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JCMN 55

CONTENTS

RANDOM TRIANGLES

J. B. Parker

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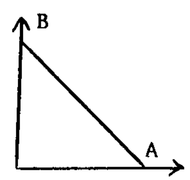
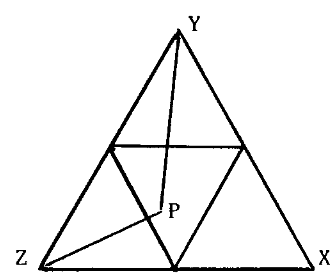
Previous contributions on the question of random triangles and whether they are acute have concentrated on defining a random triangle as three random points. A different approach is to construct the triangle from three random lines.

For example the "Pooh-sticks" method is to drop three sticks at random on the ground; the sticks are produced to infinite straight lines, which are taken as the sides of the triangle. The sticks are labelled as a, b and c; the angle A is that between sticks b and c, etc. What is the bivariate distribution of the two angles A and B of the triangle? The position of a Pooh-stick involves the position of the mid-point and the direction of the stick. The distribution of the directions is assumed to be uniform. The distribution of the mid-points is not relevant to the distribution of the angles of the random triangles that we obtain. (Proof - suppose the positions of all three mid-points and the direction of stick c to be fixed, then the probability of stick a being in an appropriate direction is $(2/\pi)dB$, and, given that, the probability of stick b being in an appropriate direction is $(1/\pi)dA$.)



The (bivariate) probability density of the angles A and B is uniform, equal to $2/\pi^2$, in the triangle defined by $0 < A < A+B < \pi$; see the diagram. The symmetry in the distribution of the three angles of the random triangle may be shown by taking an equilateral triangle XYZ of area π, and representing each triangle ABC by a point P in XYZ such that the angles A, B and C are the areal co-ordinates of P, i.e. A is the area PYZ, etc.



The distribution of the point P is uniform in the triangle XYZ.

With this definition of random triangles it is easy to see that the probability of the triangle ABC being acute is 1/4, for the acute triangles are represented by points P in the small equilateral triangle in the middle of the big one.

The same distribution of the angles A, B and C is obtained by drawing three random tangents (uniformly independently distributed) to a fixed circle.

Another method of finding random triangles is as follows. Take three random numbers, a, b and c, from the uniform distribution on (0, 1). If such a triple fails to satisfy the three triangle inequalities, that each side must be less than the sum of the other two, then reject the triple. Now we have a triangle with a, b and c as sides. What is the probability that this triangle is acute?

ANALYTIC INEQUALITY

(JCMN 51, p.5228, 53, p.5276, 54, p.6007)

Let $f(x)$ and its derivative $f'(x)$ be positive and continuous in the closed interval $[0, c]$. For what constants K can we assert that $\int_0^c x/f(x)dx < \int_0^c K/f'(x)dx$?

In JCMN 53 the result was proved with $K = 4$, and in JCMN 54 it was proved with $K = 2{\cdot}47188$. Now we shall establish the result with $K = 2$, and it will then appear that no further improvement is possible.

In the 3 lemmata below all the functions are on the closed unit interval $[0, 1]$, and all the integrals are from 0 to 1.

<u>Lemma 1</u> If $g(0) = 0$ and $g(1) = 1$ and if the derivative $g'(x)$ is positive and continuous, then

$$\int g(t)g'(t)/t\,dt \quad < \quad 2\int g'(t)^2\,dt$$

Proof Let A and B be the square roots of the integrals of the squares of $g(t)/t$ and $g'(t)$ respectively, i.e. they are the norms of these two functions in the sense of Hilbert space. Using the Cauchy-Schwarz inequality and integration by parts, we find

$$2AB \geq 2\int gg'/t\, dt = [g^2/t] + \int g^2/t^2\, dt = 1 + A^2 > A^2$$

(in evaluating the limit as $t \to 0$ in $[g^2/t]$, note that $g \to 0$ and $g/t \to g'(0)$ which is finite). Therefore $A < 2B$. Now use the same Cauchy-Schwarz inequality again.

$$\int gg'/t\, dt \leq AB < 2B^2 \qquad \text{QED}$$

Perhaps in the lemma above there is an echo of ANALYTIC INEQUALITY 2 (JCMN 53, p.5277 and 54, p.6008).

Lemma 2 If $y(0) = 0$ and $y(1) = 1$ and if dy/dx is positive and continuous, then $\int x/y\, dx < \int 2/y'\, dx$.

Proof Let the inverse function to $y = y(x)$ be $x = g(y)$, so that $g(y(x)) = x$ in the unit interval. Then $g'(y) = dx/dy$ and $\int gg'/y\, dy = \int x/y\, dx$ and $\int g'^2\, dy = \int g'\, dx = \int (dx/dy)dx$.
An application of Lemma 1 gives $\int x/y\, dx < \int 2/y'\, dx$ QED

Lemma 3 If $g(0) \geq 0$ and if $g'(x)$ is positive and continuous, then $\int x/g(x)\, dx < \int 2/g'(x)\, dx$

Proof Put $u(x) = g(x) - g(0)$. Note that $x/g \leq x/u$ for $x > 0$, and also for $x = 0$ if the limits are taken. Now apply Lemma 2 to the function $y(x) = u(x)/u(1)$.

Theorem If $f(x) \geq 0$ and $f'(x)$ is positive and continuous in the closed interval $[0, c]$, then

$$\int_0^c x/f(x)\, dx < \int_0^c 2/f'(x)\, dx$$

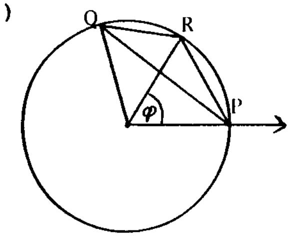
Proof Apply Lemma 3 to the function $g(x) = f(cx)$.

Finally we come to the question about the 2 in the theorem above being the best possible. Take the inequality with K instead of 2. A simple argument is as follows. Consider the function $f(x) = x^b$ (where $1 < b < 2$) in the interval $[0, 1]$. The LHS of the inequality is $1/(2-b)$, and the RHS is $K/(b(2-b))$, so that the result implies $b < K$. This is for all such b, so that $K \geq 2$. The well-trained analyst will of course see a fault in this reasoning, the function f does not satisfy the conditions of the theorem, as $f'(0) = 0$. However, the well-trained analyst will be able to remedy the fault.

GEOMETRICAL PROBABILITY 2

A. Brown
(61, Dexter Street, Cook, A.C.T 2614, Australia)

For random triangles defined by three points uniformly distributed on the unit circle, what can we say about the area?



Take points P $(1, 0)$, Q $(\cos\theta, \sin\theta)$ and R $(\cos\varphi, \sin\varphi)$, as shown. The area of the triangle PQR is $|A|$, where

$$A = (\sin\varphi + \sin(\theta-\varphi) - \sin\theta)/2$$

We can take $0 \leq \theta \leq \pi$, since there will be a mirror image triangle with the same area obtained by changing the sign of θ.

For a given value of θ we can consider values of φ from 0 to 2π, noting that A is positive when $0 < \varphi < \theta$, and negative when $\theta < \varphi < 2\pi$. The expectation for the area of the triangle is

$$\int_0^\pi \left\{ \int_0^\theta A \frac{d\varphi}{2\pi} - \int_\theta^{2\pi} A \frac{d\varphi}{2\pi} \right\} \frac{d\theta}{\pi}$$

Carrying out the integration gives the mean area as $3/(2\pi) = 0{\cdot}47746$.

Similarly the second moment of the area (the mean square) is the integral mean of the square of A over the rectangle where $0 < \theta < \pi$ and $0 < \varphi < 2\pi$, and is 3/8.

The variance of the area is therefore $(3/8)-(9/4)\pi^{-2}$, which is 0·147027, and the standard deviation is 0·38344.

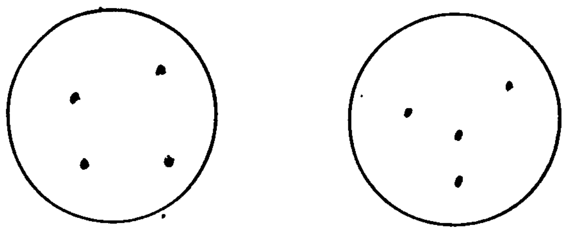
For fixed θ the triangle will have an obtuse angle at R if $0 < \varphi < \theta$, and the probability of this is $\theta/(2\pi)$. Averaging over θ, the probability of angle R being obtuse is 1/4. The probability of the triangle being acute is therefore 1-3/4=1/4.

QUOTATION CORNER 35

The tin price had dropped by more than 50% from $13000 a tonne in 1989 to about $7100 this year.

- Adelaide Advertiser (newspaper) March 7th 1991, p.35.

ARROWS IN THE TARGET
(JCMN 54, p.6019)



The 335 soldiers of King Arthur's army had each shot four arrows into a round target. How many did he expect to have made a pattern like that shown on the left, a convex quadrangle? And how many did he expect to have made a pattern like that on the right above, with one point in the triangle formed by the other three?

Suppose that the four arrows are at the points A, B, C and D, in the order in which they were shot. The four points are uniformly and independently distributed in the disc, which we may take to be of unit radius, with area π. The triangle ABC has expected area $35/(48\pi)$, as we saw in GEOMETRICAL PROBABILITY, JCMN 54, pp.6012-6017. The probability that D is in the triangle ABC is therefore $(1/\pi)(35/48\pi)$. There is the same probability that A is in BCD, or B is in ACD, or C is in ABD. These four events are mutually exclusive, and if none of them happens then the 4 points form a convex quadrangle. The probability of the latter is therefore $1 - 35/(12\pi^2)$. Consequently the number of soldiers expected to accompany King Arthur on his campaign was:

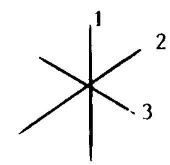
$$335 - (335\times 35/12)\pi^{-2} = 236{\cdot}001$$

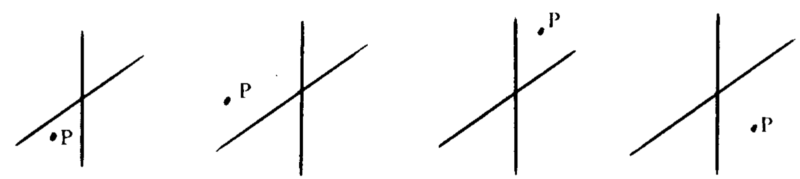
leaving 99 to garrison the castle under Sir Lancelot.

COCKED HATS (JCMN 41, p.4218)

Readers may have forgotten this 1986 note from John Parker, containing a question not yet answered. Recall how a navigator uses "position lines" on a plane map or chart. Typically a position line is obtained from an astronomical observation of the Sun or a star, or from a bearing of a beacon, and is of course liable to inaccuracy. With only two position lines, their intersection has to be taken as the estimated position, but this leaves the navigator with no check on the accuracy of the fix, and indeed with a lurking fear that there may have been a major blunder somewhere in the procedure. Therefore often a navigator gets 3 position lines, they form a triangle called a "cocked hat" (the name probably dates from the eighteenth century Royal Navy, in which officers wore cocked hats). The size of the cocked hat indicates the order of magnitude of the errors in the position lines. Can we be more precise? Yes.

<u>Theorem 1</u> (Admiralty Manual of Navigation) The probability of the true position being in the cocked hat is 1/4.
<u>Proof</u> First we shall fix the directions of the 3 position lines; we shall get the value 1/4 for any such combination of directions. They are best represented graphically (see the figure on the right), rather than symbolically. Now we draw the true position P and the position lines 1 and 2, (see below) Since each position line is equally likely to be on one side or the other of the true position (this is the only assumption that we make about the error distributions), each of the four possibilities shown below has probability 1/4.

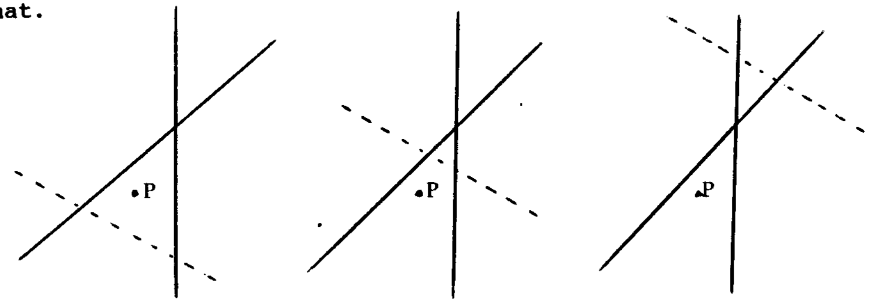




(i) (ii) (iii) (iv)

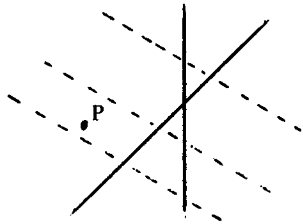
Now consider case (i). In this case there is probability 1/2 of the third position line being as in (i)(a) below, putting

P in the cocked hat, and probability 1/2 of the line being either as in (i)(b) or as in (i)(c) below, putting P outside the cocked hat.



(i)(a) (i)(b) (i)(c)

Now consider case (ii). Wherever the third position line may be, the true position P will be outside the cocked hat. See the three possible position lines 3, shown dotted.



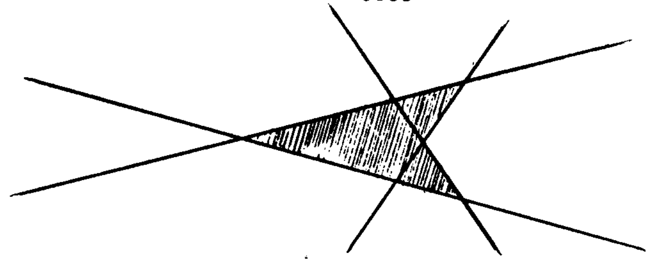
Now case (iii) may be seen to give the same result as case (i), and (iv) the same as (ii).

Collecting our results, the probability of P being in the cocked hat is

$$(1/2 + 0 + 1/2 + 0)/4 = 1/4 \qquad \text{QED}$$

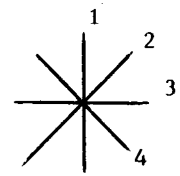
With this background we can start looking at the question asked in the previous contribution, what can be said about the situation with 4 position lines?

Firstly what do four lines in the plane look like? Any set of four lines looks very like any other set! The 4 lines intersect in 6 points (three points on each line), the convex hull of the 6 points is a triangle, two of the lines are sides of the triangle, and the other two lines meet inside the triangle. The 4 lines divide the plane into 11 polygonal regions, 3 of them bounded and the other 8 unbounded. See below. The union of these 3 bounded regions is a non-convex polygon with four sides, we might call it an "arrowhead". It is used in Theorem 2, playing the part of the cocked hat in Theorem 1. See the figure on the next page.

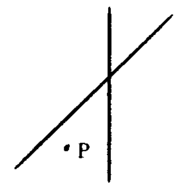


<u>Theorem 2</u> Given 4 position lines in the plane (each with probablity 1/2 of being on either side of the true position), the true position has probablity 1/2 of being in the arrowhead, shown shaded in the figure above.

<u>Proof</u> Firstly, as in Theorem 1, we fix the directions of the 4 position lines to be as shown. Now we take lines 1 and 2 and the true position, P, and distinguish the four possible cases, as in Theorem 1. As before we see that cases (i) and (iii) are essentially the same as one another, as are (ii) and (iv).



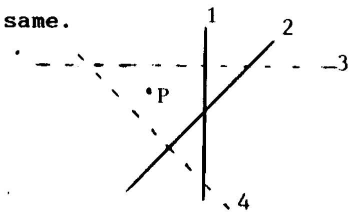
Consider Case (i). Referring to this sketch on the right, we may describe position lines 3 and 4 as being "above" or "below" the true position P. Therefore we see the four possibilities, all with the same probability:-



3 above and 4 above. P is outside the shaded region.
3 above and 4 below. P is inside.
3 below and 4 above. P is inside.
3 below and 4 below. P is inside.

Thus we see that in case (i) there is probability 3/4 that P is in the arrowhead. Case (iii) is the same.

Now consider case (ii). Again we use "above" and "below" to denote the posssible positions of lines 3 and 4. The only one of the four sub-cases in which P is in the arrow-head (in fact when it is surrounded by lines) is when line 3 is above P and line 4 is below. Thus in case (ii) (and in case (iv)) the probability of P being in the arrowhead is 1/4.



Summing up, the probability of P being in the arrowhead is

$$(3/4 + 1/4 + 3/4 + 1/4)/4 = 1/2. \qquad \text{QED}$$

DEMOCRACY FROM THE POINT OF VIEW OF MATHEMATICS

A. M. Slin'ko (translated by Irini Ozolins)

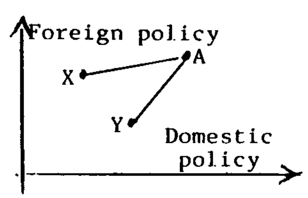
Mathematics has a responsibility. Classical pure mathematics has become too abstract to be useful, but applied mathematicians are finding a fertile soil in which to work. Much use of mathematics is made in economics, sociology, linguistics and even politics. It often happens that a simple model gives much insight into a problem.

Consider politics, or more precisely the mechanics of its realization. Our models are rough and they may be inadequate, but they show the delicacy of some democratic institutions, how they may fail to reach a conclusion and how they may be liable to manipulation. In considering democratization we must ask what benefit we can expect from it.

The one-party system. Let's look at an imaginary state that has decided to change to democracy. So far there has been a constitution based on a one-party system which has guaranteed open discussion on proposed reforms. "Open-ness" meaning that every person can declare his views, so that (in principle) all views are known to all. We suppose that the machinery for surveying the voters' opinions makes no mistakes, and that the platform of the ruling party is known, and that the platform can be changed only by referendum. Our imagined state is a rough model of the system we have tried to create in our own country (USSR), although more recently there have been moves to a many-party system.

To study the problem by mathematical methods we must change the qualitative model to a formal model. In our formal model the political views of citizens are points in a Euclidean space, which we call the space of political ideals. For simplicity we take the space as two-dimensional, imagine one dimension to be foreign policy and the other to be domestic policy.

Let point X represent the old platform of the party, and Y the proposed new one in a referendum. A citizen with political views represented by the point A will vote for the change if Y is nearer than X to

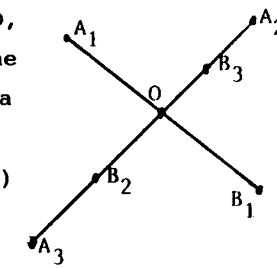


A, that is if $AY < AX$, or if the proposed new platform is nearer to his own political ideals than the old one. If $AX = AY$ the voter abstains. If the majority of the citizens vote for the new platform it will be adopted.

For convenience let's adopt two more rules : (1) The number n of citizens in the state is odd. (2) All the citizens have different political ideals.

The question is: does there exist an ideal political platform that in the best way mirrors the interests of the citizens, and which in the referendum shows preference over any other political platform? To find the answer we need

<u>Theorem 1</u> <u>Plott's Theorem</u> Suppose that $n = 2k+1$ and $k \geq 1$, then an ideal political platform exists if and only if there are points A_j and B_j for $j = 1, 2, \ldots k$, and a point O, such that for each j the line segment A_jB_j contains O, and each of these 2k+1 points represents the view of one citizen. An example of such a situation is shown in the figure.



<u>Proof</u> (But readers might do it themselves)

To prove the "if", suppose that the the points are as described. Two people such as A_j and B_j are paired, let's call them "political opponents", let A and B be two such people, so that O is between A and B in the line AB. Let X be any point of the plane except O. Since $AX + BX \geq AB = AO + BO$, it follows that either $AX > AO$ or $BX > BO$. Therefore at least one of A and B regards O as better than X, this holds for all such pairs, and so in a referendum the platform O would win against X.

Define a "median" as a straight line such that the two closed half-spaces that it defines each contains at least k+1 of the points.

<u>Properties of medians</u> Every median contains at least one of the n points. There is exactly one median parallel to any given straight line. There is at least one median through any point.

To prove the "only if" of Plott's Theorem, suppose that O is the ideal platform. Take any line through O. To prove that this line is a median, suppose not. The line must have at least k+1 points strictly on one side. Take O′, on that side, sufficiently close to O, with OO′ perpendicular to the line. The k+1 points would all be closer to O′ than to O, so that O′ would get at least k+1 votes in a referendum against O, a contradiction because we assumed O to be the ideal platform. Thus, every line through O is a median. Therefore the point O is one of our set of 2k+1. Now take any point A (not O) of our set. The two open half-spaces determined by the line AO must have the same number of the points, and therefore the line AO must have an odd number. Now consider these points on the line AO, apart from O itself; the number on one side of O must be equal to the number on the other side, and so they may be paired as in the theorem.

Now that we have established Plott's Theorem, we may derive several results from it.

<u>Theorem 1.1</u> If there exists an ideal platform O then in a referendum the platform X will beat the platform Y if and only if citizen O votes for X.

<u>Proof</u> If O abstains then O is on the perpendicular bisector of the segment XY, and so of any pair of "political opponents" A and B, either both abstained or one voted for X and the other for Y, the referendum therefore gives a drawn result. Suppose that X wins in the referendum and that O votes for Y. There must be two "political opponents" A and B, both voting for X; those who vote for X are all those in an open half-space bounded by the perpendicular bisector of XY, But since O is between A and B, O is also in this half-space, and O votes for X, a contradiction.

The point O represents a really admirable voter! In a chaotic country such a citizen would vote as the majority does.

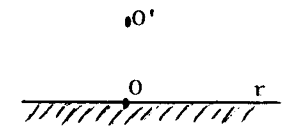
We now see that an ideal political platform is nearly impossible. The smallest change in the voters would disturb the harmony. Introduce the notation $Q \succ R$ to denote that in a referendum the platform Q will win over platform R. It may be seen that if QR is perpendicular to a median m, and if Q is nearer to m than R is, then $Q \succ R$.

<u>Theorem 1.2</u> If there is an ideal political platform then the

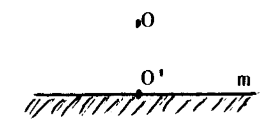
relation $\succ$ is transitive, that is $X \succ Y$ and $Y \succ Z$ together imply $X \succ Z$.

<u>Proof</u> This follows at once from Theorem 1.1.

<u>Theorem 1.3</u> There is an ideal platform O if and only if every median goes through O.



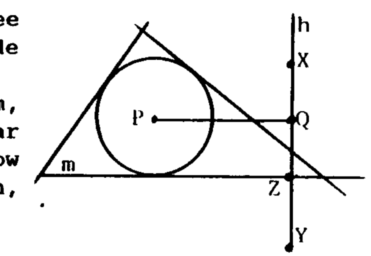
<u>Proof</u> <u>If</u> Suppose O not ideal, there exists $O' \succ O$. Let r be the median through O perpendicular to OO′. There are at least k+1 voters in the closed half-space shown, bounded by r and not containing O′. These voters are enough to ensure that $O \succ O'$, which is the required contradiction.



<u>Only if</u> Let O be the ideal platform. Let m be a median not through O. Let O′ be the foot of the perpendicular from O to m. The k+1 voters in the closed half-space bounded by m not containing O will be enough to ensure that $O' \succ O$, contradiction.

Now the reader will wonder what happens in the other case, when there is no ideal platform. In fact the paradoxical situation with $X \succ Y \succ Z \succ X$ is possible.

<u>Lemma</u> Suppose that there is no ideal platform, then (Theorem 1.3) there are three medians forming a non-trivial triangle, let P be the in-centre, and let r be the in-radius. Then for any X we can find Y such that $Y \succ X$ and $PY^2 > PX^2 + r^2$. To prove this, there is one (m) of the three medians such that X is on the same side as P and at a distance $\geq r$ from m. Drop the perpendicular h from X to m, meeting m at Z, and the perpendicular PQ from P to h (see the diagram); Now it is easy to choose a suitable Y on h, so that $YZ < XZ$ and $PY^2 > PX^2 + r^2$.



<u>Theorem 2</u> (McKelvey) If there is no ideal platform, then given any points X and Z it is possible to find a finite sequence $Y_1, \ldots Y_n$ such that $X \prec Y_1 \prec \ldots \prec Y_n \prec Z$.

<u>Proof</u> Using the lemma it is possible to choose such a sequence of points Y at increasing distances from P, and clearly $Y \prec Z$ for Y at sufficiently large distance.

Theorem 2 shows that if there is no ideal political platform the relation $\succ$ is not transitive. Each citizen separately knows what would be best, but the nation as a whole does not know. We now clearly see that with voting as the majority prefers the outcome could be nonsensical. The ruling party, using this method, could in time achieve the acceptance of any programme that they wanted. Of course this reasoning lessens the significance of democracy in an existing one-party system.

Some readers may remember how Stalin's politics were "supported by the whole nation". The awakening followed later.

To avoid this weakness, the constitution should include instructions for constant changing of the party apparatus (as the leadership of the party is fixed in the constitution) right up to the highest political echelons, so that the apparatus, while at work, would not foster its own political interests. The party's interests should not be above the interests of the nation!

Maybe the reader hopes that it may be possible to avoid the lack of transitivity of the collective decision, if the decisions were made not by a majority but, for example, by a two-thirds majority. It is easy to dispel this hope. Let's take 100 citizens in our imagined nation. Let's say the nth citizen earns n roubles per month (n = 1, 2, ... 100). Citizens are asked to vote for this proposal: for all who earn less than 100 roubles there should be a rise of 1 rouble, but for those who earn 100 roubles the future wages should be 1 rouble. However many times the voting was repeated the proposal would be accepted by a 99% majority. There is no transitivity.

Voting for a leader

First, let's look at the simplest case, where n voters have to choose one leader out of m candidates. The sample space is $A = \{a_1, \dots a_n\}$. Often this is the case: each voter chooses one candidate, and the candidate with most votes is proclaimed the winner (this system is sometimes called the "first past the post" method). In this system there is a fundamental weakness, which can be illustrated by the following example. Let there be 5 candidates, a, b, c and d are for change, and e is against change. Let's also say that 60% of the voters are for change and 40% are against. The counting of votes gives: a: 15%, b: 15%, c: 15%, d: 15% and e :40%. With such a method the winner would be e. At the same time each of the candidates a, b, c and d separately could easily beat e. As a result of such a system the worst candidate wins. This system also allows some manipulation by choosing additional candidates with a platform similar to that of e.

To find the best candidate we must compare each pair of candidates separately, without involving the other candidates. Each voter (the voter numbered k) has formed a transitive relation $<_k$ in the set A (i.e. among the candidates); it gives a complete ordering of A. However, even in the comparison of one pair of candidates we come across the same stumbling block, the collective preference by the majority is not transitive. This fact is called Condorcet's Paradox.

For example, suppose that in a collective there are n = 5 voters and 3 candidates, a, b and c. The opinions of the voters are:-

1 and 2 think	$a > b > c$
3 and 4 think	$b > c > a$
5 thinks	$c > a > b$

which means, for example, that voters 1 and 2 prefer a to b and prefer b to c.

In this case, if voting is done in pairs, a beats b with 3 votes to 2, b beats c with 4 votes to 1, and c beats a with 3 votes to 2. These facts may be summarised by $a \succ b \succ c \succ a$. Therefore from the common viewpoint of the society, there is not a best candidate - one that could beat any other candidate.

But of course a voting procedure must proclaim a winner. For example voting may be organized using the Olympic system - "with withdrawals". The voters may be asked first to compare two candidates and then to compare the winner with another, continuing in this way until all the candidates have been compared. If we denote as x*y the winner in comparing x with y, then we have a binary algebraic operation. In the example above a*b=a, a*c=c and b*c=b. We have defined the candidature together with the algebraic operation * which indicates the sympathies of the voters. The operation * is commutative (i.e. x*y=y*x) but not associative. In our example $c = (a*b)*c \neq a*(b*c) = a$. When voting is by the Olympic system there are in the example above three possible outcomes, $a*(b*c) = a$, $(a*c)*b = b$, and $(a*b)*c = c$. In this case the results of the voting are completely dependent on the order in which the candidates are compared. The person or committee that decides the order has, in a hidden way, dictated the outcome. In that case we can say that the voting has been manipulated.

The extent of the manipulation and the algebraic operation * are bound in an interesting way. Try to prove on your own that manipulation is impossible if and only if the operation * is associative, i.e. if always $(x*y)*z = x*(y*z)$.

Now, let's look at another example:-

Voters 1 and 2 think $c > a > b$
Voters 3 and 4 think $b > c > a$
Voter 5 thinks $a > c > b$

In this situation manipulation is impossible, the winner must be c, because c can beat a (4 votes to 1) and can beat b (3 votes to 2). If the committee wants b to win, what can it do? The answer is to introduce a new candidate d, who is the instrument of manipulation. Suppose the sympathies of the voters about the four candidates are:-

Voters 1 and 2 think $d > c > a > b$
Voters 3 and 4 think $b > d > c > a$
Voter 5 thinks $a > c > b > d$

The voting is then arranged as follows, $c*d = d$ (4 votes to 1), then $d*a = d$ (4 votes to 1), and finally $d*b = b$ (3 votes to 2).

As we can see, the accepted and often used systems of voting have fundamental weaknesses. Does there exist a more satisfying and faultless system, even if it is more complicated? To clarify this question, let's remember that the procedure of choosing one candidate is equivalent to ordering the whole set of candidates, and equivalent to the problem of how to choose the best of the remaining m-1 candidates. Usually besides voting for the director or leader, we also choose his deputy and other members. Therefore we should be trying to find an ordering on the set A of candidates, starting from the orderings $<_1, \ldots <_n$ of the n separate voters.

Using the language of mathematics we have to form a function $F(<_1, \ldots <_n) = \prec$. From the given n orderings which describe completely the sympathies of all the voters, we must find a new ordering relation representing the collective sympathy of the voters. Of course not every function F would describe a reasonable procedure acceptable to society. Ideally such a function must firstly satisfy the "axiom of accordance", that if $a <_i b$ for all i, then $a \prec b$. In other words if all the voters prefer b to a then the group prefers b to a. Secondly, no possibility of manipulation is to be allowed; the condition that $\prec$ must be transitive cuts out that possibility. To avoid

the setting up of misleading candidates it is necessary that the construction of the relation $\prec$ should satisfy the axiom of independence, that the cancelling of one candidature should not alter the relation $\prec$ between the other candidates.

Now we will state one of the most interesting theorems of this century, formulated by Kenneth Arrow. For his series of works on this theme he received the Nobel Prize in 1972.

<u>Theorem 3</u> (Arrow) If there are at least 3 candidates and the function F satisfies the axiom of accordance and the axiom of independence, then, for some i, $F(<_1, \ldots <_n) = <_i$.

The interpretation of this theorem is: the only procedure of voting in which manipulation is not possible is when the result is decided by one person. In the West such a person is called a "dictator", and Arrow's theorem is called "the theorem about the dictator". It is easy to give this theorem a threatening meaning, and it is welcomed by "strong men" or "outstanding personalities" and admirers of monarchies. The author does not agree with that. His viewpoint is that it is better to renounce the axiom of independence and allow the possibility of manipulation.

What can a supporter of democracy learn from this theory? First, that there is not and cannot be an easy road to democracy. As it is always possible to manipulate the views of the society, there will always be people wanting to do so. Examples are not hard to find. It would be better to learn how democratic institutions work, so that we can know when we are being manipulated. Such was the goal of this article.

[<u>Editor's note</u>] This contribution has suffered two translations (into Latvian and then into English) and my editing since Professor Slin'ko wrote it, and so I must offer apologies to him and to our readers for whatever distortions and inaccuracies have crept in.

BASIC STATISTICS

The simplest type of problem in statistics is as follows. The distribution of a random variable X is known in terms of one unknown parameter c; how can we best estimate c from a sample of values of X? The very simplest problem is perhaps:- given n values of a random variable X with a uniform distribution on (0, c), to estimate c.

Sir Harold Jeffries at the beginning of his book on statistics (of which I forget the name) gave the example of the trams: a traveller came out of the railway station in a strange town and saw a tram with the serial number x. Assuming that the trams are numbered 1, 2, 3, ... c, what is the best estimate for c? Non-statisticians tend to suggest the value 2x for c, though the more cunning might favour 2x-1, academic statisticians like to preserve their professional integrity by not offering an answer, the maximum likelihood method gives x.

Putting the problem in abstract form, with real variables instead of integers (this does not affect the principles in which we are interested), we have one sample, x, from a random variable X uniformly distributed on (0, c). The estimate of 2x for c can be justified as a matching of the first moment of the distribution to that of the data.

Now generalize the problem to that of having n samples of the same random variable X. The idea of matching first moments runs into difficulties; for example the data (1, 2, 9) by this method would indicate $c/2$ = mean of data = 4, which is impossible because obviously $c \geq$ largest value = 9.

The Bayesian approach is that we have a prior probability distribution for the unknown parameter c, and the distribution is successively changed by taking into account each of the n observations of the random variable X. The final result is not affected by the order in which the the observations are taken into account. Therefore we may suppose that the largest observation, m, was one of the first n-1. Now we shall

calculate the effect (on our prior probability) of the nth observation, let the observed value be v. The current prior probability $P(c|)$ (which takes into account the first n-1 observations but not the nth) is zero in the interval (0, m), because on the given information the true value of c must be at least m. We want $P(c|v)$, that is the probability of the parameter being between c and c+dc, given that the new observation is v. By the usual theory it is

$P(v|c)P(c|)/\Sigma P(v|c')P(c'|)$ (with summation is over c')

Since $P(v|c) = (1/c)dv$, the expression above does not depend on v. Therefore the value v is not relevant to our calculation. Thus the Bayes method tells us that from our data (the n values of the random variable X) we need only the maximum, m, and the number, n, of observations. The rest of the information does not matter. Our conclusions about the unknown parameter c must therefore depend only on m and n, and of course on whatever prior information or prejudices about c that we had before knowing any of the observations.

This Bayesian reasoning puts a constraint on any efforts to give a good estimator for c as a function of the n values, the estimate must be a function of m and n only; in fact the only possible unbiassed estimate is $m(1 + 1/n)$, which has a variance of $c^2/(n(n+2))$.

A similar Bayesian argument enables us to deal with the problem of estimating a and b from observations of a random variable X uniform on an unknown interval (a, b). The only relevant part of the data is the number of observations,and the largest and smallest values. The unbiassed estimate for a is (n.smallest - largest)/(n-1), and the estimate for b is similarly (n.largest - smallest)/(n-1). Each has variance equal to

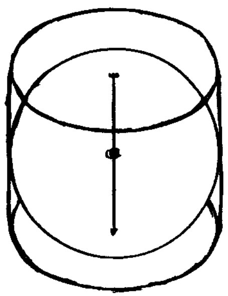
$$n(b-a)^2/((n-1)(n+1)(n+2)).$$

On this question it might be asked what we should do to estimate the parameter a if we have only the two or three smallest of the observations. Are we in a position to give a good estimate for a? Of course we cannot do as well as when we have all the observations, but the estimator (2.smallest - second smallest) is unbiassed and has variance equal to $2(b-a)^2/((n+1)(n+2))$.

Using the third smallest observation, there is a better estimator, ((3/2).smallest - (1/2).(third smallest)), which has variance equal to $(3/2)(b-a)^2/((n+1)(n+2))$. In fact when we know the three smallest observation the second smallest does not seem to be any use. Is this accidental?

These days we like to show that our mathematics is socially relevant, that it can be usefully applied in the real world. Suppose that we are trying to find the maximum of a differentiable function of two variables, using a Monte Carlo method. We have the values of f(x, y) for a sequence of random points (x, y) uniformly distributed in some given set. The largest value is an obvious estimator for the maximum of f, but it is always an under-estimate, and we should be able to do better. If we translate and rotate the x-y coordinate system, and change the scales of both x and y, then the function f, for small x and y, will be of the form $c - k(x^2+y^2)$ + smaller terms, where c is the required maximum.

Now recall that Cicero, when he was Quaestor in Sicily in 75 B.C., recorded having seen at Syracuse the tombstone of Archimedes, on which could be seen the figure:-



Archimedes had died 137 years before, in 212 B.C. This figure illustrates one of his great discoveries, that a sphere has the same area as the circumscribing cylinder. In fact radial projection perpendicular to the axis of the cylinder gives an area-preserving mapping between the two surfaces. Using the obvious Cartesian co-ordinates, with the z-axis vertical, if we have a uniform distribution of points on the sphere, their z-co-ordinates, given by $z = \pm\sqrt{(R^2 - x^2 - y^2)}$, will have a rectangular

distribution, from -R to R.

The connection with our statistical problem now becomes clear. If we confine our attention to points near the top of the sphere and to points (x, y) near the maximum of the function f, then the distribution of f(x, y) is to a first approximation like that of z(x, y) near the top of the sphere, i.e. approximately uniform until it drops to zero. We do not actually have a uniform distribution for values of the function f, but we have one that is approximately uniform near where the function attains its upper bound, which is the only part that matters. The calculation above indicates that (2.(largest)-(second largest)) or ((3/2).(largest)-(1/2).(third largest)) would be good estimators for the maximum, but of course we cannot say anything more definite without knowledge of the function f being maximized.

As far as we know from the few of his works that have survived, Archimedes had no ideas about probability, but he would have been familiar with the trading ships sailing from the port of Syracuse (his home town) to the Phœnicean colony then known as Portus Herculis (after the temple there dedicated to Hercules) or Monœci Portus, and later to be called Monaco or Monte Carlo.

GEOMETRICAL INEQUALITY

Find the bounds of the sum

$$\frac{a-b}{a+b+2c} + \frac{b-c}{b+c+2a} + \frac{c-a}{c+a+2b}$$

where a, b and c are the sides of a triangle.

BOOK REVIEW

"God and the New Physics" by Paul Davies, 235 pages. Published by J.M.Dent, 1983, Pelican 1984 and Penguin, 1990, the Penguin paperback price is Aust$15·99 or £5·99.

The author seems to have been told that there was a need for a new book on 'the conflict between science and religion', and has done his best to turn out what the public wants, but the project suffers a bit from lack of definitions; the author drifts between different meanings of the words 'science' and 'religion'. Does 'science' mean particle physics or cosmology? Does 'religion' mean the Summa Theologiae of St Thomas Aquinas, the Book of Genesis, the modern protestant theologians, or the crude ideas of God that the author attributes to some average person?

In Chapter 3 entitled 'Did God create the universe?' the author discourses in the style of the mediaeval scholastic philosophers about causes, but in the light of modern ideas it seems no more than an artificial creation of confusion. The argument, shorn of frills, goes like this:

1 There is a word 'cause' in the language and in dictionaries.
2 The question "What is the cause of .... ?" is grammatically well-formed.
3 Therefore the question has an answer.
4 Therefore, given any X, there is Y such that Y is the cause of X.
5 Given any X, there is an infinite sequence X, Y, ... in which each is caused by the next.
6 Such a sequence must have a cause (for which of course a pure mathematician would use the Hebrew letter $\aleph$).
7 And so on.

Looking at this sequence of propositions in the cold light of common sense, it is a claim that from a fact about our language we can deduce a fact about the world we live in. As a piece of reasoning it does not carry conviction. Elsewhere in the book the author does admit that there can be an event without a cause (such as radioactive disintegration), but fails to note how pointless this fact makes his chapter 3.

On the nature of God, the author quotes without comment the famous "God created man in his own image" from the book of Genesis. A pure mathematician would be tempted to observe that this proposition is singularly uninformative about God; it is like saying that the real variable is a homomorphic image of Hilbert space; true, but quite unhelpful to a beginner trying to learn about Hilbert space. Your reviewer does not know anything about the Hebrew word that Moses used for 'image', but the Greek word 'εικων' or 'ikon' always means homomorphic rather than isomorphic image. This verse from Genesis is often used by the ignorant to try to imply that God is something like man, for sadly the common people of our species are as ignorant of abstract algebra as they are of the Bible.

The book makes an admirable attempt to explain the modern theories of elementary particles in non-technical terms. But when it ventures into cosmology and metaphysics it runs into their inherent problems. The difficulty may be illustrated as follows. Consider the two questions - "How far is it to the nearest Post Office?" and "What was the diameter of the Universe a millionth of a second after its creation?" At first sight they look essentially similar, both questions about length. The careful reader responds to the first question with "I wonder what the answer is." but to the second with "I wonder what the question is." An answer to the first may, in principle, be found (we all agree) by certain procedures involving measuring rods, putting them end to end. The second is not like that, any answer to it must (at the moment) be accepted or rejected on faith alone, for there is no agreement on what an experimental scientist ought to do to make the measurement.

It is interesting to recall that thousands of years ago the astronomers of Babylonia must have faced a similar problem about non-observables. They had been studying the stars carefully for many years, and then one of them suggested that the stars were rotating on a sphere, so that during the day there were still stars scattered over the sky, but they were not visible. This was asking people to believe in things that could not be observed. What arguments went on before the suggestion was adopted? All we know is that in the end they all agreed. Scientists do not (as some would have us think) deal only with observed facts, they often take the leap of faith and try to believe in something not observed, motivated by the consideration that things are easier when everybody believes in the same dogma. That is how the Scientific Establishment arises. Will someone write a book on the resemblances between science and religion?

BCR

BANK ROBBERY

The old methods with masks and guns are still favoured by some bank robbers, but there is a growing awareness that more powerful methods of acquiring money are available. We shall outline a few.

<u>The real estate method</u> This needs some capital investment, and needs the co-operation of several people.

The opportunity arises when some person or company (denoted by A) wants to sell a fair-sized shop or office block in the big city. The operators of the scheme form two companies, (denote them by B and C), with the openly acknowledged purpose of dealing in real estate. The two companies each appoint a 'financial consultant', usually a senior official in a bank, with a small retaining fee. Company B approaches A and finds a price Q at which A is prepared to sell the property. B then offers A an "option", that is an agreement that at the end of a certain period, perhaps 2 or 3 weeks, B may either buy the property for the price Q or give A a small sum of money, the price of the option, perhaps Q/100. Such an arrangement usually suits a seller like A well, for he either sells for what he regards as a fair price, or collects a small clear profit.

Company C then approaches a land agent saying that it wants to buy such a property. Then B approaches the same land agent saying that it has this property for sale. The land agent happily remembers the enquiry from C, and there is general rejoicing, and the two companies begin to negotiate over a sale at a price of about 5Q/4. B then discuses the situation with the financial consultant, who says that B is in a good position to borrow the amount Q from a bank. This is duly arranged, B borrows the money and buys the property.

Now all is ready for the whole story to be re-enacted, with B playing the part of A, C playing the part of B, a new company D playing the part of C, and with all sums of money increased by 25%. B sells to C, making a profit of Q/4, more than enough to pay all the expenses.

At this point you (gentle reader) may be moved to comment "Yes, I see how the plotters have gained a lot of money, but who

has been robbed?" A very good question, perhaps unanswerable.

But of course the story has not finished, in real life no story is ever completely finished. The sequence of events described above cannot continue to be repeated indefinitely, for it involves the prices increasing exponentially. People say there is a "property boom". Then builders work out that office blocks can be built for less than the prices at which they are being bought and sold, and there comes what people call a "slump in the property market", and a few companies find themselves owing a lot of money to banks and unable to sell their property for what it cost. Such a company goes into liquidation, its debtors (the banks) share out what assets there are. The office blocks (or whatever) are auctioned to the highest bidder. The banks recover part of their loans and have to "write off" the rest. The operators of the scheme (even those of the liquidated companies) have all done well for themselves. At this stage the only loser seems to be the bank.

<u>The finance company method</u> This is in some ways like the real estate method, but it involves the buying and selling of companies (you "buy" a company by buying more than half the shares, thereby gaining complete control). It is more subtle than the real estate method, and more flexible, and is capable of many variations.

Typically the story starts with a "target" company A, which is carrying on its normal business of making and selling something or other. An official of the Bank approaches the General Manager and explains how the Bank's research department has identified the business as one with a great potential for profitable expansion. The company is persuaded to start new branches, advertise heavily, take on extra staff, and generally behave extravagantly. A nice little extra touch is to persuade the General Manager to have an expensively furnished office and a large Company motor car with a driver, "in keeping with the Company's new image". Before long the Company runs into difficulties, but the Bank gives an increased loan and more encouragement. But this situation cannot continue for long, the Company's shareholders may ask awkward questions at the annual meeting. Consequently the Bank proposes a "debt for equity" arrangement, (the use of trendy jargon is an important part of modern fraud). The Company issues more shares, so that the total issued is more than doubled. The friendly Bank official explains that a finance company B has great faith in Company A and will take up the new shares. Company B has of course been set up by the operators of the scheme, and is using money on loan from the Bank to buy the shares. This operation is duly carried out, and Company A repays its debt to the Bank. Company B now owns more than half the shares in A, and so is able to replace the original Board of Directors with its own people.

Another company, C, is started, by the operators of the scheme of course. The operators own all the shares in C. The new Board of Company A, having control of all the assets, starts dealing with C, but the dealings are more profitable to C than to A, and gradually most of the assets of A are transferred to C. At this stage Company A has outlived its usefulness, it is liquidated. The shareholders, consisting of the original shareholders (with a bit less than 50%) and Company B (with a bit more than 50%), share between them what the Liquidator can recover from the wreck. Now Company B has also outlived its usefulness, its debt to the Bank exceeds the assets that it was able to recover from the liquidation of A, therefore B is also liquidated, and the Bank has to write off the part of the loan that it was unable to recover.

The story now comes to its end. The Bank officials can point to two large loans that they negotiated, one was completely repaid , and the other partly. The operators of the scheme have acquired complete ownership of a company. Some of the general public have lost, of course, but that's business, isn't it?

<u>The method of the missing cheque</u> This involves the bank having two customers with similar names. When the bank has some reason for sending money to one of them, it sends the money to the other without explanation. When the mistake emerges the rightful owner is reimbursed, and the bank writes off the "lost" money as a bad debt. Eventually the other customer can usually be persuaded to return the money, in the form of a cheque payable to the bank. This cheque comes into the hands of the operators who put it through the bank's accounting system as payment for

a bank cheque in foreign currency. They duly issue such a cheque, payable of course to themselves, and the operation is complete.

<u>The method of the loan including interest</u> This is simple, but in the hands of a careful operator can be very successful. A bank loan is made to a person or company, with the provision that the interest is automatically added to the loan. There is no trouble of writing off bad debts, and once the structure is set up it needs no attention, and has the advantage that in the accounts of the bank it shows up as a steady income (from the fictitious interest being credited to the bank every year). The structure has to collapse some time, but with care it will not until after the operator has retired; and even if the collapse is earlier the operator has not broken any law, and will still be able to retire on a good pension.

<u>The unavoidable delay</u> Suppose that a financial organization of some kind is due to pay out a fairly large sum of money in the course of legitimate honest busines. It often just happens that the person who should be dealing with the matter is away on holiday, or away ill, or perhaps the office is being moved, or the computer is being reprogrammed, or the message to Head Office did not get through, or ... (earnest workers are busy thinking of new reasons all the time). What has actually happened is that the money has been put into a "suspense account" and then sent by telegraphic transfer to a merchant bank operating on the money market. Such banks will pay interest on money held for as little as one day, the rate is of the order of $3 per day on $10,000. The interest is of course paid to an account started by the operators of the scheme. Most customers take a few weeks to get dangerously insistent about overdue money, and when they do the money is recovered and paid to them, the suspense account is closed, and any written records are put in the paper-shredder. Sometimes the better operators, as a gracious gesture, send a letter of apology to the customer.

<u>Masks and guns</u> No self-respecting financier would ever get involved in such a crude and dangerous method of robbery, but it

has its place in the grand strategy of banking. The banking industry relies on an active class of low-grade criminals who will discourage the general public from keeping money in the form of cash either at home or in the pocket. The encouragement of this class is done quite easily and safely by making banks easy to rob, the actual losses from these robberies are trivial compared with the amounts of interest to bankers.

In connection with the grand strategy of banking, it may be noted that the "Finance company method" described above has a subsidiary value in discouraging the general public from owning shares in companies, thereby encouraging deposits in banks.

If ever people start to ask questions about activities such as those described above, the banking industry unites to proclaim their solemn obligation to their customers to keep all transactions secret. When any profession starts proclaiming that its members have a duty of confidentiality, one can be reasonably sure that their actual motive is to conceal their own ill-doings.

Some of the methods outlined above involve robbery of the bank, and the question may well be asked - how much can a bank afford to lose? The major Australian banks "write off" several thousands of millions of dollars every year (that is in a country of which the total population is only about 18 million). The answer is of course that banks can get money from their customers. How? That's another long story, but the main methods are so well known that there is no need to describe them here.

EQUATION TO SOLVE

Solve the set of equations:-

$$y^2 + yz + z^2 = a^2$$
$$z^2 + zx + x^2 = b^2$$
$$x^2 + xy + y^2 = c^2$$

What is wanted is a "solution in radicals", i.e. it should involve only the extraction of roots and the algebraic operations of addition, subtraction, multiplication and division.

GEOMETRICAL PUZZLE

First of all (there is no puzzle about this) consider 4 points in the plane. Drawing a square with one point on each side is an old problem. There is a solution by Euclidean methods (ruler and compasses), but it may be noted that we have to interpret a side as an infinite line, for a point may have to be on the side outside the square. We are interested in an algebraic solution, let the 4 points in the plane be represented by the complex numbers a,b,c and d. Now consider the 4 complex numbers

$$a + ib - c - id$$
$$b + ic - d - ia$$
$$c + id - a - ib$$
$$d + ia - b - ic$$

they represent the directions of the four sides of the square. In the exceptional case where all are zero, there are infinitely many squares. Analysts will note that 1, i, -1 and -i are the fourth roots of unity, which seems appropriate.

Now can we find a similar result with 5 instead of 4? Given 5 points in the plane, can we draw a regular pentagon through them? No, regular pentagons have 4 degrees of freedom, and so we cannot in general make a regular pentagon satisfy 5 conditions. Can we draw a pentagon with all its angles equal through the 5 points? Yes, but in infinitely many ways, for we may take any direction for one side, and fill in the others accordingly.

The puzzle is that we have a neat algebraic construction as follows. Let the points be a, b, c, d and e, and let τ be a primitive fifth root of unity. The 5 complex numbers

$$a + \tau b + \tau^2 c + \tau^3 d + \tau^4 e$$
$$b + \tau c + \ldots$$
$$\text{etc.}$$

give 5 directions at angles of 72° to one another (perhaps more precisely we might say 72° mod 180°). Draw lines in these directions through the 5 points. What can we say about the pentagon that we get? It surely must have some interesting property apart from having its angles equal. Or is our algebraic construction just a solution looking for a problem?

What does it mean if the 5 complex numbers are zero?