The permanent editor will take over the reins again in October. He would like to hear from you about anything connected with mathematics or James Cook, R.N.

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JAMES COOK MATHEMATICAL NOTES

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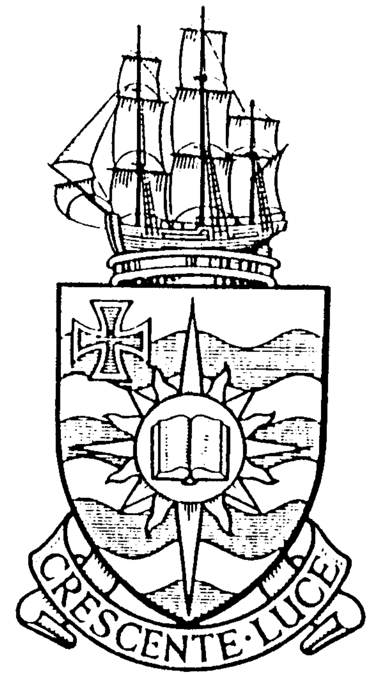


The Crest of James Cook University of North Queensland incorporating a representation of Captain Cook's ship the *Endeavour* in full sail.

KNOWING THE ANSWER

Sometimes I tell students that to solve a difficult problem it is a good idea to find the answer first. They think I am joking. And in a way I am. But there is a sense in which the idea can be taken seriously.

Last month a friend told me that in the inequality

$$\left|\int_0^1 f(x)g(x)dx - \int_0^1 f(x)dx\int_0^1 g(x)dx\right| \leq \frac{1}{8}\left(\int_0^1 (f'(x))^2dx\int_0^1 (g'(x))^2dx\right)^{1/2}$$

he doubted if 1/8 was the best possible constant. Could I find out? The problem looked difficult. I coult not even see how to prove the given inequality, and so what hope had I of discovering if there was a better one? At times like this it sometimes pay to fall back on the method of finding the answer first.

Try to find what the best possible constant is. This is a reasonably straightforward exercise in the calculus of variations, to find functions f and g on the interval (0, 1) such that f minimizes the integral of $(f')^2$ subject to the integrals of f and of fg being given, while g simultaneously satisfies a similar condition. The usual method leads to a pair of differential equations for f and g and the solution is that f(x) and g(x) are both of the form $a + b \cos m\pi x$ (with m an integer). This indicates that the relevant minimum is with $m = 1$, and that the appropriate constant in the inequality (to give equality in the critical case) is $1/\pi^2$ instead of 1/8. Such a use of the calculus of variations does not establish an inequality, but it indicates one that should be worth investigating.

Consider trying to prove:

$$\left| \int fg\,dx - \int f\,dx \int g\,dx \right| \le \pi^{-2}\left(\int (f')^2 dx \int (g')^2 dx \right)^{1/2} .$$

There is a factor π^{-2} telling us that trigonometric functions probably come into the calculation somewhere, and the fact that $\cos \pi x$ gives equality tells us the same thing. The method of proof is almost obvious. Expand each of the functions f and g in a half-wave cosine series and the Parseval identity gives what we want. The half-wave cosine series with terms in $\cos n\pi x$ is the appropriate form of Fourier series because it corresponds to a continuous periodic function with derivative having only simple discontinuities.

PERVERSE POLYNOMIALS

C.J. Smyth

Show that

(a) for all $n \geq 2$, $x^n - 2x^{n-1} + 1$ is a factor of a polynomial P_n all of whose coefficients are 0 or ± 1

(b) for no $n \geq 2$ is $x^n - 2x^{n-1} - 1$ a factor of a polynomials all of whose coefficients are 0 or ± 1.

RANK INEQUALITY

H. Kestleman

If A is a square matrix and n is a positive integer, show that the rank of A^{n+1} is at most $\frac{1}{2}$(rank of A^n + rank of A^{n+2}).

EASY HAHN-BANACH (JCMN 25, p. 3013)

If f is a real Lipschitz function on a finite subset Y of the plane, then can it be extended to a function F on the whole plane with the same Lipschitz constant?

Yes. W.A.J.L. Luxemburg writes to say that a general result of S. Banach states that if f is a function in a subset Y of a metric space (X, d) with Lipschitz constant k, then

$$F(x) = \inf_{y\in Y} f(y) + kd(x, y)$$

is an extension of f to X with the same Lipschitz constant.

The proof is not difficult: clearly $F(y) = f(y)$ for $y \in Y$, and for $x_1, x_2 \in X$, and given $\varepsilon > 0$

$$\begin{aligned} F(x_1) - F(x_2) &= \inf_{y\in Y} f(y) + kd(x_1, y) - (\inf_{y\in Y} f(y) + kd(x_2, y)) \\ &\le \inf_{y\in Y} f(y) + kd(x_1, y) - (f(y_1) + kd(x_2, y) - \varepsilon) \\ &\le f(y_1) + kd(x_1, y_1) - f(y_1) - kd(x_2, y_1) + \varepsilon \\ &\le kd(x_1, x_2) + \varepsilon . \end{aligned}$$

Since this is true for every $\varepsilon > 0$, $F(x_1) - F(x_2) \le kd(x_1, x_2)$. Interchanging x_1 and x_2 we get $|F(x_1) - F(x_2)| \le kd(x_1, x_2)$.

Other generalisations of the result for Lipschitz functions between normed spaces have appeared. See T.B. Flett (J. London Math.

Soc. (2) 7 1974, pp. 604 - 608). Banach's result appears in his book "Introduction to the theory of real functions", Monografie Mat. Tam. 17, P.W.N. Watson, 1951.

MATRIX EQUATIONS (JCMN 24, Vol. 2, p. 146)

We were given an n×n matrix M with distinct eigenvalues, and a polynomial f of degree k. We were asked to show that the equation $f(X) = M$ had between 1 and k^n solutions. *H. Kestelman's* solution is as follows:

By hypothesis we can diagonalize M so that $M = S\,\mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)S^{-1}$, where the j^{th} column of S spans the eigenvectors of M with eigenvalue λ_j. Choose complex numbers $z_1, z_2, \ldots, z_n$ so that $f(z_r) = \lambda_r$ $(r = 1, \ldots, n)$. Then

$$X = S\,\mathrm{diag}(z_1, \ldots, z_n)S^{-1}$$

satisfies $f(X) = A$. This shows $f(X) = M$ has at least one solution. To show that there are at most k^n solutions, it is enough to show that every solution X is constructed in this way, since there are k roots of $f(z_r) = \lambda_r$ for any r. Now every solution X commutes with f(X) and so with A. So $\lambda_r(Xv_r) = X(\lambda_r v_r) = X(Av_r) = A(Xv_r)$ which implies that Xv_r is a scalar multiple of v_r, say $c_r v_r$. Then $XS = S\,\mathrm{diag}(c_1, \ldots, c_n)$.

One can in fact show that $f(X) = M$ has at least k^{n-1} solutions. From the above, we know that the number of solutions is

$$\prod_{r=1}^{n} (\#\text{ of distinct solutions to } f(z) = \lambda_r).$$

Now $f(z) = \lambda_r$ and $f(z) = \lambda_s$ have no roots in common for $r \neq s$. Further, any multiple root of $f(z) = \lambda_r$ will be a root of $f'(z) = 0$. So if R_r is the number of common roots of $f(z) = \lambda_r$ and $f'(z) = 0$, counted with multiplicity, it follows that $f(z) = \lambda_r$ has at least $n - R_r$ distinct roots. Hence $f(X) = M$ has at least $\prod_{r=1}^{n} (k - R_r)$ distinct roots. Since $\sum_r R_r \leq k - 1$, it follows that the smallest value of $\prod_{r=1}^{n} (k - R_r)$ is k^{n-1}. This is attained when one of the polynomials $f(z) - \lambda_r$ is of the form $c(z - \alpha)^k$.

VARIATION ON A PROBLEM OF ERDÖS

A problem of Erdös asked whether, given $f(x) = \prod_{n=1}^{n} \sin(x - a_r)$, a_r all real, the ratio of the mean to the maximum of $|f(x)|$ is at most $\frac{2}{\pi}$. This was shown by E.B. Saff and T. Sheil-Small in J. London Math. Soc. (2), 9 (1972), pp. 16-22. They, however, posed another question: Is the ratio of the mean of $f(x)$ to the maximum of $|f(x)|$ at most $\frac{1}{2}$?

THE SURFACE AREA OF AN ELLIPSOID

Simple Simon encloses an ellipsoid of semi-axes a, b and c inside one of semi-axes $a + \delta$, $b + \delta$ and $c + \delta$. The difference in volume, he says, is $A\delta$ to first order in δ, where A is the surface area, and therefore $A = (4\pi/3)(ab + bc + ca)$.

Is this value too big or too small?

TRIGONOMETRIC FUNCTIONS (JCMN 25, p. 3013)

We had $f(x) = \sum_1^N a_r \exp(ib_r x)$ (with a_r complex and b_r real), $f(n) = O(1/n)$ for n large, and were asked if this implied that $\sin \pi x$ is a factor of $f(x)$.

Alf van der Poorten comments that the fact that $f(n) = 0$ for n sufficiently large follows from Turan's Second Main Theorem for power sums. Then, since $\{f(n)\}$ is a linear recurrence sequence of order at most N, it follows that $f(n) = 0$ for all integers n. Finally, from a theorem of Ritt which states that any quotient of exponential sums which is entire is actually an exponential sum, we have that $f(x)/\sin \pi x$ is an exponential sum, or $\sin \pi x$ is a factor of $f(x)$. A relevant reference to this circle of ideas is the survey article "On the growth of recurrence sequences", Math. Proc. Camb. Phil. Soc. <u>81</u> (1977), 369-376 by J.H. Loxton and A.J. van der Poorten.

Van der Poorten suggests that a 'first principles' solution should be possible. Here is one, based on solutions of *J.B. Parker* and *Chris Smyth*.

Put $\omega_r = \exp(ib_r)$ $(r = 1, \ldots, n)$. Then, if not all the a_r are 0,

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \omega_1 & \omega_2 & \cdots & \omega_N \\ \omega_1^2 & \omega_2^2 & \cdots & \omega_N^2 \\ \vdots & & & \\ \omega_1^{N-1} & \omega_2^{N-1} & \cdots & \omega_N^{N-1} \end{pmatrix} \begin{pmatrix} a_1\omega_1^n \\ a_2\omega_2^n \\ \vdots \\ a_N\omega_N^n \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{pmatrix} \quad \text{as } n \to \infty$$

say $V\underline{v}_n \to 0$. However $\det V = \prod_{r<s} (\omega_r - \omega_s)$, and the $\underline{v}_n$ lie in a compact region of $C^N \backslash \{\underline{0}\}$. Hence we can find an infinite subset of the $\underline{v}_n$ which tend to a limit $\underline{v} \neq 0$. But then $V\underline{v} = 0$, so $\det V = 0$, i.e. the ω_r are not all distinct.

However, if $\omega_r = \omega_s$, i.e. $b_r = b_s + 2\pi k$ say, then

$$a_r \exp(ib_r x) + a_s \exp(ib_s x) = (a_r + a_s) \exp(ib_r x) + a_s \exp(ib_s x)\,(1 - \exp(2\pi ikx)).$$

Since the second summand is divisible by $\sin \pi x$, this identity enables us to write $f(x)$ as

$$f(x) = f^*(x) + \text{terms divisible by } \sin \pi x$$

where

$$f^*(x) = \sum_1^{N^*} a_r^* \exp(ib_r^* x)$$

with the $\omega_r^* = \exp(ib_r^*)$ all distinct. But $f^*(n) \to 0$ as $n \to \infty$, so by the above argument all the a_r^* must be 0.

<u>Generalisation</u>. Given $z_r = \rho_r e^{i\theta_r}$ $(i = 1, 2, \ldots, N)$ all at least one in modulus, we shall show that if

$$f(x) = \sum_{r=1}^N a_r z_r^x \quad \left(= \sum_r a_r \rho_r^x \exp(i\theta x)\right)$$

and $f(n) \to 0$ as $n \to \infty$, then $f(x)$ is divisible by $\sin \pi x$. (Of course if f contained terms $a_r z_r^x$ with $|z_r| < 1$ the result would no longer be true).

The idea of the proof is to first use the previous result to show that the $\sum a_r z_r^x$ summed over the z_r of maximum modulus is divisible by $\sin \pi x$. We then 'peel off' these terms, and show that the same is true for $\sum a_r z_r^x$ summed over z_r of second largest modulus, and so on.

Assume as we can that the a_r are all non-zero, and let $R_1 > R_2 > \dots > R_k \geq 1$ be the distinct moduli of the z_r's. Then defining $f_j(x) = \sum_{|z_r| \leq R_j} a_r z_r^x$, we have

<u>Lemma</u>. If $f_j(n) \to 0$ $(n \to \infty)$ then $f_j(x) - f_{j+1}(x)$ is divisible by $\sin \pi x$, and $f_{j+1}(n) \to 0$ $(n \to \infty)$, for $j = 1, 2, \dots, k$.

<u>Proof</u>. Let $z_{rj} = |z_r/R_j|$. Then $R_j^{-x} f_j(x) = \sum_{|z_{rj}|=1} a_r z_{rj}^x + \sum_{|z_{rj}|<1} a_r z_{rj}^x$. Since $R_j \geq 1$, $R_j^{-n} f_j(n) \to 0$ $(n \to \infty)$. Clearly $\sum_{|z_{nj}|<1} a_r z_{rj}^n \to 0$ $(n \to \infty)$, so $\sum_{|z_{rj}|=1} a_r z_{rj}^n \to 0$ $(n \to \infty)$. By the previous argument, this implies that $\sum_{|z_{rj}|=1} a_r z_{rj}^x = R_j^{-x}(f_j(x) - f_{j+1}(x))$ is divisible by $\sin \pi x$. Hence $f_{j+1}(n) - f_j(n) = 0$, and $f_{j+1}(n) \to 0$ $(n \to \infty)$.

Now $f(x) = \sum_{j=1}^{k} (f_j(x) - f_{j+1}(x))$ since $f_1(x) \equiv f(x)$ and $f_{k+1}(x) \equiv 0$, so the result follows straight from the lemma.

TRIGONOMETRIC INEQUALITY

Find the best possible value (if any) of the constant k in the proposition that if y is real and n is a natural number, then at least one of $\sin y, \sin 2y, \dots, \sin ny$ has modulus less than k/n.

SERIES BUSINESS

C.J. Smyth

Let $f(t) = [\prod_{i=1}^{k} (1 - t^{n_i})]^{-1}$, where the n_i are positive integers. Show that if $f(t) = \sum_{n=0}^{\infty} c_n t^n$ for $|t| < 1$, then

$$f(t^{-1}) = \sum_{n=1}^{\infty} c_{-n} t^n \quad \text{for } |t| < 1 ,$$

where c_j are defined for all j by the recurrence satisfied by the c_n for large enough positive n. This is a lemma of Chabauty.

REAL FUNCTION THEORY

If the continuous real function $f(x)$ has derivative zero for all irrational x, does it follow that f is a constant?

QUOTATION CORNER (11)

The following comes from *J.B. Douglas*, with the cryptic remark that it has Alf van der Poorten's approbation:

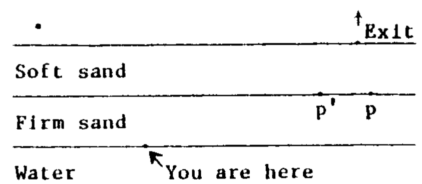
> Bartoline was as dull at drawing
> inferences from the occurences
> of common life as any Dutch
> professor of Mathematics.
>
> *The Heart of Midlothian*
> *W. Scott*

I put two and two together and it came out IV.
Wash the blood from my toga, Wayne and Shuester.

FERMAT ON THE BEACH

C.J. Smyth

You are standing at the water's edge of a straight beach which, like many beaches, consists of a strip of damp firm sand in front of a strip of soft dry sand. You want to head for the exit, which is some way along the beach. What route do you take?



Firm sand being easier to walk on than soft sand, you most likely don't go in a straight line, but tend to keep more to the firm sand. On the other hand, you don't head straight for the point p, the point on the firm sand nearest the exit, which would give the shortest path across the soft sand. The reason of course is that by heading towards some point p' instead, we can make our firm path length significantly shorter while only very slightly increasing the soft path length. One is clearly minimising something: perhaps total walk time, hassle, or sand in the shoes. In any event, whatever it is, it is of the form $c_1 \times$ (firm path length) + $c_2 \times$ (soft path length), and so the situation is quite analogous to a light ray travelling in two media of different refraction index. The interesting point is that by observing walkers in such situations, their 'refractive index' of sin (angle of incidence)/sin (angle of refraction) gives the ratio c_2/c_1. Hence we can estimate the walker's relative preference for the two surfaces.

THE RIG OF A ROWING BOAT

A racing boat with n oars has the crew seated on the centre-line of the hull with oars numbered 1, 2, ..., n from bow to stern. The "side" of the oar numbered j may be represented by the function s(j) equal to one if the oar is one the starboard side and -1 if on the port side. There are 2^n ways of choosing this function , that is of rigging a boat, and it is often possible by moving the riggers to change from one rig to another.

Most people agree that a boat should have the same number of oars on the two sides, that is $\sum s(j) = 0$, and in fact this rule is almost universal but not quite. Fishermen in parts of the South coast of England have traditionally used 5-oared boats, even in races.

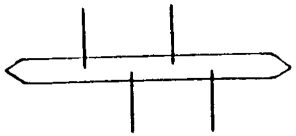
However racing boat builders now all follow the principle that $\sum s(j) = 0$, with the consequence that n must be even. The usual values for n are 2, 4 and 8, though 6-oared boats were popular in the nineteenth century. The constraint of having equal numbers on the two sides reduces the possible ways of rigging an eight from 256 to 70.

Another consideration about the choice of a rig arises as follows. For perhaps three-quarters of the time during a one-stroke cycle the oars are out of the water; consequently their weights together with those of the riggers set up a twisting moment in the hull, about a fore-and-aft axis. In fact on the section of hull between oars j and j + 1 the twisting moment is proportional to the partial sum $a(j) = \sum_{r=1}^{j} s(r)$. As racing boats are lightly built this moment has an appreciable effect, partly the temporary one that the boat twists according to Hooke's law of elasticity, and partly the long-term result that after long use a boat acquires a permanent twist. How can the sequence $\{s(j)\}$ be chosen to minimize the harmful effects? Of course the partial sums $a(j)$ cannot all be made zero, for the values at adjacent points must differ by one. The twists in the sections of the boat are right-handed or left-handed according to whether $a(j)$ is positive or negative, and their effect is cumulative, so that we must consider their partial sums. The quantity $b(j) = \sum_{r=0}^{j-1} a(r) = \sum_{t=1}^{j-1} (j - t)s(t)$ in fact measures how much the part of the boat at position j is listing when the bow section is on an even keel, with positive sign for a list to port and negative for starboard.

A few examples of rig are given below. In each case the three functions $s(j)$, $a(j)$ and $b(j)$ for $j = 1, 2, \ldots, n$ are set out in rows, and staggered so that each table is like a difference table upside down, with $b(j)$ vertically below $s(j)$ for each j.

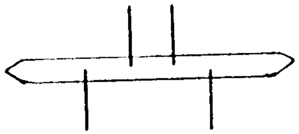
Conventional four-oar

s	1		-1		1		-1	
a		1		0		1		0
b	0		1		1		2	



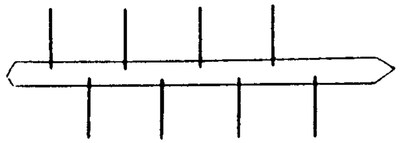
"Italian" four-oar

s	-1		1		1		-1	
a		-1		0		1		0
b	0		-1		-1		0	



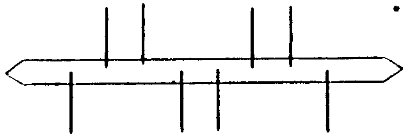
Conventional eight-oar

s	1		-1		1		-1		1		-1		1		-1	
a		1		0		1		0		1		0		1		0
b	0		1		1		2		2		3		3		4	



"German" eight-oar

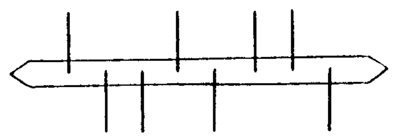
s	-1		1		1		-1		-1		1		1		-1	
a		-1		0		1		0		-1		0		1		0
b	0		-1		-1		0		0		-1		-1		0	



Looking at these examples, a few generalizations suggest themselves. There is a tendency for $b(j)$ and $s(j)$ to be of opposite sign: in fact $\sum_1^n b(j)\, s(j)$ is always negative. Proving this inequality is a good exercise for the student (remember that we are considering only the case where $\sum s(j) = 0$). Is it possible to go further and show that $\sum b(j)\, s(j) \leq -n/2$?

Another phenomenon sometimes thought important by rowing theoreticians is the fact that the force exerted by a blade on the water is not exactly fore-and-aft, there is a component athwartships and so to prevent a resulting turning moment (in the horizontal plane) on the boat we should make $b(n)$ as well as $a(n)$ zero. A more practical consideration is that a racing eight does not travel more than about 30 feet per stroke, so that if s(1) and s(8) are of the same sign (that is bow and stroke on the same side) stroke's blade will be put into the water stirred up by bow's, that is water which not only is turbdent but also has a mean velocity astern. This would be mechanically inefficient as well as disturbing to the crew. Trial and error indicates that the only rig for an eight to satisfy all the constraints discussed above is the following (or of course the one obtained by changing all the signs).

s	1	-1	-1	1	-1	1	1	-1
a	1	0	-1	0	-1	0	1	0
b	0	1	1	0	0	-1	-1	0



DISTINGUISHED INVERSE

H. Kestleman

If the rows of an m×n matrix A are linearly independent and $m < n$, then AA^+ is invertible (A^+ being the hermitean transpose of A); the matrix X_0 equal to $A^+(AA^+)^{-1}$ is a right inverse of A, i.e. $AX_0 = I_m$. By adding to each column of X_0 an arbitrary vector $\underline{v}$ satisfying $A\underline{v} = 0$ we can obtain infinitely many right inverses of A. What distinguishes X_0 from the others?

THE GROUP OF ROTATIONS OF A SPHERE

C.F. Moppert

Let S denote the surface of a sphere with fixed centre. If S_1, S_2 are two positions of S then S_1 can be transformed into S_2 by rotation, i.e. there is an axis A and an angle such that $A^\alpha S_1 = S_2$. Every rotation of S has the form A^α where A is uniquely determined and the orientated angle α is determined up to multiples of 2π. It follows then that given two rotations A^α, B^β there is C^γ such that $A^\alpha B^\beta = C^\gamma$ where C is uniquely determined and γ is determined up to a multiple of 2π : the rotations of the sphere form a group G, say.

Everybody knows that. It is therefore rather surprising that none of the colleagues I have asked : which are the generators? was able to give me an answer.

I shall prove the (probably well-known)

<u>Theorem</u> Every element C^γ of G has a representation

$$C^\gamma = A^{\alpha_1}B^\beta A^{\alpha_2} = B^{\beta_1'}A^{\alpha'}B^{\beta_2'}$$

provided the axes A and B are perpendicular. The six angles involved are unique up to multiples of 2π. If the axes A, B are not perpendicular, then not all elements C of the group can be represented in either manner.

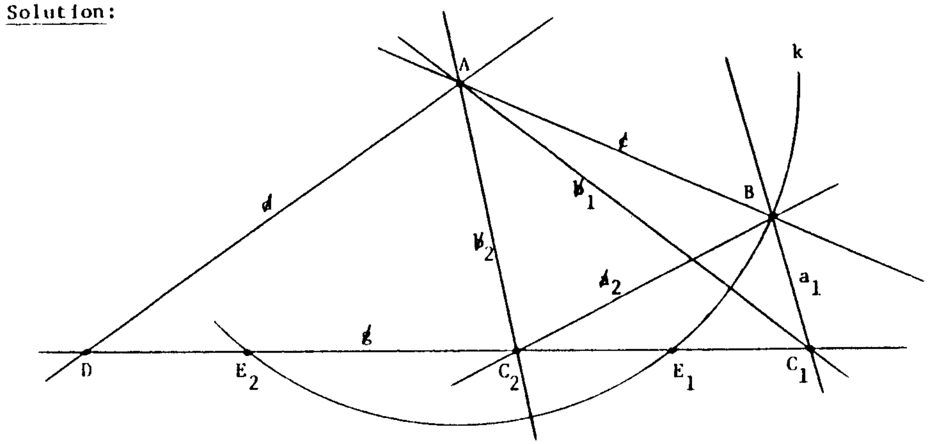
For the proof we look first at the case where A^α, B^β are rotations of the Euclidean plane about the points A, B by the angles α, β. Any rotation A^α can be written in the form $A^\alpha = bc$ where b, c are reflections on the lines $\not{b}$, $\not{c}$ respectively and where $\not{b}$ and $\not{c}$

intersect in A under an angle $\frac{1}{2}\alpha$.

Let A, B, D denote distinct points in the Euclidean plane and δ an orientated angle.

Problem: to find angles α, β, α' such that $D^{\delta} = A^{\alpha}B^{\beta}A^{\alpha'}$.

Solution:



k is the circle about A through B. g is the line through D such that $D^{\delta} = dg$ where d is the line DA. E_1, E_2 are the points of intersection of k and g. The points C_i are the points of intersection of the perpendiculars from A to BE_i with g (i = 1, 2). Put $a_i = BC_i$, $b_i = AC_i$. The line AC_i bisects the angle BC_iE_i and thus $a_ib_i = b_ig$. Putting $A^{\alpha_i'} = cb_i$, $B^{\beta_i} = a_ic$, $A^{\alpha_i} = db_i$ we have $B^{\beta_i}A^{\alpha_i'} = a_iccb_i$. As c is a reflection, $c^2 = 1$ and thus $B^{\beta_i}A^{\alpha_i'} = a_ib_i = b_ig$. It follows that $A^{\alpha_i}B^{\beta_i}A^{\alpha_i'} = db_ib_ig = dg = D^{\delta}$. The angles α_i, β_i, α_i', δ are the doubles of the angles C_iAD, C_iBA, C_iAB, C_iDA.

The problem has two, one or no solutions according to whether k intersects, touches or avoids g. In order to represent D^{δ} in the form $B^{\beta}A^{\alpha}B^{\beta'}$ we replace k by the circle about B through A.

Proof of the Theorem On the surface S of the sphere, lines are replaced by great circles. Reflections can be taken as reflections on the planes containing these great circles. Circles on S are great circles provided their radius is $\frac{\pi}{2}$. Any two lines intersect in two diametrically opposite points.

The Euclidean construction described above holds - mutatis mutandis - throughout. The construction is again possible if and only if the circle k intersects the line g and this is always the case if and only if k is a line, i.e. iff $|AB| = \frac{\pi}{2}$. Then the circle about B through A is also a line and the theorem is proved.

QUOTATION CORNER (12)

C.F. Moppert sends the following:

> Newton ist tot, Einstein ist tot und mir ist es auch schon so komisch (and I have a queer sensation too).
>
> *Students' lavatory,*
> *Mathematics Dept.,*
> *Heidelberg.*