} \frac{u^n V_n^{(s)}(x)}{\sqrt{n!}} \exp - \frac{x^2}{2}\, dx$$

which because of equation *(1)* may be expressed:

$$\int_{-\infty}^{\infty} t^s \exp(tx - t^2/2)\, u^s \exp(ux - u^2/2) \exp - \frac{x^2}{2}\, dx$$

$$= t^s u^s \exp(tu) \int_{-\infty}^{\infty} \exp - (x - u - t)^2/2 \, dx = \sqrt{2\pi}\,(tu)^s \exp(tu) \quad .$$

It is equal to the coefficient of $t^{j-s} u^{k-s}$ in $\sqrt{2\pi}\exp(tu)$, and is therefore $\sqrt{2\pi}/(j-s)!$ when $j = k \geq s$, and zero otherwise.

This establishes the orthogonality equation:

$$(2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} V_j^{(s)}(x)\, V_k^{(s)}(x) \exp -\frac{x^2}{2}\, dx =$$

$$= \begin{cases} \dfrac{j!}{(j-s)!}\,\delta_{jk} & \text{if } j \geq s \\ 0 & \text{if } j < s \end{cases} \qquad (2)$$

Now side-track to establish a result on partial binomial sums:

$$\sum_{s=0}^{N} (-1)^s \binom{n}{s} = (-1)^N \binom{n-1}{N} \quad . \qquad (3)$$

This can be regarded as holding for all positive n and N, but for $n \leq N$ there is the more convenient form:

$$\sum_{s=0}^{N} = \sum_{s=0}^{n} (-1)^s \binom{n}{s} = 0 \quad . \qquad (4)$$

Proof is by induction on N.

Now consider any polynomial f. It is expressible as

$$f(x) = \sum_0^\infty B_n V_n(x)$$

where only finitely many of the coefficients B_n are non-zero.

Equation *(2)* above leads to

$$(2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} (f^{(s)}(x))^2 \exp -\frac{x^2}{2}\, dx = \sum_{n=s}^{\infty} \frac{n!}{(n-s)!} B_n^2 \quad .$$

Now divide by $(-1)^s s!$ and take the sum for s = 0, 1, 2, ..., 2N

$$(2\pi)^{-\frac{1}{2}} \sum_{s=0}^{2N} \frac{(-1)^s}{s!} \int (f^{(s)}(x))^2 \exp -\frac{x^2}{2}\, dx =$$

$$= B_0^2 + \sum_1^\infty B_n^2 + \sum_{s=1}^{2N} (-1)^s \sum_{n=s}^{\infty} \binom{n}{s} B_n^2$$

$$= B_0^2 + \sum_{n=1}^{\infty} B_n^2 \left[1 + \sum_{s=1}^{\mathrm{Min}(2N,n)} (-1)^s \binom{n}{s}\right]$$

$$= B_0^2 + \sum_{n=1}^{2N} B_n^2 \left[1 + \sum_{s=1}^{n} (-1)^s \binom{n}{s}\right] + \sum_{n=2N+1}^{\infty} B_n^2 \binom{n-1}{2N}$$

$$= B_0^2 + \sum_{n=2N+1}^{\infty} \binom{n-1}{2N} B_n^2 \geq 0 \quad .$$

There is strict inequality unless the polynomial f(x) is of degree $\leq 2N$ and has the property that $\int_{-\infty}^{\infty} f(x) \exp -x^2/2\, dx = 0$.

It is tempting to speculate that this inequality might be extended to some wider class of functions, and that there may be a proof not involving orthogonal polynomials.

A good exercise for the student would be an investigation of the inequality:

$$\sum_{r=0}^{2N} \int_0^\infty (-1)^r \frac{x^r}{r!} (f^{(r)}(x))^2 \exp(-x)\, dx \geq 0 \quad .$$

Another speculation is obtained by letting $N \to \infty$, the identity

$$(2\pi)^{\frac{1}{2}} \sum_0^\infty \frac{(-1)^s}{s!} \int_{-\infty}^{\infty} f^{(s)}(x) g^{(s)}(x) \exp -\frac{x^2}{2} dx = \int_{-\infty}^{\infty} f(x) \exp -\frac{x^2}{2} dx \int_{-\infty}^{\infty} g(x) \exp -\frac{x^2}{2} dx.$$

It certainly holds for f and g any polynomials, as proved by applying the formula found above to f + g and to f − g. The speculation may be verified in simple cases where f and g are exponentials or sines or cosines.

PERVERSE POLYNOMIALS (JCMN 26, p. 3028)

J.B. Parker

(a) To show that $Q_n(x) = x^n - 2x^{n-1} + 1$ is a factor of some polynomial P in which all non-zero coefficients are ± 1. Put $F_n(x) = \sum_0^{n-1} x^i$, then the coefficient of x^r in $P = F_n Q_n$ is 1 for $0 \le r \le n-2$, and -1 for $n-1 \le r \le 2n-2$ and 1 when $r = 2n-1$.

(b) To show that $R_n(x) = x^n - 2x^{n-1} - 1$ is not a factor of any polynomial P in which all non-zero coefficients are ± 1. If possible put $P = R_n F$, and express F in the form $F(x) = \sum_0^m a_i x^i$. Clearly each of the coefficients a_i must be an integer. For simplicity of notation let $a_i = 0$ for $i < 0$ or $> m$. The coefficients satisfy $a_i - 2a_{i+1} - a_{i+n}$ = (One of the values -1, 0, 1). Now look for the coefficient that has largest magnitude, and if several have the same take the one with largest suffix, that is $a_k = M$ and

$$|a_i| \le |M| \quad \text{for all } i < k$$

$$|a_i| \le |M| - 1 \quad \text{for all } i > k \quad .$$

Since $2M = 2a_k$ is between $a_{k-1} - a_{k+n-1} \pm 1$ it follows that $a_{k-1} = M$ and $a_{k+n-1} = 1 - M$. Similarly $a_{k-2} = M$, and inductive reasoning leads to $a_{k-n+1} = \ldots = a_{k-1} = a_k = M$.

But the condition $a_{k-n} - 2a_{k-n+1} - a_k$ = (One of -1, 0, 1) leads to a_{k-n} having one of the values $3M - 1$, $3M$, $3M + 1$ which are all of modulus $\ge 2|M|$.

OF THE EARTH MURPHY

You have four plants of each of two varieties of potato, P and S. Eight plots of ground, each big enough for one plant are available. They are arranged in a straight line behind the boatshed, and are numbered 1, 2, ... 8 in order. Put $s(j) = 1$ if an S plant is in plot j and $= -1$ if a P plant. How would you arrange the plants to make the moments $M(r) = \sum i^r s(i)$ zero for $r = 0, 1, 2$? What is the smallest $n = n(r)$ such that (using n plants instead of 8) we can make $M(0), M(1), M(2), \ldots M(r)$ all zero?

THE RIG OF A ROWING BOAT (JCMN 26, p. 3037)

H.O. Davies

Recall that $s(j) = \pm 1$ for $j = 1, 2, \ldots n$ and $a(i) = \sum_1^i s(j)$ and $b(i) = \sum_1^j a(i)$. Take the a(i) as independent variables, the table of values becomes

s	a(1)	a(2)-a(1)	a(3)-a(2)	a(4)-a(3)	...
a	a(1)	a(2)	a(3)		...
b		a(1)	a(1)+a(2)	a(1)+a(2)+a(3)	...

The query was about $\sum b(j)\, s(j)$. There is an algebraic identity

$$a(1)\big(a(2) - a(1)\big) + \big(a(1) + a(2)\big)\big(a(3) - a(2)\big) + \ldots$$

$$+ \big(a(1) + a(2) + \ldots + a(r-1)\big)\big(a(r) - a(r-1)\big)$$

$$= a(r)\big(a(1) + \ldots + a(r)\big) - \big(a(1)^2 + a(2)^2 + \ldots + a(r)^2\big)$$

easily proved by induction on r. Taking r = n it follows that if $a(n) = 0$ then $\sum_1^n b(j)\, s(j) = -\sum_1^n a(j)^2 \leq -n/2$ (because each a(j) is an integer and two adjacent terms cannot both be zero).

THE SURFACE AREA OF AN ELLIPSOID (JCMN 26, p. 3031)

P.A.P. Moran writes that the surface area S of the ellipsoid $E(x^2/a^2 + y^2/b^2 + z^2/c^2 = 1)$ satisfies

$$4\pi(ab + bc + ca)/3 \leq S \leq 4\pi\left\{(a^2b^2 + b^2c^2 + c^2a^2)/3\right\}^{\frac{1}{2}}$$

and that these inequalities are proved in a paper that he has written for the C.R. Rao Festschrift to appear soon. The "Simple Simon" idea may be refined as follows. Let V be the volume enclosed between E and the larger ellipsoid with semi-axes $a+\delta$, $b+\delta$ and $c+\delta$. Every point of V satisfies (for some t between 0 and δ) and equation

$$x^2/(a + t)^2 + y^2/(b + t)^2 + z^2/(c + t)^2 = 1$$

and is therefore at distance $t \leq \delta$ from the point $\left(\frac{ax}{a+t}, \frac{by}{b+t}, \frac{cz}{c+t}\right)$ of E. V is therefore a subset of the set V^* of all points outside E but at distance $\leq \delta$ from E, and $\mu V \leq \mu V^*$. Also

$$\mu V^* = S\delta + 0(\delta^2) .$$

(You can think of this as intuitively obvious or as part of the more exact result known as Steiner's Formula).

Finally $S\delta + 0(\delta^2) = \mu V^* \geq \mu V = 4\pi\delta(ab + bc + ca + (a + b + c)\delta + \delta^2)/3$

so that $S \geq 4\pi(ab + bc + ca)/3$.

WORKED EXAMPLES IN SOCIAL MECHANICS

A visitor to England in 1981 may wonder why this country, which generated the Industrial Revolution, has sunk from a condition of power and prosperity to the verge of chaos and bankruptcy. The mineral wealth and technical skill are still there. Why has a system that used to work well now failed?

A mathematician might think that insight could be gained by using the modern techniques of operational research, control theory, time series analysis, stochastic processes, and so on. But is there a stream of papers on the English Disease in the mathematical journals? No! The reason is that the problems are too easy, they usually succumb to very simple reasoning, with mathematics of no more than matriculation standard. Let us investigate a few simple questions.

(a) Why did Government loans in the nineteenth century offer about three per cent interest? The typical potential investor was a married man aged between 40 and 50. What was a £100 bond at 3% worth to him? He and his wife could expect from it £3 per year for 20 years and they would pay two shillings in the pound income tax on it, so that the interest would be worth £54 to them. Then they would leave the bond, still worth £100, to their children, what was this prospect worth to them? The value is hard to estimate, but we take a stab at it, call the value £50. This means that the value of the bond was £104, so that the investor is willing to pay £100 for it, but not much more.

(b) How will the calculation above be modified if the era is changed to the late twentieth century? It indicates a bond interest rate of 15%. The potential investor considers the value of an income of £15 per year. With inflation running at 10% and income tax at 30% the value is £10·5 $(1 + 0.9 + 0.9^2 + \ldots + 0.9^{19})$ = £95. The prospect of leaving the bond to the investor's children in twenty year's time can be estimated as before to be worth £5, so that the present value of the bond is £100, which verifies that the appropriate interest rate is 15%.

(c) Consider the setting up of an industry in the nineteenth century. What return on capital would the business have to generate in order to attract the necessary investment? The potential shareholder would be comparing the return from shares in an industrial company with that from fixed-interest Government bonds, and so the business would be viable only if it gave at least 3% return on capital. Another consideration would be that the business might lose heavily through the advance of technology. Near Sheffield is an abandoned quarry in which are bits of stone partly shaped into grindstones; the wheels of natural stone used for centuries here, driven by water power, for the knife-grinding industry, were suddenly unwanted when the synthetic materials for grindstones came on the market. Suppose that we estimate at 1% the probability per year of an industrial concern losing half its capital value by becoming outdated, then we conclude that a business would have to generate at least 3½% return on its capital in order to be viable.

(d) Now translate (c) to the late twentieth century. The calculation is a little more complicated, we allow for 10% inflation per year and reckon all values in terms of the present day pound. Suppose that the

business generates 18% annually on the invested capital. The plant and equipment maintained at their initial real value will have their nominal value increased 10% per year and so on each £100 of initial investment the income as assessed for tax will be £28. After paying corporation tax at 40% the income available for distribution to shareholders is £7 and so the shareholder paying income tax at 30% gets £5. The value of enjoying this for 20 years (subject to a 4% per year probability of semi-bankruptcy) is £5 $(1 + .98 + \ldots + .98^{19})$ = £83. The expected value after 20 years is £67 and the value of leaving this to ones children paying 40% death duties is £20. These calculations give the return on £100 invested as £103, so that the business is just viable. That is why a wise financial adviser these days will tell clients that a business is not worth starting (or continuing) unless it gives 18% return on its capital, five times what was necessary in the nineteenth century.

Try doing a few sums like this yourself, and you will find it easier to understand the world around you.

COMPLEX FUNCTION THEORY

If a sequence of entire complex functions converges pointwise then is the limit entire? What else can you say about the limit?

This is a pretty little question but, it must be admitted, not new. *I.N. Baker* writes that it is set as an exercise (without any hint) in Rudin's "Real and Complex Analysis".

CAPTAIN COOK AND THE LOCH NESS MONSTER

J.H. Loxton

A little known story* from the early life of James Cook is that, before setting out to explore Terra Australis, he spent many fruitless years combing the waters of Scotland for the Terror of the North. According to some accounts, his most consistent sightings were obtained from the grounds of a small distillery not a stone's throw from Inverness. The advent of recreational computers has led to major advances in the discipline of transcendental holography, the science of reconstructing visions and displaying them in glorious three-dimensional technicolour. To date, only a few random bits of this marvellous new technique have been received from the margins of outer space and the results do not quite live up to some of the claims that have been made. It is nonetheless possible to make a partial reconstruction of Cook's vision of the Loch Ness Monster; it appears in Figure 1, perhaps for the very first time.

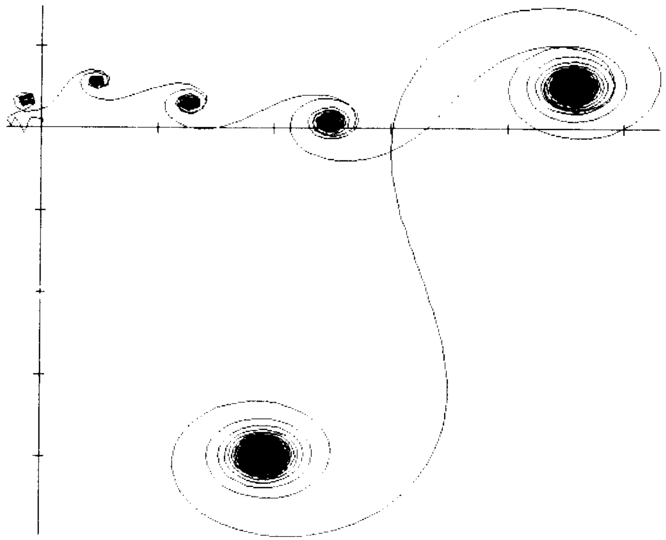

Figure 1. The Monster

* I certainly had not heard it before.

It turns out to be quite simple. The figure is in fact the graph of the first 5000 terms of the sequence

$$L_N = \sum_{n=1}^{N} \exp(2\pi i(\log n)^4)$$

that is, the graph is obtained by plotting the points $L_1, L_2, \ldots, L_{5000}$ in the complex plane represented by this page, and joining successive points by straight line segments. It is possible to analyse the shape of this monster by considering the angle between successive line segments namely

$$\theta_n = 2\pi\left\{(\log(n+1))^4 - (\log n)^4\right\} \approx 8\pi(\log n)^3/n$$

for large n. The graph of this function is shown in Figure 2, to illustrate the point that a picture is worth a thousand words.

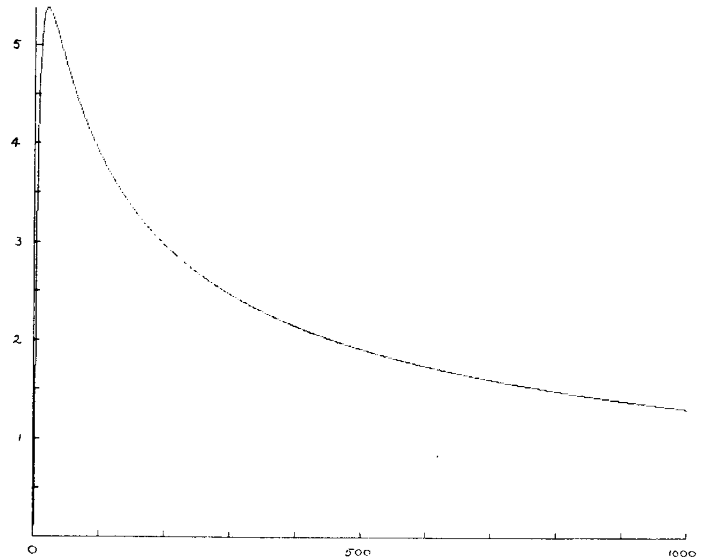

Figure 2. Graph sans paroles

The anatomy of the monster has three parts. For very large n, say $n \geq 100000$, θ_n is small and monotonically decreasing to 0, and the monster approximates a spiral slowly spreading out to cover the whole plane from its ultimate limit point. This can be seen in the graph of the baby monster in Figure 3; the spiral behaviour takes over much earlier because the angle corresponding to θ_n is already small for $n \geq 500$. When $\theta_n/2\pi$ is close to an integer, the monster approximates

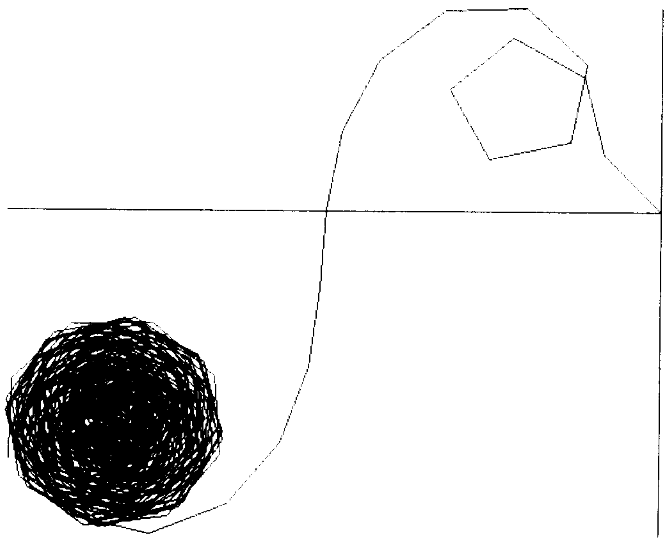

Figure 3. The Baby Monster

$$B_N = \sum_{n=1}^{N} \exp(2\pi i(\log n)^{2.7}), \quad 1 \leq N \leq 500$$

a smooth slowly turning curve and this accounts for the links between the blobs in Figure 1. When the distance from $\theta_n/2\pi$ to the nearest integer is not small, the line segments of the graph make rapid changes

in direction and the graph is confined to a very restricted region until $\theta_n/2\pi$ reaches the next integer. The enlarged picture in Figure 4. is the graph of L_N as N runs from 3200 to 3700, giving a rare view of a black

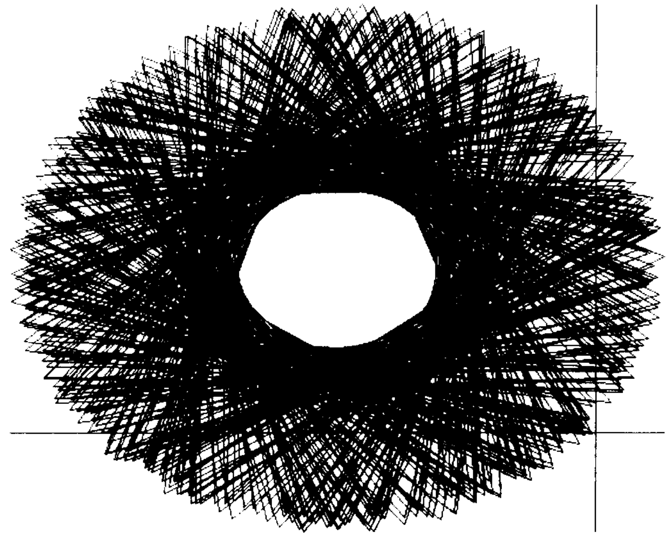

Figure 4. The Black Hole

hole in the making. The sizes of the blobs increase as N increases because the rate of change of θ_n decreases steadily after its initial hiccup.

As the above analysis might suggest, the family of graphs of the sequences

$$F_N = \sum_{n=1}^{N} \exp(2\pi i(\log n)^k)$$

becomes more and more random as k increases. The Milky Way in Figure 5 was obtained with k = 6. It must, however, be admitted that, as a vision of heaven, it still leaves something to be desired.

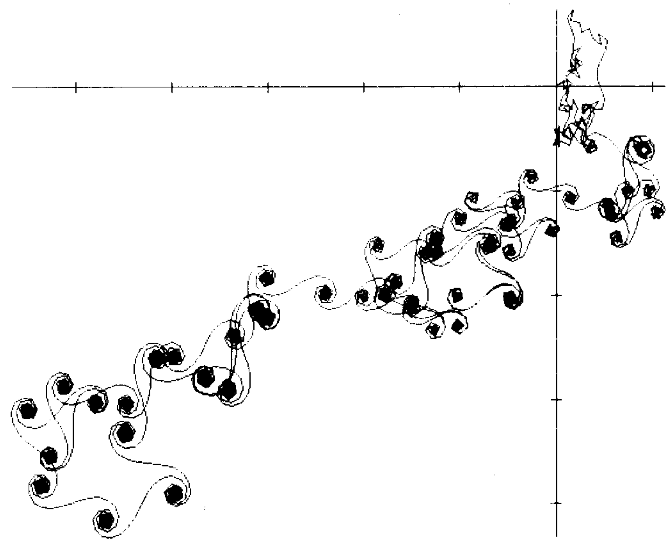

Figure 5. The Milky Way

I would like to thank Graeme Cohen of the School of Mathematical Sciences at the New South Wales Institute of Technology for help in producing the figures.

LATTICE COLOURINGS - A COUNTEREXAMPLE

Jamie Simpson

Suppose we are given a set of lattice points

$$S = \{(n_1, n_2, n_3) | \lambda_1 n_1 + \lambda_2 n_2 + \lambda_3 n_3 \leq \Lambda, n_1, n_2, n_3 \geq 0\}$$

where n_i are integers and the Λ_i and Λ are fixed positive numbers. How do you colour as many as possible of the points of S, but so that no two adjacent points $\underline{n}$, $\underline{n}'$ ($|n_1 - n_1'| + |n_2 - n_2'| + |n_3 - n_3'| = 1$) are coloured?

For the generalised k-dimensional problem, there are always two non-adjacent colourings called parity colourings - those obtained by colouring all points $\underline{n}$ with $n_1 + \dots + n_k$ even, or all those with $n_1 + \dots + n_k$ odd. For the two-dimensional case, Chris Smyth has shown (Black and White Cubes, JCMN 23 p. 125, JCMN 25 p. 3017) that one of the parity colourings is a maximal non-adjacent colouring. However, for k = 3, we give an example below for which the parity colourings are not maximal.

Specifically, we take

$$S = \{(n_1, n_2, n_3) | 5n_1 + 5n_2 + 6n_3 \leq 36\} .$$

Put $s = n_1 + n_2$ and

$$\left.\begin{array}{l} E(s) \\ O(s) \end{array}\right\} = \#\text{ of points } (s, n_3) \text{ in } R^2 \text{ with } (n_1, n_2, n_3) \text{ of } \left\{\begin{array}{l}\text{even} \\ \text{odd}\end{array}\right. \text{ parity in } S.$$

Then for each s there are s + 1 points (n_1, n_2) with $n_1 + n_2 = s$, $n_1, n_2 \geq 0$, so that the last two columns of the following table give the total number of points (n_1, n_2, n_3) of S of each parity, for each value of s.

s	E(s)	0(s)	(s + 1)E(s)	(s + 1)0(s)
0	4	3	4	3
1	3	3	6	6
2	3	2	9	6
3	2	2	8	8
4	2	1	10	5
5	1	1	6	6
6	1	1	7	7
7	0	1	0	8
		Totals	50	49

Thus the even-parity colouring colours 50 points and the odd-parity colouring 49 points. However, we see from the table that the colouring obtained by colouring all even-parity points for s = 0, 1, ..., 5, no points with s = 6, and all odd-parity points with s = 7, colours 51 points.

This problem arose from a problem of Combinatorial Number Theory (JCMN24 p. 130, JCMN 25 p. 3019), where it was necessary to obtain the cardinality of the maximal non-adjacent colouring of sets of points of the form

$$T = \{(n_1, n_2, n_3) \mid p^{n_1} q^{n_2} r^{n_3} \leq N\}$$

where p, q, r and N are positive integers with (p, q, r) = 1. If we take p = 148, q = 147, r = 405, $N = 405^6$, then S = T. Hence our example shows that parity colourings need not be maximal, for sets of this type also.

BOUND VOLUMES

Both Volume 1 (containing issues 1 - 17) and Volume 2 (issues 18 - 24) have now been reprinted. They are for sale in Australia for $5.00 each, postage included. Subscribers in Australia should send cheques payable to James Cook University. There is some doubt when the new reprint of Volume 1 will be available (January or February 1982 perhaps).

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