

Analysis Club: Littlewood-Paley Theory DRAFT 2

Singular Integrals

Thm $Tf(x) = K * f(x)$

(i) T bounded on L^2

(ii) Hörmander condition

$$\int_{|x| > 2|y|} |K(x-y) - K(x)| dx \leq B \quad \forall y \in \mathbb{R}^n$$

(may be deduced from $|\nabla K(x)| \leq C|x|^{p-n-1}$)

$\Rightarrow T$ is bounded on L^p , $1 < p < \infty$

Proof Marina next week!

Remark Can also be proved for operators T ~~between~~ between Hilbert spaces (i.e. vector-valued analogue).

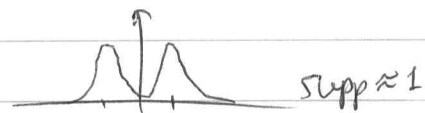
Littlewood-Paley

Given $f = \sum a_k e^{2\pi i k} \in L^p$, when is $\sum \pm a_k e^{2\pi i k} \in L^p$ for random choices of \pm ?

Almost never (although counterexample ?!)

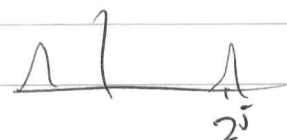
But Littlewood-Paley shows that choosing the same sign on dyadic blocks of indices, i.e. $k \in [2^j, 2^{j+1})$, keeps $f \in L^p$.

Let $\psi \in \mathcal{S}(\mathbb{R}^n)$ with $\psi(0) = 0$.



Define S_j by $\widehat{S_j f}(\xi) = \psi_j(\xi) \widehat{f}(\xi)$

$$\psi_j(\xi) = \psi(2^{-j}\xi)$$



Theorem (L-P). For $1 < p < \infty$,

$$(a) \quad \left\| \left(\sum |S_j f|^2 \right)^{1/2} \right\|_p \lesssim \|f\|_p$$

(b) If for $\xi \neq 0$ we have $\sum |\psi(2^{-j}\xi)|^2 = C$ then

$$\|f\|_p \lesssim \left\| \left(\sum |S_j f|^2 \right)^{1/2} \right\|_p$$

i.e. $\|f\|_p \approx \|S_j f\|_{L^p(\mathbb{R}^2)}$.

Remark This proves Tim's statement from last week, that $F_{p,2}^0 \approx L^p$.

Proof We consider the operator $f \mapsto \{S_j f\}$.

(a) It is bounded from L^2 to $L^2(\ell^2)$ since

$$\begin{aligned} \left\| \left(\sum |S_j f|^2 \right)^{1/2} \right\|_2^2 &= \int_{\mathbb{R}^n} \underbrace{\sum_j |\psi_j(\xi)|^2}_{\leq C \text{ since } \psi \in \mathcal{S} \text{ and } \psi(0) = 0} |\widehat{f}(\xi)|^2 d\xi \quad \text{by Plancherel} \\ &\leq C \|f\|_2^2 \end{aligned}$$

The Hörmander condition $\|\nabla \psi_j(x)\|_{L^2} \leq C|x|^{-n-1}$ is true, again using $\psi \in \mathcal{S}$.

So (a) is proved by singular integral theorem.

(b) The Plancherel argument gives $\|S_j f\|_{L^2(\mathbb{R}^2)} = \sqrt{c} \|f\|_{L^2}$

Using polarization ($\langle u, v \rangle = \frac{1}{4} (\|u+v\|^2 - \|u-v\|^2)$) we convert norms to inner products and get

$$\int_{\Sigma} S_j f \overline{S_j g} = \sqrt{c} \int f \overline{g}$$

$$\text{So } \left| \int f \overline{g} \right| \lesssim \int (\sum |S_j f|^2)^{1/2} (\sum |S_j g|^2)^{1/2} \quad (\text{B on } \Sigma)$$

$$\lesssim \|S_j f\|_{L^p(\mathbb{R}^2)} \|S_j g\|_{L^{p'}(\mathbb{R}^2)} \quad (\text{Hölder})$$

$$\lesssim \|S_j f\|_{L^r(\mathbb{R}^2)} \|g\|_{L^{r'}} \quad (\text{part (a)})$$

Taking the supremum over $g \in L^{r'}$ with $\|g\|_{r'} = 1$ gives the result.

Remark

This is one approach to dyadic decomposition, where we have cut up the operator on the FT side using smooth functions supported on annuli where $|S| \sim 2^j$.

We could also have used characteristic functions of dyadic intervals (or smooth cutoffs of them).

But not χ_{annuli} - related to disc multiplier result.

