Littlewood-Paley Theory and Multipliers

George Kinnear

September 11, 2009

Definitions

How does an operator T behave with respect to function spaces, e.g. L^p?

Definition

- We say that T is **bounded** from L^p to L^p if

$$\|Tf\|_p \leqslant C \|f\|_p \,.$$

  We may also say T satisfies a **strong** (p, p) inequality.
- A weaker condition is the **weak** (p, p) inequality,

$$|\{x : |Tf(x)| > \alpha\}| \leqslant C \left(\frac{\|f\|_p}{\alpha} \right)^p \,.$$

Indeed, if T is strong (p, p), then it is weak (p, p).

Interpolation

We can often deduce that T is bounded for intermediate values of p just by considering end points.

e.g. Marcinkiewicz interpolation:

Theorem

weak (p_0, p_0) *and* $(p_1, p_1) \implies$ *strong* (p, p), $p_0 < p < p_1$

Maximal functions

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\, dy$$

- M is weak $(1,1)$.
- M is weak (∞,∞).

Theorem

M is strong (p,p), $1 < p < \infty$

Singular integrals

$$Tf(x) = K * f(x) = \int_{\mathbb{R}^n} K(x-y) f(y)\, dy$$

where K is locally integrable away from the origin, and

(i) $|\hat{K}(\xi)| \leqslant B$,

(ii) $\int_{|x|>2|y|} |K(x-y) - K(x)| dx \leqslant B, y \in \mathbb{R}^n$.
(or $|\nabla K(x)| \leqslant C|x|^{-n-1}$)

- T is strong $(2, 2)$, from (i).
- T is weak $(1, 1)$ – use (ii) and the Calderón-Zygmund decomposition.
- Thus T is strong (p, p) for $1 < p < 2$.
- By duality, also for $2 < p < \infty$.

Theorem

T is strong (p, p), $1 < p < \infty$

Littlewood-Paley Theorem

Take $\psi \in \mathcal{S}(\mathbb{R}^n)$ with $\psi(0) = 0$ and define S_j by

$$\widehat{(S_j f)}(\xi) = \psi_j(\xi)\hat{f}(\xi) \quad \text{where} \quad \psi_j(\xi) = \psi(2^{-j}\xi).$$

Theorem

For $1 < p < \infty$,

(a) $\left\| \left(\sum_j |S_j f|^2 \right)^{1/2} \right\|_p \leqslant C \|f\|_p$.

(b) *If for $\xi \neq 0$ we have $\sum_j |\psi(2^{-j}\xi)|^2 = C$, then also*

$$\|f\|_p \leqslant C \left\| \left(\sum_j |S_j f|^2 \right)^{1/2} \right\|_p .$$

Littlewood-Paley Theorem – Proof

Consider the operator $f \mapsto \{S_j f\}$.

- It is bounded from L^2 to $L^2(\ell^2)$:

$$\left\| \left(\sum_j |S_j f|^2 \right)^{1/2} \right\|_2^2 = \int_{\mathbb{R}^n} \sum_j |\psi_j(\xi)|^2 |\hat{f}(\xi)|^2 \, d\xi \leqslant C \, \|f\|_2^2 \, .$$

- Other L^p follow from the Hörmander condition, which is satisfied since

$$\left\| \nabla \check{\psi}_j(x) \right\|_{\ell^2} \leqslant C|x|^{-n-1}.$$

So we have part (a).

Littlewood-Paley Theorem – Proof

Now if $\sum_j |\psi(2^{-j}\xi)|^2 = C$ we actually have

$$\left\| \left(\sum_j |S_j f|^2 \right)^{1/2} \right\|_2 = \sqrt{C} \, \|f\|_2 \, .$$

So by polarization,

$$\sqrt{C} \int_{\mathbb{R}^n} f\overline{g} = \int_{\mathbb{R}^n} \sum_j S_j f \overline{S_j g}.$$

Hence

$$\begin{aligned} \left| \int f\overline{g} \right| &\leqslant C' \int \left(\sum_j |S_j f|^2 \right)^{1/2} \left(\sum_j |S_j g|^2 \right)^{1/2} \\ &\leqslant C' \left\| \left(\sum_j |S_j f|^2 \right)^{1/2} \right\|_p \left\| \left(\sum_j |S_j g|^2 \right)^{1/2} \right\|_{p'} \\ &\leqslant C' \left\| \left(\sum_j |S_j f|^2 \right)^{1/2} \right\|_p \|g\|_{p'} \, . \end{aligned}$$

Multipliers

Given $m \in L^\infty(\mathbb{R}^n)$ we can define an operator T_m by

$$\widehat{T_m f}(x) = m(x)\hat{f}(x).$$

We say m is a **multiplier for** L^p if T_m is bounded on L^p.

The class of multiplers for L^p is $\mathcal{M}_p$.

Example

$\mathcal{M}_2 = L^\infty$.

Theorem

If $\frac{1}{p} + \frac{1}{p'} = 1$, $1 \leqslant p \leqslant \infty$, *then* $\mathcal{M}_p = \mathcal{M}_{p'}$.

Sobolev spaces

For positive integers k,

$$L_k^p = \{f \in L^p : D^\alpha f \in L^p, |\alpha| \leqslant k\}$$

with norm $\|f\|_{L_k^p} = \sum_{|\alpha| \leqslant k} \|D^\alpha f\|_p$.
There is an alternative definition of L_a^p for general $a > 0$.
When $p = 2$, this is

$$L_a^2 = \{g \in L^2 : (1 + |\xi|^2)^{a/2} \hat{g}(\xi) \in L^2\},$$

with norm $\|g\|_{L_a^2} = \left\|(1 + |\cdot|^2)^{a/2} \hat{g}\right\|_2$.

Theorem

If $m \in L_a^2$ with $a > \frac{n}{2}$ then $m \in \mathcal{M}_p$ for $1 \leqslant p \leqslant \infty$.

Hörmander multiplier theorem

Let $\psi \in C^\infty$ be radial, supported on $\frac{1}{2} \leqslant |x| \leqslant 2$, and s.t.

$$\sum_{j=-\infty}^{\infty} |\psi(2^{-j}x)|^2 = 1, \quad x \neq 0.$$

Theorem

If m is such that, for some $k > \frac{n}{2}$,

$$\sup_j \left\| m(2^j \cdot)\psi \right\|_{L^2_k} < \infty$$

then $m \in \mathcal{M}_p$ for all $1 < p < \infty$.

Hörmander multiplier theorem – Proof

Let $\tilde{\psi} \in C^{\infty}$, be supported on $\frac{1}{4} \leqslant |\xi| \leqslant 4$ and equal to 1 on $\operatorname{supp} \psi$. The operators $\tilde{S}_j$ with multipliers $\tilde{\psi}(2^{-j}\xi)$ satisfy

$$\left\| \left(\sum_j |\tilde{S}_j f|^2 \right)^{1/2} \right\|_p \leqslant C_p \, \|f\|_p \, .$$

Now if S_j has multiplier $\psi(2^{-j}\xi)$,

$$\|Tf\|_p \leqslant C \left\| \left(\sum_j |S_j T f|^2 \right)^{1/2} \right\|_p = C \left\| \left(\sum_j |S_j T \tilde{S}_j f|^2 \right)^{1/2} \right\|_p .$$

Hörmander multiplier theorem – Proof

$$\int_{\mathbb{R}^n} |S_j Tf|^2 u \leqslant \left\| m_j \right\|_{L^2_k}^2 C \int_{\mathbb{R}^n} |f(y)|^2 Mu(y)\, dy$$

For $p > 2$, Riesz representation gives some $u \in L^{(p/2)'}$ s.t.

$$\left\| \left(\sum_j |S_j Tf_j|^2 \right)^{1/2} \right\|_p^2 = \left\| \sum_j |S_j Tf_j|^2 \right\|_{p/2} = \int_{\mathbb{R}^n} \sum_j |S_j Tf_j|^2 u.$$

Putting these together,

$$\begin{aligned} \left\| \left(\sum_j |S_j Tf_j|^2 \right)^{1/2} \right\|_p^2 &\leqslant C \int_{\mathbb{R}^n} \sum_j |f_j|^2 Mu \\ &\leqslant C \left\| \sum_j |f_j|^2 \right\|_{p/2} \|Mu\|_{(p/2)'} \\ &\leqslant C' \left\| \sum_j |f_j|^2 \right\|_{p/2} \end{aligned}$$

Hörmander multiplier theorem – Proof

Combining this with the Littlewood-Paley estimates,

$$\begin{aligned}\|Tf\|_p &\leqslant C \left\| \left(\sum_j |S_j T \tilde{S}_j f|^2 \right)^{1/2} \right\|_p \\ &\leqslant C \left\| \left(\sum_j |\tilde{S}_j f|^2 \right)^{1/2} \right\|_p \\ &\leqslant C \|f\|_p \,.\end{aligned}$$

So $m \in \mathcal{M}_p$ for $p > 2$.
The result for $1 < p < 2$ follows by duality, and for $p = 2$ by interpolation.