JAMES COOK MATHEMATICAL NOTES

Volume 7 Issue number 69

August 1996

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The history of the James Cook Mathematical Notes (JCMN) is that the first issue (a single foolscap sheet) appeared in September 1975, then others at irregular intervals, to number 17 in November 1978. JCMN settled into the routine of three issues per year from 1979 to 1994; but from Issue 66 (April 1995) at the start of Volume 7, it has been irregular, appearing when enough contributions are available.

The issues up to number 31 (May, 1983) were produced and sent out free by the Mathematics Department of the James Cook University of North Queensland, of which I was then the Professor. The arrangement was beginning to be unsatisfactory, and in October 1983 I started producing the JCMN myself and asking readers to pay subscriptions. In October 1992 it had become clear that the paying of subscriptions by readers is an inefficient operation. Bank charges for changing currency and for international transfers, with postage, together absorb most of the initial input of money. Therefore we abandoned subscriptions as from issue number 60 (January, 1993). I now ask readers only to tell me every two years if they still want to have JCMN. To those who want to give something in return for the JCMN, I ask them to make a gift to an animal welfare society in their own country. The animals of the world will be grateful and so will I.

Contributors, please tell me if and how you would like your address printed.

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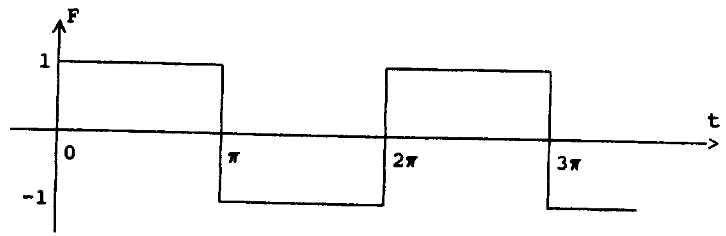
CONTENTS

IN THE FOOTSTEPS OF PAPPUS (JCMN 68, p.7052)

A. Brown

To answer the question in the previous issue about the area of the family of circles, it is necessary first to find the sums of some series.

Consider the odd function F(t) of period 2π with value 1 in $(0, \pi)$



Fourier theory gives $F(t) = \sum_{m=0}^{\infty} \frac{4}{(2m+1)\pi} \sin(2m+1)t$

Now take the Laplace transform term-by-term. The Laplace transform f(s) of F(t) is $f(s) = \frac{4}{\pi} \sum_{m=0}^{\infty} \frac{1}{s^2 + (2m+1)^2}$

But also we may find the Laplace transform directly,

$$f(s) = \int_0^{\pi} e^{-st}dt - \int_{\pi}^{2\pi} e^{-st}dt + \int_{2\pi}^{3\pi} e^{-st}dt - \ldots$$

$$= \frac{1}{s}\left(-1 + \frac{2}{1 + \exp(-s\pi)}\right) = \frac{1}{s} \tanh \frac{s\pi}{2}.$$

Equate the two expressions for f(s), and put s = 2u.

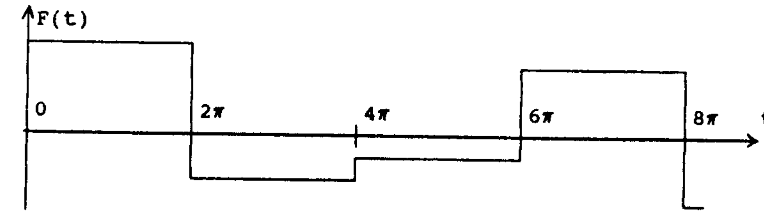
Thus we have the formula $\frac{\tanh \pi u}{\pi u} = 2\pi^{-2} \sum_{0}^{\infty} \frac{1}{u^2 + (m+\frac{1}{2})^2}.$

It is a little more complicated to find

$$S_1 = \sum_{n=-\infty}^{\infty} \frac{1}{s^2 + (n+b)^2} \qquad \text{(where } 0 < b < 1)$$

but the ideas are the same.

Let F(t) have the constant value $\sin(2n+1)\pi b$ in each of the intervals $(2n\pi, 2n\pi+2\pi)$



It is convenient to use a complex-valued function F*, taking the values $F^*(t) = \exp (2n+1)i\pi b$ in each interval so that F is the imaginary part of F*.

Let $G(t) = F^*(t) \exp{-ibt}$. Then G takes the value $\exp ib(2n\pi+\pi-t)$ in each interval, so that G is periodic,

Consequently $G(t) = \sum_{n=-\infty}^{\infty} c_n e^{int}$, where

$c_n = \frac{1}{2\pi} \int_0^{2\pi} G(t)\, e^{-int}dt = \frac{\sin \pi b}{\pi(b+n)}$, and therefore

$$F^*(t) = \frac{\sin \pi b}{\pi} \sum_{-\infty}^{\infty} \frac{\exp i(b+n)t}{b + n}$$

and $F(t) = \frac{\sin \pi b}{\pi} \sum_{-\infty}^{\infty} \frac{\sin(b+n)t}{b + n}$

Its Laplace transform is $f(s) = \frac{\sin \pi b}{\pi} \sum_{-\infty}^{\infty} \frac{1}{s^2 + (b+n)^2}$

But we may calculate the transform directly, the contribution from the typical interval is $\sin(2n+1)\pi b \int_{2n\pi}^{2n\pi+2\pi} e^{-st}\, dt$

$= \frac{2}{s} \sin(2n+1)\pi b \sinh \pi s\, e^{-(2n+1)\pi s}$, the sum ($n = 0$ to ∞) is

$f(s) = \frac{\sinh 2\pi s \sin \pi b}{s(\cosh 2\pi s - \cos 2\pi b)}$; equate this to the f(s) above.

$$S_1 = \sum_{n=-\infty}^{\infty} \frac{1}{s^2 + (n + b)^2} = \frac{\pi \sinh 2\pi s}{s(\cosh 2\pi s - \cos 2\pi b)}.$$

This is a periodic function of b, and so it may be expressed as the sum of its Fourier series:

$$S_1 = \frac{\pi}{s} + \frac{2\pi}{s} \sum_{n=1}^{\infty} e^{-2\pi ns} \cos 2\pi nb$$

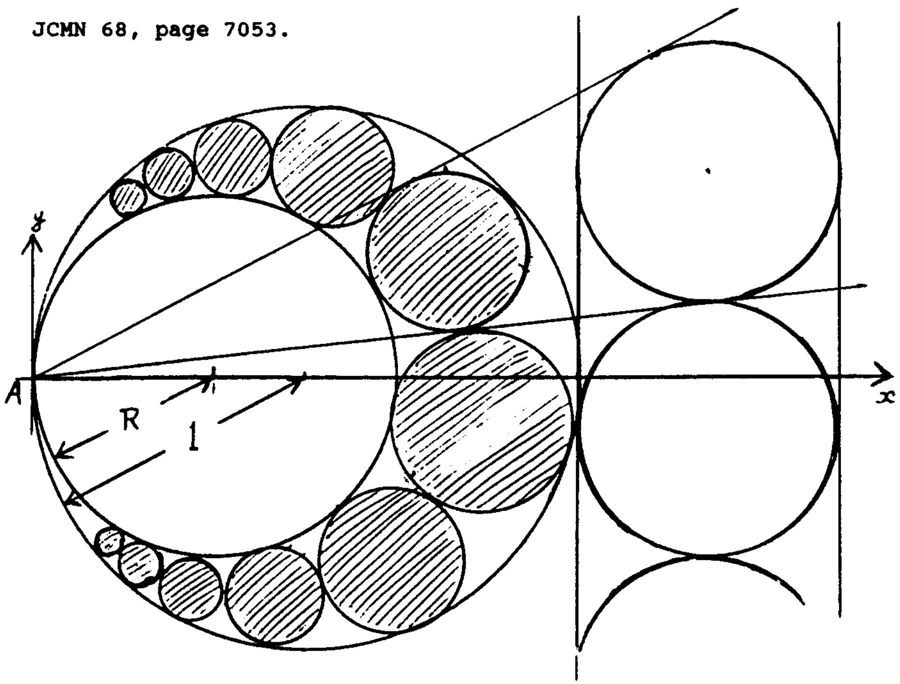
Differentiation with respect to the parameter s gives:-

$$\sum_{n=-\infty}^{\infty} (s^2 + (n+b)^2)^{-2} =$$

$$\pi \frac{\sinh 4\pi s - 4\pi s + 2 \cos 2\pi b\,(2\pi s \cosh 2\pi s - \sinh 2\pi s)}{4\,s^3\,(\cosh 2\pi s - \cos 2\pi b)^2}$$

$$= \frac{\pi}{2s^3} + \pi s^{-3} \sum_{n=1}^{\infty} (1 + 2\pi ns)\, e^{-2\pi ns} \cos(2\pi nb)$$

An interesting point to observe is that if s is at all large, this sum is almost independent of b. If $s > \surd 6$ then $\exp(-2\pi s) < 2{\cdot}07 \times 10^{-7}$. The reason for considering this value of $\surd 6$ for s will emerge soon. Recall the figure from JCMN 68, page 7053.



Circle number n has centre at (x_n, y_n) and diameter d_n.

$x_n = kd_n$ and $y_n = (n + b)d_n$ where

$k = \dfrac{1 + R}{2 - 2R}$ and $b = y_1/d_1 - 1$.

The lines $x = 2$ and $x = 2/R$ and the circles between them are obtained by inverting the figure from the origin A; they make more obvious the equations for x_n/d_n and y_n/d_n. From the equation $x_n^2 + y_n^2 - d_n^2/4 = 2x_n - d_n$, which says that the circle number n touches the outer circle, we may find

$$d_n = (2k-1)/(k^2 - \tfrac{1}{4} + (n+b)^2).$$

The sum of the areas of the circles is therefore

$$\pi R^2(1-R)^{-2} \sum_{n=-\infty}^{\infty} (s^2 + (n+b)^2)^{-2}$$

where $s = \surd R/(1-R)$. In the drawing above we made $R = \frac{2}{3}$, giving $s = \surd 6$, and leading to the comment about the area being nearly independent of b.

Other calculations, such as $\displaystyle\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{s^2 + (n+b)^2}$ may be left as problems for the interested reader.

CONGRATULATIONS

John Parker has been awarded the 1996 Gold Medal of the Royal Institute of Navigation.

TRIGONOMETRIC IDENTITY (JCMN 68 p.7066)

P. H. Diananda

(49 Jalan Raja Udang # 10-55, Singapore 329210)

The identity was $\Sigma \cos A \cos B \sin (A-B) + \Pi \sin (A-B) = 0$, for any three angles A, B and C, the symbols Σ and Π meaning sums and products over the three cyclic permutations of (A, B, C).

A possible proof is as follows. Let $e^{iA} = a$, $e^{iB} = b$, $e^{iC} = c$. Then L.H.S. =

$$\Sigma \frac{1}{2}\left(a + \frac{1}{a}\right) \frac{1}{2}\left(b + \frac{1}{b}\right) \frac{1}{2i}\left(\frac{a}{b} - \frac{b}{a}\right) + \Pi \frac{1}{2i}\left(\frac{a}{b} - \frac{b}{a}\right).$$

$$8ia^2b^2c^2 \times \text{L.H.S.} = \Sigma (a^2+1)(b^2+1)(a^2-b^2)c^2 - \Pi (a^2 - b^2)$$

$$= \Sigma a^2b^2c^2(a^2-b^2) + \Sigma (a^4-b^4)c^2 + \Sigma (a^2-b^2)c^2$$

$$- (a^2b^2c^2 - a^4b^2 - c^4a^2 + c^2a^4 - b^4c^2 + a^2b^4 + b^2c^4 - a^2b^2c^2)$$

$$= 0 + \Sigma (a^4 - b^4)c^2 + 0 - \Sigma (a^4 - b^4)c^2 = 0.$$

SUMS GIVEN BY ZETA FUNCTIONS

(JCMN 65 p.6360, 66 p.7010, 67 p.7030, 68 p.7058)

Chris Smyth, (University of Edinburgh)

It was conjectured in the previous issue that

$$HD(2k) = \sum_{n=1}^{\infty} n^{-2k} \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{2n}\right) =$$

$$\frac{2^{2k+1} + 2k + 1}{4} \zeta(2k+1) - \sum_{r=1}^{k-1} 2^{2r} \zeta(2k-2r)\zeta(2r+1).$$

This is confirmed by results in *Explicit Evaluation of Euler Sums* by David Borwein, Jonathan M. Borwein & Roland Girgensohn in *Proceedings of the Edinburgh Math Soc,* (1995) 38, 277-294.

COCKED HATS AGAIN

(JCMN 41 p.4218, 55 p.6033, 56 p.6076, 62 p.6024)

A theorem in the Admiralty Manual of Navigation tells us that, under very weak assumptions, the probability of the true position being in the "cocked hat" formed from three position lines is 1/4. It is tempting to ask how suitably stronger assumptions would lead to a stronger conclusion.

To describe the error distribution of a position line, let p be the perpendicular distance of the true position from the position line, reckoned positive or negative according to some rule. The original assumption of AMN was only that p could be positive or negative, each with probability 1/2. We strengthen this in two ways as follows:-

(a) The distribution is symmetrical about zero, i.e. for any $x > 0$, the probability of p being in the interval $(0, x)$ is equal to that of its being in the interval $(-x, 0)$. The distribution may be described by the infinitesimal $f(p)\,dp$.

(b) All three position lines have the same error distribution.

Then we can draw two conclusions:-

(c) Let the cocked hat be the triangle ABC, draw the internal bisectors of the angles, they meet at the incentre, they divide the inside of the triangle into six regions, as shown below (fig.1). Then the probability of the true position being in each of these regions is 1/24.

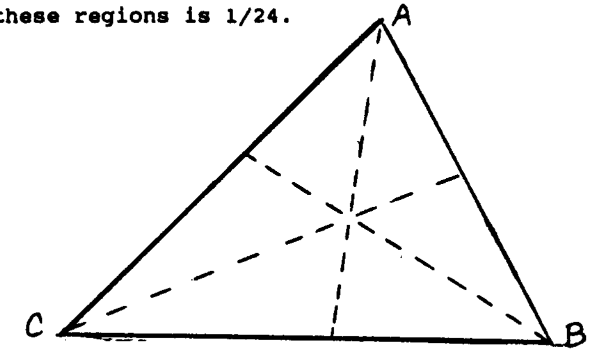


Figure 1

(d) In the cocked hat ABC suppose that angle A > angle B. Then there is probability 1/8 of the true position being in the triangular region shown below (fig. 2), bounded by the side AC, the side AB (produced) and the external bisector of the angle at C. (In the limiting case of angle A = angle B the region becomes infinite, but the conclusion still holds.)

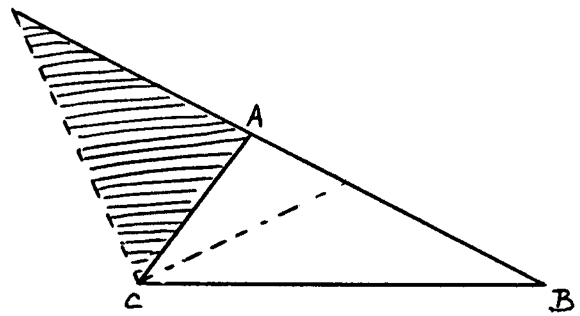


Figure 2

Proof of (c) We have to consider the following random process, which is modelled on what happens when a navigator sets out to estimate a position from three position lines.

Take three directions in the plane, the directions of the position lines. Consider a triangle ABC as in Fig. 3 below, with the sides in these directions; note that our cocked hat has sides parallel to those of fig. 3, but it also may be like fig. 4 (obtained by 180° rotation).

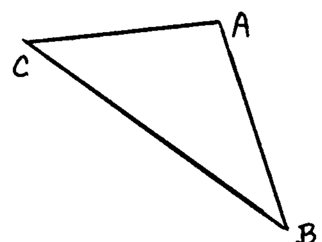


Figure 3

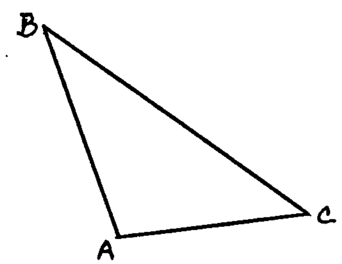


Figure 4

For each position line we distinguish a positive and a negative side by taking the inside of the triangle ABC in fig. 3 as being on the positive side of all three lines, then in fig. 4 the inside will be on the negative side of all three lines.

Next, from the error distribution for the position lines, choose one error (positive or negative) for each position line (this is taking three values of a random variable, in the usual sense of probability theory). Denote the values by x for the line BC, y for the line AC and z for the line AB. These choices enable us to draw a triangle, the "cocked hat", by starting with an origin, the "true position", and drawing each position line to be in the assigned direction at the assigned perpendicular distance. If x, y and z are all positive the cocked hat will be like the triangle ABC of fig. 3 and the true position will be inside; if they are all negative the cocked hat will be like fig. 4, with the true position inside.

The proof of the theorem from the Admiralty Manual of Navigation is now clear, the probability of the true position being inside the cocked hat is the probability of x, y and z all having the same sign, therefore equal to 1/4.

Now, we seek a more detailed result. Consider the data of just two position lines, the lines AC and BC. Take them to be at an angle of 2α, see Fig 5 below. With origin at the true position, the intersection of the two position lines is as shown, at perpendicular distances x and y from the two lines. Its probability density is $f(x)\,f(y)\,dx\,dy$, or

$$\sin 2\alpha\; f(X \sin\alpha - Y\cos\alpha)\; f(X\sin\alpha + Y\cos\alpha)\; dX\,dY$$

where X and Y are the rectangular Cartesian coordinates as shown.

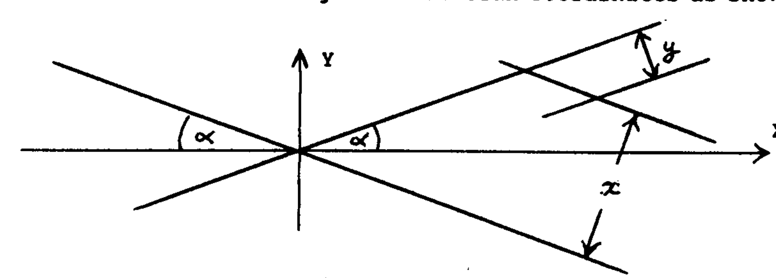


Figure 5

Instead of regarding fig. 5 as showing where the intersection of the position lines is relative to the true position, we may regard it as showing the probability distribution of the true position relative to the intersection C of the two position lines CA and CB. These two lines and the internal and external bisectors of the angle at C divide the plane into eight infinite sectors, and the data from the two position lines tell us that there is probability 1/8 of the true position being in any one of the sectors.

Suppose that the true position (TP) is in the cocked hat. This can be in two ways, as shown in fig. 6 and fig. 7 below.

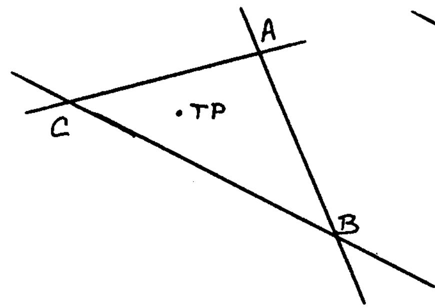


Figure 6

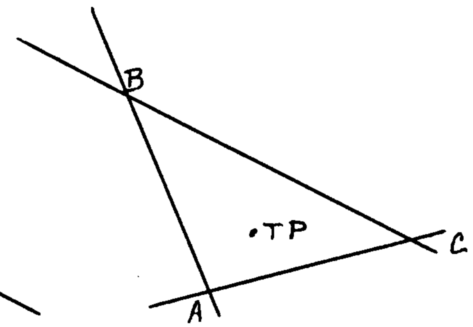


Figure 7

These two cases are essentially the same, so that it will be sufficient to consider only the case of fig. 6. The perpendicular distances x, y and z of the true position from the three position lines are all positive, and they are independent random variables drawn from the same probability distribution described by a certain density on the interval $(0, \infty)$. Each of their six possible relative orderings:-

$0 < x < y < z$	$0 < y < x < z$
$0 < x < z < y$	$0 < z < y < x$
$0 < y < z < x$	$0 < z < x < y$

has probability 1/6. That is to say, if the true position is in the cocked hat, there is probability 1/6 of its being in each of the six regions shown in fig. 1 above, separated by the angle bisectors. The case of fig. 7, where x, y and z are all negative, clearly leads to the same result.

Proof of (d)

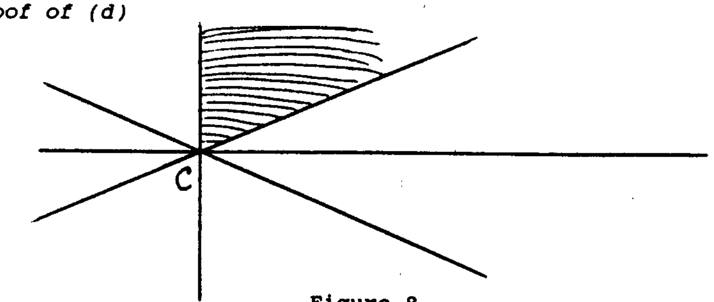
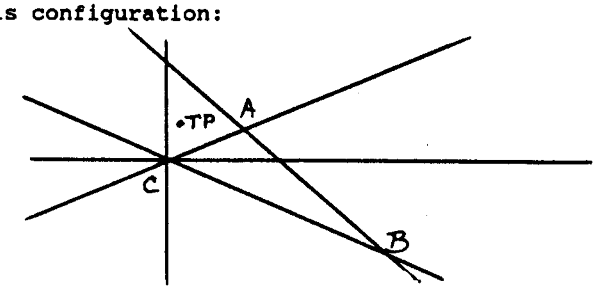


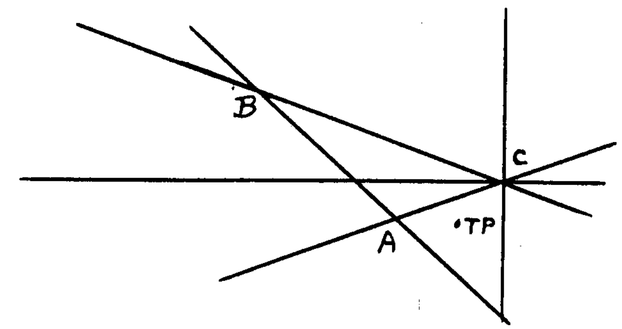
Figure 8

Consider the two position lines through C and the bisectors of the angle between them, as shown above (fig. 8). The other position line AB may be on either side of C, take one of the two cases, that of fig. 3 (or fig. 6).

With probability 1/8 the true position TP will be in the infinite sector shown shaded in fig. 8 above; if this is so then the third position line AB will have probability 1/2 of giving this configuration:



A similar calculation gives probability 1/16 for this:



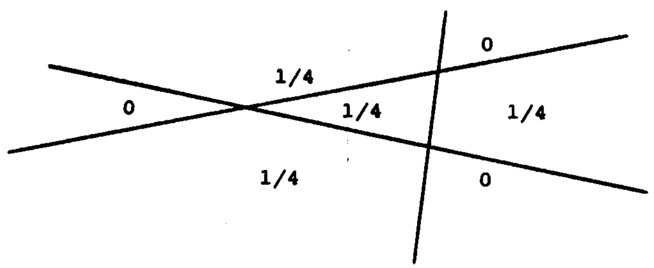
Thus the result (d) is proved.

EXAMPLES OF COCKED HATS

(JCMN 41 p.4218, 55 p.6033, 56 p.6076, 62 p.6284, 69 p.7085)

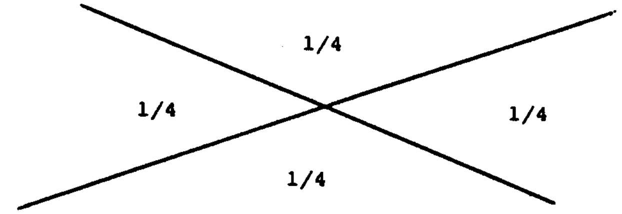
Recall how three position lines in the plane form a triangle called a "cocked hat". The mathematics is conveniently visualised as being concerned with a problem of navigation — finding a point in the plane given three straight lines that approximately go through the point; but more generally the work is on the statistical problem of estimating two unknowns, x and y, from three linear equations such as ax + by = c (with c inaccurate) connecting them. In what follows we shall use the language and imagery of the navigational problem.

Example 1 Suppose that each position line has the error distribution that it must be at unit perpendicular distance from the true position. Then all three position lines are tangents to the unit circle round the true position. This circle is therefore either the inscribed circle or one of the escribed circles of the cocked hat, these four possibilities are equally probable. Because the true position must be at one of the tritangent centres of the triangle, the probability of the true position being in any one of the seven regions into which the three position lines divide the plane, is as shown below.

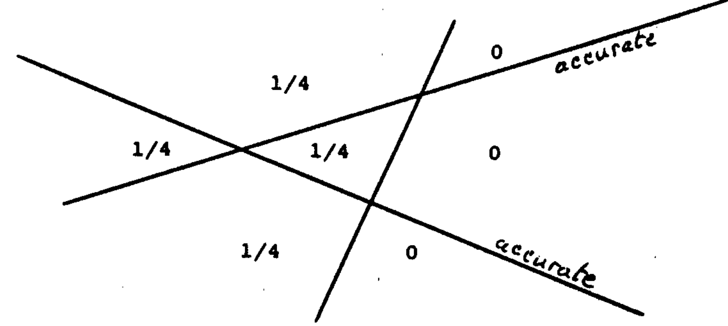


Example 2 Consider the case of two position lines, so that we have a cross instead of a cocked hat. For the error distribution, make only the assumption that each position line is as likely to be on one side of the true position as on the other. The two position lines divide the plane into four

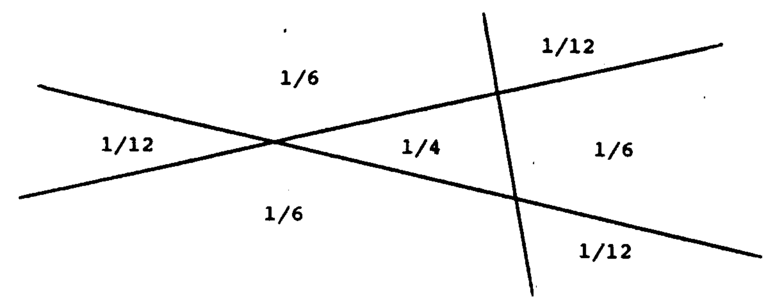
sectors, and the probability of the true position being in any one of them is 1/4.



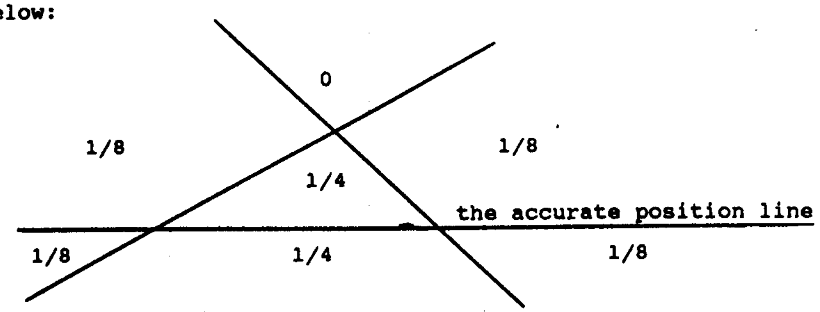
Example 3 With three position lines, we know the cocked hat, and we know two of the position lines to be very accurate and the other to be inaccurate. For each of the seven regions into which the position lines divide the plane, the probability of the true position being in the region is as shown below, because the true position is very close to where the two accurate position lines meet, and we may use Example 2 above.



Example 4 The error distributions are as in Example 3 above, but we do not know which position line is the inaccurate one. The probabilities for the seven regions are then:-

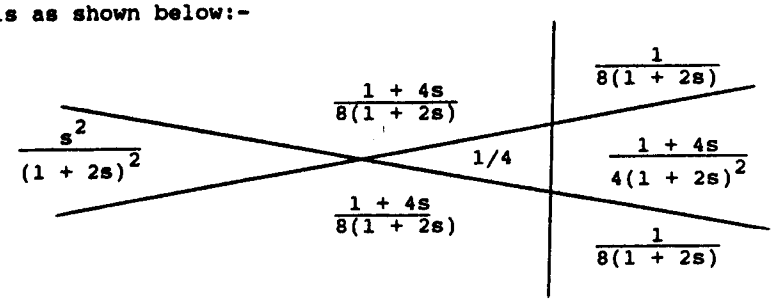


Example 5 Of the three position lines we know that one is very accurate, and the two others are not. If we know which of the position lines in a cocked hat is the accurate one then the probability distribution of the true position among the seven regions into which the cocked hat divides the plane, is as shown below:

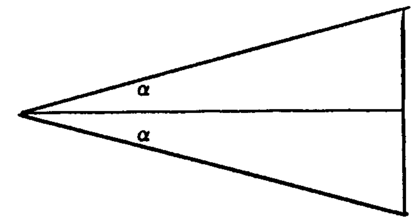


Example 6 Modify Example 5 by saying that in any cocked hat we do not know which position line is the accurate one. Then the probability distribution of the true position becomes as in Example 4 above.

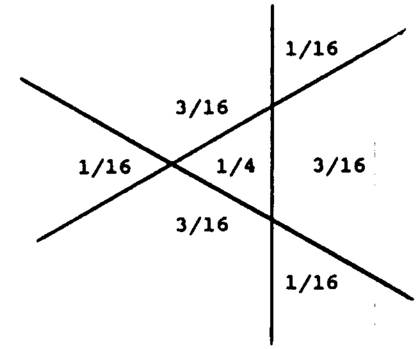
Example 7 All three position lines have the same error distribution, which is negative exponential, the probability of the true position being at a perpendicular distance between p and p+dp from the position line is exp(-2p)dp. If the cocked hat is an isosceles triangle the probability of the true position being in each of the seven regions of the plane is as shown below:-



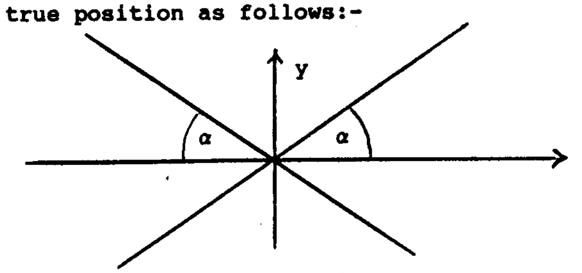
where s = sin α and α is half the angle at the vertex of the triangle, as shown below:



A simple special case is that of the equilateral triangle, where s = ½, with the probabilities as below:



The proofs of these results about an isosceles triangle cocked hat with negative exponential error distribution are not interesting enough to be printed in full. They depend on the fact that from two position lines we obtain a probability density for the true position as follows:-



(Thinking of the y-axis as pointing North, and the x-axis as pointing East, taking as origin the intersection of the two position lines)

In the Northern sector: sin 2α exp(-4y cos α) dx dy

In the Eastern sector: sin 2α exp(-4x sin α) dx dy.

SPHERICAL TRIANGLE GEOMETRY

(JCMN 47, p.5136, JCMN 68, pp.7069-7072)

In Euclidean 3-space take three linearly independent unit vectors, $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, and $\boldsymbol{\gamma}$; they specify a non-degenerate triangle ABC on the unit sphere. Let p, q and r denote the cosines of the sides BC, CA and AB. As (normalized) coordinates of any point P on the sphere take x, y and z, the cosines of PA, PB and PC, respectively. The coordinates may be treated as homogeneous, treating $(\lambda x, \lambda y, \lambda z)$ as representing the same point, for any non-zero λ. Another way of defining the coordinates is to say that a non-zero vector $\mathbf{V} = u\boldsymbol{\alpha} + v\boldsymbol{\beta} + w\boldsymbol{\gamma}$, whether of unit length or not, represents the point $\mathbf{V}/\|\mathbf{V}\|$ on the sphere. The (homogeneous) coordinates (x, y, z) of this point are related to the coefficients (u, v, w) by

$$x = \boldsymbol{\alpha}.(u\boldsymbol{\alpha} + v\boldsymbol{\beta} + w\boldsymbol{\gamma}) = u + rv + qw$$
$$y = \boldsymbol{\beta}.(u\boldsymbol{\alpha} + v\boldsymbol{\beta} + w\boldsymbol{\gamma}) = ru + v + pw$$
$$z = \boldsymbol{\gamma}.(u\boldsymbol{\alpha} + v\boldsymbol{\beta} + w\boldsymbol{\gamma}) = qu + pv + w$$

and these equations may be written (u, v, w) = (x, y, z)M

where $$M^{-1} = \begin{bmatrix} 1 & r & q \\ r & 1 & p \\ q & p & 1 \end{bmatrix}$$

$$\det M^{-1} = 1 + 2pqr - p^2 - q^2 - r^2$$

which, in terms of the sides a, b, c, and angles A, B, C, may by the cosine rule: $\cos a = \cos b \cos c + \sin b \sin c \cos A$ be written $\sin^2 A \sin^2 b \sin^2 c$ or $\sin^2 a \sin^2 B \sin^2 c$, etc.

$$M = \begin{bmatrix} 1-p^2 & pq-r & pr-q \\ pq-r & 1-q^2 & qr-p \\ pr-q & qr-p & 1-r^2 \end{bmatrix} (1+2pqr-p^2-q^2-r^2)^{-1}$$

Both M and its inverse are symmetric and positive definite.

In terms of the sides a, b, c, and angles, A, B, C, we have

$M \sin A \sin B \sin C \sin a \sin b \sin c =$

$$\begin{bmatrix} \sin A & 0 & 0 \\ 0 & \sin B & 0 \\ 0 & 0 & \sin C \end{bmatrix} \begin{bmatrix} 1 & -\cos C & -\cos B \\ -\cos C & 1 & -\cos A \\ -\cos B & -\cos A & 1 \end{bmatrix} \begin{bmatrix} \sin a & 0 & 0 \\ 0 & \sin b & 0 \\ 0 & 0 & \sin c \end{bmatrix}$$

The middle one of these three matrices has an intriguing relation to the inverse of M, because $-\cos C = \cos(\pi - C)$ is the cosine of a side of the polar triangle (whose sides are the great circles orthogonal to the vertices of the original triangle).

If $\mathbf{x}$ and $\mathbf{y}$ are the normalised coordinates of any two points on the sphere, then the cosine of the distance between them is the bilinear form

$$\mathbf{x}^T M \mathbf{y}$$

which is zero if the points are at 90° apart. It would be interesting to factorize M as the product of some matrix and its transpose.

If we have homogeneous coordinates (x, y, z) (as described above) for a point on the sphere, we can obtain from them the normalized coordinates, as follows. From (x, y, z) we find the coefficients (u, v, w) = (x, y, z)M as above. The magnitude of the vector $\mathbf{V} = u\boldsymbol{\alpha} + v\boldsymbol{\beta} + w\boldsymbol{\gamma}$ is given by the scalar product:

$$\|\mathbf{V}\|^2 = (u\boldsymbol{\alpha} + v\boldsymbol{\beta} + w\boldsymbol{\gamma}).(u\boldsymbol{\alpha} + v\boldsymbol{\beta} + w\boldsymbol{\gamma})$$

$$= u^2 + v^2 + w^2 + 2pvw + 2qwu + 2ruv$$

$$= \mathbf{u}^T M^{-1} \mathbf{u} = \mathbf{x}^T M \mathbf{x}, \quad \text{because } \mathbf{u} = M\mathbf{x}.$$

where we write $\mathbf{x}$ for the column vector that is the transpose of the row (x, y, z), and similarly $\mathbf{u}$.

Thus to normalize any homogeneous coordinates (x, y, z) we divide them all by $\sqrt{(\mathbf{x}^T M \mathbf{x})}$. For instance the cosine of the distance from the point (x, y, z) to the vertex B is equal to $y(\mathbf{x}^T M \mathbf{x})^{-1/2}$.

As always, there are analogies between plane and spherical geometry. In plane geometry we take as (normalized) trilinear coordinates of any point the distances (x, y, z) from the sides of the triangle of reference (the signs positive for points inside); and having done that, we can change to using homogeneous coordinates, treating $(\lambda x, \lambda y, \lambda z)$ as representing the same point (for any $\lambda \neq 0$). Normalised coordinates satisfy the relation $ax + by + cz = 2\Delta$ (where Δ is the area of the triangle of reference); and so from the homogeneous coordinates (x, y, z) it is easy to recover the normalized coordinates, just multiply by 2Δ and divide by $ax+by+cz$.

This algebra draws our attention to the line $ax+by+cz = 0$. It is the line at infinity. What is the analogue in spherical geometry? Is it the conic

$$\mathbf{x}^T M \mathbf{x} = 0 \ ?$$

This is the conic such that the polar of any point is the great circle at right angles to the point. You might call it

a non-conic, it has no points on the sphere, just as the line at infinity in the Euclidean plane has no points. See the note below on the grin of the Cheshire Cat.

The circumcentre of the triangle has the (normalised) coordinates $x = y = z = \cos R$, where R is the radius of the circumcircle. From this fact, by using the quadratic form given by the matrix M, it may be calculated that:

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C} = 2 \tan R \cos \frac{a}{2} \cos \frac{b}{2} \cos \frac{c}{2}$$

which is the analogue of the sine rule:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

for plane triangles.

For a plane triangle the centroid is where the sum of squares of distances to the vertices is a minimum, equal to one third of the sum of squares of the sides. The analogous result for a spherical triangle is that the centroid is where the sum of cosines of distances to the vertices is a maximum, this maximum is equal to $\sqrt{(3 + 2 \cos a + 2 \cos b + 2 \cos c)}$.

SPHERICAL GEOMETRY AND THE CHESHIRE CAT

Those familiar with Lewis Carroll's book *Alice's Adventures in Wonderland* will recall the account in Chapter 6 of Alice's meeting with the Cheshire Cat. In particular:-

.... "You'll see me there," said the Cat, and vanished.

Alice was not much surprised at this, she was getting used to queer things happening. While she was looking at the place where it had been, it suddenly appeared again.

"By-the bye, what became of the baby?" said the Cat, "I'd nearly forgotten to ask."

"It turned into a pig," Alice quietly said, just as if it had come back in a natural way.

"I thought it would," said the Cat, and vanished again.

Alice waited a little, half expecting to see it again, but it did not appear, and after a minute or two she walked on in the direction in which the March Hare was said to live. "I've seen hatters before," she said to herself; "the March Hare will be much the most interesting, and perhaps as this is May, it won't be raving mad —— at least not so mad as it was in March." As she said this she looked up, and there was the Cat again, sitting on a branch of a tree.

"Did you say pig, or fig?" said the Cat.

"I said pig," said Alice; "and I wish you wouldn't keep appearing and vanishing so suddenly: you make one quite giddy."

"All right," said the Cat; and this time it vanished quite slowly, beginning with the end of the tail, and ending with the grin, which remained some time after the rest of it had gone.

"Well! I've often seen a cat without a grin," thought Alice; "but a grin without a cat! It's the most curious thing I ever saw in all my life!"

Consider the foundations of plane and spherical geometry, a topic which Lewis Carroll, alias Charles Dodgson, would no doubt have spent a lot of time explaining to his undergraduate pupils. Modern mathematicans would start with a field F, but

in those days it was the real variable. Consider elements labelled by 3 real numbers not all zero, such as (x, y, z), and take $(\lambda x, \lambda y, \lambda z)$ for any $\lambda \neq 0$ to represent the same element. Our first geometrical picture of these elements is as the lines through the origin in Cartesian 3-space. Our second picture is as the real projective plane, obtained by taking any plane in the 3-space not through the origin, and taking its intersection with one of the lines to be a point of the projective plane. Our third geometrical picture comes from taking the unit sphere where $x^2+y^2+z^2 = 1$, and taking (x, y, z) to represent the two diametrically opposite points where the line (from the first picture) meets the sphere. The relation between the second and third pictures is simply radial projection from the origin.

A linear homogeneous equation in (x. y, z) represents a great circle on the sphere or a straight line in the projective plane. A quadratic homogeneous equation (by definition, perhaps) represents a conic in the projective plane, and so we use the word "conic" to denote the corresponding locus on the sphere. A small circle on the sphere is a familiar example of a conic.

An interesting case is the conic on the sphere given by:

$$x^2 + y^2 + z^2 = 0.$$

It has no points and no tangents, but it has the property that it behaves like the familiar conic in setting up a structure of poles and polars. It is what may be called an "elliptic polarity" (see H.S.M. Coxeter, *Introduction to Geometry*, §14.7, page 252). It is something like a grin without a cat.

FOURIER TRANSFORM

What is the Fourier transform of $\log|x|/\sqrt{|x|}$? This function is not absolutely integrable, and not a member of the Hilbert space $L^2(-\infty, \infty)$, so that most of the usual theory found in text-books does not apply. However, this function is integrable in any bounded interval, and is an "ordinary function" in the sense of Lighthill's book on "generalised functions", and therefore has a Fourier transform.

The formal expression $F(x) = \int_{-\infty}^{\infty} \frac{\log|y|}{\sqrt{|y|}} e^{-2\pi ixy}\, dy$

is an integral not absolutely convergent at infinity; in these circumstances we can often get the right answer by using

$$\lim_{A \to \infty} \cdot \lim_{B \to \infty} \int_{-A}^{B} \quad \text{for} \quad \int_{-\infty}^{\infty}$$

or by using Abel or Cesàro summation, i.e. replacing

$$\int_0^{\infty} \varphi(x)\, dx \quad \text{by} \quad \lim \int_0^{\infty} \varphi(x)\, e^{-x/n}\, dx \quad \text{or} \lim \int_0^{n} (1 - \tfrac{x}{n})\varphi(x)\, dx.$$

A serious analyst may say that we need to plunge into the theory of generalized functions, and find a transform that gives the correct inner products with the appropriate test functions. But never mind theory, what is the answer?

There are indications that the transform may be equal to

$$-\frac{\log|x|}{\sqrt{|x|}} - \frac{K}{\sqrt{|x|}} \qquad \text{with } K = 5{\cdot}3721....$$

If this is so, then what is K exactly?

Printed by Economy Press, Adelaide, South Australia