Linear Programming and Numerical Analysis
MATH08037
Lecture 4
Vertex enumeration
and introduction to the simplex method

Julian Hall

January 20th 2011
Geometric view of slack variables

For the simple LP problem

maximize \( f = 3x_1 + 5x_2 \)
subject to
\[
2x_1 + 3x_2 \leq 6
\]
\( x_1 \geq 0 \) and \( x_2 \geq 0 \)

After adding slack variable \( x_3 \), the problem in standard form is

maximize \( f = 3x_1 + 5x_2 \)
subject to
\[
2x_1 + 3x_2 + x_3 = 6
\]
\( x_1 \geq 0, \ x_2 \geq 0 \) and \( x_3 \geq 0 \)
Basic and nonbasic variables

After adding slack variables:

- Each vertex corresponds to a choice of \( n \) nonbasic variables to be set to zero.
- The \( m \) equations can be used to solve for the values of the remaining \( m \) basic variables.
Vertex enumeration for the blending problem

\[
\text{maximize} \quad f = 8x_1 + 10x_2 \\
\text{subject to} \quad 0.3x_1 + 0.5x_2 + x_3 = 120 \\
\quad 0.7x_1 + 0.5x_2 + x_4 = 210 \\
\quad x_1 \geq 0 \quad x_2 \geq 0 \quad x_3 \geq 0 \quad x_4 \geq 0
\]

Consider all 6 choices of \( n = 2 \) from \( n + m = 4 \) variables

<table>
<thead>
<tr>
<th>Vertex</th>
<th>Nonbasic</th>
<th>Basic</th>
<th>Feasible</th>
<th>( f )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( O )</td>
<td>( x_1 \ \ x_2 )</td>
<td>( x_3 = 120 \ \ x_4 = 210 )</td>
<td>Y</td>
<td>0</td>
</tr>
<tr>
<td>( A )</td>
<td>( x_1 \ \ x_3 )</td>
<td>( x_2 = 240 \ \ x_4 = 90 )</td>
<td>Y</td>
<td>2400</td>
</tr>
<tr>
<td>( B )</td>
<td>( x_1 \ \ x_4 )</td>
<td>( x_2 = 420 \ \ x_3 = -90 )</td>
<td>N</td>
<td>-</td>
</tr>
<tr>
<td>( C )</td>
<td>( x_2 \ \ x_3 )</td>
<td>( x_1 = 400 \ \ x_4 = -70 )</td>
<td>N</td>
<td>-</td>
</tr>
<tr>
<td>( D )</td>
<td>( x_2 \ \ x_4 )</td>
<td>( x_1 = 300 \ \ x_3 = 30 )</td>
<td>Y</td>
<td>2400</td>
</tr>
<tr>
<td>( E )</td>
<td>( x_3 \ \ x_4 )</td>
<td>( x_1 = 225 \ \ x_2 = 105 )</td>
<td>Y</td>
<td>2850</td>
</tr>
</tbody>
</table>

Vertex \( E \) has the greatest objective value so is optimal
Vertex enumeration: limitation

- For a problem with $n$ (original) variables and $m$ inequalities, there are

\[ \frac{(m + n)!}{m!n!} \]

ways of choosing $n$ of the $n + m$ original+slack variables to be nonbasic
- For $n = m = 35$ there are more than $10^{20}$ choices!
- Large LP problems may have $n = m = 10^6$
- Vertex enumeration is a prohibitively expensive method for solving large problems
- The **simplex method** identifies an optimal vertex by considering only a relatively small number of vertices
A 3-dimensional LP problem

maximize \[ f = x_1 + 2x_2 + 3x_3 \]
subject to
\[ x_1 + 2x_3 \leq 3 \]
\[ x_2 + 2x_3 \leq 2 \]
\[ x_1 \geq 0 \quad x_2 \geq 0 \quad x_3 \geq 0 . \]

- Feasible region is the polyhedron \( OBCFEG \)
- At each vertex there are 3 constraints/bounds satisfied exactly
The simplex method

- The simplex method starts from the vertex $O$ with all original variables nonbasic
- It identifies whether the objective improves along any of the $n$ edges leaving the vertex
  - If there is no improving edge the vertex is optimal
  - Otherwise the simplex method chooses the “best” improving edge
- The simplex method identifies the next vertex along the improving edge and moves to it
- The above steps are repeated until the method terminates
A 3-dimensional LP problem: edges at vertex $O$

\[ x_2 + 2x_3 = 2 \]
\[ x_1 + 2x_3 = 3 \]

Edges at vertex $O$:
- $f = 0$
- $f = 3$
- $f = 4$
A 3-dimensional LP problem: edges at vertex $B$

![Diagram showing a 3-dimensional linear programming problem with edges at vertex B.](image-url)

- $x_1 + 2x_3 = 3$ along $EG$
- $x_2 + 2x_3 = 2$ along $BF$
- $x_2 = 0$ along $OB$
- $x_1 = 0$ along $OE$
- $f = 0$ along $OB$
- $f = 3$ along $BO$
- $f = 4$ along $BF$
- $f = 4$ along $EG$
A 3-dimensional LP problem: edges at vertex $F$
A 3-dimensional LP problem: edges at vertex $G$

\begin{align*}
\mathbf{x} & = 0 \\
\mathbf{x}_1 + 2\mathbf{x}_3 & = 3 \\
x_2 + 2x_3 & = 2
\end{align*}
The simplex method

\[
\begin{align*}
\text{maximize} & \quad f = 8x_1 + 10x_2 \\
\text{subject to} & \quad 0.3x_1 + 0.5x_2 + x_3 = 120 \\
& \quad 0.7x_1 + 0.5x_2 + x_4 = 210 \\
& \quad x_1 \geq 0 \quad x_2 \geq 0 \quad x_3 \geq 0 \quad x_4 \geq 0
\end{align*}
\]

- The simplex method starts from \( O \)
  - \( x_1 = x_2 = 0 \) nonbasic
  - \( x_3 = 120 \), \( x_4 = 210 \) basic
- Increase \( x_2 \) from zero since it has greater objective coefficient
- As \( x_2 \) is increased from zero
  - \( x_3 \) is zeroed at \( A \) when \( x_2 = 120/0.5 = 240 \)
  - \( x_4 \) is zeroed at \( B \) when \( x_2 = 210/0.5 = 420 \)
- Increase \( x_2 \) to 240: move from \( O \) to \( A \)
- \( x_2 \) becomes basic; \( x_3 \) is zeroed and becomes nonbasic
The simplex method

maximize \[ f = 8x_1 + 10x_2 \]
subject to
\[ 0.3x_1 + 0.5x_2 + x_3 = 120 \]
\[ 0.7x_1 + 0.5x_2 + x_4 = 210 \]
\[ x_1 \geq 0 \quad x_2 \geq 0 \quad x_3 \geq 0 \quad x_4 \geq 0 \]

- At vertex \( A \)
  - \( x_1 = x_3 = 0 \) nonbasic
  - \( x_2 = 240, \ x_4 = 90 \) basic
- Does the objective improve if \( x_1 \) is increased from zero?
- **Problem:** \( x_2 \) changes if \( x_1 \) is increased from zero
- How much can \( x_1 \) be increased by before \( x_2 \) or \( x_4 \) is zeroed?

Use the first equation to eliminate \( x_2 \) from the objective and second equation
The simplex method

maximize \( f = 2x_1 - 20x_3 + 2400 \)
subject to
\[
0.6x_1 + x_2 + 2x_3 = 240 \\
0.4x_1 - x_3 + x_4 = 90 \\
x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0
\]

- At vertex \( A \)
  - \( x_1 = x_3 = 0 \) nonbasic
  - \( x_2 = 240, x_4 = 90 \) basic
- Increase \( x_1 \) from zero since it has positive objective coefficient
- As \( x_1 \) is increased from zero
  - \( x_2 \) is zeroed at \( C \) when \( x_1 = 240/0.6 = 400 \)
  - \( x_4 \) is zeroed at \( E \) when \( x_1 = 90/0.4 = 225 \)
- Increase \( x_1 \) to 225: move from \( A \) to \( E \)
  - \( x_1 \) becomes basic; \( x_4 \) is zeroed and becomes nonbasic

Use the second equation to eliminate \( x_1 \) from the objective and first equation
The simplex method

maximize \( f = -15x_3 - 5x_4 + 2850 \)
subject to
\[
\begin{align*}
  x_2 + 3.5x_3 + 2.5x_4 &= 105 \\
  -2.5x_3 - 1.5x_4 &= 225 \\
  x_1 &\geq 0 \\
  x_2 &\geq 0 \\
  x_3 &\geq 0 \\
  x_4 &\geq 0
\end{align*}
\]

- At vertex \( E \)
  - \( x_3 = x_4 = 0 \) nonbasic
  - \( x_1 = 225, \ x_2 = 105 \) basic
- Both \( x_3 \) and \( x_4 \) have negative objective coefficient
- Objective gets smaller if \textit{either} \( x_3 \) or \( x_4 \) is increased from zero
- Vertex is optimal