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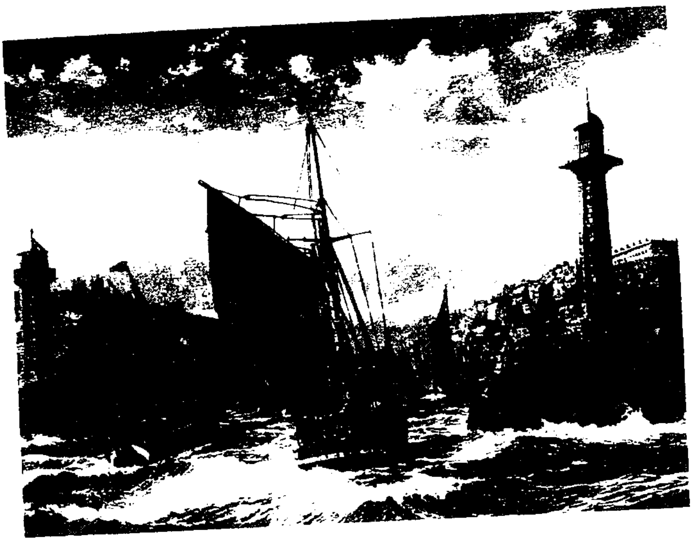

Entering Whitby Harbour, *by Peter Leath*.

In Captain Cook's younger days this view must have been welcome, the entrance to his home port. The reproduction is by courtesy of the Shipwrecked Fishermen and Mariners Royal Benevolent Socity of 1, North Pallant, Chichester, Sussex, U.K.

TRIANGLES FROM FOUR LINES (JCMN 22, Vol.2,p.88)

M.J.C. Baker and S.R. Mandan

C.F. Moppert asked whether the orthocentres of the four triangles formed by four lines are collinear. The answer is yes. This is proved in theorem 62 (p. 134) of C.V. Durell's Modern Geometry, Macmillan, London 1920. Durell's proof depends on various theorems about co-axal circles. Another way is by showing that each orthocentre is on the directrix of the parabola that touches the four lines, for a proof by coordinate methods see Smith's Conics, page 126, or by synthetic methods page 133 of Askwith's Pure Geometry, this result is Exercise 7 on page 7 of E.M. Lockwood's Book of Curves.

Four lines in general position make four triangles.

(i) The circumcircles of these triangles have a common point P.

(ii) The feet of the perpendiculars from P to the four lines are collinear.

(iii) The orthocentres of the triangles lie on a line which is parallel to the line of feet and twice as far from P.

(iv) P is the focus and the line of orthocentres is the directrix of the parabola that touches all four lines.

(v) The circumcentres of the triangles are concyclic together with P.

(vi) The centroids, nine-point centres, incentres of the triangles are not in general either collinear or concyclic.

Proposition 1 The line joining the feet of the perpendiculars to two of the sides of a triangle from a point on its circumcircle bisects the line joining that point to the orthocentre of the triangle.

Given a triangle ABC with circumcentre O, altitudes AD, BE meeting at H, and a point P on the circumcircle of the triangle; if PL is the $\perp$ from P to BC, and Q is the mid-point of PH, and if LQ meets AB and AC in K and M respectively, to prove that PK, PM are $\perp$ to AB, AC respectively.

Construction Let A' be the mid-point of BC. Join OB, OA', OC, OP. Let N be the mid-point of OH. N is thus the centre of the nine-point circle, which passes through A', D, E, and whose radius is half the

circumradius. Join NA', NQ, ND. Let NU, QV be $\perp$ s from N, Q to BC. Join A'E, PB, PC, DE, QD. (See Figure 1).

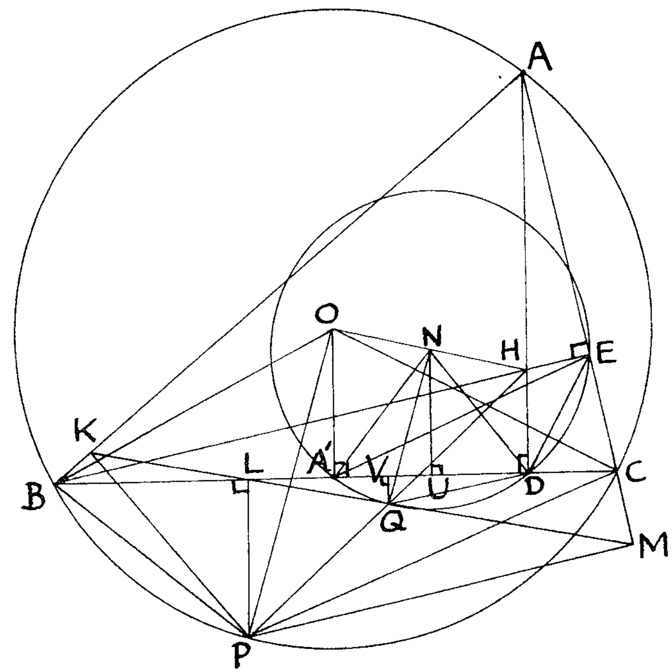

Figure 1.

Proof $HN = \frac{1}{2}HO$, $HQ = \frac{1}{2}HP$, so $NQ = \frac{1}{2}OP$. Therefore Q lies on the nine-point circle.

A' is the centre of circle BEC, so $A'E = A'C$ and $\angle A'EC = C$. AEDB is cyclic, so $\angle DEC = B$. Thus $\angle A'ED = C - B$. But $\angle A'NU = \frac{1}{2}\angle A'ND = \angle A'ED$ (nine-point circle). So $\angle A'NU = C - B$.

Now $\angle PCM = \angle PBA$ (PBAC cyclic), so $\angle PBC = \angle PCM - B$. Thus $\angle POC = 2\angle PCM - 2B$. But $\angle A'OC = \frac{1}{2}\angle BOC = \angle BAC = A$; so $\angle POA' =$

$= 2\angle PCM - 2B - A$. $\angle QNU = \angle POA'$ (pairs of parallel lines) and therefore $\angle A'NQ = \angle A'NU - \angle QNU = A + B + C - 2\angle PCM = \pi - 2\angle PCM$. Thus $\angle A'DQ = \frac{\pi}{2} - \angle PCM$ (nine-point circle). Now V is clearly the mid-point of LD (PQ = QH; intercept theorem); and so $\angle QLV = \angle QDV =$ $= \frac{\pi}{2} - \angle PCM$. Therefore $\angle PLM = \angle PCM$. Thus PLCM is cyclic, and so $\angle PMC$ is a right angle. Similarly $\angle PKB$ is a right angle.

Q.E.D.

We have established not only the Simson's Line theorem (that the feet of the perpendiculars to the sides of a triangle from a point on its circumcircle are collinear, Durrell's theorem 27, p. 46), but also that the Simson Line of P bisects PH (Durell's rider no. 67, p. 48).

Lemma (Converse of the Simson Line theorem, see Durell p. 46). If the feet of the perpendiculars from a point P to the sides of a triangle ABC are collinear, then P lies on the circumcircle of ABC.
For $\angle PBA = \angle PBK = \pi - \angle PLK = \angle PLM = \angle PCM$, so PBAC is cyclic.

Proposition 2 The orthocentres of the four triangles formed by four lines are collinear. For let the circumcircles of two of the triangles meet at P. The feet of the perpendiculars from P to the four lines are on a line since this is the Simson Line of P with respect to the two chosen triangles. It is thus the Simson Line of P with respect to the other two triangles. By Proposition 1 the lines joining P to the orthocentres of the four triangles are all bisected by the Simson Line, and must therefore themselves all lie on a line parallel to it and twice as far from P.

Corollary 1 The circumcircles of the four triangles formed by four lines have a common point (Durell's rider no. 64, p. 48). By the converse of the Simson Line theorem the point P in Proposition 2 is on the circumcircles of the other two triangles as well.

Corollary 2 Consider the parabola that touches the four lines. It is an elementary fact that the feet of the perpendiculars from the focus to the tangent lines all lie on the tangent at the parabola's vertex.

Thus, by the converse of the Simson Line theorem, the focus must lie on all the circumcircles and so be identical with P. It follows that each orthocentre is on the directrix of the parabola.

The fact that the four circumcircles all pass through the focus P of the parabola may also be shown projectively (see Figure 2).

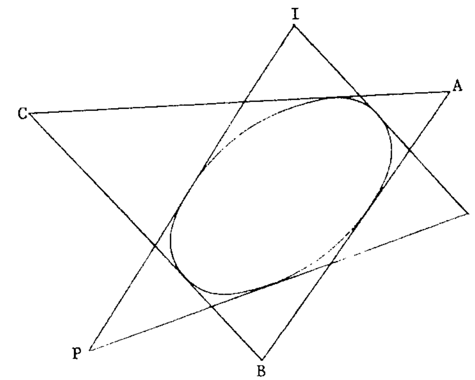

Figure 2.

The circular points are I and J, the line IJ touches the parabola, and the other tangents from I and J meet at the focus P, ABC is one of the four triangles formed by the given lines. If two triangles circumscribe a conic the six vertices lie on another conic (Poncelet's porism), and so P is on the circumcircle of ABC.

Proposition 3 The circumcentres of the four triangles formed by four lines lie on a circle together with the point common to the four circumcircles.

Proof Let the configuration be as shown in Figure 3 where O, O_1, O_2, O_3 are the circumcentres of triangles ABC, AYZ, BZX, CXY respectively, and P is the point on all four circumcircles.

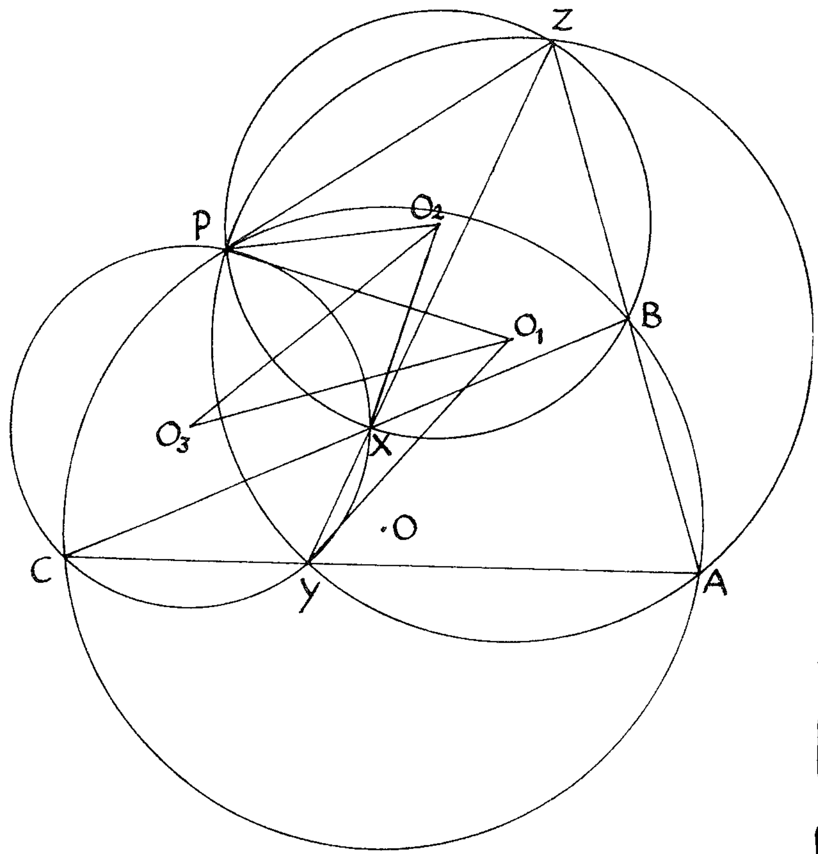

Figure 3.

We have $\angle PO_1O_3 = \frac{1}{2}\angle PO_1Y$ (PY is common chord of circles centres O_1, O_3)

$= \angle PXY$ (O_1 is centre of circle PYAZ)

$= \angle PZX$

$= \frac{1}{2}\angle PO_2X$ (O_2 is centre of circle PZBX)

$= \angle PO_2O_3$ (PX is common chord of circles centres O_2, O_3).

Thus $PO_1O_2O_3$ is cyclic. Similarly O lies on the same circle.

Q.E.D.

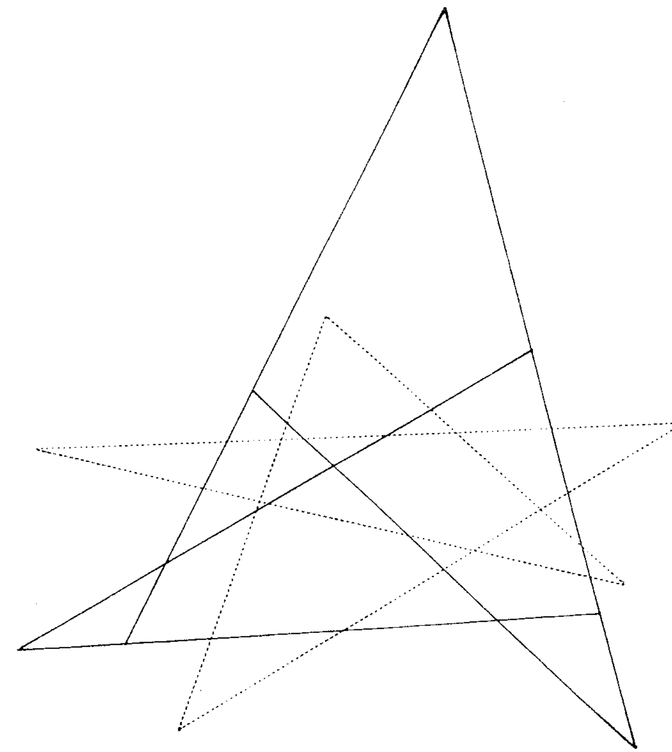

Figure 4.

Observation 1 That the centroids, or the nine-point centres, or the incentres of the four triangles formed by four lines are not in general either collinear or concyclic can be seen by letting three of the lines form an equilateral triangle and the fourth be one of its medians.

Observation 2 Four lines yield one line of four orthocentres: so five lines yield five lines each containing four orthocentres. In general there seems to be no special relationship between the two figures (see Figure 4); but if the original lines form a regular pentagram, then the lines of orthocentres form a congruent pentagram which is the reflexion of the original one in its centre (see Figure 5).

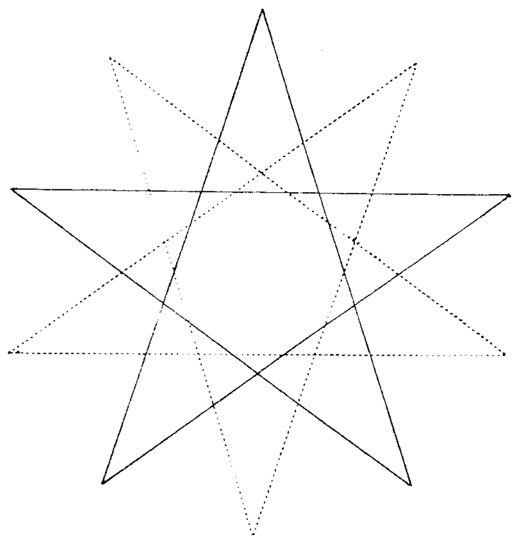

Figure 5.

(See also page 128 for further comments).

ON UNIVERSAL COVERING SETS AND TRANSLATION COVERS IN THE PLANE

G.F.D. Duff

1. Introduction

A universal covering set (UC for short) is a plane set, or figure, that can contain any plane set of unit diameter. In the first instance we allow translations and rotations in the plane, and reflections, for all sets of unit diameter, which we must be able to fit into the UC. The problem of defining a UC of minimum area was posed by Lebesgue in 1914 and has not yet been fully solved. In this note I describe the results to date and also give a probable solution for a restricted version of the problem, the universal translation cover (TC) relative to which rotations are not permitted for the sets being covered.

2. The hexagon and its subsets

The diameter of a plane set is the largest distance between two of its points (or least upper bound of such distances, if the set is not closed). Thus a set of unit diameter may be placed on a (closed) unit strip defined by two parallel straight lines at unit separation, and this is true whatever the direction of the straight lines. Considering two such unit strips at right angles, we can easily deduce that any set of unit diameter can be placed in a unit square. Thus a unit square is a UC of area 1.

An important further step was taken by J. Pál [6] in 1920. He took three strips at 60° angles to each other and enclosed the typical set of unit diameter in a hexagon with three parallel pairs of sides. In general, the middle or centre lines of these three strips do not all intersect at one point (Figure 1). However, Pál showed that by rotation of the given set, such an intersection could be achieved. To see this, consider two unit strips at a 60° angle. They intersect in a parallelogram that is itself a UC. Within the parallelogram we can rotate the set of unit diameter provided we also translate it so as to keep the centre of the parallelogram fixed. Suppose that the centre of the parallelogram was initially on one side of the middle line of the third

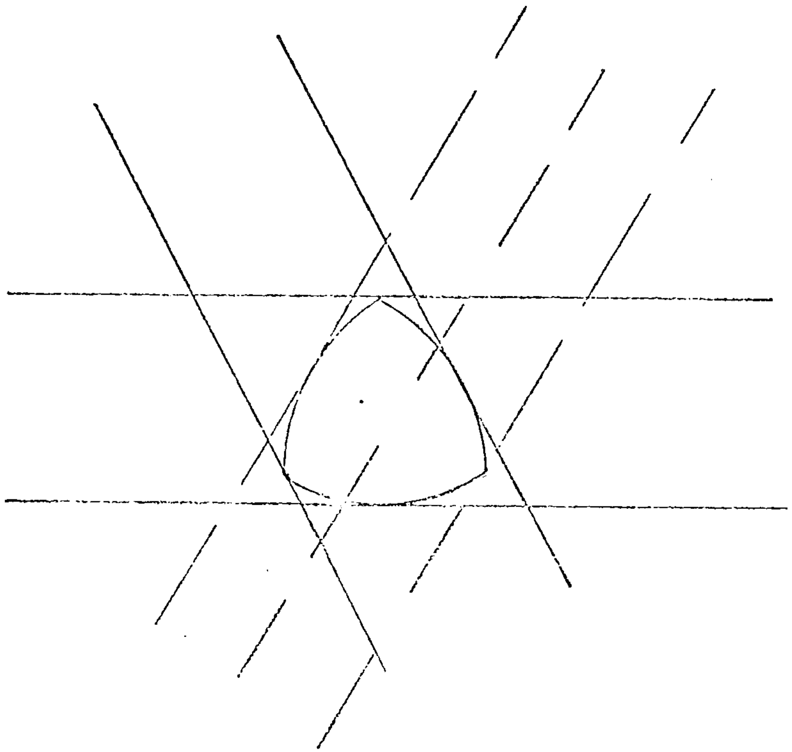

Figure 1.

strip that also moves to contain the rotating set. After a rotation of 180° the centre of the parallelogram must be on the other side of the third middle line. By continuity, there exists an angle of rotation at which the third middle line meets the centre of the parallelogram - that is, the three middle lines are concurrent.

This shows that every set of unit diameter can lie within a regular hexagon with opposite sides at unit distance. The side length of this regular hexagonal UC is actually $\frac{1}{\sqrt{3}}$ and its area is $\frac{1}{2}\sqrt{3} = 0{\cdot}866025404$.

Consider now the six little corners cut off by the sides of a concentric congruent regular hexagon rotated 30° to the first (Figure 2).

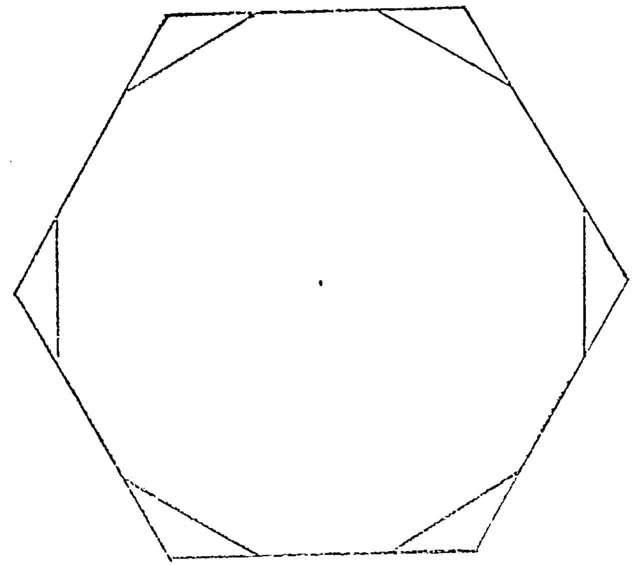

Figure 2.

Since the parallel cuts are at unit distance, every point in on such corner triangle is more than one unit distance from every point within the opposite corner triangle. Thus either a given one or its opposite triangle contains no points of a given set of unit diameter. Hence at least three of the corner triangles, one of each opposite pair, are unoccupied by the given set. Of the eight cases thus created, six yield three consecutive unoccupied corners, and two yield equally spaced alternate unoccupied corner triangles. Therefore, there are always two corner triangles, neither adjacent nor opposite, that are not occupied. By a rotation we can bring these to a standard position. In this way, Pál showed that two corners can be removed and this gives a UC of area

$2 - \frac{2}{\sqrt{3}} = 0\cdot84529945\ldots$. Simultaneously, in 1920 he showed that a UC must have area at least $\frac{\pi}{8} + \frac{\sqrt{3}}{4} = 0\cdot825711786\ldots$, by considering the superposition of a circle and a Rouleaux triangle which is a curve of constant width. A concise account of these results is also given in [8].

3. Sets of constant width unity

The width, or breadth, of a set or curve in a given direction is the distance between its two extreme lines of support having the perpendicular direction. The circle has constant width, the same in all directions. Surprisingly enough, there are many other closed plane curves with the same property of constant width. We consider unit width curves with the same property of constant width. We consider unit width for simplicity. Best known beside the circle is the Rouleaux triangle with three arcs of unit radius and three vertices with tangent angle of 60° (Figure 3). Of all curves of constant width, the Rouleaux triangle has the sharpest corners and least area. The Rouleaux triangle has been used as the rotor in the Wankel engine and in other mechanical applications. Elegant accounts of these curves are given by Bonneson and Fenchel [1, Chap. 15], Meshkowski [5, Chap. 5] and Rademacher and Toeplitz [7, Chap. 25].

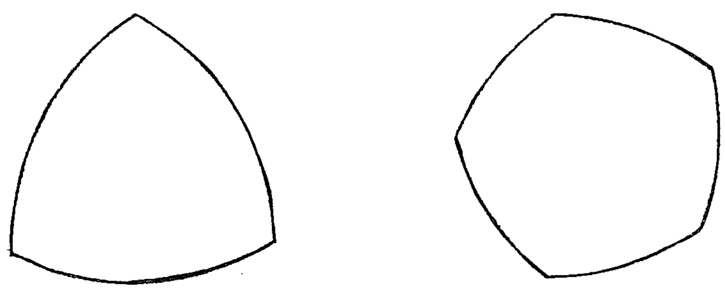

Figure 3.

Imagine a compass with pencils mounted on both legs: if this compass is walked around a closed path with each step a unit circular arc, then the closed curve so drawn is a Rouleaux polygon of constant width containing an odd number of arcs and vertices. By a limiting process, a class of other curves of constant width having curvature greater than unity can be constructed (Figure 4). A curve parallel to a curve of constant width and more generally, the Minkowski sum curve of two such curves, is again of constant width.

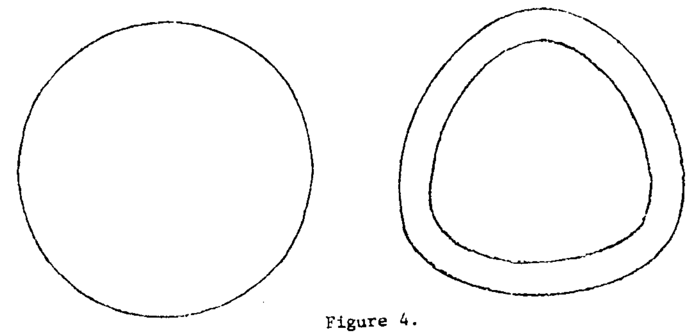

Figure 4.

Useful properties of curves CW of constant width include [7]:

I CW has exactly one point of intersection with each supporting line (including tangents).

II CW with constant width 1 has diameter 1 and arc length π.

III If a straight line joins the two contact points of two parallel and opposite supporting lines of CW, the line is perpendicular to the supporting lines.

IV There is at least one supporting line through every point of CW.

V Through every point of CW, a unit circle can be drawn that encloses CW and is tangent to any given supporting line at the point.

VI Every set of diameter one can be extended to a set of constant width one [6, see also 2].

In view of VI, a sufficient test for UC is that it contains every set of constant width 1. From Figure 5 with corner triangles GBG', FHH' removed from regular hexagon ABCDEF, we see that every curve of constant width 1 must have diameter 1 parallel to EC and must meet CG' and EH' in one point each. Likewise, the curve must meet AG and AH each in exactly one point. With centres G and H and radius 1, draw circular arcs IK and JK intersecting at K. The small area IKJD can now be excluded as every point interior to it is more than 1 unit distant from the intersection points of the curve with AG or AH. This reduction to area 0·84413770... was demonstrated by Sprague [9] in 1936.

4. Recent further reductions

For many years the UC set problem then languished, but in 1975 Hansen [4] showed that two tiny corners of area 10^{-19} could be removed around G' and H' in Figure 5. He reasoned that a curve of constant width CW with points in one of these corners, could be rotated by 120° within the hexagon, and would then not cover these corners. In effect he used properties III, IV and V in his proof.

Hansen also proved that his figure is a "set theoretic" minimum UC, i.e. that no further points can be removed from it without destroying the UC property. He achieved this result by constructing three curves of constant width 1 that "span" the UC, in the sense that their vertices occupy all vertices of the UC. The three curves are the Rouleaux triangle, the pentagon HGLKM, and a 9-gon symmetric about vertical axis AK, and having short (circular) sides bounding the two excluded corners.

Despite its set theoretic minimum property, Hansen did not consider his UC to have minimal area. His view was confirmed recently [3]. A UC of smaller area was found with an arc V'K' sliced off but a still smaller area bounded by an arc β' restored to excluded triangle FHH' (Figure 5). The arc V'K' has centre V on AG at distance $\varepsilon = 0{\cdot}00056$ from G. The arc β' (also of unit radius) has centre W on LG' at distance ε from L. The area "saved" by this construction is $2{\cdot}00 \times 10^{-6}$ so the new minimum known area is 0·84413570... . The proof involves a 60° counterclockwise rotation for curves C with points in the excluded area, and a further 60° rotation for those curves C that meet HA below W', where $W'N = \varepsilon$.

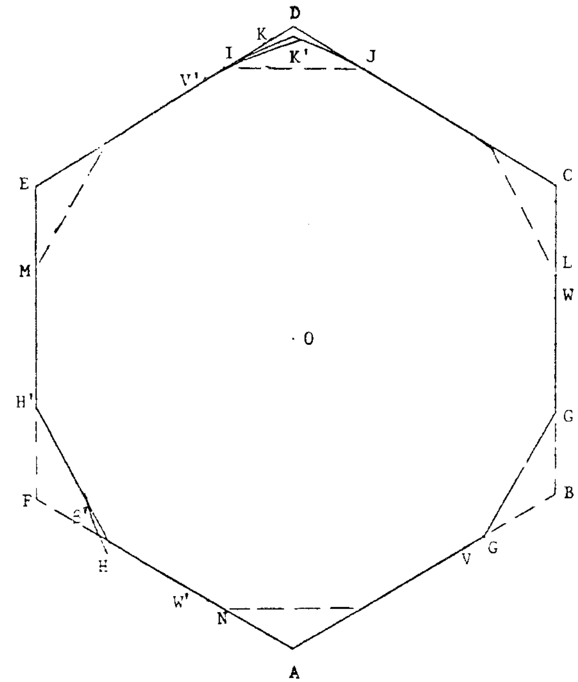

Figure 5.

Since this latest UC is no longer symmetrical about axis AD, and is not convex near H, one may question whether a minimal UC, if it can be found, would have these properties. Also the effect of allowing or not allowing reflections might then make a difference. Nor do we know whether a UC of minimal area is unique.

5. The universal translation cover

The modified problem of finding a set of minimal area able to cover any plane set of unit diameter moved by translation only has been suggested by Rennie. Clearly this translation cover TC must have larger area than the earlier UC. Here we shall find a set theoretic minimum TC within a unit square. It seems probably that this TC also has minimum area, but this has not been proved.

Clearly a unit square is a TC, for no rotation is needed to enclose an arbitrary curve of constant width in a unit square of any orientation. We first determine the minimal set within the square that encloses all Rouleaux triangles of differing orientations (Figure 6).

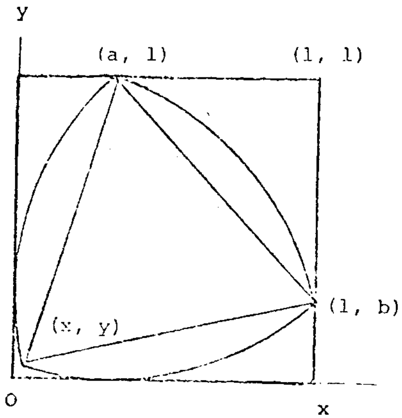

Figure 6.

Two of the vertices always lie on adjacent sides of the square, say at (a, 1) and (1, b) in Figure 6. Let the vertex near 0 have co-ordinates (x, y) and let ϕ be the angle the unit segment joining this vertex to (1, b) makes with a parallel to the x-axis. Then

$$b - y = \sin\phi \quad \text{and} \quad 1 - x = \cos\phi \ .$$

Now the unit segment joining (x, y) and (a, 1) makes an angle $\phi + \frac{\pi}{3}$ with the x-axis.

Hence

$$\begin{aligned} 1 - y &= \sin(\phi + \tfrac{\pi}{3}) \\ &= \sin\phi \cdot \tfrac{1}{2} + \cos\phi \cdot \tfrac{\sqrt{3}}{2} \\ &= (b - y) \cdot \tfrac{1}{2} + (1 - x) \cdot \tfrac{\sqrt{3}}{2} \end{aligned}$$

and

$$\begin{aligned} a - x &= \cos(\phi + \tfrac{\pi}{3}) \\ &= \cos\phi \cdot \tfrac{1}{2} - \sin\phi \cdot \tfrac{\sqrt{3}}{2} \\ &= (1 - x) \cdot \tfrac{1}{2} - (b - y)\tfrac{\sqrt{3}}{2} \ . \end{aligned}$$

We thus find

$$a - 1 = 1 - x + \sqrt{3}(y - 1)$$

and

$$b - 1 = 1 - y + \sqrt{3}(x - 1) \ .$$

Since the distance from (a, 1) to (1, b) is also unity, we have

$$1 = (a - 1)^2 + (b - 1)^2 = [1 - x + \sqrt{3}(y - 1)]^2 + [1 - y + \sqrt{3}(x - 1)]^2$$

which reduces to

$$(1 - x)^2 + (1 - y)^2 - \sqrt{3}(1 - x)(1 - y) = \frac{1}{4} \ .$$

This locus of (x, y) is an ellipse centred at (1, 1) that touches the axes of co-ordinates at $(1 - \frac{\sqrt{3}}{2}, 0)$ and $(0, 1 - \frac{\sqrt{3}}{2})$. The major axis lies on $y = x$, the minor axis on $x + y = 2$.

We translate and rotate the axes to new co-ordinates

$$\xi = \frac{2 - x - y}{2}, \qquad \eta = \frac{y - x}{2} \ .$$

The ellipse becomes

$$\frac{\xi^2}{1 + \frac{\sqrt{3}}{2}} + \frac{\eta^2}{1 - \frac{\sqrt{3}}{2}} = 1 \ .$$

To calculate the corner area excluded we first calculate the area cut off this ellipse by the chord joining the two points of contact with the x and y axes. This area is found to be $\frac{(\pi - 3)}{24}$. It follows that one corner area cut off the square is

$$\frac{1}{2}(1 - \frac{\sqrt{3}}{2})^2 - \frac{\pi - 3}{24} = 0{\cdot}003074902\ldots \ .$$

The TC with four corners removed has therefore area 0·987700392... .

To show that our figure is also a translation cover for all other curves of constant width, we require the following lemma (Figure 7

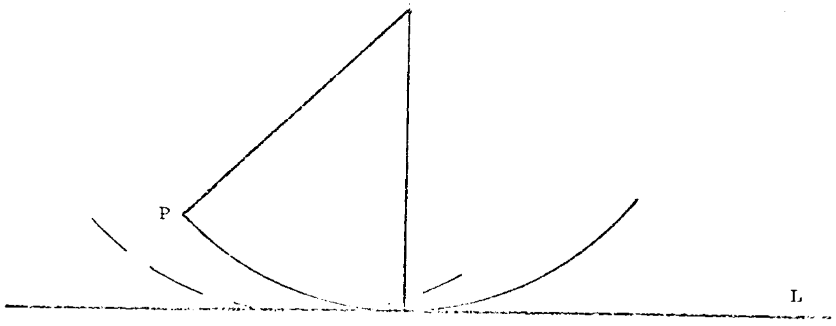

Figure 7.

Lemma If P is a point within or on a curve CW of constant width and L a support line of CW, then the common point of C and L (meeting L to the right of P) lies on segment QR, where R is the foot of the perpendicular through P to L, and Q is the contact point of the circle of radius 1 through P touching L.

Proof Let Q' be the unique point of CW that lies on support line L. Then CW lies within the unit circle touching L at Q' on the side shown. If Q' were to the right of Q, P could not belong to CW. Q.E.D.

Theorem No curve of constant width embedded in the square has points between the corner O and the elliptic locus of vertices of the Rouleaux triangles.

Proof If P is such a point in the excluded corner, then the curve of constant width CW must meet the ellipse in two distinct points A, D. Construct the Rouleaux triangles ABC, DEF, touching the sides of the square at B', C', E', F' where BB', CC', EE' and FF' are each parallel

to the appropriate sides of the square (Figure 8).

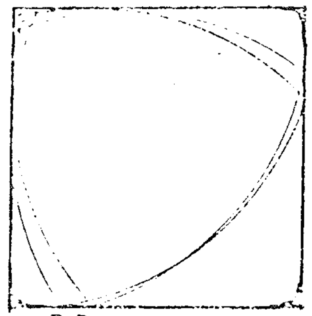

Figure 8.

By the lemma, since CW passes through A, CW meets side OF' of the square above C' (i.e. closer to the corner than C'), hence also (by III) meets side CF above C. Likewise, since CW passes through D, CW meets side OB' to the left of E' (on segment OE') and thus also meets side EB to the left of E. Since distance BC = 1, points on the side above C are further than unit distance from E and from points on the side to the left of E. This gives a contradiction for the curve CW of constant width one, and the result is established.

Therefore, the unit square with four corners trimmed to the ellipse is a universal translation cover TC. Since every point of this set is needed to cover Rouleaux triangles in the square, it is a set theoretic minimum. However, this does not yet prove that it is a TC of minimum area.

Is there a TC of smaller area? Consideration of the possible alternatives suggests that this is not likely. Of the TC's bounded by two pairs of parallel sides, the square has the least area. Any departure from embedding in the square seems to involve penalties of area for the Rouleaux triangles alone. However, a complete proof must essentially include a demonstration that the minimal TC lies within some unit square. As in the problem of the UC, this general minimum property remains an open problem.

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QUOTATION CORNER (6)

A good message for the working week, *"Take things easy"*. — A disc jockey of the Australian Broadcasting Commission, 18th August, 1980.

BINOMIAL IDENTITY NUMBER NINE (JCMN 22, Vol.2, p.88)

C.S. Davis

Putting $b/2 = t$, we may write the equation to be proved as

$$\sum_0^\infty \binom{2n}{n} t^n/n! = e^{2t} \sum_0^\infty t^{2n}(n!)^{-2},$$

or, say, $\quad f(t) = e^{2t}\,\phi(t),$

with $f(t) = \sum_0^\infty a_n t^n$, where $a_n = \binom{2n}{n}/n!$. Then

$$(n+1)^2 a_{n+1} - 2(2n+1)a_n = 0,$$

so $f(t)$ satisfies the differential equation

$$t\ddot{f} + (1-4t)f - 2\dot{f} = 0,$$

and hence $\phi(t)$ satisfies

$$\ddot{\phi} + \dot{\phi}/t - 4\phi = 0 \quad .$$

Noting that $\phi(0) = f(0) = 1$ and that ϕ is analytic, the only solution is $\phi(t) = \sum_0^\infty t^{2n}/(n!)^2$, which in the usual notation for Bessel functions is $I_0(2t)$. See also page 127.

BINOMIAL IDENTITY NUMBER TEN (JCMN 22, Vol.2, p.90)

C.S. Davis

$$n^2\binom{2n}{n}\sum_{r=0}^{n-1}\frac{1}{(2r+1)(2n-2r-1)}\binom{2r}{r}\binom{2n-2r-2}{n-r-1} = 2^{4n-3}$$

Denoting the sum by S, and writing $a_r = \dfrac{1}{2r+1}\dbinom{2r}{r}$ and $r + s = n - 1$, this is $n^2\binom{2n}{n}S = 2^{4n-3}$.

Now $\displaystyle\sum_{r=0}^\infty \binom{2r}{r}x^{2r} = (1-4x^2)^{-\frac12}$, so $\displaystyle\sum_{r=0}^\infty \binom{2r}{r}\frac{t^{2r+1}}{2r+1} = \int_0^t \frac{dx}{\sqrt{(1-4x^2)}}$,

i.e. $\displaystyle\sum_{r=0}^\infty a_r t^{2r+1} = \frac12 \sin^{-1}(2t) = f(t)$, say. Then if $\{f(t)\}^2 =$

$$= \sum_{n=1}^{\infty} c_n t^{2x}, \quad \text{we have } c_n = \sum_{r=0}^{n-1} a_r a_s = S.$$

$$\text{But } \{f(t)\}^2 = \tfrac{1}{4}\{\sin^{-1}(2t)\}^2 = \frac{1}{2}\sum_{n=1}^{\infty} \frac{2^{n-2}(2t)^{2n}}{n^2\binom{2n}{n}}$$

(see, e.g., Bromwich, Infinite Series (1926), 197 (Ex. 4)), so

$$S = \frac{1}{2}\,\frac{2^{2n-2}2^{2n}}{n^2\binom{2n}{n}}, \quad \text{i.e.} \quad n^2\binom{2n}{n}S = 2^{4n-3}.$$

Another solution came from *J.B. Parker*.

A MATRICULATION PROBLEM GIVEN IN MOSCOW

Dieter K. Ross

In a recent edition of the Notices of the American Mathematica Society there appears, under the heading "Jewish problems", the following question:

Let $ab = 4$, $c^2 + 4d^2 = 4$. Prove the inequality

$$(a - c)^2 + (b - d)^2 \geq 1{\cdot}6 \quad .$$

This is one example of the type of problem given by examiners to high school candidates seeking admission to the Mechanics and Mathematics Department of Moscow University.

Clearly what is needed is an expression for the square of the shortest distance between the hyperbola $xy = 4$ and the ellipse $x^2 + 4y^2 = 4$. The line representing this shortest distance is normal to both curves and this leads to the relation $ac = 4bd$.

Since the points (a, b) and (c, d) can be taken to lie in the first quadrant of the cartesian plane it follows by simple algebra tha

$$b = 4/a, \quad c = 16/\sqrt{a^4 + 64}, \quad d = a^2/\sqrt{a^4 + 64}$$

and that

$$(a - c)^2 + (b - d)^2 = (a - 16/\sqrt{a^4 + 64})^2 + (4/a - a^2/\sqrt{a^4 + 64})^2$$

This expression has its least value when a is the positive root of the equation

$$12a^3 = (a^4 - 16)\sqrt{a^4 + 64} \quad .$$

The appropriate root is $a = 2{\cdot}390978\ldots$ which leads to the final inequality

$$(a - c)^2 + (b - d)^2 \geq 1{\cdot}774796\ldots \quad .$$

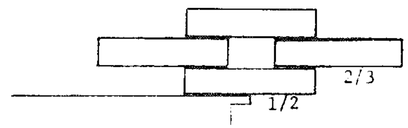
The question which remains is how a Matriculation student could be expected to do this problem in the twenty minutes specified?

PILES OF BRICKS (JCMN 22, Vol.2, p.89)

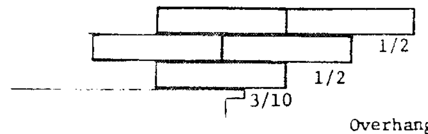
H.O. Davies

How much overhang can be obtained piling bricks on the edge of a table? The obvious way gives $\frac{1}{2}(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n})$ with n bricks. Some better ways are as follows.

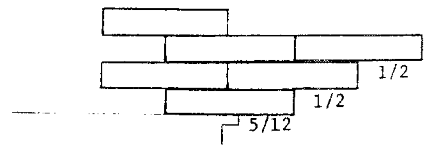
4 bricks Overhang = $1\frac{1}{6}$ lengths



5 bricks Overhang = $1\frac{3}{10}$



6 bricks Overhang = $1\frac{5}{12}$



There does not appear to be any general pattern emerging that could be extended to n bricks.

The percentage improvements on the overhang given by the obviou: method are -

Bricks	3	4	5	6
Percentage	9•1	12	13•9	15•6

S.U.M.S. COMPETITION 1980

By kind permission of *D. Cartwright* we reprint Question 3 of this year's Sydney University Mathematical Society Competition.

Consider finite strings of a's and b's (for example aabab). We use the letters σ and τ to stand for strings. A string is called primitive if it is of the form $\sigma\sigma$ (for example abbabb). There is an operation of deriving a string from two other strings: if we already have σ and $\sigma\tau$ we can derive τ. (For example if we already have abb an abbbabb, we can derive babb). A string is called good if it can be obtained, starting from the primitive strings, by applying the deriving operation a finite number of time. Is ab a good string? Describe all good strings.

ADVANCED EDUCATION

Here is a sample from the College of Advanced Education of an Australian State. In the first year calculus course a homework question set in May 1980 took the problem of the motion of a rocket fired vertically up, and told the candidates that the equation of motion was

$$\frac{d}{dt}(mv) = F - mg$$

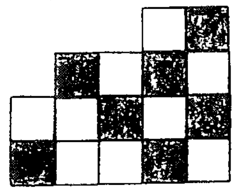
(where m is the instantaneous mass of the rocket, v is the velocity, F is the engine thrust and g is gravity). From *L. Bode* comes the

information that he found the same mistake in a book on gyroscopes, but there the author always kept m constant and consequently was not led into getting wrong answers.

BLACK AND WHITE CUBES

C.J. Smyth

A boy has a large number of black cubes and white cubes, and is trying to arrange them into adjacent piles of given heights $h_1, \ldots, h_n$, using as many black cubes as possible, but so that no two adjacent cubes (horizontally or vertically) are black (see diagram).

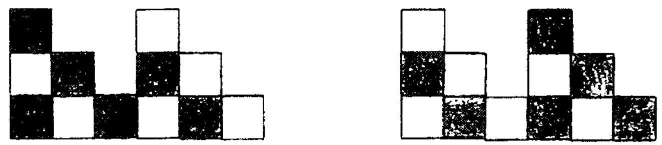


Show that if the heights satisfy

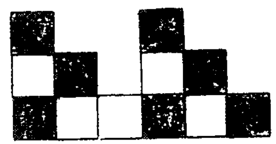
(*) $\qquad h_1 \le h_2 \le \cdots \le h_J \ge h_{J+1} \ge \cdots \ge h_n$ for some $J \ge 1$,

then one of the two ways of arranging the cubes alternately (chess-board style) will use the maximum number of black cubes (other arrangements may also use the maximum number).

The result is not true if condition (*) is removed: take the h_i to be 3, 2, 1, 3, 2, 1.



Chess-board style: 6 black cubes



Optimal: 7 black cubes.

ORTHOGONAL MATRICES

If in a real $n\times n$ matrix every row and every column has sum of squares equal to one, is it possible to make the matrix orthogonal by appropriate changes of sign among the components?

THE DREADED ZETA THREE AGAIN (JCMN 20, Vol.2, p.47)

C.S. Davis

Show that $$10\int_0^{\log\frac{1+\sqrt5}{2}} t^2\coth t\,dt = \sum_{n=1}^{\infty}\frac{1}{n^3}\ .$$

The proof depends on a couple of lemmas, in which, for real x,

$$g(x) = \int_0^x \log^2|1-t|\frac{dt}{t}\ .$$

<u>Lemma 1</u> For $0 < x \le 1$,

$$g(1-x) + g(1+x) = \tfrac14 g(1-x^2) + \tfrac74 g(1)\ ; \qquad \ldots\ldots\ldots(1$$

$$g(1-x) + g(\tfrac1x) = 2g(1) - g(x) + \log^2 x\,\log(1-x) - \tfrac13\log^3 x \quad ..(2$$

(These identities are essentially due to Spence (1807), perhaps even earlier to Euler or Abel. They are proved by elementary manipulation of integrals).

<u>Lemma 2</u> $g(1) = 2\zeta(3)\ ;$ $\qquad\ldots\ldots\ldots(3$

if $\theta = \frac12(\sqrt5 - 1)$, $g(\theta) = \frac15 g(1) + \frac43\log^3\theta$ $\qquad\ldots\ldots\ldots(4$

For (3), we have

$$\zeta(3) = \frac12\int_0^\infty \frac{x^2e^{-x}dx}{1-e^{-x}} = \frac12\int_0^1 \log^2(1-t)\frac{dt}{t} = \frac12 g(1)\ .$$

Observing that $\theta^2 + \theta - 1 = 0$, so that $1 + \theta = \frac1\theta$, putting $x = \theta$ in (1) and (2) makes the l.h.ss. equal. Since $1 - \theta^2 = \theta$ and $1 - \theta = \theta^2$, equating the r.h.ss. with $x = \theta$ gives

$$\frac14 g(\theta) + \frac74 g(1) = 2g(1) - g(\theta) + \log^2\theta\,\log(\theta^2) - \frac13\log^3\theta\ ,$$ and (4) follows.

Now to the stated result. Putting $\xi = e^{-t}$ in the integral, we have

$$I = \int_0^{\log\frac{1+\sqrt5}{2}} t^2\coth t\,dt = \int_1^\theta \log^2\xi\ \frac{1+\xi^2}{1-\xi^2}\ \frac{-d\xi}{\xi}$$

$$= \int_\theta^1\left(\frac1\xi + \frac1{1-\xi} - \frac1{1+\xi}\right)\log^2\xi\,d\xi = -\frac13\log^3\theta + g(1-\theta) + g(1+\theta) - g(2)$$

$$= -\frac13\log^3\theta + \frac14 g(1-\theta^2) + \frac74 g(1) - g(2)$$, on using (1)

$$= -\frac13\log^3\theta + \frac14 g(\theta)$$, using $1 - \theta^2 = \theta$ and taking $x = 1$ in (1)

$$= \frac1{20} g(1) = \frac1{10}\zeta(3)$$, on using (3) and (4).

BINOMIAL IDENTITY NUMBER NINE (JCMN 22, Vol.2, p.88)

J.B. Parker

If ϕ is a random angle uniformly distributed on the circle there are two ways to calculate the expectation of $\exp(2b\cos^2\phi)$.

On the one hand it is $\sum_{s=0}^{\infty}\frac{(2b)^s}{s!}E(\cos^{2s}\phi)$ and $E(\cos^{2s}\phi) = 2^{-2s}\binom{2s}{s}$ so that the expectation is the given left hand side.

On the other hand the value is $e^b\sum_0^\infty (b^s/s!)\,E(\cos^s 2\phi)$ where the odd numbered terms may be left out and so the expectation is $e^b\sum_0^\infty \frac{b^{2s}}{(2s)!}\ 2^{-2s}\binom{2s}{s}$ which is the right hand side of the given identity. See also page 121.

BINOMIAL IDENTITY NUMBER ELEVEN

J.B. Parker

$$2^{2n-2} = \sum_{s=0}^{n-1} \binom{2s}{s} \binom{2n-2s-2}{n-s-1} .$$

STOP PRESS

TRIANGLES FROM FOUR LINES (Page 102 above)

When answers to this problem of *C.F. Moppert* came from both *M.J.C. Baker* and *S.R. Mandan* the editor asked the former to write a joint paper, but collaboration has been hampered by slowness of the mail service with Bombay. The following comments by *Dr. Mandan* arrived too late to be put in the article.

For Proposition 1 there is a proof in Court's "College Geometry" (1952) pp. 142-143. For Proposition 2 proofs may be found in the book "Geometry Revisited" by Coxeter and Greitzer (1967) p. 39 and in "Reflections on a Triangle" by S.N. Collings, Math. Gaz. 57 (1973) pp. 291-293. To Observation 2 may be added the comment that from five lines you get five parabolas and their foci lie on the Miquel (1845) circle.

Another reference of interest is "The Real Projective Plane" (1955) by H.S.M. Coxeter, page 158, there are two proofs by synthetic methods, one by involutions as in Holgate's "Projective Pure Geometry" (1930) p. 209 and the other by Brianchon's Theorem, as in Salmon's "Conic Sections" (6th edition 1879) p. 247, due to J.C. Moore.

Your editor would like to hear from you anything connected with mathematics or with James Cook.

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