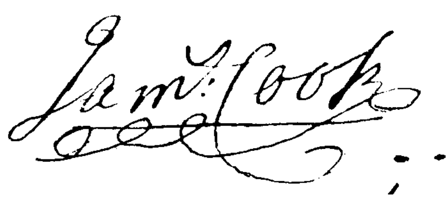
The New

JAMES COOK MATHEMATICAL NOTES

Volume 4, Issue number 34

June 1984



Editor : B. C. Rennie,
69 Queens Road,
Hermit Park, N.Q. 4812,
Australia.

Three issues per year will be published, in February, June and October.

Subscription rates, in Singapore dollars, per year (covering three issues).

In Singapore (including postage)	\$20
Outside Singapore (including air mail postage)	\$30

Subscribers in countries with no exchange control (such as Australia, U.K. and U.S.A.) may send ordinary cheques in their own currency to the Editor, the amount calculated at the current exchange rate. The rates at the time of writing are approximately

\$1 (Singapore) = 52 cents (Australia) = 34 pence (U.K.)
= 48 cents (U.S.A.)

Otherwise send whatever kind of currency you can, but do not send a personal cheque unless it can be cleared from Singapore. Back numbers (when available) are \$12 (Singapore) each (sent by surface mail from Australia).

Please make cheques payable to B. C. Rennie.

CONTENTS

Centres of Acute Triangles : an unexplained coincidence? - A. P. Guinand 4048
Binomial Identity 17 4052
Binomial Identity 18 - Marta Sved 4052
Function from Blasius 4053
Complete Difference Triangles - George Berzsenyi 4054
Adding Numbers - C. J. Smyth 4055
Model for Taxation 4056
Linear Equations and Latin Squares - J. B. Parker 4059
Goat and Compasses 4060
Complex Inequality 4061
Correction 4061
Point in a Triangle 4062
Point in a Triangle - I. B. Tabov 4063

CENTRES OF ACUTE TRIANGLES : AN UNEXPLAINED COINCIDENCE?

A. P. Guinand

In reference [1], and in more detail in reference [2] it was shown that, given the Euler line of a triangle (including circumcentre O, centroid G, and orthocentre H) then the incentre I may lie anywhere inside the "critical circle" on diameter GH. From this fact extrema of various angles formed by the centres are easily deduced; for example, the lower bound of angles GIH is 90° and the upper bound of angles IOH is 30°, but the actual bounds are only attained for degenerate triangles with a zero angle (Figure 1).

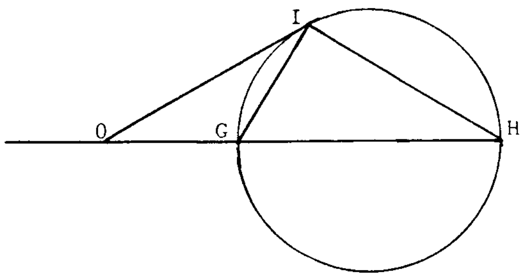


Figure 1

If we take cartesian coordinates with O as origin, H as (1,0) then the locus of incentres I for right triangles in [1] or [2] reduces to

$$(x^2 + y^2 - 2)^2 = 5 - 4x. \qquad (1)$$

This is a limacon with node at H, and only its inner loop lies inside the critical circle (Figure 2). Positions of I inside this loop then correspond to acute triangles. From this we can investigate the extrema of angles formed by centres of acute triangles, the actual extrema only being

attained for right triangles.

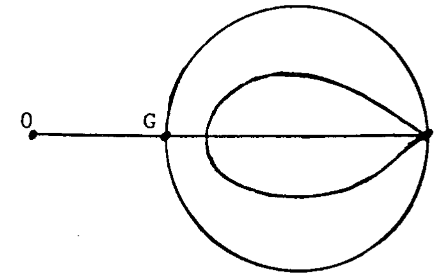


Figure 2

In order to carry this out, introduce polar coordinates (Figure 3) with H as origin and HO as axis.

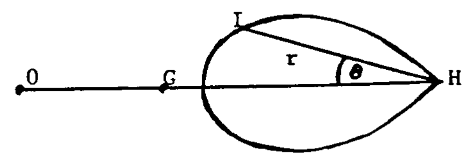


Figure 3

Then the polar equation of the inner loop of (1) reduces to

$$r = 2\cos\theta - \sqrt{2} \qquad (-\tfrac{1}{4}\pi \le \theta \le \tfrac{1}{4}\pi). \qquad (2)$$

(For the outer loop, $r = 2\cos\theta + \sqrt{2}$)

Consider, first, the angle GIH ($=\alpha$, say, Figure 4). By the sine rule

$$\frac{\sin\alpha}{GH} = \frac{\sin(\pi - \alpha - \theta)}{IH}.$$

That is $\quad (3/2)\sin\alpha = \dfrac{\sin(\alpha + \theta)}{2\cos\theta - \sqrt{2}}$, whence $\tan\alpha = \dfrac{2\sin\theta}{4\cos\theta - 3\sqrt{2}}$.

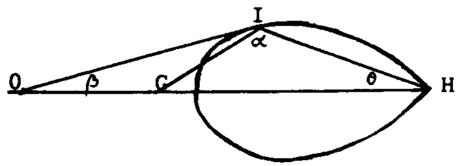


Figure 4

Differentiating with respect to θ, a critical value of α occurs where

$$\cos\theta(4\cos\theta - 3\sqrt{2}) + 4\sin^2\theta = 0,$$

whence $\cos\theta = 2\sqrt{2}/3$, $\sin\theta = 1/3$,

and $\tan\alpha = -\sqrt{2}$,

$$\alpha = \pi - \arctan\sqrt{2} = 125.26^\circ.$$

From Figure 4, this is the minimum value of α.

Now consider the angle IOH (= β, say). Again, by the sine rule

$$\frac{\sin\beta}{IH} = \frac{\sin(\pi - \beta - \theta)}{OH},$$

or $$\frac{\sin\beta}{2\cos\theta - \sqrt{2}} = \sin(\beta + \theta),$$

whence $$\tan\beta = \frac{\sin 2\theta - \sqrt{2}\sin\theta}{\sqrt{2}\cos\theta - \cos 2\theta},$$

and differentiation with respect to θ indicates a critical value where

$$(\sqrt{2}\cos 2\theta - \cos\theta)(\sqrt{2}\cos\theta - \cos 2\theta)$$
$$= (\sqrt{2}\sin 2\theta - \sin\theta)(\sin 2\theta - \sqrt{2}\sin\theta)$$

After some calculation this, surprisingly, also reduces to $\cos\theta = 2\sqrt{2}/3$, $\sin\theta = 1/3$,

whence $\tan\beta = \sqrt{2}/5$,

and so $\beta = \arctan(\sqrt{2}/5) = 15.79^\circ$.

In this case, from Figure 4, this is the maximum of β.

Thus the same right triangles give minimum angle GIH and maximum angle IOH. If there is any reason for this other than blind chance coincidence, I have not discovered it.

Setting $\cos\theta = 2\sqrt{2}/3$ in (2) gives $r = \sqrt{2}/3$, leading to cartesian coordinates for I of

$$x = 1 - r\cos\theta = 5/9,\quad y = r\sin\theta = \sqrt{2}/9.$$

Putting these values in the cubic (14) of [1] it is found that the cosines c of the angles of the triangle satisfy

$$c(18c^2 - 24c + 7) = 0$$

whence $c = 0,\ (4 + \sqrt{2})/6,\ (4 - \sqrt{2})/6$.

That is, for non-obtuse triangles the right triangle with sides in the ratio $4 - \sqrt{2} : 4 + \sqrt{2} : 6$ corresponds both to the lower bound of angles GIH and the upper bound of angles IOH.

Other angles OIH, IGH, IHO, IGO can be investigated in the same way, but do not seem to yield any further surprises.

REFERENCES.

[1] A. P. Guinand, Triangles from centres out, JCMN 30, pp. 3127-32.

[2] -------- Euler lines, tritangent centers, and their triangles, American Math. Monthly, 91 (1984) (to appear).

BINOMIAL IDENTITY 17

This question was raised in Crux Mathematicorum, volume 10, no. 1, January 1984, by the Editor (Léo Sauvé), who traced it back to Ian Bruce, "Binomial Identities" in The Mathematical Gazette, 65, 1981, pages 282-285.

$$\sum_{r=0}^{n} \binom{n}{r}^2 \binom{m+r}{2n} = \binom{m}{n}^2$$

for all integers $m > n > 0$.

BINOMIAL IDENTITY 18

Marta Sved

(a) $$\sum_{i=0}^{[n/2]} \binom{2i}{i} \binom{n}{2i} 2^{n-2i} = \binom{2n}{n}$$

(b) $$\sum_{i=0}^{[k/2]} \binom{n}{i} \binom{n-i}{k-2i} 2^{k-2i} = \binom{2n}{k}$$

FUNCTION FROM BLASIUS

In the boundary layer theory for a flat plate in a uniform stream at high Reynolds number there arises a certain function $f(x)$ which is well-behaved, smooth, positive and increasing for all real positive x; it may be defined as the solution of the differential equation $ff'' + f''' = 0$ for which $f(0) = f'(0) = 0$ and $f''(0) = 1$. For small x the function may be expanded in a power series

$$f(x) = x^2/2! - x^5/5! + 11x^8/8! - 375x^{11}/11! + 27987x^{14}/14! - \ldots$$

The coefficients are given by the recursion

$$(3n+3)(3n+4)(3n+5)c_{n+1} = \sum_{r=0}^{n} (3r+1)(3r+2)c_r\, c_{n-r}$$

where c_n denotes the coefficient of $(-x)^{3n+2}$.

Numerical studies of the coefficients (see "Analysis and Improvement of Perturbation Series" by Milton Van Dyke, Q.J.M.A.M., volume 27, 1974, page 443) suggest that the radius of convergence of the series is about 3.127 and that the function has a singularity where the circle of convergence meets the negative real axis. If this is so, then what kind of singularity is it? Pole? Essential singularity? Branch point?

COMPLETE DIFFERENCE TRIANGLES

George Berzsenyi

For each $n = 1, 2, 3, \ldots$, consider a finite sequence of positive integers $k_1 < k_2 < \ldots < k_n$, and construct a complete difference table thereof. If all of the entries of this difference table are positive integers distinct from one another and from the members of the original sequence, we term the difference triangles complete. It is not difficult to show that complete difference triangles exist for each $n = 1, 2, 3, \ldots$, and hence K_n, the smallest possible value of k_n, is well-defined. The author found that the first few values of the sequence $\langle K_n \rangle$ are 1, 3, 8, 20, 43 and 98, as exemplified by the complete difference triangles given below.

```
1        1        4         4
            2        1         2
         3        5    2    6     1
                     3         3     7
                  8         9     8
                               11
                            20
```

```
7                             10
   2                              2
9     1                       12     3
   3     4                        5      1
12    5     6                 17     4      6
   8    10                        9      7      8
20   15                       26    11     14
   23                             20    21
43                            46    32
                                  52
                              98
```


He communicated the problem to Stanley Rabinowitz of Digital Equipment Corporation (Nashua, NH), who found the next three values of the sequence are 212, 465 and 1000 (resulting, for example, from $\langle$ 13, 19, 29, 44, 67, 110, 212 $\rangle$, $\langle$ 16, 24, 38, 59, 91, 143, 241, 465 $\rangle$ and $\langle$ 21, 34, 53, 82, 126, 193, 303, 515, 1000 $\rangle$, respectively). One of his colleagues (at D.E.C.) claims that the next member of the sequence is 2144. Can you verify this find? What are the next few members of the sequence? What about nice, non-trivial upper and/or lower bounds? (The problem arose while the author was preparing an examination question in numerical analysis.)

ADDING NUMBERS

C. J. Smyth

Suppose that the set $\{1,2,3,\ldots,3n\}$ is divided into three disjoint sets A, B and C, each containing n of the numbers. Is it necessarily possible to choose x in A, y in B and z in C, such that one of these three numbers is the sum of the other two (or in other words, that one of the four sums $x \pm y \pm z$ is zero)?

MODEL FOR TAXATION

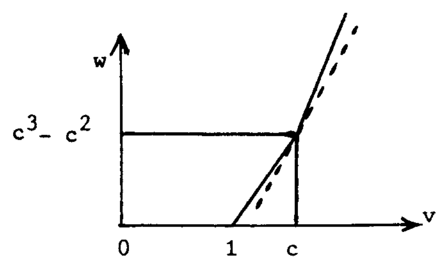
This model is a game played by the beloved leader, (BL for short) and any number (n) of plebeians, $P_1, P_2, \ldots P_n$. The game is that BL chooses a "tax policy", that is, a functional relation, giving v as a monotonic increasing function of u for $0 \leq u < \infty$; it is subject to the constraint $v \geq 1$. We interpret u as the income earned by a plebeian, and v as the amount left after deduction of tax. The constraint $v \geq 1$ may be regarded as a provision of "dole" or "pension" or "negative tax" for those not earning enough. It is convenient to put $w = uv$ and to define the tax policy by the function $w = G(v)$. Each plebeian, given the tax policy, chooses u and v (satisfying the relation) to maximise $v(b-u) = bv - w$ (where b is a positive parameter, b_1 for P_1, b_2 for P_2, etc.). We may interpret this function as a measure of quality of life, equal to take-home pay multiplied by leisure time. The payoff to BL is the total tax collected $\sum u_r - v_r$ and the objective of BL is to maximise it.

The case $n = 1$ is reasonably simple. Let $P = P_1$ have parameter $b = b_1 > 1$. The optimum strategy for BL can be approached but not attained, as follows. Take c just a little smaller than $\sqrt{b}$, then for the function G take

$$G(v) = c^2(v - 1) \text{ for } 1 \leq v \leq c$$

$$G(v) = c^2(c - 1) + B(v - c) \text{ for } v > c$$

where B is any number > b.



The choice to be made by P is to maximise bv−w, which roughly is the finding of a tangent of slope b to the curve shown, the result is that $u = c(c - 1)$ and $v = c$. The payoff to BL is $u - v = c^2 - 2c = (c - 1)^2 - 1$, which may be negative but is greater than −1. The payoff therefore can be arbitrarily near to $b - 2\sqrt{b}$.

Note that BL cannot do any better than this, because if so there would be u and v such that $u - v > b - 2\sqrt{b}$, and in this case

$$v(b - u) < v(2\sqrt{b} - v) = b - (v - \sqrt{b})^2 \leq b,$$

and this option would not be acceptable to P, who can always attain the value b for the objective function $v(b - u)$ by choosing $u = 0$, $v = 1$.

If $b < 1$ the analysis above does not apply, and we proceed as follows. Take any u and v with $0 < u < b < 1$, and observe that $(b - u)(1 + u) = b - u(1 - b + u) < b$.

If P were to prefer earning u (and taking home v) to a life of idleness (with $u = 0$ and $v = 1$) then

$$v(b - u) > b.$$

By the inequality above it would follow that

$$v > b/(b - u) > 1 + u.$$

The payoff to BL would then be $u - v < -1$; to prevent this, BL must choose the tax policy (any positive tax rate will do) to ensure that P is not tempted to work.

To sum up, in the case $n = 1$, the payoff to BL is −1 for $0 < b < 1$ and $b - 2\sqrt{b}$ for $b > 1$. These calculations also apply to the case of $b_1 = b_2 = \ldots = b_n = b$.

Now, what can be done about the case of $n = 2$?

The cases where $0 < b_1 < 1 \leq b_2$, or $1 < b_1^2 < b_2$, present no real difficulty, for P_2 is treated as a solitary citizen and BL can obtain (nearly) $b_2 - 2/b_2$ in tax from P_2, but pays out the dole to P_1, with the resulting payoff to BL being $b_2 - 2/b_2 - 1$.

For $1 < b_1 < b_2 < b_1^2$ there is again the situation that the optimum for BL may be approached arbitrarily closely but not attained; the limiting function $uv = w = G(v)$ is piecewise linear with the gradients b_1 and b_2 in the first two linear segments. The arithmetic reduces to solving a cubic, $v_2 = b_1 - u_1 = b_2 - u_2$ is the positive root of $x^3 = b_1 x + b_1(b_2 - b_1)$. For example, if $b_1 = 8$ and $b_2 = 12$, then $u_1 = 4$, $v_1 = 2$, $u_2 = 8$, $v_2 = 4$, and the total tax gathered is 6.

If instead of a finite number of plebeians we have a continuous distribution specified by a density function, the optimising problem for BL becomes an interesting problem of mathematical programming.

LINEAR EQUATIONS AND LATIN SQUARES (JCMN 33, p. 4033)

J. B. Parker

Given the positive integers a, b and c, we are interested in families of solutions (x_1, x_2, x_3) such that if S_i (for i = 1, 2, 3) is the set of $|x_i|$, the three families S_i are permutations of one another.

1. (a) Since S_3 has a largest member, there must be one solution in which $|x_3| \geq$ both $|x_1|$ and $|x_2|$. Then $c|x_3| = |ax_1 + bx_2| \leq a|x_1| + b|x_2| \leq (a + b)|x_3|$, and so $c \leq a + b$.

   (b) Suppose that the prime p divides a and b but not c, and that there is a solution set. We may suppose that not all of the x_{ij} are divisible by p, for otherwise we could divide them all by p to get a new solution set. Then one member of S_3 is not divisible by p; the corresponding solution has x_3 not divisible by p. This is a contradiction because $cx_3 = -ax_1 - bx_2$ which is divisible by p.

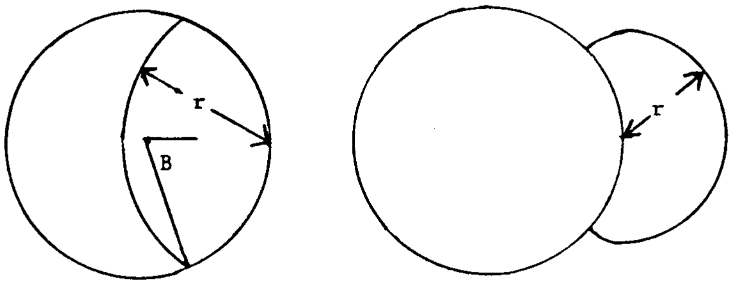
2. For a = 6, b = 11 and c = 13, the set of six solutions : (1, 3, -3), (3, 9, -9), (5, -11, 7), (7, -5, 1), (9, 1, -5) and (11, 7, -11) gives S_1, S_2 and S_3 all permutations of (1, 3, 5, 7, 9, 11).

3. If $a^2 + b^2 = c^2$ then the two solutions (b, c - a, -b) and (c - a, -b, c - a) form a solution set with S_1, S_2 and S_3 all permutations of (b, c - a).

GOAT AND COMPASSES (JCMN 33, p.4034)

A goat is tethered to a point on a circular fence of unit radius, what length r of the tether will let the goat graze over an area of $\pi/2$?

Those experienced in pastoral matters will know that a goat may be inside or outside a field, and that the latter is more likely.



(a) (b)

In case (a) we may find $r = 1.1587$ as $r = 2 \sin B/2$ where B satisfies the transcendental equation

$$\sin B + (\pi - B) \cos B = \pi/2.$$

Clearly B/π is irrational, but is r transcendental?

In case (b) $r = 0.9151$ is the positive root of the cubic

$$2r^3/\pi + 3r^2 = 3$$

and r is transcendental because π is.

COMPLEX INEQUALITY (JCMN 32, p.4017)

Given three complex numbers u^2, v^2 and w^2, all of modulus one, each of u, v and w is defined only up to an ambiguity of sign. There are two of eight possible combinations of sign for which

$$|2u^2 + 2v^2 + 2w^2 + 3uv + 3vw + 3wu| \le |u^2 + v^2 + w^2|.$$

This is just translation into algebra of a geometrical result pointed out by A.P. Guinand (JCMN 30,p.3127). Interpreting u^2, v^2 and w^2 as the vertices of a triangle ABC, the circumcentre is the origin, the orthocentre H is $u^2 + v^2 + w^2$, and the centroid G is $(u^2 + v^2 + w^2)/3$. The points $-vw-wu-uv$ (four of them because of the ambiguities of sign) are the centres of the inscribed circle and the three escribed circles. The inequality is the assertion that one of these four points (the incentre) is at distance less than $|u^2 + v^2 + w^2|/3$ from the point $(2/3)(u^2 + v^2 + w^2)$. There is equality only for the case of an equilateral triangle, with $u^2 + v^2 + w^2 = 0$, and the degenerate cases when two of the original numbers (or points) are the same. If $v^2 = w^2$ then put $w = -v$, the two sides of the inequality are $|2u^2 + v^2|$ and $|u^2 + 2v^2|$ which are clearly equal because of u and v having unit modulus.

CORRECTION

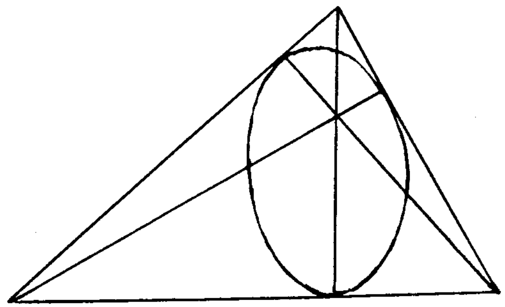
In the previous issue on page 4031 there was given a formula

$$\frac{2\Delta\,(\cos A,\ \cos B,\ \cos C)}{R(\sin 2A + \sin 2B + \sin 2C)}$$

for the circumcentre in trilinear coordinates. The editor regrets having perpetrated this monstrosity; it is not wrong, but should be more neatly expressed as

$$R(\cos A,\ \cos B,\ \cos C)\ .$$

POINT IN A TRIANGLE (JCMN 32, p.4008 and 33, p.4030)



Another characterization of the symmedian point of a triangle is that it is the centre of the conic that touches the sides at the feet of the altitudes. This fact is given as Exercise 41 on page 191 of E. H. Askwith's "Pure Geometry" (C.U.P, second edition, 1917).

To prove this, note (using trilinear coordinates as before) that the conic with line equation

$$mn \cos A + ln \cos B + lm \cos C = 0$$

touches the line (1, 0, 0) at the point with equation

$$n \cos B + m \cos C = 0.$$

This point, with point coordinates (0, cos C, cos B), is the foot of the altitude from A, and the other two points of contact may be verified similarly. The line at infinity has coordinates (a, b, c), and its pole, the centre of the conic, has equation

$$(mc + nb)\cos A + (lc + na)\cos B + (lb + ma)\cos C = 0$$

$$\text{or} \quad la + mb + nc = 0.$$

The centre of the conic has point coordinates (a, b, c) and it is the symmedian point.

A related fact is that the imaginary conic with line equation $l^2 + m^2 + n^2 = 0$ (with respect to which the triangle is self-polar) is confocal with the conic under discussion, because the line equation

$$l^2 + m^2 + n^2 = 2mn \cos A + 2ln \cos B + 2lm \cos C$$

of the pair of circular points is linearly dependent on the line equations of the two conics.

Are the principal axes of this conic (touching the sides at the feet of the altitudes) the same as the principal axes of inertia of the system of particles of masses a^2, b^2 and c^2 at the vertices A, B and C respectively?

POINT IN A TRIANGLE (JCMN 32, p. 4008, and 33, p. 4030)

I. B. Tabov

Consider the triangle ABC with sides a, b, and c in the usual notation. The plane $ax + by + cz = 2\Delta$ (in three-dimensional Cartesian coordinates) cuts the axes in points A', B' and C'. The foot of the perpendicular from the origin on to the plane (which is the orthocentre of A'B'C') has Cartesian coordinates the same as the trilinear coordinates of the symmedian point of ABC. This construction, treating Cartesian coordinates in the A'B'C' plane as trilinear coordinates in the ABC plane, gives the (unique) projective mapping between ABC and A'B'C' such that the symmedian point of ABC maps into the orthocentre of A'B'C'. This mapping has the property that the centroids of the two triangles map into one another.

EDITORIAL

Reprints of issues 1 - 31 (published by the University) are still available from

Mathematics Department,
James Cook University of North Queensland,
Post Office James Cook,
Townsville, N.Q. 4811, Australia,

bound as paperback volumes;

Volume 1 (Issues 1-17)
Volume 2 (Issues 18-24)
Volume 3 (Issues 25-31).

The price is $10 (Australian) per volume (including surface mail postage), and cheques for these volumes should be made payable to James Cook University.

Since Issue 32 (October 1983) the JCMN has been published by me (the Editor). Subscriptions and contributions will be welcomed. My address is either at the University (see above or at home (see page 4046).

Basil Rennie