Analysis Club: Littlewood-Paley Theory DRAFT 2

Singular Integrals

Thm $Tf(x) = K * f(x)$

(i) T bounded on L^2

(ii) Hörmander condition

$$\int_{|x|>2|y|} |K(x-y) - K(x)|\,dx \leq B \quad \forall y \in \mathbb{R}^n$$

(may be deduced from $|\nabla K(x)| \leq C|x|^{-n-1}$)

$\Rightarrow T$ is bounded on L^p, $1 < p < \infty$

Proof Marina next week!

Remark Can also be proved for operators T ~~betw~~ between Hilbert spaces (i.e. vector-valued analogue).

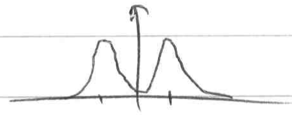
Littlewood-Paley

Given $f = \sum a_k e^{2\pi i k} \in L^p$, when is $\sum \pm a_k e^{2\pi i k} \in L^p$ for random choices of $\pm$?

Almost never (although counterexample ?!)

But Littlewood-Paley shows that choosing the same sign on dyadic blocks of indices, i.e. $k \in [2^j, 2^{j+1})$, keeps $f \in L^p$.

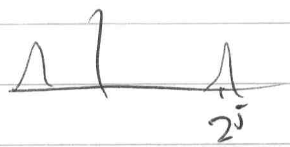
Let $\psi \in S(\mathbb{R}^n)$ with $\psi(0) = 0$.



supp ≈ 1

Define S_j by $\widehat{S_j f}(\xi) = \psi_j(\xi)\hat{f}(\xi)$

$$\psi_j(\xi) = \psi(2^{-j}\xi)$$



<u>Theorem</u> (L-P). For $1 < p < \infty$,

(a) $\| (\sum |S_j f|^2)^{1/2} \|_p \lesssim \|f\|_p$

(b) If for $\xi \neq 0$ we have $\sum |\psi(2^{-j}\xi)|^2 = C$ then

$$\|f\|_p \lesssim \| (\sum |S_j f|^2)^{1/2} \|_p$$

i.e. $\|f\|_p \approx \| S_j f \|_{L^p(\ell^2)}$.

<u>Remark</u> This proves Tim's statement from last week, that $\dot{F}_{p,2} \approx L^p$.

<u>Proof</u> We consider the operator $f \mapsto \{S_j f\}$.

(a) It is bounded from L^2 to $L^2(\ell^2)$ since

$$\| (\sum |S_j f|^2)^{1/2} \|_2^2 = \int_{\mathbb{R}^n} \underbrace{\sum_j |\psi_j(\xi)|^2}_{\leq C \text{ since } \psi \in S \text{ and } \psi(0)=0} |\hat{f}(\xi)|^2 \, d\xi \quad \text{by Plancherel}$$

$$\lesssim \|f\|_2^2$$

The Hörmander condition $\| \nabla \check{\psi}_j(x) \|_{\ell^2} \leq C|x|^{-n-1}$ is true, again using $\psi \in S$.

So (a) is proved by singular integral theorem.

(b) The Plancherel argument gives $\| S_j f \|_{L^2(\ell^2)} = \sqrt{c}\, \| f \|_{L^2}$

Using polarization $\left(\langle u, v \rangle = \frac{1}{4} \|u+v\|^2 - \|u-v\|^2 \right)$ we convert norms to inner products and get

$$\int \sum S_j f \overline{S_j g} = \sqrt{c} \int f \bar{g}$$

$$\text{So } \left| \int f \bar{g} \right| \lesssim \int \left(\sum |S_j f|^2 \right)^{1/2} \left(\sum |S_j g|^2 \right)^{1/2} \qquad (\text{CS on } \Sigma)$$

$$\lesssim \| S_j f \|_{L^p(\ell^2)} \| S_j g \|_{L^{p'}(\ell^2)} \qquad (\text{Hölder})$$

$$\lesssim \| S_j f \|_{L^r(\ell^2)} \| g \|_{L^{p'}} \qquad (\text{part (a)})$$

Taking the supremum over $g \in L^{r'}$ with $\|g\|_{r'} = 1$ gives the result.

<u>Remark</u>

This is one approach to dyadic decomposition, where we have cut up the operator on the FT side using smooth functions supported on annuli where $|\xi| \sim 2^j$.

We could also have used characteristic functions of dyadic intervals (or smooth cutoffs of them).

But not χ_{annuli} — related to disc multiplier result.

Multipliers

Given $m \in L^\infty(\mathbb{R}^n)$ we define T_m by $\widehat{T_m f}(x) = m(x)\,\hat{f}(x)$.

Def m is an L^p multiplier if T_m is bounded on L^p

The class of L^p multipliers is denoted M_p.

Example $M_2 = L^\infty$

- $m \in L^\infty$. $\|T_m f\|_2 \overset{Pl.}{\leq} \|m\hat{f}\|_2 \leq \|m\|_\infty \|f\|_2$
- $\|T_m f\|_2 \leq A\|f\|_2 \Rightarrow \int |m\hat{f}|^2 \leq \int |A\hat{f}|^2$

So $\|m\|_\infty \leq A$ since if $|m(x)| > A$ on E with $|E|>0$, taking $\hat{f} = \chi_E$ gives a contradiction.

Theorem $M_p = M_{p'}$ $\quad (1 \leq p \leq \infty,\ \frac{1}{p} + \frac{1}{p'} = 1)$

Proof Stein (IV §3.1)

Def The Sobolev space $L^p_k = \{ f \in L^p : D^\alpha f \in L^p,\ |\alpha| \leq k \}$ for positive integers k. L^p_a can be defined for general $a>0$, e.g.

$$L^2_a = \{ g \in L^2 : (1+|\xi|^2)^{a/2}\,\hat{g}(\xi) \in L^2 \}$$

with $\|g\|_{L^2_a} = \|(1+|\cdot|^2)^{a/2}\,\hat{g}\|_2$.

Theorem $m \in L^2_a,\ a > \frac{n}{2} \Rightarrow m \in M_p$ for $1 \leq p \leq \infty$

Proof $T_m f = \check{m} * f$, so $\|T_m f\|_p \overset{Young}{\leq} \|\check{m}\|_1 \|f\|_p$.

$m \in L^2_a$ means $(1+|\xi|^2)^{a/2}\,\hat{m}(\xi) = h(\xi) \in L^2$, so

$$\|\check{m}\|_1 = \|\hat{m}\|_1 = \left\| \frac{h(\xi)}{(1+|\xi|^2)^{a/2}} \right\|_1$$

$$\leq \underbrace{\|h\|_2}_{=\|m\|_{L^2_a}} \underbrace{\left(\int (1+|\xi|^2)^{-a} \, d\xi \right)^{1/2}}_{=c_n < \infty \text{ if } a > \frac{n}{2}} \qquad (C\text{-}S)$$

$$< \infty.$$

Another sufficient condition for being an L^p multiplier:

Theorem (Hörmander)

- $m \in C^k(\mathbb{R}^n \setminus \{0\})$ for some integer $k > \frac{n}{2}$
- $|D^\alpha m(x)| \leq C|x|^{-|\alpha|}$ for $|\alpha| \leq k$

$\Rightarrow$ $m \in M_p$ for all $1 < p < \infty$

This is a consequence of

Theorem

Let $\varphi \in C^\infty$ be radial with support on $\frac{1}{2} \leq |\xi| \leq 2$, and

$$\sum_{j=-\infty}^{\infty} |\varphi(2^{-j} x)|^2 = 1 \qquad (x \neq 0).$$

If $\exists k > \frac{n}{2}$ s.t. $\sup_j \|m(2^j \cdot)\varphi\|_{L^2_k} < \infty$, then $m \in M_p$ for $1 < p < \infty$.

Remark

This is saying each "piece" $m(2^j \cdot)\varphi$ is in L^2_k, uniformly in j.

So each piece is M_p by the previous result, but we have some extra information which lets us add them together.

Proof

First let $\tilde{\varphi} \in C^\infty$ be supported on $\frac{1}{4} \leq |\xi| \leq 4$ and equal to 1 on $\operatorname{supp} \varphi$. We can use (a) of L-P, so the operators $\tilde{S}_j$ with multipliers $\tilde{\varphi}(2^{-j}\xi)$ satisfy

$$\| (\sum |\tilde{S}_j f|^2)^{1/2} \|_p \lesssim \|f\|_p.$$

If S_j has multiplier $\varphi(2^{-j}\xi)$ we use (b) of L-P:

$$\|T_m f\|_p \lesssim \| (\sum |S_j T f|^2)^{1/2} \|_p$$

$$= \| (\sum |S_j T \tilde{S}_j f|^2)^{1/2} \|_p \qquad \text{(due to } \tilde{\varphi} = 1 \text{ on } \operatorname{supp}\varphi\text{)}$$

The goal is to strip out this $S_j T$. Now

- $\| (\sum |S_j T f_j|^2)^{1/2} \|_p^2 = \| \sum |S_j T f_j|^2 \|_{p/2}$

$$= \int_{\mathbb{R}^n} \sum |S_j T f_j|^2 u$$

$u \in L^{(p/2)'}$ Riesz repⁿ $(p > 2)$

- $S_j T$ has mult. $m_j = \varphi(2^{-j}\cdot) m(\cdot) \in L^2_k$ by hypothesis. So

$$\int |S_j T f|^2 u = \int \left| \int \check{m}_j(x-y) (1+|x-y|^2)^{k/2} \frac{f(y)}{(1+|x-y|^2)^{k/2}} dy \right|^2 u(x)\, dx$$

$$\overset{C\text{-}S}{\lesssim} \underbrace{\|m_j\|_{L^2_k}^2}_{C_k \text{ indep of } j} \int |f(y)|^2 Mu(y)\, dy$$

So we can strip out the $S_j T$

$$\| (\sum |S_j T f_j|^2)^{1/2} \|_p^2 \lesssim \int \sum |f_j|^2 Mu \lesssim \| \sum |f_j|^2 \|_{p/2} \|Mu\|_{(p/2)'}$$

$$\| (\sum |S_j T f_j|^2)^{1/2} \|_p \lesssim \| \sum |f_j|^2 \|_{p/2}^{1/2} = \| (\sum |f_j|^2)^{1/2} \|_p .$$

Hence $\|T_m f\|_p \lesssim \|(\sum |\tilde{S}_j f|^2)^{1/2}\|_p \lesssim \|f\|_p$,

i.e. $m \in M_p$ $(p > 2)$. The other p come for free.