# Reading Summary

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## KAKEYA MAXIMAL CONJECTURE
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This is [1], mainly §2.

## 1 Notation

$\mathbb{F}$ is a finite field, $|\mathbb{F}|$ is its size.

We shall consider $f : \mathbb{F}^n \to \mathbb{R}$, with the norm

$$\|f\|_n = \|f\|_{\ell^n(\mathbb{F}^n)} = \left( \sum_{v \in \mathbb{F}^n} |f(v)|^n \right)^{1/n}.$$

**Definition 1.** *The **maximal function** $f^* : \mathbb{F}^{n-1} \to \mathbb{R}$ is given by*

$$f^*(w) = \sup_{\gamma \ni w} \sum_{v \in \gamma(\mathbb{F}) \setminus \mathbb{F}^{n-1}} |f(v)| \tag{1.1}$$

*where the supremum is over all lines $\gamma$ in $\mathbb{F}^n$ which pass through $w$. [In the paper, lines are replaced with algebraic curves of degree at most $d$].*

## 2 Result

**Theorem 2** (Kakeya maximal conjecture)**.**

$$\|f^*\|_{\ell^n(\mathbb{F}^{n-1})} \lesssim |\mathbb{F}|^{\frac{n-1}{n}} \|f\|_{\ell^n(\mathbb{F}^n)}.$$

*Remark.* We will tend to write this more succinctly as $\|f^*\|_n \lesssim |\mathbb{F}|^{\frac{n-1}{n}} \|f\|_n$.

### 2.1 Why is this the "correct" conjecture?

Specifically, where does the $\frac{n-1}{n}$ come from? Well, we want an identity like

$$\|f^*\|_n \lesssim |\mathbb{F}|^a \|f\|_n$$

for some $a$. We can easily see that $\frac{n-1}{n}$ would be optimal by considering the example

$$f(x) = \begin{cases} 1 & \text{if } x = x_0 \notin \mathbb{F}^{n-1} \\ 0 & \text{otherwise,} \end{cases}$$

for which $\|f\|_n = 1$. We have $f^*(w) = 1$ for all $w$, since the supremum is achieved by taking $\gamma$ to be the line through $w$ and $x_0$. Hence $\|f^*\|_n = (\mathbb{F}^{n-1})^{1/n}$, showing that we need $a \geq \frac{n-1}{n}$ for our conjectured inequality to have any chance of being true.

## 3 Proof

We prove the maximal conjecture using the following:

**Proposition 3** (Distributional estimate (Prop 2.3)). *There exists a constant* $K = K_n$ *s.t. if*

*(i)* $A > 0$

*(ii)* $f : \mathbb{F}^n \to \{0\} \cup [A, \infty)$

*(iii)* $K \|f\|_n \leq \lambda \leq A|\mathbb{F}|$

*then*

$$|\{w \in \mathbb{F}^{n-1} : f^*(w) \geq \lambda\}| \lesssim \frac{|\mathbb{F}|^{n-2}}{A\lambda^{n-1}} \|f\|_n^n.$$

We will assume this for now, and look at how it is used to prove the maximal conjecture.

I just want to give a flavour of this bit, as it's quite technical.

*Proof of Theorem 2.* We have $f : \mathbb{F}^n \to \mathbb{R}$, but we only need to consider $f$ non-negative, and not identically zero. We also normalize so $\|f\|_n = 1$. The desired result is then

$$\|f^*\|_n \lesssim |\mathbb{F}|^{(n-1)/n}.$$

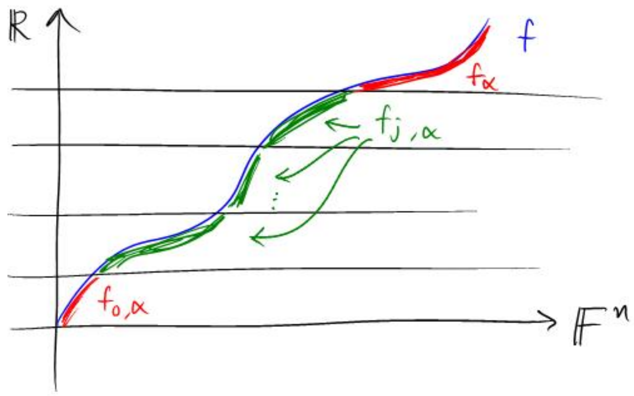
Using FTC[1] then Fubini, we get the (familiar?) identity

$$\|f^*\|_n^n = n \underbrace{\int_0^\infty |\{w \in \mathbb{F}^{n-1} : f^*(w) \geq \alpha\}|\alpha^{n-1}\, d\alpha}_{\text{so, want this } \lesssim |\mathbb{F}|^{n-1}.}$$

We split the integral at $C_0$, some large constant to be fixed later. The $\int_0^{C_0}$ part is easily dealt with, since the $|\{w \cdots\}| \leq |\mathbb{F}|^{n-1}$. So we are reduced to showing

$$\int_{C_0}^\infty |\{w \in \mathbb{F}^{n-1} : f^*(w) \geq \alpha\}|\alpha^{n-1}\, d\alpha \lesssim |\mathbb{F}|^{n-1}.$$

Now for each fixed $\alpha > C_0$ we split $f$ up into pieces of various sizes:




This actually uses Proposition 3; specifically the fixed value of $K$. The "top" piece $f_\alpha$ is defined in terms of $j_\alpha$, the largest integer s.t.

$$\frac{\alpha}{2^{j_\alpha+1}} \geq K, \quad \text{so } K \approx \frac{\alpha}{2^{j_\alpha}}.$$

So we write

$$f = f_{0,\alpha} + \sum_{j=1}^{j_\alpha - 1} f_{j,\alpha} + f_\alpha$$

$$f^*(w) \overset{\Delta\text{-ineq}}{\leq} f^*_{0,\alpha}(w) + \sum_{j=1}^{j_\alpha-1} f^*_{j,\alpha}(w) + f^*_\alpha(w)$$

$$\leq \sum_{j=1}^{j_\alpha-1} f^*_{j,\alpha}(w) + f^*_\alpha(w) + \frac{\alpha}{2}$$

since $f^*_{0,\alpha} \leq \alpha/\sqrt{C_0}$, and just take $C_0$ large enough.

[1] $\|f^*\|_n^n = \sum_{v \in \mathbb{F}^{n-1}} \int_0^{f(v)} n\alpha^{n-1}\, d\alpha$

So to have $f^*(w) \geq \alpha$ we must have either

$$f^*_{1,\alpha}(w) \geq \frac{\alpha}{4} \quad \text{or} \quad \sum_{j=2}^{j_\alpha - 1} f^*_{j,\alpha}(w) + f^*_\alpha(w) \geq \frac{\alpha}{4}.$$

"the first term $\geq \frac{\alpha}{4}$, or all the other terms $\geq \frac{\alpha}{4}$"

Proceeding in this way, at least one of the following is true:

$$f^*_{j,\alpha}(w) \geq \frac{\alpha}{2^{j+1}}, \quad 1 \leq j \leq j_\alpha - 1, \quad \text{or} \quad f^*_\alpha(w) \geq \frac{\alpha}{2^{j}_{\alpha}} \geq K$$

Hence

$$\begin{aligned} &|\{w \in \mathbb{F}^{n-1} : f^*(w) \geq \alpha\}| \\ &\leq |\{w \in \mathbb{F}^{n-1} : f^*_\alpha(w) \geq K\}| + \sum_{j=1}^{j_\alpha - 1} |\{w \in \mathbb{F}^{n-1} : f^*_{j,\alpha}(w) \geq \frac{\alpha}{2^{j+1}}\}| \end{aligned}$$

The idea is then to apply Proposition 3 to each of these terms.

To illustrate: for $f_\alpha$ we use $\lambda = K$ and $A = 100^{n(j_\alpha - 1)} \frac{\alpha}{\sqrt{C_0 |\mathbb{F}|}}$. This yields

$$\begin{aligned} |\{w \in \mathbb{F}^{n-1} : f^*_\alpha(w) \geq K\}| &\lesssim \frac{|\mathbb{F}|^{n-2}}{AK^{n-1}} \\ &\vdots \\ &\lesssim_{\sqrt{C_0}} \frac{|\mathbb{F}|^{n-1}}{\alpha^{2n}} \end{aligned}$$

The $f_{j,\alpha}$ are treated similarly, it's just a little bit more technical. □

To prove the estimate in Proposition 3, we make simplifications:

1. take $A = 1$ (by dividing $f$ and $\lambda$ by $A$),
2. let $\{w \in \mathbb{F}^{n-1} : f^*(w) \geq \lambda\} = \{w_1, \ldots, w_J\}$,
3. let $\gamma_j$ be the line attaining the supremum in the definition of $f^*(w_j)$.

So we now want to prove

**Proposition 4** (Distributional estimate, simplified (Prop 2.4))**.** *Let $w_1, \ldots, w_J \in \mathbb{F}^{n-1}$ be distinct, with $\gamma_j$ ($1 \leq j \leq J$) lines through $w_j$ not in $\mathbb{F}^{n-1}$. Then there is a constant $K = K_n$ s.t. if*

*(i)* $f : \mathbb{F}^n \to \{0\} \cup [1, \infty)$

*(ii)* $K \|f\|_n \leq \lambda \leq |\mathbb{F}|$

*(iii)* $\displaystyle\sum_{v \in \gamma_j(\mathbb{F}) \setminus \mathbb{F}^{n-1}} f(v) \geq \lambda, \forall 1 \leq j \leq J$

*then*

$$J \lesssim \frac{|\mathbb{F}|^{n-2}}{\lambda^{n-1}} \|f\|_n^n.$$

It actually suffices to consider just a special case of this estimate.

**Proposition 5** (Distributional estimate, special case)**.** *Let $w_1, \ldots, w_J \in \mathbb{F}^{n-1}$ be distinct, with $\gamma_j$ ($1 \leq j \leq J$) lines through $w_j$ not in $\mathbb{F}^{n-1}$. Then there is a constant $K_0$ depending on $n$ s.t. if*

*(i)* $f : \mathbb{F}^n \to \{0\} \cup [1, \infty)$

*(ii)* $K_0 \|f\|_n \leq |\mathbb{F}|$

*(iii)* $\displaystyle\sum_{v \in \gamma_j(\mathbb{F}) \setminus \mathbb{F}^{n-1}} f(v) \geq K_0 \|f\|_n, \forall 1 \leq j \leq J$

*then*

$$J \lesssim \frac{|\mathbb{F}|^{n-2}}{(K_0 \|f\|_n)^{n-1}} \|f\|_n^n \quad \text{i.e.} \quad J \lesssim_{K_0} |\mathbb{F}|^{n-2} \|f\|_n^n.$$

**Proposition 6** (Reduction (Prop 2.5))**.** *It suffices to prove the special case.*

*Proof.* The idea is to take $f$ satisfying the hypotheses of the full result, and produce a related function $f_M$ to which we can apply the special case; this then allows us to deduce the conclusion for $f$.

We define $M \geq 1$ for a particular choice of $f$ and $\lambda$ satisfying the hypotheses in the full result. The detail of $M$ is not important here.

The definition of $f_M$ comes from the probabilistic method. We select $M$ points $u_1, \ldots, u_M \in \mathbb{F}^{n-1}$ independently and uniformly at random, and set

$$\Omega = \{w_j + u_m : 1 \leq j \leq J, 1 \leq m \leq M\}.$$

For each $w \in \mathbb{F}^{n-1}$, $\mathbb{P}(w \in \Omega) = 1 - \left(1 - \frac{J}{|\mathbb{F}|^{n-1}}\right)^M \approx \min\left(\frac{MJ}{|\mathbb{F}|^{n-1}}, 1\right)$, so

$$\mathbb{E}|\Omega| \approx \min(MJ, |\mathbb{F}|^{n-1}).$$

Thus for a particular choice of $u_1, \ldots, u_M$ we have

$$|\Omega| \gtrsim \min(MJ, |\mathbb{F}|^{n-1})$$

and set

$$f_M(v) = \left( \sum_{m=1}^{M} f(v - u_m)^n \right)^{1/n}.$$

We have, by changing the order of summation,

$$\|f_M\|_n^n = \sum_{m=1}^{M} \|f(\cdot - u_m)\|_n^n = M \|f\|_n^n$$

and we can check that $f_M$ in fact satisfies the requirements (*i*)–(*iii*) of the special case. So applying the result to $f_M$ (with the set of points $\Omega$), we have

$$|\Omega| \lesssim \frac{|\mathbb{F}|^{n-2}}{(K_0 \|f_M\|_n)^{n-1}} \|f_M\|_n^n$$

but the denominator can be replaced by $\lambda^{n-1}$ since the definition of $M$ gives us $\lambda \lesssim K_0 \|f_M\|_n$.

Combining this with the lower bound on $|\Omega|$ and $\|f_M\|_n^n = M \|f\|_n^n$ we get

$$\min(MJ, |\mathbb{F}|^{n-1}) \lesssim \frac{|\mathbb{F}|^{n-2}}{\lambda^{n-1}} M \|f\|_n^n.$$

If $MJ$ is smallest, then we're done. Otherwise, we can force a contradiction by taking $K_0$ large enough. □

We now come to the use of Dvir's polynomial method, or at least a variant of it, to prove this special case of the distributional estimate.

*Proof of Proposition 5.* We simplify even further by rounding $f$ down to the nearest integer[2], and then replacing it with $\min(f, |\mathbb{F}|)$.

So we want to show

$$J \lesssim |\mathbb{F}|^{n-2} \|f\|_n$$

for $f$ taking values in $\{0, 1, \ldots, |\mathbb{F}|\}$.

- There exists a nonzero poly $p \in V_f = \{q \text{ poly on } \mathbb{F}^n : \deg(q) \leq D, \text{mult}(q, v) \geq f(v)\}$, i.e.

  $p$ is a polynomial on $\mathbb{F}^n$ of degree $\leq D$ (to be set later) which vanishes to order at least $f(v)$ at $v$.

[2] valid since $\frac{1}{2} f \leq \lfloor f \rfloor \leq f$ gives $\|f\|_n \approx \|\lfloor f \rfloor\|_n$

*Pf.* Note that $\dim_{\mathbb{F}}(\mathcal{P}_D) = \binom{D+n}{n} \approx D^n$. The multiplicity condition imposes $\binom{n+f(v)-1}{n}$ constraints on the coefficients of $p$, at each $v$. So

$$\dim_{\mathbb{F}} \mathcal{P}_D - \dim_{\mathbb{F}} V_f \leq \text{num. constraints} \lesssim \sum_{v \in \mathbb{F}^n} f(v)^n = \|f\|_n^n .$$

Taking $D = k \|f\|_n$ for large enough $k$ ensures $\dim_{\mathbb{F}} V_f > 0$, so there is a nonzero $p \in V_f$. □

- With $\mathbb{F}^{n-1} = \{x \in \mathbb{F}^n : x_n = 0\}$, we factor $p$ as

$$p = x_n^j q$$

taking $j \geq 0$ as large as possible, so that the polynomial $q$ has no $x_n$ factor. This $q$ is a poly of degree $\leq D$ and $\text{mult}(q, v) \geq f(v)$ for $v \in \mathbb{F}^n \setminus \mathbb{F}^{n-1}$.

- For each line $\gamma_j$, we have $q|_{\gamma_j} = 0$.

*Pf.* Otherwise, $\{v \in \gamma_j : q(v) = 0\}$ has dimension 0.

But by Bezout's Theorem, this set has degree $O(1.D) = O(\|f\|_n)$. More precisely, counting multiplicity on the LHS, we have

$$|\{v \in \gamma_j \;\; q(v) = 0\}| \lesssim \|f\|_n$$

but the LHS is larger than $\sum_{v \in \gamma_j \setminus \mathbb{F}^{n-1}} \text{mult}(q, v)$, and by construction of $q$ and by the hypothesis (*iii*),

$$K_0 \|f\|_n \leq \sum_{v \in \gamma_j \setminus \mathbb{F}^{n-1}} f(v) \leq \sum_{v \in \gamma_j \setminus \mathbb{F}^{n-1}} \text{mult}(q, v) \lesssim \|f\|_n$$

giving a contradiction if $K_0$ is chosen large enough. □

- So $q(w_j) = 0$ for each $j$, giving

$$J \leq |\{w \in \mathbb{F}^{n-1} : q(w) = 0\}|.$$

- On the other hand, the restriction of $q$ to $\mathbb{F}^{n-1}$ is nontrivial, and has degree $\leq D$. So by Schwartz-Zippel,

$$J \leq |\{w \in \mathbb{F}^{n-1} : q(w) = 0\}| \lesssim D|\mathbb{F}|^{n-2}$$

and since $D \lesssim \|f\|_n$ we have the desired estimate

$$J \lesssim |\mathbb{F}|^{n-2} \|f\|_n .$$

□

## References

[1] J. Ellenberg, R. Oberlin, and T. Tao. The Kakeya set and maximal conjectures for algebraic varieties over finite fields, 2009. http://arxiv.org/abs/0903.1879.