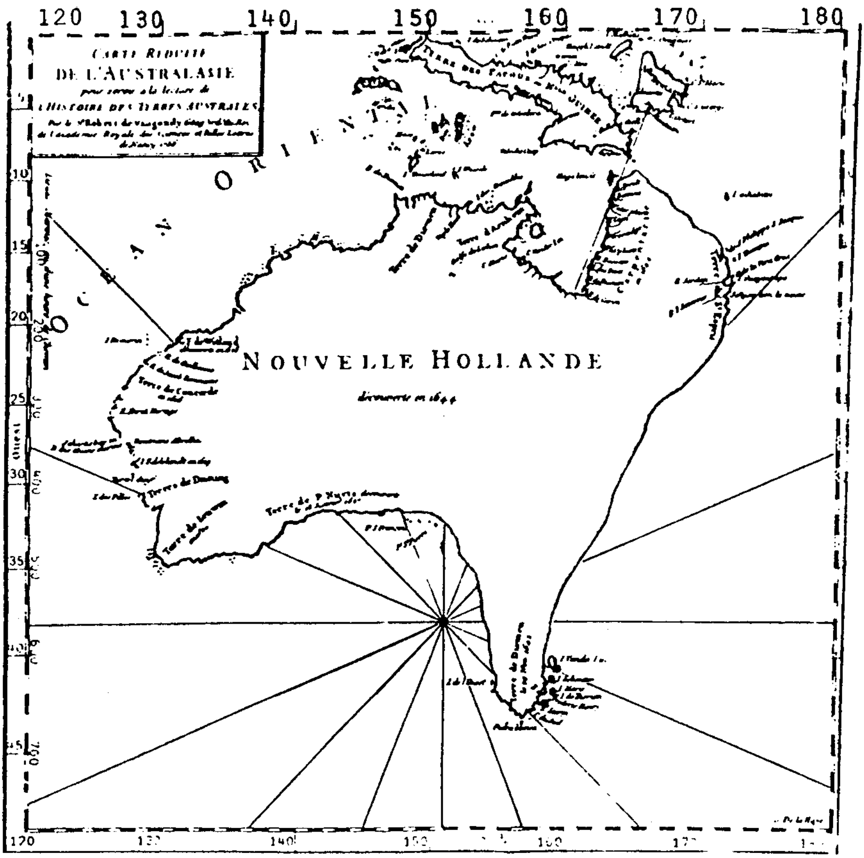
JAMES COOK MATHEMATICAL NOTES

Issue No. 30, Vol. 3

December, 1982.



A map published in France in 1756, 22 years before James Cook's first voyage of exploration in HMS Endeavour. Note how the longitudes, measured East from Paris, are all too large, the errors are about 14° for Tasmania, 18° for Cape Leeuwin in the South West, and 27° for the North Queensland coast near Townsville.

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TRIANGLES FROM CENTRES OUT

A.P. Guinand

Introduction. In a recent note W. Wernick [4] tabulates 139 re-construction problems for triangles ABC, given three related points. Among the 41 problems still unsolved is that where the in-centre I is given, together with any two from the circum-centre O, the centroid G, and the orthocentre H.

Since OGH forms the Euler axis and G divides OH in the ratio 1:2, any two of the last three points determine the third. Also, by considering angles subtended at a vertex it is easily seen that if any two of O, G, H, I coincide then they all coincide, and any equilateral triangle with that centre is a solution. So let us exclude that case, and assume that these four points are all distinct.

Next, it seems intuitively obvious that the in-centre I cannot be too far separated from all of the other centres; it certainly cannot lie outside the circum-circle. On investigation it turns out that I must lie inside the circle on GH as diameter, and consequently the angle GIH cannot be acute. I have not been able to find this result in the literature. On the contrary, there is a problem in Hobson's "Plane Trigonometry" [3] about triangles ABC for which GIH is equilateral. This is impossible unless ABC is itself equilateral, and GIH a mere point, but this slip has apparently remained unnoticed in all seven editions and one re-printing from 1891 to 1939.

As for the re-construction problem, a cubic equation can be found whose roots are the cosines of the angles of the triangle, whence it can be shown that a general 'ruler-and-compass' solution cannot exist.

Nevertheless this cubic leads to a curious family of quartic loci for I when O, H, and one angle of ABC are given, and then these curves can be used to map solutions of the problem.

The critical circle. If R denotes the circum-radius, and r the in-radius of the triangle ABC, then known results are: [3]

$$\cos A + \cos B + \cos C = 1 + 4 \sin \tfrac{1}{2}A \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C, \quad (1)$$

$$\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2 \cos A \cos B \cos C, \quad (2)$$

$$r = 4R \sin \tfrac{1}{2}A \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C = R(\cos A + \cos B + \cos C - 1) \quad (3)$$

$$OI^2 = R(R - 2r), \quad (4)$$

$$OH^2 = R^2(1 - 8 \cos A \cos B \cos C), \quad (5)$$

$$IH^2 = 2r^2 - 4R^2 \cos A \cos B \cos C. \quad (6)$$

The first three formulae are simple trigonometric exercises, the last three follow by applying the cosine rule in the triangles AOI, AOH, AIH, respectively. Since OG:GH = 1:2 it follows from Stewart's theorem [2] that

$$2.OI^2 + IH^2 = 3.IG^2 + 2.OG^2 + GH^2 = 3.IG^2 + (3/2).\ GH^2.$$

Hence by (4), (5), and (6)

$$GH^2 - (IG^2 + IH^2) = (2/3)(OH^2 - OI^2 - 2.IH^2)$$

$$= (2/3)R^2(1 - 8 \cos A \cos B \cos C) - (2/3)R(R - 2r)$$

$$- (4/3)(2r^2 - 8R^2 \cos A \cos B \cos C)$$

$$= (4/3)r(R - 2r) = \frac{4r}{3R} \times OI^2 \geq 0,$$

with equality if and only if O and I coincide (equilateral ABC) or r = 0 (degenerate ABC with one angle zero). Hence for all other ABC we have $GH^2 > IG^2 + IH^2$, whence:

Theorem 1. For all non-degenerate, non-equilateral triangles the in-centre I lies inside the circle on diameter GH.

We shall refer to this circle as the 'critical circle'.

The cubic for angle cosines and its consequences. From (4), (5), (6)

$$2.OI^2 + 2.IH^2 - OH^2 = (R - 2r)^2, \text{ and } OH^2 - OI^2 - 2.IH^2 = 2r(R - 2r).$$

Writing $\alpha = OH^2/OI^2$, and $\beta = IH^2/OI^2$ (7)

it follows successively that

$$R = OI^2\{2.OI^2 + 2.IH^2 - OH^2\}^{-\frac{1}{2}}, \quad (8)$$

$$2r = \{OH^2 - OI^2 - 2.IH^2\}\{2.OI^2 + 2.IH^2 - OH^2\}^{-\frac{1}{2}},$$

$$\cos A \cos B \cos C = \frac{1}{8}\{1 - \frac{OH^2}{R^2}\} = \frac{1}{8}\{1 - \frac{OH^2}{OI^4}(2.OI^2 + 2.IH^2 - OH^2)\}$$

$$= \frac{1}{8}(\alpha^2 - 2\alpha\beta - 2\alpha + 1), \quad (9)$$

$$\cos A + \cos B + \cos C = 1 + \frac{r}{R} = 1 + \{OH^2 - OI^2 - 2.IH^2\}/(2.OI^2)$$

$$= \tfrac{1}{2}(\alpha - 2\beta + 1). \quad (10)$$

Then by (2) $\cos B \cos C + \cos C \cos A + \cos A \cos B$

$$= \tfrac{1}{2}(\cos A + \cos B + \cos C)^2 - \tfrac{1}{2}(\cos^2 A + \cos^2 B + \cos^2 C)$$

$$= \frac{1}{8}\{(\alpha - 2\beta + 1)^2 - 4 + (\alpha^2 - 2\alpha\beta - 2\alpha + 1)\}$$

$$= \tfrac{1}{4}(\alpha^2 - 3\alpha\beta + 2\beta^2 - 2\beta - 1). \quad (11)$$

From (9), (10), and (11) we have:

Theorem 2. If α, β are defined by (7) then the cosines of the angles of the triangle are the roots of the cubic in λ

$$\lambda^3 - \tfrac{1}{2}(\alpha - 2\beta + 1)\lambda^2 + \tfrac{1}{4}(\alpha^2 - 3\alpha\beta + 2\beta^2 - 2\beta - 1)\lambda - \frac{1}{8}(\alpha^2 - 2\alpha\beta - 2\alpha + 1) = 0. \quad (12)$$

It is known that the classical problems of cube duplication, angle trisection, and construction of a regular heptagon cannot be solved by ruler-and-compass methods alone because they correspond to cubic equations with rational coefficients but no rational roots. [1] The same considerations apply here. For example, if $OI = \sqrt{3}$, $OH = \sqrt{6}$, and IH = 1 then I is within the critical circle and (12) can be arranged as a cubic in 6λ

$$(6\lambda)^3 - 7(6\lambda)^2 + 5(6\lambda) + 9 = 0. \quad (13)$$

This has no integral root in 6λ. Suppose that it has a rational root $6\lambda = p/q$, with co-prime integers p, q and $q \geq 2$. Then

$$p^3 - q(7p^2 - 5pq - 9q^2) = 0.$$

Hence $q | p^3$, contradicting the hypothesis that p, q are co-prime. That is, (13) has no rational root.

Theorem 3. It is not, in general, possible to re-construct a triangle from its in-centre, circum-centre, and orthocentre (or centroid) by ruler-and-compass methods alone.

However, if the angles are calculated with sufficient accuracy from the cubic (12), then we can construct a triangle with these angles, together with its various centres, and then adjust its scale, location, and orientation to fit the given centres.

Alternatively, the circum-radius R can be calculated from (8), and then the ray from O to A located by the formula

$$\tan AOH = \frac{\sin 2B - \sin 2C}{1 + \cos 2B + \cos 2C},$$

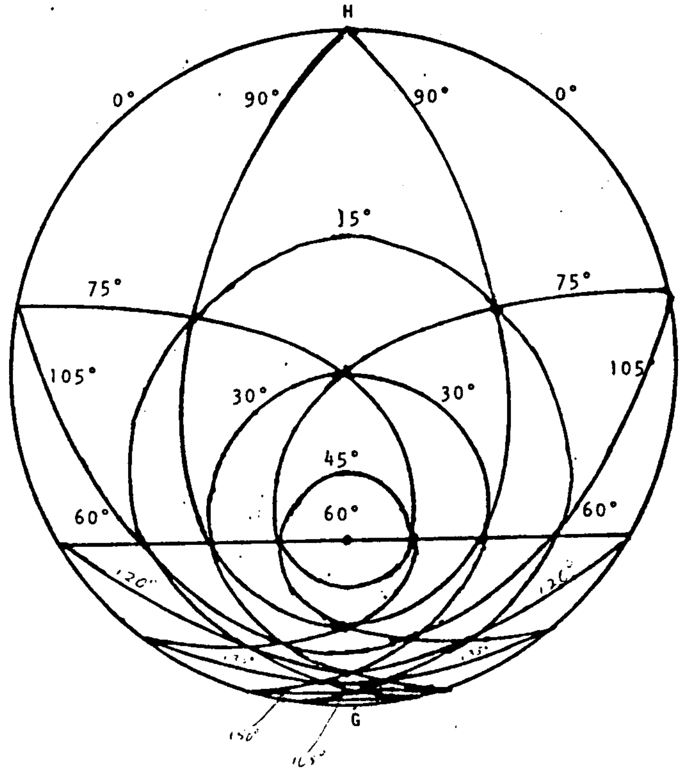
and similarly for the other vertices. Proof is left to the reader.

Graphical methods. If O is taken as origin of cartesian coordinates, H as (1, 0), and I as (x, y), then $OH = 1$, $OI^2 = x^2 + y^2$, and $IH^2 = (x-1)^2 + y^2$. Substituting these in Theorem 2 and putting $\lambda = \cos\theta$ then gives the locus of in-centres I for triangles with one angle equal to θ. Clearing the resulting equation of fractions, we get:

Theorem 4. For triangles with circum-centre (0, 0) and orthocentre (1, 0) and one angle equal to θ the locus of in-centres consists of that part of the bicircular quartic

$$\begin{aligned}&(2\cos\theta - 1)(2\cos\theta + 1)^2(x^2+y^2)^2\\ &+ 4\cos\theta(2\cos\theta + 1)(x^2+y^2)(1-2x)\\ &- 2(2\cos\theta - 1)(\cos\theta + 2)(x^2+y^2) + 4\cos\theta(1-2x)^2\\ &- 2(3\cos\theta - 1)(1-2x) + (2\cos\theta - 1) = 0 \qquad (14)\end{aligned}$$

which lies inside the critical circle.



For given θ and x the equation (14), formidable though it looks, is only a quadratic in y^2. So it is easy to programme a pocket computer to help draw the curves concerned. The figure shows the curves for $\theta = 0°, 15°, 30°, \ldots, 165°, 180°$. The critical circle (repeated) corresponds to degenerate triangles and to $\theta = 0°$. For $\theta = 60°$ the locus is the line $x = \frac{1}{2}$ together with a point circle at $(\frac{1}{2}, 0)$, the nine-point centre. For $\theta = 120°$ there is part of a hyperbola, and for $\theta = 180°$ there is only the point $(1/3, 0)$ on the critical circle at G.

In general, for points inside the critical circle there are three curves through the point, corresponding to three values of θ whose sum is 180°, so a sufficiently fine grid on the pattern of the figure could be used to read off the angles of the triangle from the position of I.

Ex-centres. Similar methods show that all centres of escribed circles lie outside the critical circle, and that their loci correspond to the remaining parts of the family of curves (14).

REFERENCES

[1] R. Courant & H. Robbins, What is Mathematics?, (Oxford, 1941), pp. 134-9.

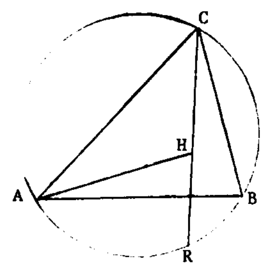
[2] H.S.M. Coxeter & S.L. Greitzer, Geometry re-visited, (New York, 1967), pp. 6, 18.

[3] E.W. Hobson, A Treatise on Plane Trigonometry, (Cambridge, 1939), Chapter XII and p. 201 (9).

[4] W. Wernick, Triangle Constructions with Three Located Points, Mathematics Magazine, 55, (1982), 227-30.

GEOMETRY BY NUMBERS

A question in a recent Hungarian Mathematical Olympiad reminds us how useful complex numbers are in plane geometry. To discover the Euler line and nine-point circle of a triangle, take the origin at the circumcentre and let the complex numbers α, β, and γ, all of modulus one, represent the vertices A, B and C. Then $(\alpha + \beta + \gamma)/3$ is the centroid, and $(\alpha + \beta + \gamma)/2$ is the centre of the nine-point circle because it is at distance $\frac{1}{2}$ from the mid-points $(\alpha + \beta)/2$, etc. of the sides. Up to now the argument could have used vectors instead of complex numbers. The multiplication of complex numbers shows that $(\alpha - \beta)/(\alpha + \beta)$ has zero real part, and so the line AB is perpendicular to the line from C to the point H represented by $\alpha + \beta + \gamma$. Therefore H is the orthocentre. Then it is easy to observe that the nine-point circle also passes through the point represented by $\gamma + (\alpha + \beta)/2$, the mid-point of CH, and the two similar points. Now consider the point R represented by $-\alpha\beta/\gamma = -\alpha\beta\bar{\gamma}$. The ratio $(\gamma + \alpha\beta\bar{\gamma})/(\alpha + \beta)$ is real, and so R is on CH. Also $AR = |\alpha + \alpha\beta\bar{\gamma}| = |\beta + \gamma| = AH$ and so the foot of the perpendicular from C to AB is the mid-point of HR and represented by $(\alpha + \beta + \gamma)/2 - \alpha\beta\bar{\gamma}/2$, clearly also on the nine-point circle.



Problem - find the quartic equation whose roots are the centres of the incircle and the three escribed circles.

USELESS INFORMATION

A.P. Guinand

$$(2143/22)^{1/4} = \pi - 10^{-9} \times 1.007\ldots$$

MATRIX PROBLEMS JCMN 28/3072 AND 29/3121

H.M. Finucan

It is a pleasure to report that this Kestleman pure problem with the Brown applied solution has a statistical aspect too. Their matrix C, in the form $\frac{1}{2}C$ actually, occurs in the theory of time series; the eigenvectors *and* eigenvalues of $\frac{1}{2}C$ are used.

The $n \times n$ matrix C has elements $c_{rs} = 1$ if $r = s \pm 1$ and $c_{rs} = 0$ otherwise. Using only traditional school trigonometry it is easy to half-discover the scalar λ and corresponding (column) vector v which make $Cv = \lambda v$. The "trick" is to guess $v' = (\sin\theta, \sin 2\theta, \ldots, \sin n\theta)$ and $\lambda = 2\cos\theta$, then the third element of Cv is $\sin 2\theta + \sin 4\theta$ while the third element of λv is $2\cos\theta \sin 3\theta$ and these are obviously equal; similar considerations take care of all intermediate rows of C and the first row is even easier. So far θ has been arbitrary, but equality of the last elements of Cv and λv requires that $\sin(n+1)\theta = 0$. So the n distinct values of $\cos\theta$, for $\theta = \pi/(n+1)$ and its multiples, are the n eigenvalues of $\frac{1}{2}C$ and the eigenvectors are v's as above.

The quadratic form $b'\frac{1}{2}Cb$ is clearly $b_1b_2 + b_2b_3 + \ldots b_{n-1}b_n$ and this is the first serial covariance $\gamma_x(1)$ of the moving average series $\{X_t\}$

$$X_t = b_1Z_t + b_2Z_{t-1} + \ldots + b_nZ_{t-n+1} ,$$

where the Z's are uncorrelated "innovations" with zero means and unit variances. The result being discussed concerns the first serial *correlation* $\rho_x(1) = \gamma_x(1)/b'b$. This is a Rayleigh quotient and its maximum and minimum as b varies are the extreme eigenvalues of $\frac{1}{2}C$, the matrix of the numerator; these are $\cos\{\pi/(n+1)\}$ and $-\cos\{\pi/(n+1)\}$.

The extreme values of the higher serial correlations ρ_2, ρ_3, etc. may be studied by using the above bookwork for ρ_1 with no fresh appeal to matrix theory or trigonometry. As an example

take ρ_2 in the case $n = 5$ and denote $[b_1, b_2, b_3, b_4, b_5]$ by $[b, c, d, e, f]$ for ease. Then $\rho_2 = \dfrac{bd + ce + df}{bb + cc + dd + ee + ff}$ and the reader must not worry, bb etc. are *just* space-saving versions of b^2 etc. So

ρ_2 is a weighted average of

$$\frac{bd + df}{bb + dd + ff} \quad \text{and} \quad \frac{ce}{cc + ee}$$

with weights $bb + dd + ff$ and $cc + ee$, which are positive. Now the first fraction here is the $\rho(1)$ of a moving average with coefficients, b, d, f and can attain a maximum of $\cos(\pi/4)$ by choice of $b{:}d{:}f$ - while the second can ONLY attain $\cos(\pi/3)$. So we give all the weight to the first, that is take $c = e = 0$, then maximise ρ_2 by taking $b{:}d{:}f = \sin(\pi/4) : \sin(\pi/2) : \sin(3\pi/4)$. For the minimum, change the sign of d only.

MATRIX PROBLEM (JCMN 28, p. 3072)

After seeing the comments on question 3 from the points of view of applied mathematics (A. Brown, JCMN 29, p. 3121) and of statistics (H. Finucan, above) an analyst is tempted to observe that to solve the eigenvalue problem no trick or guess work is needed. By introducing fictitious elements x_0 and x_{n+1}, both zero, the eigenvalue equations may be written

$$x_{r-1} + x_{r+1} = \lambda x_r \quad \text{for } r = 1, 2, \ldots, n .$$

The recursion has a solution $x_r = p^r$ if p is a root of the quadratic $p^2 - \lambda p + 1 = 0$. The roots are $p = \lambda/2 \pm (\lambda^2/4 - 1)^{\frac{1}{2}}$ and so it is appropriate to note (as on page 3121) that each eigenvalue λ is between ± 2 and may be expressed as $\lambda = 2\cos\phi$ where $0 < \phi < \pi$. Then $p = \cos\phi \pm i \sin\phi$ and the general solution of the recursion is $x_r = a\cos r\phi + b\sin r\phi$. The conditions $x_0 = x_{n+1} = 0$ lead to $\phi = k\pi/(n+1)$ $(k = 1, 2, \ldots, n)$.

MATHEMATICAL BLINDNESS

Carl Moppert,
Monash University.

I am writing this paper after reading Morris Kline's book: Mathematics, The Loss of Certainty (Oxford University Press 1980). What I want to say here is an enhancement of the statements in the book, a shifting of the weights of some of the arguments, putting forward some simpler and more basic criticism of mathematics and - last but not least - making the whole thing shorter and thus (I hope) more readable.

In calling the paper "blindness in mathematics" I claim to be less blind than others. This is, of course, a presumption; I am not aware of my own blind spots. However, it is obvious that a dialogue of those people who are dissatisfied is very necessary indeed. Where one man is blind, another might see and together they might be able to move forward.

Let us then start with geometry. One of the most startling facts about the history of geometry has been pointed out by Sophus Lie some hundred years ago: every thinking mathematician should have seen that the parallel postulate does not hold for great circles on the sphere. The question whether this postulate was dependent on the others should then never have been put. To put it differently: unless other axioms (betweenness etc.) which Euclid had omitted were considered, the question was obviously pointless.

I claim that the average mathematician of today is still not aware of the implications of Boliay and Lobatchewski's work. If I ask a mathematical colleague: "How accurate is Pythagoras?" I meet with an uncomprehending stare. Neither the average educated man nor the average professional philosopher is aware that one of the essential pillars of Kant's philosophy has been knocked over. At a dinner party with distinguished mathematicians in Germany I mentioned

that hyperbolic lines are "more than straight". The discussion caused the dinner to get cold.

Some more about this, my pet topic. Nobody seems to bother about the fact that Einstein assumes that the geometry in a Galileian environment is euclidean. More important: nobody worries about the existence of a cartesian coordinate system. We still live in a Cartesian paradise: numbers have been made visible. Few people are aware that Hilbert has reversed the process: the only way in which lines "exist" is as linear equations.

We are products of history, indoctrinated far beyond our awareness. In the last century, projective geometry was taken to be the key to the universe. I blame Klein with his enormous influence for a lot of the damage. It was again Hilbert who saw further: he recognised what a degenerate object the projective plane in fact is.

Morris Kline ranks Hamilton's discovery of the quaternions to be as important as the discovery of non-euclidean Geometry. I disagree. Operations that are not commutative are found all over the place, only mathematicians could have been so blind as not to see this. Every cook knows that the order in which he "adds" the ingredients to a soup is essential. I remember how deeply I was impressed as a student forty years ago when Heinz Hopf demonstrated bodily that the two operations: taking a step forward and making a quarter turn - are not commutative.

Instead, I rank the discovery of the p-adic numbers on a par with non-euclidean geometry. Hensel himself probably did so too: he expected to do with them quite unreasonable things. The theory of p-adic numbers shares the fate of non-euclidean geometry: who bothers? Do they belong to the knowledge of the average mathematician?

Let us look a bit more at analysis. In talking with a mathematical logician I mentioned that Dedekind in his paper: "What are numbers and what are they good for" proves that root two times root three equals root six. He says that none of his colleagues knew why this is so. Who of us is even aware that there is a problem? My logician friend was most surprised. Are we aware of the fact that if we say that the solution of the equation $x^2 = 2$ is root two we say exactly nothing?

There is no point in being alarmist; a mathematician can live happily ever after without bothering about geometry or number systems. What about analysis? Surely there everything is alright, Karl Weierstrass did not live in vain. The vague percepts of Newton and Leibniz have been put on a sure footing. A student who understands epsilon and delta knows what he is talking about. No wonder that mathematics can be used in such a magnificent way in engineering. Euler and other workers who found results by their smell were, after all, right.

I maintain that this is nothing but an illusion, and that we are as far from a proper foundation as ever, and my reason for this opinion is the following. Every application of analysis to a problem in physics starts with a differential equation. I think it is safe to say that every discovery of a physical law was in fact the discovery of a differential equation. Newton's laws are differential equations, Maxwell's equations are differential equations. The art is then to find differential equations. Once they are there it is the analyst's job to find solutions, or at least to establish their existence, and he does this with all of Weierstrass's paraphernalia. But - where do the differential equations come from? Can we justify a differential equation with epsilon and delta? I think that, red in the face, we have to admit that we can never arrive at a differential equation without thinking of small changes, i.e. of Leibniz's differentials.

I am afraid I was unable to respond to about a third of Morris Kline's book: to the section dealing with sets and Goedel's work. I must justify this rash statement. In my long career I have taught many subjects. There is one subject I refuse to teach, set theory. I am unable to understand the distinction between a set consisting of

one element and this element itself. Now in mathematics, if you don't understand A then you can't understand B. The whole of set theory including Goedel is therefore a closed book for me. I dare exposing my ignorance as I am sure that I am far from being alone. I mentioned my problem once to Paul Erdös. He said he could live happily without bothering about such questions. (If Prof. Erdös ever happens to read this he might sue me).

I used to tell my students that in my life I have applied mathematics twice. The first time was in my schooldays when I helped my mother making a conical lampshade. The second time was many years later when I made some steps in my house.

During the last ten years or so I have tried to apply my wide mathematical knowledge to problems in engineering. The result was dismal: almost always I found that trying to overcome a related mathematical problem I had overlooked a physical problem which was straight in front of my nose. Together with a friend I built a Foucault pendulum and I think I am justified in saying that it is the best one in existence (see Quarterly Journal of the Royal Astronomical Society 1980). However, it does not behave as it should. Moreover I found out that nobody is able to give a full analysis of the spherical Foucault pendulum. Now, if such a simple problem is not solved - ?

There is the famous record where you can hear Hilbert speaking. His last sentence is "wir müssen wissen, wir werden wissen." (we must know, we shall know). After that, you hear a weird cackling. I think Hilbert was much too intelligent to make such a stupid statement. He feared the cry of the Boeotians. In my opinion, any thinking man is not ashamed to say: ignoramus, ignorabimus.

COMPLEX GEOMETRY

H. Kestelman

If A is a complex $n \times n$ matrix write S(A) for the set of all the complex numbers (Rayleigh quotients) $\underline{x}^*A\underline{x}/\underline{x}^*\underline{x}$ (for all complex column vectors $\underline{x} \neq 0$). It is known to be a convex set. For each $n \geq 2$ find A such that S(A) is the unit disc $\{z;\ |z| \leq 1\}$, and show that no such A is diagonalizable (in the sense TA = DT with T non-singular and D diagonal).

BEAR STORY

C.F. Moppert

After reading about Cook's constant (JCMN 29, p. 3116) I am encouraged to point out to you Bear's constant.

Mummy Bear, Daddy Bear and Little Bear went for a walk. Little Bear said "Here we are, all five of us.". Why? Here are three solutions.

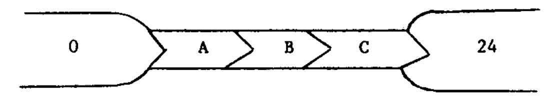
(i) Little Bear couldn't count.

(ii) He was thinking of his forbears.

(iii) Little Bear had been going to a course on set theory. The adder who sat next to him had explained that the lecturer was making cardinal numbers unnecessarily difficult, and that to count animals it was best to count the front feet, add on the number of noses and subtract the number of ears. Little Bear had done this and found the answer $6 + 3 - 4 = 5$.

STAIRCASE LOCKS (JCMN 28, pp. 3073-3075)

J.B. Parker



What can be said about the levels in the three lock chambers with ordinary usage?

If the levels are initially (a, b, c) then when a boat goes up they will be changed to (b/2, b/4 + c/2, 24), and after a long sequence of boats going up the limiting values will be (8, 16, 24). Similarly after a long sequence of boats going down the limiting levels are (0, 8, 16). It is easy to check that these limits are in fact bounds, in the sense that if the initial levels are between these bounds, and a boat goes either up or down then the final levels will also be between the bounds.

The important question is of course the level when there is a boat in the chamber. The level in chamber B has bounds 4 and 20, the lower bound approached when a boat goes up after a long sequence of boats going down, and vice versa.

BINOMIAL IDENTITY 15

C.J. Smyth

Prove that for $0 \leq k < n$

$$\sum_{j=k}^{n} \frac{2n}{n+j} \binom{n+j}{n-j} \binom{2j}{j-k} (-1)^{j-k} = 0.$$

SPECIAL FUNCTIONS (JCMN 29, p. 3103)

H. Kestelman

Let $\phi(x) = \int_{-\infty}^{x} \exp(-t^2/2)dt$. The problem was to show that $x + \phi'/\phi$ is positive and increasing.

Clearly ϕ and ϕ' are positive, and $\phi'' = -x\phi'$. Put $g(x) = x\phi(x) + \phi'(x)$, then $g' = \phi + x\phi' + \phi'' = \phi > 0$. Since $g(-\infty) = 0$ it follows that $0 < g/\phi = x + \phi'/\phi$. It remains to show that the derivative $1 + \phi''/\phi - (\phi'/\phi)^2$ is also positive. Put $f = \phi^2 + \phi\phi'' - (\phi')^2 = \phi^2 - x\phi\phi' - (\phi')^2$. Since $f(-\infty) = 0$ it will suffice to show that f' is positive, that is:

$$0 < f' = 2\phi\phi' - \phi\phi' - x(\phi')^2 - x\phi\phi'' - 2\phi'\phi''$$
$$= (\phi + x^2\phi + x\phi')\phi'$$

which is equivalent to showing that $\phi + x\phi'/(1+x^2)$ is positive. Since this expression tends to zero at minus infinity it will be sufficient to show that its derivative is positive, that is

$$0 < \phi' + (\phi' + x\phi'')/(1 + x^2) - 2x^2\phi'/(1 + x^2)^2$$
$$= ((1 + x^2)^2 + (1 + x^2)(1 - x^2) - 2x^2)(1 + x^2)^{-2}\phi'$$
$$= 2(1 + x^2)^{-2}\,\phi'$$, which is clearly positive.

GEOMETRICAL EXERCISE

H. Kestelman

If b is a given positive number, describe the set of points in the argand plane that correspond to complex numbers

$$\cos\phi + b \sin\phi \exp(i\theta)$$

for all real θ and ϕ.

MONOTONIC TRIADS IN A SEQUENCE

B.C. Rennie

A sequence of n unequal real numbers must contain at least $N(n)$ monotonic subsequences of length 3, where

$$N(2m) = m(m-1)(m-2)/3 = \frac{1}{4}\binom{n}{3}(1 - 3/(n-1))$$

and $$N(2m + 1) = m(m-\tfrac{1}{2})(m-1)/3 = \frac{1}{4}\binom{n}{3}(1 - 3/n)$$

We prove this and determine the cases where the bounds are attained. Some of the results are not new, they were given by H. Burkill and L. Mirsky, see Reference [1]. An earlier investigation of this problem is to be found in [2].

<u>The case of even $n = 2m$</u>

Take $\{-m + \tfrac{1}{2}, -m + 3/2, \ldots, -\tfrac{1}{2}, \tfrac{1}{2}, \ldots, m - \tfrac{1}{2}\}$ both as index set and as set of values for the sequence $s(r)$.

For any such sequence s and any r let

$\alpha = \alpha(r)$ = the number of t for which $t > r$ and $s(t) > s(r)$
$\beta = \beta(r)$ = the number of t for which $t < r$ and $s(t) > s(r)$
$\gamma = \gamma(r)$ = the number of t for which $t < r$ and $s(t) < s(r)$
$\delta = \delta(r)$ = the number of t for which $t > r$ and $s(t) < s(r)$.

In other words, α, β, γ and δ are the numbers of points of the graph of $s(t)$ as a function of t that are in the four quadrants meeting at the point $(r, s(r))$. The number of monotonic triads that have $(r, s(r))$ as middle member is $\alpha\gamma + \beta\delta$, and this number may be expressed in terms of any one of the four numbers α, β, γ and δ because

β	α
γ	δ

$$\left.\begin{aligned} \alpha+\beta &= m - s(r) - \tfrac12 \\ \gamma+\delta &= m + s(r) - \tfrac12 \\ \alpha+\delta &= m - r - \tfrac12 \\ \beta+\gamma &= m + r - \tfrac12 \end{aligned}\right\} \quad \dots (1)$$

Partition the 2m points of the graph into the four quadrants of the plane, that is, Q1 where r and $s(r)$ are both positive, Q2 where r is negative and $s(r)$ positive, and so on.

Q2	Q1
Q3	Q4

For points in Q1 the number $\alpha\gamma + \beta\delta$ may (using (1)) be expressed as

$$\alpha(\alpha + s + r) + (m - s - \alpha - \tfrac12)(m - r - \alpha - \tfrac12)$$

$$= \tfrac12(2\alpha - m + r + s + \tfrac12)^2 + (m^2 - r^2 - s^2 - m)/2 + 1/8$$

where $s(r)$ has been abbreviated to s.

Similarly for points in Q2 we write the number as

$$\tfrac12(2\beta - m - r + s + \tfrac12)^2 + (m^2 - r^2 - s^2 - m)/2 + 1/8$$

and there are similar expressions for points in Q3 and Q4 expressed in terms of γ and δ respectively. Because $\Sigma r^2 = \Sigma s^2 = m(2m - 1)(2m + 1)/6$, the total number of monotonic triads, which is the sum of all $\alpha\gamma + \beta\delta$, is

$$(4m^3 - 12m^2 + 5m)/12 + \tfrac12 B \qquad (2)$$

where

$$B = \sum_{Q1} (2\alpha - m + r + s + \tfrac12)^2 + \sum_{Q2} (2\beta - m - r + s + \tfrac12)^2$$

$$+ \sum_{Q3} (2\gamma - m - r - s + \tfrac12)^2 \quad \sum_{Q4} (2\delta - m + r - s + \tfrac12)^2 \qquad (3)$$

Since each of α, β, γ, δ, m and $r \pm s$ is an integer, each of the 2m terms in B is at least 1/4, and $B \geq m/2$.

The number of monotonic triads is therefore at least

$$(4m^3 - 12m^2 + 5m)/12 + m/4$$

$$= m(m-1)(m-2)/3 \qquad (4)$$

For this bound (4) to be attained it is necessary and sufficient that each term of (3) be 1/4. Define $\theta(r)$ to be α, β, γ, or δ, according to whether the point $(r, s(r))$ is in the quadrant Q1, Q2, Q3 or Q4. In fact $\theta(r)$ is the number of points $(t, s(t))$ of the graph that are in the same quadrant as $(r, s(r))$ and that have $|t| > |r|$ and $|s(t)| > |s(r)|$. Each term of B in (3) may be expressed as $(2\theta - m + |r| + |s| + \tfrac12)^2$, and for this to equal 1/4 it is necessary that

$$|r| + |s| = m - 2\theta - \tfrac12 \pm \tfrac12 \leq m.$$

Since $|r| + |s|$ has mean = m this relation implies $\theta = 0$ and

$$|r| + |s| = m \qquad (5)$$

Finally (5) implies that the points of the graph are in a diamond-shaped pattern and all $\theta(r) = 0$, and the bound (4) is attained. The 2^m sequences satisfying (5) are given by

$$s(r) = -s(-r) = \pm(m-r) \quad \text{for} \quad r = \tfrac12, 3/2, \dots, m - \tfrac12 .$$

The case of odd n = 2m + 1

The notation and reasoning are very much like those in the case of even n, but there are differences of detail. Take $\{-m, -m+1, \dots -1, 0, 1, \dots m\}$ as both index set and set of values for a sequence s with members $s(r)$. The numbers α, β, γ and δ are defined as before, but (1) is replaced by

$$\left.\begin{aligned} \alpha+\beta &= m - s \qquad & \alpha+\delta &= m - r \\ \gamma+\delta &= m + s \qquad & \beta+\gamma &= m + r \end{aligned}\right\} \quad \dots (6)$$

The four quadrants Q1, .. Q4 are as before but the specification does not define the quadrant to which we assign the points for which r = 0 or s(r) = 0, however this ambiguity does not matter, as long as each of the 2m + 1 points is assigned to one quadrant. For the total number $\Sigma\alpha\gamma + \beta\delta$ of monotonic triads, we have, instead of formulae (2) and (3),

$$(2m^3 - 3m^2 - 2m)/6 + \tfrac{1}{2}A \tag{7}$$

$$\text{where } A = \sum_{Q1} (2\alpha + r + s - m)^2 + \sum_{Q2} (2\beta - r + s - m)^2 + \sum_{Q3} (2\gamma - r - s - m)^2 + \sum_{Q4} (2\delta + r - s - m)^2 \tag{8}$$

Introducing $\theta(r)$ as before,

$$A = \sum (2\theta + |r| + |s| - m)^2 \tag{9}$$

Consider the contribution to A from the terms for which $|r| + |s| \geq m$. Since $\theta \geq 0$ the total is at least the sum of $(|r| + |s| - m)^2$ for all such r, and (since an integer is not greater than its square) the contribution is at least the sum of $|r| + |s| - m$ (for all such r), and therefore $\geq \sum_r (|r| + |s| - m) = m$.

This has shown that $A \geq m$, and the number of monotonic triads is at least

$$(2m^3 - 3m^2 - 2m)/6 + m/2$$
$$= m(2m-1)(m-1)/6 \tag{10}$$

It is clear that the bound is attained, for instance by the sequence (-1, -2, ... -m, m, m-1, ... 1, 0). If the bound is attained then $|r| + |s| - m$ takes the value 1 at m points and zero at the other m+1 points. Conversely if $|r| + |s| - m$ takes only the values 0 and 1 then all $\theta(r) = 0$ and the bound is attained. It can be proved by induction that the number of such sequences is 2^{2m}. For m = 1 the four possible such sequences are (0, 1, -1), its reversal, and their

negatives. We set up a 1-4 correspondence between sequences of length 2m + 1 with the property and those of length 2m + 3, as follows. Take any sequence s of this kind of length 2m + 1, add 1 to each positive value and subtract 1 from each negative value. Denote these new values by S(r). For the r for which s(r) = 0, put S(r) = ± 1. One of the two values ± 1 is still not allocated, assign it to one of the two end-points r = ±(m+1), and give the value zero to the other end-point. This constructs, in four different ways, a sequence S(r) with $|r| + |S(r)| = m+1$. Conversely, given such a sequence S(r) of length 2m + 3, the corresponding s(r) is constructed by omitting the two end members, reducing each positive value by 1 and increasing each negative value by 1, for it may be noted that of the two end members omitted, one must have value 0 and the other ± 1.

The author's thanks are due to H. Burkill for his help.

References

[1] H. Burkill and L. Mirsky, Monotonicity, J. Math. Anal. Appl. 41 (1973) 391-410.

[2] B.G. Eke, Monotonic Triads, Discrete Mathematics, 9 (1974) 359-363.

WRECK OF H.M.S. ENDEAVOUR

There is some doubt about the story of Captain Cook's ship after she was sold out of the Royal Navy in 1775. The author Clive Cussler when in Townsville in November gave a newspaper interview in which he said that the American National Underwater and Marine Agency was hoping to locate and identify the wreck of the former H.M.S. Endeavour on the mud flats off Rhode Island.

HARBOUR MASTER'S DILEMMA

For how long does a harbour master have to observe the water level in the harbour before being able to predict tides?

The corresponding pure mathematical problem is as follows. Suppose that we are given a countable set $\{\lambda_n;\ n = 1, 2, \ldots\}$ of positive numbers, we call it the spectrum. Consider all uniformly almost periodic real functions

$$f(t) = \tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos \lambda_n t + b_n \sin \lambda_n t)$$

with the given spectrum. Find the least upper bound K of all k such that there is a function f of the set, not identically zero, but vanishing in the interval from 0 to k. Alternatively K is the greatest lower bound of the set of b such that knowledge of the values of f in an interval of length b is sufficient to determine all the Fourier coefficients and so to determine f(t) for all real t. For example, if the spectrum is the set of positive integers then clearly $K = 2\pi$. The problem seems difficult, and solvers should feel free to impose on the spectrum any condition that appears helpful, such as that the sequence $\{\lambda_n\}$ should increase.

In the practical problem of tide prediction there are complications. One is that the level of the sea is affected by weather, which is notoriously unpredictable, and consequently the calculation is statistical. Another difficulty is that there is no generally agreed spectrum. The spectrum used in the numerical work must be finite. The frequencies are obtained from three or six fundamental astronomical data, but which of these, their multiples, and their sums and differences, are worth including, is a difficult question, often decided by study of the data itself. Lord Kelvin used 5 or 10 frequencies, but modern workers use as many as 400.

MATRIX INVERSION

R.B. Potts

Let p be any positive integer.

Let n be the integer part of (p-1)/2.

Let δ be 0 if p is even and 1 if p is odd.

Let $\alpha = \pi/p$.

Let $k = 2 \cos \alpha$.

Find the inverse of the $n \times n$ matrix:-

$$K = \begin{pmatrix} k & -1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & k & -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & k & -1 & \cdots & 0 & 0 \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \\ 0 & 0 & 0 & 0 & & k & -1 \\ 0 & 0 & 0 & 0 & & -1 & k+\delta \end{pmatrix}$$

AN EQUATION

C.J. Smyth

Equations of the form

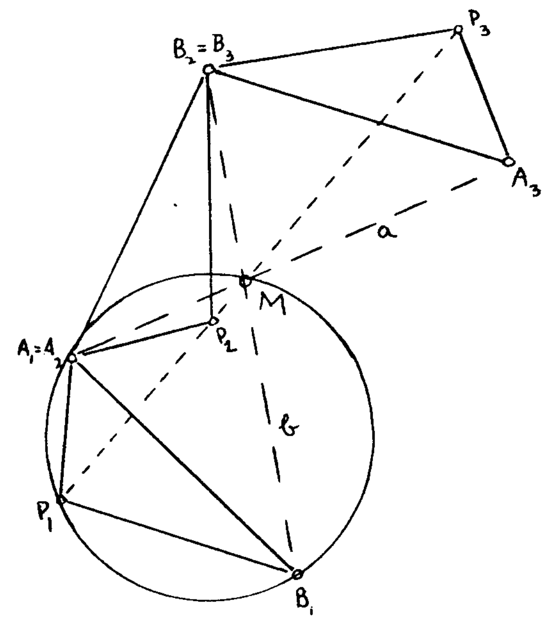
$$\prod_i (x^{n_i} - 1) = \prod_j (x^{m_j} - 1)$$

where the n_i, m_j are positive integers (not necessarily distinct), have arisen recently in connection with some identities relating to the dilogarithm function.

Show that 2 and 3 are roots of (different) equations of this type. Is 4? 5?

PLANE SAILING (JCMN 25, p. 3016)

C.F. Moppert



The plane L moves over the fixed plane E, taking three positions L_1, L_2 and L_3. Every point P of L has three positions P_1, P_2 and P_3 on E. Which points P in L have P_1, P_2 and P_3 collinear?

The answer is that the locus in L is a circle.

The transition from L_1 to L_2 is a rotation about some point A of L. Of the three positions A_1, A_2 and A_3 of A in E we know that $A_1 = A_2$. Similarly there is B with $B_2 = B_3$ corresponding to the move from L_2 to L_3. Let a be the line in E from $A_1 (=A_2)$ to A_3 and let b be the line from B_1 to $B_2 (=B_3)$. The transitions $L_1 \to L_2 \to L_3$ may be made continuous by constraining the plane L to move so that A remains on the line a and B on b. As L moves to every possible position with this constraint, any point P of L traces out on E an ellipse with centre at the intersection M of a and b (see figure). This ellipse degenerates into a straight line segment if and only if P is on the circle shown (which in the position L_1 is the circle through A_1, B_1 and M).

The locus in the moving plane L is the circumcircle of the triangle ABC. In the fixed plane E the three positions $A_i\ B_i\ C_i$ (i = 1, 2, 3) of the triangle ABC are the three reflections in its own sides of the triangle whose vertices are $A_1 = A_2$, $B_2 = B_3$ and $C_3 = C_1$.

BINOMIAL IDENTITY 14 (JCMN 29, p. 3111)

C.S. Davis

$$\sum_{j=m}^{k}\binom{j}{m}\binom{2k}{2j} = 4^{k-m}\,\frac{k}{2k-m}\binom{2k-m}{m} \quad \text{for } k>0 \text{ and } 0\le m\le k$$

The result may be generalized to half-integer k. Write n for the 2k of the original formula. Observe that the sum may be taken over all j, and indeed all sums below may, for convenience, be taken over all integers, except that sums over m are for $0 \le m \le n/2$.

Let $z = r\exp(i\theta)$ and compare the expansions in powers of r of $2\log(1-z)$ and of $\log|1-z|^2 = \log(1-2r\cos\theta + r^2)$

$$2(\cos n\theta)/n = \text{Coefficient of } r^n \text{ in } \sum_m (2r\cos\theta - r^2)^{n-m}/(n-m)$$

For brevity write c for $\cos\theta$ and s for $\sin\theta$

$$\cos n\theta = \sum_m (-1)^m \binom{n-m}{m}\frac{n}{n-m}\,2^{n-2m-1}\,c^{n-2m}$$

On the other hand by considering

$$\cos n\theta + i\sin n\theta = (c+is)^n = \sum \binom{n}{r} c^{n-r}(is)^r$$

$$\cos n\theta = \sum_j \binom{n}{2j} c^{n-2j}(c^2-1)^j$$

$$= \sum_m\sum_j \binom{n}{2j}\binom{j}{m}(-1)^m c^{n-2m}$$

Equating the two expressions for the coefficient of c^{n-2m} in $\cos n\theta$,

$$\sum_j \binom{j}{m}\binom{n}{2j} = 2^{n-2m-1}\frac{n}{n-m}\binom{n-m}{m} \quad \text{for } 0 \le 2m \le n \ne 0 .$$

A DOUBLY DEFINED INTEGER SEQUENCE
(JCMN 29, p. 3101)

C.S. Davis

For a sequence $S_1 = 1$, $S_2 = 5$, $S_3 = 40$, etc. we want a proof that the recursive definition

$$S_k = 2\sum_{m=1}^{k-1}(-1)^{k-m-1}\binom{2k}{k-m}S_m + (-1)^{k-1}\binom{2k}{k}/2 \qquad (k\ge 1)$$

is equivalent to

$$S_k = \sum_{m=1}^{k-1}\frac{2k}{k+m}\binom{k+m}{k-m}S_m + 1 . \qquad (k \ge 1)$$

We extend the first definition to all integer k by putting

$$S_0 = 1/2 \qquad \text{and} \qquad S_{-k} = S_k .$$

The first recurrence can then be written

$$S_k = \sum_{\nu=1}^{k-1}(-1)^{\nu-1}\left[\binom{2k}{\nu}S_{\nu-k}\binom{2k}{2k-\nu}S_{k-\nu}\right] + \tfrac{1}{2}(-1)^{k-1}\binom{2k}{k}$$

$$= \sum_{\nu=1}^{k-1}(-1)^{\nu-1}\binom{2k}{\nu}S_{\nu-k} + \sum_{\mu=k+1}^{2k-1}(-1)^{\mu-1}\binom{2k}{\mu}S_{\mu-k} + (-1)^{k-1}\binom{2k}{k}S_0$$

$$= \sum_{\nu=1}^{2k-1}(-1)^{\nu-1}\binom{2k}{\nu}S_{\nu-k}$$

$$= \sum_{\nu=0}^{2k}(-1)^{\nu-1}\binom{2k}{\nu}S_{\nu-k} + S_k + S_k .$$

Therefore $$S_k = \sum_{\nu=0}^{2k}(-1)^{\nu}\binom{2k}{\nu}S_{\nu-k}$$

Using what is sometimes called "umbral calculus" the equation may be written symbolically as

$$S_k \doteq S^{-k}(1-S)^{2k} \qquad \ldots (1)$$

where the sign $\doteq$ indicates that in the expansion of any polynomial in S and S^{-1}, S^{ν} is to be replaced by S_{ν}.

Defining ϕ_k for $k = 1, 2, \ldots$ as the function

$$\phi_k = \phi_k(S) = S^k + S^{-k} - 2,$$

the relation (1) becomes

$$S_k \doteq (S^{-1}(1-S)^2)^k = (S + S^{-1} - 2)^k$$

$$S_k \doteq \phi_1^{\,k} \qquad (2)$$

Now use the polynomial identity

$$\phi_k = \sum_{m=1}^{k} 2b_{km}\,\phi_1^{\,m}, \qquad (3)$$

where $b_{km} = \frac{k}{k+m}\binom{k+m}{k-m}$. Writing this as

$$\phi_k - \phi_1^{\,k} = \sum_{m=1}^{k-1} 2b_{km}\,\phi_1^{\,m}$$

and using (2), observing that $\phi_k \doteq S_k + S_k - 2S_0 = 2S_k - 1$, we obtain the second recurrence.

No doubt the identity (3), like all identities, is trivial and can be established in a variety of ways; it is practically (3) on p. 3110. It seems most natural to appeal to the expansion of $\cos n\theta$ in the note above (p. 3152) with $n = 2k$.

$$\cos 2k\theta = \sum_{m=0}^{k} (-1)^m \frac{k}{2k-m}\binom{2k-m}{m}(2\cos\theta)^{2k-2m}$$

Putting $\theta = \pi/2 - \phi$ and $m = k-n$ we obtain

$$\cos 2k\phi = \sum_{n=0}^{k} (-1)^n \frac{k}{k+n}\binom{k+n}{k-n}(2\sin\phi)^{2n} = \sum_{n=0}^{k} b_{kn}(2i\sin\phi)^{2n}.$$

Putting $\exp(2i\phi) = z$, so that $(2i\sin\phi)^2 = z + z^{-1} - 2$, this is

$$z^k + z^{-k} = 2 + \sum_{n=1}^{k} 2\,b_{kn}\,(z + z^{-1} - 2)^n$$

which is essentially (3).

DIVISION IN OLD GERMANY (JCMN 29, p. 3102)

P.H. Diananda

The first example, 1152450/325 is done as follows. The first digit of the answer must be 3, and so we write

```
Step 1      2
        1 1 5 2 4 5 0     (3
            3
```

The two above the line comes from $11 - 3\times 3 = 2$. The figures can be taken as indicating that $1152450 - 3000 \times 300 = 252450$. Next we put 2 after the 3 in the lower line because 32 is the start of 325.

```
Step 2      1
            2 9
        1 1 5 2 4 5 0     (3
            3 2
```

The 1 and 9 are obtained from $25 - 3\times 2 = 19$. The progress so far is that $1152250 - 3000\times 320 = 192450$. Next we put 5 (the third digit of 325) after 32 in the lower line

```
Step 3      1 7
            2 9 7
        1 1 5 2 4 5 0     (3
            3 2 5
```

The 1, 7 and 7 come from $192 - 3\times 5 = 177$. So far then, we have found that $1152250 - 3000 \times 325 = 177450$. This concludes the work on the first digit, 3, of the answer. Now we have to divide 177450 by 325, the first digit will be 5 and so we put this 5 after the 3 on the right, and proceed as before

```
Step 4        2
            1 7
            2 9 7
        1 1 5 2 4 5 0     (35
            3 2 5
              3
```

The 2 comes from 17 − 5 × 3 and we have 177450 − 500 × 300 = 27450.

```
Step 5       1
             2
           1 7
           2 9 7
         1 1 5 2 4 5 0      (35
           3 2 5
             3 2
```

The 1 is written because 27 − 5 × 2 = 17, and we have
1152450 − 3000 × 325 − 500 × 32 = 177450 − 500 × 32 = 17450.

The complete division does not seem to have appeared in the manuscript, it should have some zeros added as follows,

```
            0 0 0
            1 1 1
            2 2 9
          1 7 4 1 0
          2 9 7 9 3
        1 1 5 2 4 5 0        (3546
          3 2 5 5 5 5        (remainder zero)
            3 2 2 2
              3 3
```

The second example was dividing 1817437 by 276 and (with some zeros added) would be

```
              0
              1
              7 2
          0 2 3 5 5
          1 6 6 3 7
          6 9 1 8 5 3        (6584
        1 8 1 7 4 3 7
          2 7 6 6 6 6        (Remainder 253)
            2 7 7 7
              2 2
```


DEFINITIONS

P.A.M. Dirac said at the first lecture of his Quantum Mechanics course long ago that the difference between pure and applied mathematics was that a pure mathematician would start the lecture course by writing on the blackboard that a + b = b + a. But quantum mechanics, he said, was the exception. And he wrote up on the blackboard a + b = b + a.

Alternatively one can classify mathematicians by their attitudes to infinity. An applied mathematician is fond of infinity, using it as a good approximation for any inconveniently large number, such as the number of molecules in a bottle of water. A numerical analyst, on the other hand, avoids infinity like the plague, always using a large number as a good approximation to it. Different from both of these creatures is the logician, who cannot see the slightest resemblance between infinity and a hundred thousand million. Classical analysis is the art of being able to treat both infinity and a very large number with the same easy familiarity.

We should be sorry for pocket calculators, usually paralysed by meeting infinity, and for the superstitious undergraduates who regard infinity as a demon sent to punish those who divide by zero.

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EDITORIAL

We would like to hear from you anything entertaining or puzzling or significant, connected with either mathematics or Capt. James Cook.

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