School of Mathematics

Interior Point Methods: Second-Order Cone Programming

Jacek Gondzio
Email: J.Gondzio@ed.ac.uk
URL: http://www.maths.ed.ac.uk/~gondzio

Outline

- **Self-concordant Barriers**
- **Second-Order Cone Programming**
  - example cones
  - example SOCP problems
  - logarithmic barrier function
  - IPM for SOCP
- **Final Comments**

Self-concordant Barrier

Def: Let $C \in \mathcal{R}^n$ be an open nonempty convex set.
Let $f : C \mapsto \mathcal{R}$ be a 3 times continuously diff'able convex function.
A function f is called **self-concordant** if there exists a constant $p > 0$ such that

$$|\nabla^3 f(x)[h,h,h]| \leq 2p^{-1/2}(\nabla^2 f(x)[h,h])^{3/2},$$

$\forall x \in C, \forall h : x+h \in C$. (We then say that f is p-self-concordant).

Note that a self-concordant function is always well approximated by the quadratic model because the error of such an approximation can be bounded by the 3/2 power of $\nabla^2 f(x)[h,h]$.

Self-concordant Barrier

Lemma The barrier function $-\log x$ is self-concordant on $\mathcal{R}_+$.
Proof:
Consider $f(x) = -\log x$ and compute
$f^{'}(x) = -x^{-1}, \; f^{''}(x) = x^{-2}$ and $f^{'''}(x) = -2x^{-3}$
and check that the self-concordance condition is satisfied for $p = 1$.

Lemma
The barrier function $1/x^{\alpha}$, with $\alpha \in (0, \infty)$ is not self-concordant on $\mathcal{R}_+$.

Lemma
The barrier function $e^{1/x}$ is not self-concordant on $\mathcal{R}_+$.

Use self-concordant barriers in optimization

Second-Order Cone Programming (SOCP)

SOCP: Second-Order Cone Programming

- Generalization of QP.
- Deals with conic constraints.
- Solved with IPMs.
- Numerous applications:
  quadratically constrained quadratic programs,
  problems involving sums and maxima/minima of norms,
  SOC-representable functions and sets,
  matrix-fractional problems,
  problems with hiperbolic constraints,
  robust LP/QP,
  robust least-squares.

SOCP: Second-Order Cone Programming

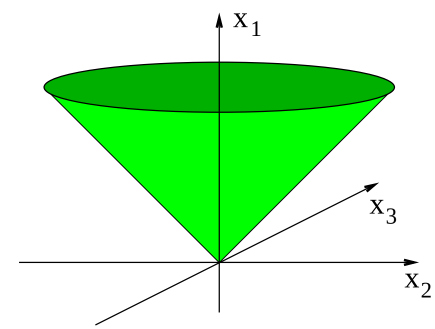
This lecture is based on three papers:

- **M. Lobo, L. Vandenberghe, S. Boyd and H. Lebret**, Applications of Second-Order Cone Programming, *Linear Algebra and its Appls* 284 (1998) pp. 193-228.
- **L. Vandenberghe and S. Boyd**, Semidefinite Programming, *SIAM Review* 38 (1996) pp. 49-95.
- **E.D. Andersen, C. Roos and T. Terlaky**, On Implementing a Primal-Dual IPM for Conic Optimization, *Mathematical Programming* 95 (2003) pp. 249-273.

Cones: Background

Def. A set $K \in \mathcal{R}^n$ is called a cone if for any $x \in K$ and for any $\lambda \geq 0$, $\lambda x \in K$.

Convex Cone:



Example:

$$K = \{x \in \mathcal{R}^n : x_1^2 \geq \sum_{j=2}^{n} x_j^2,\ x_1 \geq 0\}.$$

Example: Three Cones

R_+:

$$R_+ = \{x \in \mathcal{R} : x \geq 0\}.$$

Quadratic Cone:

$$K_q = \{x \in \mathcal{R}^n : x_1^2 \geq \sum_{j=2}^{n} x_j^2,\ x_1 \geq 0\}.$$

Rotated Quadratic Cone:

$$K_r = \{x \in \mathcal{R}^n : 2x_1x_2 \geq \sum_{j=3}^{n} x_j^2,\ x_1, x_2 \geq 0\}.$$

Matrix Representation of Cones

Each of the three most common cones has a matrix representation using orthogonal matrices T and/or Q.
(Orthogonal matrix: $Q^TQ = I$).

Quadratic Cone K_q. Define

$$Q = \begin{bmatrix} 1 & & & & \\ & -1 & & & \\ & & -1 & & \\ & & & \ddots & \\ & & & & -1 \end{bmatrix}$$

and write:

$$K_q = \{x \in \mathcal{R}^n : x^TQx \geq 0,\ x_1 \geq 0\}.$$

Example: $x_1^2 \geq x_2^2 + x_3^2 + \cdots + x_n^2.$

Matrix Representation of Cones (cont'd)

Rotated Quadratic Cone K_r. Define

$$Q = \begin{bmatrix} 0 & 1 & & & \\ 1 & 0 & & & \\ & & -1 & & \\ & & & \ddots & \\ & & & & -1 \end{bmatrix}$$

and write:

$$K_r = \{x \in \mathcal{R}^n : x^TQx \geq 0,\ x_1, x_2 \geq 0\}.$$

Example: $2x_1x_2 \geq x_3^2 + x_4^2 + \cdots + x_n^2.$

Matrix Representation of Cones (cont'd)

Consider a linear transformation $T : \mathcal{R}^2 \mapsto \mathcal{R}^2$:

$$T_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}.$$

It corresponds to a rotation by $\pi/4$. Indeed, write:

$$\begin{bmatrix} z \\ y \end{bmatrix} = T_2 \begin{bmatrix} u \\ v \end{bmatrix}$$

that is

$$z = \frac{u+v}{\sqrt{2}}, \quad y = \frac{u-v}{\sqrt{2}}$$

to get

$$2yz = u^2 - v^2.$$

Matrix Representation of Cones (cont'd)

Now, define

$$T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & & & \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}$$

and observe that the rotated quadratic cone satisfies

$$Tx \in K_r \quad \text{iff} \quad x \in K_q.$$

Example: Conic constraint

Consider a constraint:

$$\frac{1}{2}\|x\|^2 + a^T x \leq b.$$

Observe that $g(x) = \frac{1}{2}x^T x + a^T x - b$ is convex hence the constraint defines a convex set.
The constraint may be reformulated as an intersection of an affine (linear) constraint and a quadratic one:

$$\begin{aligned} a^T x + z &= b \\ y &= 1 \\ \|x\|^2 &\leq 2yz, \;\; y, z \geq 0. \end{aligned}$$

Example: Conic constraint (cont'd)

Now, substitute:

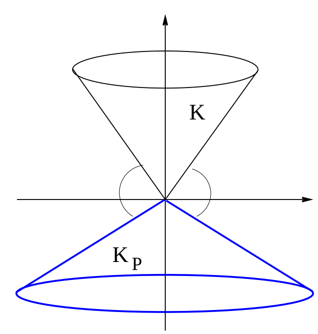
$$z = \frac{u+v}{\sqrt{2}}, \quad y = \frac{u-v}{\sqrt{2}}$$

to get

$$\begin{aligned} a^T x + \frac{u+v}{\sqrt{2}} &= b \\ u - v &= \sqrt{2} \\ \|x\|^2 + v^2 &\leq u^2. \end{aligned}$$

Dual Cone

Let $K \in \mathcal{R}^n$ be a cone.
Def. The set: $K_* := \{s \in \mathcal{R}^n : s^T x \geq 0, \, \forall x \in K\}$
is called the **dual** cone.
Def. The set: $K_P := \{s \in \mathcal{R}^n : s^T x \leq 0, \, \forall x \in K\}$
is called the **polar** cone (Fig below).



Conic Optimization

Consider an optimization problem:

$$\begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax = b, \\ & x \in K, \end{array}$$

where K is a convex closed cone.

We assume that

$$K = K^1 \times K^2 \times \cdots \times K^k,$$

that is, cone K is a product of several individual cones each of which is one of the three cones defined earlier.

Primal and Dual SOCPs

Consider a **primal** SOCP

$$\begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax = b, \\ & x \in K, \end{array}$$

where K is a convex closed cone.

The associated **dual** SOCP

$$\begin{array}{ll} \max & b^T y \\ \text{s.t.} & A^T y + s = c, \\ & s \in K_*. \end{array}$$

Weak Duality:
If (x, y, s) is a primal-dual feasible solution, then

$$c^T x - b^T y = x^T s \geq 0.$$

IPM for Conic Optimization

Conic Optimization problems can be solved in polynomial time with IPMs.

Consider a quadratic cone

$$K_q = \{(x,t) : x \in \mathcal{R}^{n-1}, t \in \mathcal{R},\ t^2 \geq \|x\|^2,\ t \geq 0\},$$

and define the (convex) **logarithmic barrier function** for this cone $f : \mathcal{R}^n \mapsto \mathcal{R}$

$$f(x,t) = \begin{cases} -\ln(t^2 - \|x\|^2) & \text{if } \|x\| < t \\ +\infty & \text{otherwise.} \end{cases}$$

Logarithmic Barrier Fctn for Quadratic Cone

Its derivatives are given by:

$$\nabla f(x,t) = \frac{2}{t^2 - x^T x} \begin{bmatrix} x \\ -t \end{bmatrix},$$

and

$$\nabla^2 f(x,t) = \frac{2}{(t^2 - x^T x)^2} \begin{bmatrix} (t^2 - x^T x)I + 2xx^T & -2tx \\ -2tx^T & t^2 + x^T x \end{bmatrix}.$$

Theorem:
$f(x,t)$ is a self-concordant barrier on K_q.

Exercise: Prove it in case $n = 2$.

Examples of SOCP

LP, QP use the cone $\mathcal{R}_+$ (positive orthant).

SDP uses the cone $\mathcal{SR}_+^{n\times n}$ (symmetric positive definite matrices).

SOCP uses two quadratic cones K_q and K_r.
Quadratically Constrained Quadratic Programming (QCQP) is a particular example of SOCP.

Typical trick to replace a quadratic constraint as a conic one!!!
Consider a constraint:

$$\frac{1}{2}\|x\|^2 + a^T x \le b.$$

Rewrite it as:

$$\|x\|^2 + v^2 \le u^2.$$

QCQP and SOCP

Let $P_i \in \mathcal{R}^{n\times n}$ be a symmetric positive definite matrix and $q_i \in \mathcal{R}^n$. Define a quadratic function $f_i(x) = x^T P_i x + 2q_i^T x + r_i$ and an associated ellipsoid $\mathcal{E}_i = \{x \mid f_i(x) \le 0\}$.
The set of constraints $f_i(x) \le 0,\ i = 1, 2, \ldots, m$ defines an intersection of (convex) ellipsoids and of course defines a convex set.

The optimization problem

$$\begin{array}{ll} \min & f_0(x) \\ \text{s.t.} & f_i(x) \le 0,\ i = 1, 2, \ldots, m, \end{array}$$

is an example of quadratically constrained quadratic program (QCQP).

QCQP can be reformulated as SOCP.

QCQP can be also reformulated as SDP.

SOCP Example: Linear Regression

The **least squares solution** of a linear system of equations $Ax = b$ is the solution of the following optimization problem

$$\min_x \quad \|Ax - b\|$$

and it can be recast as:

$$\begin{array}{ll} \min & t \\ \text{s.t.} & \|Ax - b\| \le t. \end{array}$$

Ellipsoids: Background

Sphere with $(0,0)$ centre:

$$x_1^2 + x_2^2 \le 1$$

Ellipsoid, centre at $(0,0)$, radii a, b:

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} \le 1$$

Ellipsoid, centre at (p,q), radii a, b:

$$\frac{(x_1 - p)^2}{a^2} + \frac{(x_2 - q)^2}{b^2} \le 1$$

General ellipsoid:

$$(x - x_0)^T H (x - x_0) \le 1,$$

where H is a positive definite matrix. Let $H = LL^T$. Then we can rewrite the ellipsoid as

$$\|L^T(x - x_0)\| \le 1.$$

SOCP Example: Robust LP

Consider an LP:

$$\begin{array}{ll} \min & c^T x \\ \text{s.t.} & a_i^T x \le b_i, \; i = 1, 2, \ldots, m, \end{array}$$

and assume that the values of a_i are uncertain.

Suppose that $a_i \in \mathcal{E}_i$, $i = 1, 2, \ldots, m$, where $\mathcal{E}_i$ are given ellipsoids

$$\mathcal{E}_i = \{\bar{a}_i + P_i u : \|u\| \le 1\},$$

where P_i is a symmetric positive definite matrix.

SOCP Example: Robust LP (cont'd)

Observe that

$$a_i^T x \le b_i \; \forall a_i \in \mathcal{E}_i \quad \text{iff} \quad \bar{a}_i^T x + \|P_i x\| \le b_i,$$

because for any $x \in \mathcal{R}^n$

$$\begin{aligned} \max\{a^T x : a \in \mathcal{E}\} &= \bar{a}^T x + \max\{u^T P x : \|u\| \le 1\} \\ &= \bar{a}^T x + \|P x\|. \end{aligned}$$

Hence **robust LP** formulated as SOCP is:

$$\begin{array}{ll} \min & c^T x \\ \text{s.t.} & \bar{a}_i^T x + \|P_i x\| \le b_i, \; i = 1, 2, \ldots, m. \end{array}$$

SOCP Example: Robust QP

Consider a QP with "uncertain" objective:

$$\min_x \max_{P \in \mathcal{E}} \; x^T P x + 2 q^T x + r$$

subject to linear constraints. "Uncertain" symmetric positive definite matrix P belongs to the ellipsoid:

$$P \in \mathcal{E} = \{P_0 + \sum_{i=1}^{m} P_i u_i : \|u\| \le 1\},$$

where P_i are symmetric positive semidefinite matrices.
The definition of ellipsoid $\mathcal{E}$ implies that

$$\max_{P \in \mathcal{E}} x^T P x = x^T P_0 x + \max_{\|u\| \le 1} \sum_{i=1}^{m} (x^T P_i x) u_i.$$

SOCP Example: Robust QP (cont'd)

From Cauchy-Schwartz inequality:

$$\sum_{i=1}^{m} (x^T P_i x) u_i \le \left(\sum_{i=1}^{m} (x^T P_i x)^2 \right)^{1/2} \|u\|$$

hence

$$\max_{\|u\| \le 1} \sum_{i=1}^{m} (x^T P_i x) u_i \le \left(\sum_{i=1}^{m} (x^T P_i x)^2 \right)^{1/2}.$$

We get a reformulation of robust QP:

$$\min_x \; x^T P_0 x + \left(\sum_{i=1}^{m} (x^T P_i x)^2 \right)^{1/2} + 2 q^T x + r.$$

SOCP Example: Robust QP (cont'd)

This problem can be written as:

$$\begin{array}{ll} \min & t + v + 2q^T x + r \\ \text{s.t.} & \|z\| \leq t,\ x^T P_0 x \leq v,\ x^T P_i x \leq z_i,\ i = 1, \ldots, m. \end{array}$$

SOCP reformulation:

$$\begin{array}{ll} \min & t + v + 2q^T x + r \\ \text{s.t.} & \|z\| \leq t, \\ & \|(2P_i^{1/2} x, z_i - 1)\| \leq z_i + 1,\ z_i \geq 0,\ i = 1..m, \\ & \|(2P_0^{1/2} x, v - 1)\| \leq v + 1,\ v \geq 0. \end{array}$$

Interior Point Methods:

- Logarithmic barrier functions for SDP and SOCP
  Self-concordant barriers
  → polynomial complexity (predictable behaviour)
- Unified view of optimization
  → from LP via QP to NLP, SDP, SOCP
- Efficiency
  - good for SOCP
  - problematic for SDP because solving the problem of size n involves linear algebra operations in dimension n^2
    → and this requires n^6 flops!

Use IPMs in your research!