To determine when we have the Fourier inversion formula,

\[ f(x) = \int_{\mathbb{R}^n} \hat{f}(\xi) e^{2\pi i \xi \cdot x} d\xi \]

we consider the partial sum operators \( S_R \),

\[ S_R f(x) = \int_{|\xi| \leq R} \hat{f}(\xi) e^{2\pi i \xi \cdot x} d\xi \]

and then by some functional analysis we have that \( \lim_{R \to \infty} S_R f = f \) in \( L^p \) if and only if

\[ \|S_R f\|_p \lesssim \|f\|_p, \]

so we have turned the problem into studying the boundedness of certain operators on \( L^p \) spaces. In this case, we have a complete answer:

- for \( n = 1 \), we can write \( S_R \) in terms of the Hilbert transform, and obtain \( L^p \)-boundedness for \( 1 < p < \infty \).
- for \( n \geq 2 \), Fefferman [3] showed the \( S_R \) are only bounded on \( L^2 \).

We now consider the family of operators \( T_\lambda \), \( \lambda \geq 0 \), defined on \( \mathbb{R}^n \) by

\[ T_\lambda f(\xi) = m_\lambda \hat{f}(\xi), \text{ where } m_\lambda(\xi) = (1 - |\xi|^2)^{\frac{\lambda}{4}}. \]

These are the **Bochner-Riesz multipliers**, which can be viewed as an attempt to smooth out the singularity of the disc multiplier to see if we can obtain boundedness on a wider range of \( L^p \) spaces. Note that when \( \lambda = 0 \) we obtain \( S_1 \), and that as \( \lambda \) increases, the multiplier \( m_\lambda \) becomes smoother hence more likely to produce a bounded operator.

The main references for the following discussion are [2, Ch 8, §5] and [1, pp143-157].
1 Useful tools

The following is a useful “duality” result, which allows us to consider only $p < 2$ or $p > 2$ as necessary.

**Theorem 1.** [5, Ch IV, §3.1] If $\frac{1}{p} + \frac{1}{p'} = 1$, $1 \leq p \leq \infty$, then $\mathcal{M}_p = \mathcal{M}_{p'}$ with equality of norms.

**Proof.** Let $\sigma$ denote the involution $\sigma(f)(x) = \overline{f}(-x)$. We see that $\sigma^{-1}T_m\sigma = T_{\overline{m}}$, and since $\sigma$ is an isometry of $L^p$ this means that $\|m\|_{\mathcal{M}_p} = \|\overline{m}\|_{\mathcal{M}_{p'}}$.

Now by Plancherel, and the definition of $T_m$,

$$\int T_m f \overline{g} = \int T_m f \overline{g} = \int f \overline{T_m g} = \int f \overline{T_m g},$$

so

$$\|m\|_{\mathcal{M}_p} = \sup_{\|f\|_p = \|g\|_{p'}} \left| \int T_m f \overline{g} \right| = \sup_{\|f\|_p = \|g\|_{p'}} \left| \int f \overline{T_m g} \right| = \|\overline{m}\|_{\mathcal{M}_{p'}}.$$

Combining this with $\|m\|_{\mathcal{M}_p} = \|\overline{m}\|_{\mathcal{M}_p}$ we have $\|m\|_{\mathcal{M}_p} = \|m\|_{\mathcal{M}_{p'}}$. □

We will also need to make use of interpolation. Generally it will be enough to use Riesz-Thorin or Marcinkiewicz interpolation, but we note the following “complex interpolation” result due to Stein [4].

**Theorem 2.** For a nice\(^1\) family of operators $T_z$, $0 \leq \text{Re } z \leq 1$, suppose

$$\|T_{iy}f\|_{q_0} \lesssim \|f\|_{p_0} \quad \text{and} \quad \|T_{1+iy}f\|_{q_1} \lesssim \|f\|_{p_1}.$$  

Then for $0 < \theta < 1$ we have

$$\|T_{\theta+iy}f\|_{q} \lesssim \|f\|_p$$

where

$$\frac{1}{p} = \frac{1}{p_0} + \frac{\theta}{p_1} \quad \text{and} \quad \frac{1}{q} = \frac{1}{q_0} + \frac{\theta}{q_1}.$$

\(^1\)see [4] for the full details, or [2, pp22-23] for a summary.
2 Known results

Following [2, pp171-172], the kernel of $T_\lambda$ is

$$K_\lambda(x) = \pi^{-\lambda} \Gamma(\lambda + 1) |x|^{-\frac{d}{2} - \lambda} J_{\frac{d}{2} + \lambda}(2\pi|x|)$$

where $J_\mu$ is the Bessel function

$$J_\mu(t) = \left(\frac{t}{2}\right)^\mu \frac{\Gamma(\mu + \frac{1}{2})}{\Gamma(\frac{1}{2})} \int_{-1}^{1} e^{its} (1 - s^2)^{\mu - \frac{1}{2}} ds.$$  

Applying the known behaviour $J_\mu = O(t^\mu)$ as $t \to 0$ and $J_\mu \approx t^{-1/2}$ as $t \to \infty$, we have

$$|K_\lambda(x)| \left\{ \begin{array}{ll}
\leq C & \text{as } |x| \to 0 \\
\approx |x|^{-\left(\frac{d+1}{2} + \lambda\right)} & \text{as } |x| \to \infty.
\end{array} \right.$$

**Theorem 3.** If $\lambda > \frac{n-1}{2}$ then $T_\lambda$ is bounded on all $L^p$.

**Proof.** For $\lambda > \frac{n-1}{2}$, the bounds for $K_\lambda$ above show that $K_\lambda \in L^1$. So by Young’s inequality,

$$\|T_\lambda f\|_p = \|K_\lambda * f\|_p \leq \|K\|_1 \|f\|_p \lesssim \|f\|_p$$

i.e. $T_\lambda$ is bounded on $L^p$. \hfill $\Box$

**Theorem 4.** If $\left| \frac{1}{p} - \frac{1}{2} \right| \geq \frac{\lambda + 1}{2n}$ then $T_\lambda$ is unbounded.

**Proof.** Now if $m \in \mathcal{M}_p$ has compact support, it follows\(^2\) that $\hat{m} \in L^p$. This shows that a necessary condition for $T_\lambda$ to be bounded on $L^p$ is that $K_\lambda \in L^p$.

Again using the bounds on $K_\lambda$, we have $K_\lambda \in L^p$ only if $p \left(\frac{n+1}{2} + \lambda\right) > n$.

By duality, this becomes $\left| \frac{1}{p} - \frac{1}{2} \right| \geq \frac{\lambda + 1}{2n}$. \hfill $\Box$

**Remark.** We can interpolate the result for $\lambda > \frac{n-1}{2}$ with the disc multiplier result to get a whole region of boundedness, shaded in blue, as well as the region of unboundedness in red.

\(^2\)Choose $f \in \mathcal{S}$ such that $\hat{f} = 1$ on the support of $m$. Then $f \in L^p$, so $T_m f \in L^p$ by assumption. But $\hat{T_m f} = m \hat{f} = m$, so $\hat{m} = T_m f \in L^p$ hence $\hat{m} \in L^p$. 

3
We now essentially re-prove the result about boundedness; but the method of proof is interesting.

**Theorem 5.** $T_\lambda$ is bounded on $L^p$ when $\left| \frac{1}{p} - \frac{1}{2} \right| < \frac{\lambda}{n-1}$.

**Proof.** We follow the proof in [2, pp170-171], with some modifications.

Decompose $T_\lambda$ on dyadic anulli as follows. Take a partition of unity $\phi_k$ subordinate to the open cover $(1 - 2^{-k+1}, 1 - 2^{-k-1})$ of $[0, 1]$ \(^3\), so that

$$ (1 - |\xi|^2)^\lambda = \sum_{k=0}^{\infty} (1 - |\xi|^2)^\lambda \phi_k(|\xi|). $$

Now, $\phi_k$ is supported near $1 - 2^{-k}$ (on an annulus of width $\sim 2^{-k}$), where we have

$$ (1 - |\xi|^2)^\lambda = \left( (1 + |\xi|)(1 - |\xi|) \right)^\lambda \sim (2 \times 2^{-k})^\lambda \sim 2^{-k\lambda}. $$

We define $\tilde{\phi}_k(|\xi|) = 2^{k\lambda}(1 - |\xi|^2)^\lambda \phi_k(|\xi|)$ so that $\tilde{\phi}_k \lesssim 1$, and then

$$ T_\lambda f = \sum_{k=0}^{\infty} 2^{-k\lambda} T_k f $$

where $T_k$ is the operator with multiplier $\tilde{\phi}_k$.

We apply Minkowski’s inequality to get

$$ \|T_\lambda f\|_p \leq \sum_{k=0}^{\infty} 2^{-k\lambda} \|T_k f\|_p $$

and then estimating each of these norms, we will see that the series converges for the hypothesised range of $\lambda$ and $p$.

\(^3\)this only makes sense for $k \geq 1$, so we need to just add in the $\phi_0$ manually
The estimate for these norms is produced in a Lemma in [2], but we get them via a slightly different method. There is the trivial $L^2$ boundedness, $\|T_kf\|_2 \lesssim \lambda \|f\|_2$ using the fact that $\tilde{\phi}_k \lesssim \lambda 1$. Then by Young's inequality, for $q = 1, \infty$ we have

$$\|T_kf\|_q = \|\tilde{\phi}_k * f\|_q \leq \|\tilde{\phi}_k\|_1 \|f\|_q$$

so the problem reduces to estimating $\|\tilde{\phi}_k\|_1$.

We do this by decomposing $\tilde{\phi}_k$ smoothly in segments of the annulus; if the annulus is $\delta$ thick then all other dimensions of the segments are $\delta^{1/2}$. Each segment $v$ supports a piece $\phi_v$ of $\tilde{\phi}_k$.

- If $\phi_v$ is one of the pieces, then $\|\tilde{\phi}_v\|_1 \lesssim 1$.

  Each piece has the same norm since they are all rotations of each other, so we may assume $\phi_v$ is perpendicular to the $\xi_1$ axis; then $\phi_v$ is a translate of $\Psi(\frac{\xi_1}{\delta}, \frac{\xi'}{\delta^{1/2}})$, for some fixed $\Psi \in S$, hence

$$\|\tilde{\phi}_v\|_1 = \|\tilde{\phi}_v\|_1 = \left\|\Psi\left(\frac{\xi_1}{\delta}, \frac{\xi'}{\delta^{1/2}}\right)\right\|_{L^1(\xi)} = \|\Psi\|_1 = C.$$

- Hence by the triangle inequality, $\|\tilde{\phi}_k\|_1 \lesssim \text{num. segments}$.

Now each segment will have surface area $(\delta^{1/2})^{n-1}$, and since the radius is bounded by 1, the total surface area of the outside of the annulus is $O(1)$; hence there are $\approx \delta^{-(n-1)/2}$ segments, and $\delta \approx 2^{-k}$, giving $\|\tilde{\phi}_k\|_1 \lesssim 2^{k(n-1)/2}$.

Finally, interpolation gives $\|T_kf\|_p \lesssim 2^{k\frac{n}{p-1}} \|f\|_p$ and putting this into the summation we see that the geometric series converges if $\left|\frac{1}{p} - \frac{1}{2}\right| < \frac{\lambda}{n-1}$.

There is another important known result, which uses restriction theory.

3 Using restriction estimates

There is an intimate connection\(^4\) between estimates for Bochner-Riesz operators and estimates on the size of the Fourier transform of a function when

\(^4\)See [8] for more on this.
restricted to a hypersurface (generally the sphere), i.e. “restriction theo-
rems”. This is illustrated in the proof of Theorem 7 below, which makes
use of the following.

**Theorem 6** (Tomas-Stein). For all \( 1 \leq p \leq \frac{2n+2}{n+3} \) we have
\[
\left( \int_{S^{n-1}} |\hat{f}(\xi)|^q d\sigma(\xi) \right)^{1/q} \lesssim \|f\|_{L^p(\mathbb{R}^n)}
\]
where \( q = \left( \frac{n-1}{n+1} \right) p' \), \( \frac{1}{p} + \frac{1}{p'} = 1 \).

**Proof.** See [6, p386] for the full proof. The following is a sketch proof of the
“restricted” result (i.e. with \( f = \chi_E \) for some set \( E \)) in the endpoint case
\( p = \frac{2n+2}{n+3} \), where \( q = 2 \) — it is this endpoint case which we will later make
use of.

We write the \( L^2 \) norm as
\[
\int \hat{f}^2 \, d\sigma = \int \mathcal{F}(\sigma^* f) = \int \mathcal{F}(\sigma_1^* f) + \int \mathcal{F}(\sigma_2^* f)
\]
where we have split \( \sigma(\xi) = \sigma(\xi)\phi(\frac{\xi}{\lambda}) + \sigma(\xi)(1 - \phi(\frac{\xi}{\lambda})) \) with \( \phi \) a standard
bump. Thus \( \sigma_1^* \) is supported on \( |\xi| \lesssim \lambda \) and \( \sigma_2^* \) on \( |\xi| \gtrsim \lambda \). The appropriate
choice of \( \lambda \) will be made later.

Using the fact that \( |\sigma(\xi)| \lesssim |\xi|^{-(n-1)/2} \), we have
\[
|\sigma_2(\xi)| \lesssim \lambda^{-\frac{n-1}{n+1}}.
\]

On the other hand, \( \sigma_1 = \sigma^* \tilde{\phi}_{1/\lambda} \), so \( \sigma_1 \) is \( \sigma \) spread out on scale \( \frac{1}{\lambda} \) maintain-
ing mass 1. So \( \|\sigma_1\|_{\infty} \sim \lambda \).

So applying Hölder and Plancherel to the first term, and Hölder and Young
on the second, we get
\[
\int \hat{f}^2 \, d\sigma = \int \mathcal{F}(\sigma_1^* f) + \int \mathcal{F}(\sigma_2^* f)
\leq \|f\|_2 \|\sigma_1\|_\infty \|f\|_2 + \|f\|_1 \|\sigma_2\|_\infty \|f\|_1
\leq C \|f\|_2^2 \lambda + \|f\|_1^2 \lambda^{-\frac{n+1}{n+2}}.
\]

Taking \( f = \chi_E \) we have
\[
\int \hat{f}^2 \, d\sigma \lesssim |E|\lambda + |E|^2 \lambda^{-\frac{n-1}{n+2}}
\]
and choosing \( \lambda \) so that both terms are the same (i.e. taking \( \lambda = |E|^{2/(n+1)} \))
we get
\[
\int \hat{f}^2 \, d\sigma \lesssim |E|^{\frac{n+3}{n+2}} = \|f\|_{L^2(S^{n-1})}^2
\]
i.e. that the restriction estimate holds from \( p = \frac{2(n+1)}{n+3} \) to \( L^2(S^{n-1}) \).  #
Theorem 7. If \( \lambda > \frac{n-1}{2(n+1)} \) then \( T_\lambda \) is bounded on \( L^p \) when \( \left| \frac{1}{p} - \frac{1}{2} \right| < \frac{2\lambda+1}{2n} \).

Proof. We again decompose \( T_\lambda \) on annuli, putting

\[
K_\lambda = \sum_{k=0}^{\infty} K_k \quad \text{where} \quad K_k(x) = K_\lambda(x) \eta(2^{-k}x) \quad (k \geq 1),
\]

with \( \eta \in C^\infty \) radial and supported on \( \frac{1}{4} \leq |x| \leq 1 \), so that each \( K_k \) is supported on \( |x| \in [2^k-2, 2^k] \).

We want \( \|T_\lambda f\|_p \lesssim \|f\|_p \) for suitable \( p, \lambda \). If we can show

\[
\|f \ast K_k\|_{p_0} \leq A 2^{-\epsilon k} \|f\|_{p_0} \quad (\exists \epsilon > 0) \quad (3.1)
\]

for some \( p_0 \leq 2 \) then with the triangle inequality we get

\[
\|T_\lambda f\|_{p_0} = \left\| \sum_{k=0}^{\infty} f \ast K_k \right\|_{p_0} \leq \sum_{k=0}^{\infty} \|f \ast K_k\|_{p_0} \overset{(3.1)}{\leq} \sum_{k=0}^{\infty} A 2^{-\epsilon k} \|f\|_{p_0} \lesssim \|f\|_{p_0}
\]

because the geometric series is summable. Then by interpolation with the obvious \( L^2 \) estimate, we have that \( T_\lambda \) is bounded on \( L^p \) for \( p_0 \leq p \leq 2 \).

Now the multiplier corresponding to \( K_k \) is

\[
m_k(\xi) = \overline{K_k}(\xi) = (\overline{K_\lambda} \ast \eta(2^{-k} \cdot)) (\xi)
\]

and we can show\(^5\), using the compact support and smoothness of \((1 - |\xi|^2)^\frac{1}{2}\) away from \( |\xi| = 1 \), that for \( 1 - |\xi| > \frac{1}{2} \),

\[
|m_k(\xi)| \leq A N 2^{-Nk}(1 + |\xi|)^{-N} \quad \forall N \geq 0.
\]

\(^5\)We exploit the fact that \( \int \xi^a \eta = (D^a \eta)(0) = 0 \ \forall a \) to introduce extra terms in the integrand. For large \( |\xi| \) we replace \( m_\lambda(\xi - y) \) with \( (m_\lambda(\xi - y) - m_\lambda(\xi)) \) and use the mean value theorem, while for small \( |\xi| \) we subtract the degree \( N \) Taylor approximation of \( m_\lambda \). Then use the fact that \( \eta \in S \).

7
Now since $K_k$ is supported in the ball of radius $2^k$, we only need to consider $f$ with this support\footnote{This is a general principle; see Lemma 1.6 in Lecture 3 of [7]. It is basically because the convolution with $K_k$ will kill any contribution from outside the ball anyway.} in (3.1). But for such $f$ we can perform a neat trick with Hölder’s inequality (since $p_0 \leq 2$):

$$\|f * K_k\|_{p_0} = \left\| \chi_{B(0,2^k)} |f * K_k| \right\|_{1/p_0}^{1/p_0} \leq \left\| \chi_{B(0,2^k)} \right\|_{1/2}^{1/2} \|f * K_k\|_{1/2}^{1/2} = C 2^{nk} \left( \frac{1}{p_0} - \frac{1}{2} \right) \|f * K_k\|_2$$

and now we can use Plancherel to write

$$\|f * K_k\|_2^2 = \int_{|\xi| \leq 2} |\hat{f}(\xi)|^2 |m_k(\xi)|^2 d\xi + \int_{|\xi| > 2} |\hat{f}(\xi)|^2 |m_k(\xi)|^2 d\xi$$

We use the estimate on $|m_k(\xi)|$ as well as the fact that (by Hölder),

$$|\hat{f}(\xi)| \leq \int |f(x)| dx \leq \left\| \chi_{B(0,2^k)} \right\|_{1/\lambda} \|f\|_{p_0} = 2C_{n,p_0} \|f\|_{p_0}$$

to get the second term

$$\lesssim \|f\|_2^2 \frac{2^{nk}}{2^{2nk}} \int_{|\xi| > \frac{1}{2}} \frac{1}{(1 + |\xi|)^{2N}} d\xi \lesssim 2^{-k(1+2\lambda)} \|f\|_{p_0}^2$$

by taking $N$ large enough.

For the first term, we use polar coordinates and then apply the $L^{p_0} - L^2$ restriction theorem:

$$\int_{1/2}^{3/2} \int_{S^{n-1}} |\hat{f}(ru)|^2 |m_k(r)|^2 r^{n-1} dr d\xi \lesssim \left( \sup_{1/2 \leq r \leq 3/2} \int_{S^{n-1}} |\hat{f}(ru)|^2 dr \right) \int_{1/2}^{3/2} |m_k(r)|^2 r^{n-1} dr \lesssim \left( \|f\|_{p_0}^2 \right) \|K_k\|_2^2 \lesssim 2^{-k(1+2\lambda)} \|f\|_{p_0}^2$$

where $p_0 = \frac{2n+2}{n+3}$ here (see [6, IX§2.1]). Putting this together, we have (3.1) when with $\epsilon = \lambda + \frac{1}{2} - n \left( \frac{1}{p_0} - \frac{1}{2} \right)$, and since we require $\epsilon > 0$ to ensure convergence of $T_\lambda$, this means we need

$$\frac{1}{p_0} - \frac{1}{2} < \frac{2\lambda + 1}{2n}$$
in order to get boundedness of $T_\lambda$ on $L^p$ with $p_0 \leq p \leq 2$. Since we must have $p_0 = \frac{2n+2}{n+3}$ from the use of the restriction theorem, this implies

$$\lambda > \frac{n - 1}{2(n + 1)}.$$ 

So for $\lambda$ in this range, we have boundedness of $T_\lambda$ on $L^p$ if

$$\frac{2n}{2\lambda + n + 1} < p_0 \leq p \leq 2$$

and by duality this means for $p$ such that $\left| \frac{1}{p} - \frac{1}{2} \right| < \frac{2\lambda + 1}{2n}$. □

4 More recent progress

The main reference here is Tao’s set of notes, [8, Lecture 5].

Lee has shown that bilinear restriction estimates can imply Bochner-Riesz. Combining this with bilinear restriction estimates due to Tao, we have that the Bochner-Riesz conjecture is true for

$$p \geq \frac{2(n + 2)}{n} \quad \text{and} \quad p \leq \frac{2(n + 2)}{n + 4},$$

so the range $\lambda > \frac{n - 1}{2(n + 1)}$ of the previous theorem has been improved to $\lambda > \frac{n - 2}{2(n + 2)}$. 

9
References


