JAMES COOK MATHEMATICAL NOTES

Issue No. 18 April, 1979

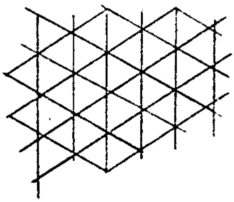
BICENTENARY = (CENTENARY)²

This issue celebrates the hundredth anniversary of the Captain Cook statue in Hyde Park, Sydney, N.S.W. set up for the centenary of Captain Cook's death on 14th February 1779. The picture is courtesy of the N.S.W. Government Department of Tourism.

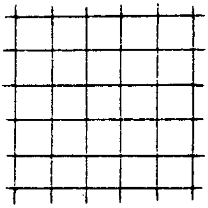
A NOTE ON PERCOLATION THEORY

B.C. Rennie

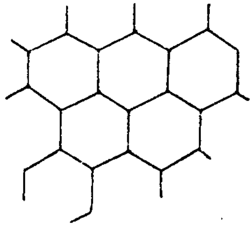
First I must thank *P.A.P. Moran* and *J.M. Hammersley* for introducing me to the delights of this subject, and to the Mathematics Department of University College London for giving me a room in which I could enjoy them.



Triangular network
$n = 6$



Cartesian or square network
$n = 4$



Hexagonal or honeycomb network
$n = 3$

Imagine a plane, or at least a large portion of a plane, covered with one of the three patterns shown above. In each case the parameter n is the number of line segments meeting at each node. Each line segment represents an electrical resistor, with resistance equal to r ohms. On the large scale such a plane network behaves like a uniformly resistive sheet, with specific conductivity in the three cases:-

$$\sigma = \sqrt{3}/r \qquad\qquad \sigma = 1/r \qquad\qquad \sigma = 1/(r\sqrt{3})$$

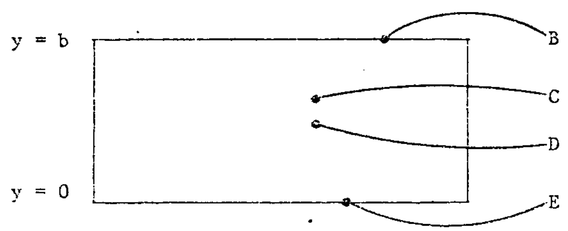
respectively. In each case the large-scale specific conductivity will be isotropic, i.e. the same in all directions.

Percolation theory asks what will happen to the large-scale specific conductivity if some of the resistors, chosen independently at random, are modified from r to R ohms.

What we hope to find is (for each network and for each p with $0 < p < 1$ and for each R/r) a ratio $\sigma(p)/\sigma(0)$ in which the conductivity will be changed. This will not be exact, because we are dealing with a random process. Actually, for a fixed finite network, and a fixed proportion of modified resistors, the ratio of the conductivities before and after the modification can vary over a wide range, depending on how the modifications are distributed. However we expect that for most of the distributions the ratio will be close to the ideal ratio that we hope to calculate, and that the approximation will tend to exactness as the size of the network tends to infinity.

Little progress has been made with finding the ratio $\sigma(p)/\sigma(0)$, but numerical estimates are available for the Cartesian network with R infinite. This note calculates the first-order approximation for small p.

Suppose that one of these networks occupies the region where $0 \le x \le a$ and $0 \le y \le b$, where a and b are large, and suppose that the nodes on the line $y = b$ are all connected to a terminal B. Choose a resistor inside this rectangle oriented in the y-direction (vertically in the diagrams above) and connect its two end-points to terminals C and D (to get our signs right the upper end goes to C). Suppose also that the nodes on the line $y = 0$ are connected to another terminal E.



Now what we have may be regarded as a network of resistors with four external terminals B, C, D and E. The resistance between B and E is $b/(a\sigma)$. We may suppose that the chosen resistor is a long way from the boundary of the rectangle (as it probably is if a and b are large) and therefore, by the method explained in the note "An electrical circuit" in JCMN 17, we may calculate the resistance between C and D to be $2r/n$ ohms.

When one ampere of current passes through the network from B to E the potential difference of B above E is $b/(a\sigma)$ volts, and so the potential difference of C above D is $c/(a\sigma)$, where c is the length of the line segment corresponding to one resistor. Because of Kelvin's reciprocal theorem for resistance networks, this quantity $c/(a\sigma)$, called the mutual resistance, is also the potential difference between B and E that will be caused by one ampere of current passing in to the network at C and out at D.

In order to find the effect of changing the chosen resistor CD from r to R ohms, we imagine one ampere constrained to pass from B to E, and try to find how the voltage of B above E is altered when the resistor is modified.

We do this by a thought-experiment. Across the terminals CD connect a battery of v volts (C above D) passing a current of x amperes (out of the battery and in to the network at C). This extra current in the network leads to an extra potential difference $2rx/n$ of C above D, superimposed on the existing $c/(a\sigma)$. Therefore we have the equation:

$$v = c/(a\sigma) + 2rx/n.$$

As far as the network is concerned, the resistor CD now has a potential difference v across it and is taking a current of $v/r - x$ amperes. We require it to be behaving like R ohms, and therefore another equation is imposed:

$$v = R(v/r - x).$$

Eliminating v and solving for x leads to

$$x = \frac{n(R-r)}{(n-2)R + 2r}\,\frac{c}{a\,r\,\sigma}\ .$$

Because the mutual resistance of CD and BF is $c/(a\sigma)$, the effect of modifying the one resistor (or of putting the battery in CD) is to increase the potential difference of B above E from $b/(a\sigma)$ to $b/(a\sigma) + cx/(a\sigma)$. The proportionate increase is $1 + xc/b$. This calculation has been for the case where the modified resistor is chosen to be of vertical orientation, and it increases the vertical component of specific resistance in the ratio above, leaving the horizontal component unchanged. If the resistor to be modified is chosen with random orientation then the expected effect on the specific conductivity will be to change it in the ratio $1 - xc/(2b)$.

The number of resistors in our rectangle of size $a \times b$ will be (in the three cases being considered) $2\sqrt{3}\,ab/c^2$, $2ab/c$ and $2ab/(\sqrt{3}c^2)$ respectively, and for all three cases this number is $2abr\sigma/c^2$. The proportion p of modified resistors is one in this number, that is $p = c^2/(2abr\sigma)$. The proportionate change in specific conductivity is therefore

$$1 - x\,a\,r\,p\,\sigma/c = 1 - \frac{pn(R-r)}{(n-2)R + 2r}\ .$$

Confining ourselves now to the case of R infinite, which is of practical interest, the situation is that we want the graph of $\sigma(p)/\sigma(0)$, and we know that at $p = 0$ the value is 1 and the slopes for the triangular, square and hexagonal networks are $-3/2$, -2 and -3 respectively. The function obviously decreases, and the value of p for which it vanishes is of interest (the 'threshhold concentration'), for the square network it is 1/2, as can be shown by a pretty little argument left as an exercise for

References:

S.R. Broadbent and J.M. Hammersley, Percolation Processes I, Crystals and Mazes, Proc. Camb. Phil. Soc. (53) 1957, pp. 629-641.

Scott Kirkpatrick, Percolation and Conduction, Rev. Mod. Phys. (45) 1973, pp. 574-588.

P.D. Seymour and D.J.A. Welsh, Annals of Discrete Mathematics, (3) 1978, pp. 227-245.

These papers illustrate the gap now between physics and mathematics. Scientists know numerical values of parameters for which mathematicians are discussing definitions.

INTEGRAL ROOTS (JCMN 17)

The question, from *C.J. Smyth*, was when do the two quadratics $x^2 + ax + b = 0$ and $x^2 + bx + a = 0$ both have both roots integral?

J.B. Parker writes that a and b must clearly both be integers, and (decreasing both roots by one) it is sufficient to ask the same question about the two quadratics:-

$$x^2 + x(a+2) + a + b + 1 = 0 \text{ and } x^2 + x(b+2) + a + b + 1 = 0.$$

An infinite set of solutions then appears, a and b any two integers such that $a + b + 1 = 0$. There are other solutions, $(a, b) = (0, -n^2)$ or $(-n^2, 0)$ for any integer n, or $(4, 4)$ or $(5, 6)$ or $(6, 5)$, but no more spring to the mind.

FINDING AN INTEGRAL (JCMN 15)

J.B. Parker

The problem is to find $J_n^m = \frac{1}{2}\int_{-1}^{1} P_n^m(\mu)\,d\mu$ where we have defined

$$P_n^m(\mu) = \frac{(n-m)!}{2^n.n!m!}\sin^m\theta \frac{d^{n+m}}{d\mu^{n+m}}(\mu^2-1)^m, \quad (\mu = \cos\theta).$$

A well known recurrence relation (Arfken, Mathematical Methods for Physicists, page 436, equation 12.87) yields

$$(n+m+1)P_n^m = \frac{2(m+1)\mu}{\sqrt{1-\mu^2}} P_n^{m+1} - (n-m-1)P_n^{m+2}$$

and on integration

$$(n+m+1)J_n^m + (n-m-1)J_n^{m+2} = -(m+1)\int_{-1}^{1} P_n^{m+1}(\mu)\frac{d}{d\mu}[\sqrt{1-\mu^2}]\,d\mu$$

$$= (m+1)\int_{-1}^{1}\sqrt{1-\mu^2}\frac{d}{d\mu}[P_n^{m+1}(\mu)]\,d\mu \quad \text{——(1)}$$

Again using Arfken, equations 12.87 and 12.90,

$$\frac{d}{d\mu}[P_n^{m+1}(\mu)] = \frac{n-m-1}{\sqrt{1-\mu^2}}P_n^{m+2}(\mu) - \frac{(m+1)}{1-\mu^2}P_n^{m+1}(\mu)$$

Hence $\int_{-1}^{1}\sqrt{1-\mu^2}\frac{d}{d\mu}P_n^{m+1}(\mu)\,d\mu = 2(n-m-1)J_n^{m+2} \int_{-1}^{1}(m+1)\sqrt{1-\mu^2}\frac{d}{d\mu}[P_n^{m+1}(\mu)]\,d\mu$

(after an integration by parts)

giving $(m+2)\int_{-1}^{1}\sqrt{1-\mu^2}\frac{d}{d\mu}P_n^{m+1}(\mu)\,d\mu = 2(n-m-1)J_n^{m+2}$

Substituting in (1) leads to the recurrence relation

$$J_n^m = \frac{m(n-m-1)}{(m+2)(n+m+1)}J_n^{m+2} \quad \text{——(2)}$$

Now $$P_n^n(\mu) = \frac{(2n)!}{2^n.n!m!}\sin^n\theta$$

and this is easily integrated to give

$$J_n^n = \frac{\pi^{\frac{1}{2}}(n+\frac{1}{2})}{2\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{n+3}{2}\right)} \quad \text{——(3)}$$

We now use (2) repeatedly, setting m first n-2, then n-4, and so on up to n-2k, getting

$$J_n^{n-2k} = \frac{n-2k}{n} \frac{\Gamma\left(\frac{2k+1}{2}\right)\Gamma(n-k+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(n+\frac{1}{2})} J_n^n$$

Using (3) and writing $k = \frac{1}{2}(n-m)$, k being necessarily integral ($J_n^m \equiv 0$) unless n-m is even) we find, for one final result

$$J_n^m = \frac{m}{2n} \frac{\Gamma\left(\frac{n+m+1}{2}\right)\Gamma\left(\frac{n-m+1}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{n+3}{2}\right)} \qquad \text{n-m even}$$

$$= 0 \qquad \text{n-m odd.}$$

ALGORITHMS WANTED

Suppose that the depth of the water is measured at n points chosen at random in a rectangular area. How could a computer draw a contour map? This question would have been of particular interest to James Cook if a computer had been available when he was surveying the St. Lawrence River under enemy fire in 1758.

From the same data how could we best estimate the mean depth? The mean of the data points is one answer, but it would be biased if as often happens the boat taking soundings concentrated on the shallower parts.

A.C. Rennie.

A PARTY GAME

The game described in JCMN14 was one where the players in turn added 0 or X to a sequence of such symbols, trying to avoid leaving the sequence with a repeated segment of two or more symbols at the end. In JCMN15 it was reported by *C.J. Smyth* that the game can go for 18 moves but no longer without one player forfeiting a point.

As the result comes only from a computer search it gives us little help

with the following related problem.

Find out if it is possible to have an infinite sequence made up of three symbols (such as A , B and C) such that nowhere in the sequence is there a segment of length two or more followed by a copy of itself.

Using the computer result that 0X00XX000XXX00XX0X is a possible sequence for the first party game, we can easily construct a repitition-free sequence of three symbols of length several hundred; and this rather suggests that the three-symbol problem would be too much for a computer.

POPULAR FALLACY

Most mathematicians, though I cannot vouch for intuitionists, would agree that

$$\int_{-1}^{2} f(x)\,dx = \int_{0}^{1} f(-x) + 2f(2x)\,dx$$

or, to be quite precise, if the integral on the left exists in the absolute proper or improper Riemann sense, or in the Lebesgue sense, or in any other absolute sense, then the integral on the right exists in the same sense, and they are equal.

In spite of this many text-book writers and University lecturers allow and even encourage their helpless first-year students to believe in the formula $\int dx/x = \log|x|$. Those that survive first year and grow up to do complex function theory will of course meet the more correct formula $\int dx/x = \log x$.

As it is hard to stop students from memorizing formulae (they think it is the way to pass examinations) it would be better to let them have the formula $\int dx/x = \log x$, which, in spite of its faults, will not lead them to wrong answers in elementary calculations, and will not have to be discarded later.

ANOTHER LIMIT

Prove that $\sqrt{\left(1+\sqrt{\left[1+2\sqrt{(1+3\sqrt{(1+\ldots+(m-1)\sqrt{(1+m)}\ldots)})}\right]}\right)} \to 2$.
This problem comes from *J.M. Hammersley* who cannot give the original source, but thinks that the question once came in the Putnam competition.

GRAPHS FOR GROUPS

A theorem of Frucht (Comp. Math. 6 (1938) 239-250) states that for any finite group G there is a finite graph (undirected and without loops) whose automorphism group is isomorphic to G . Find such a graph for the cyclic group C_3 . (Top marks for the smallest number of nodes.)

C.J. Smyth

ANOTHER IDENTITY FOR BINOMIAL COEFFICIENTS (JCMN 17)

$$(-1)^n \binom{2n+1}{n} = 2^{2n+1}(2n+1)\binom{\frac{1}{2}}{n+1}$$

<u>First Solution</u> *(H. Kestelman)*

Set $u_n = \text{LHS}/(2n+1) = (-1)^n \dfrac{(2n)!}{n!\,(n+1)!}$

and $v_n = \text{RHS}/(2n+1) = 2^{2n+1}\dbinom{\frac{1}{2}}{n+1}$

Then $u_0 = v_0$ and for $n > 0$

$$u_{n+1}/u_n = -2(2n+1)/(n+2) \quad \text{and} \quad v_{n+1}/v_n = -4\,\frac{n+\frac{1}{2}}{n+2}$$

and so $\quad u_{n+1}/u_n = v_{n+1}/v_n \quad$ and $\quad u_n = v_n$.

<u>Second Solution</u> *(J.B. Parker)*

We need to show that

$$K = (-)^n \frac{\Gamma(2n+2)}{\Gamma(n+1)\Gamma(n+2)} \; \frac{\Gamma(n+2)\Gamma(\frac{1}{2}-n)}{2^{2n+1}(2n+1)\Gamma(3/2)} \equiv 1$$

We have $K = (-1)^n \dfrac{\Gamma(2n+1)\Gamma(\frac{1}{2}-n)}{\Gamma(n+1)\,2^{2n+1}\,\Gamma(3/2)}$

Now Legendre's Duplication formula (I think that's what it's called: Copson is in the office) gives

$$\Gamma(2n+1)\sqrt{\pi} = 2^{2n}\,\Gamma(n+\tfrac{1}{2})\Gamma(n+1)$$

Feeding this in gives $\quad K = (-1)^n \dfrac{\Gamma(n+\frac{1}{2})\Gamma(\frac{1}{2}-n)}{\pi}$

Now there is another, distantly remembered, property of Gamma functions to the effect that $\Gamma(x)\Gamma(1-x) = \dfrac{\pi}{\sin \pi x}$.

The equality $K \equiv 1$ now follows by putting $x = n + \frac{1}{2}$.

<u>Third Solution</u> *(C.J. Smyth)*

The following derivation, though certainly not elegant, does at least avoid expanding the binomial coefficients out in full. (While it may seem pointless to do this, it does illustrate a standard technique which can be used with other coefficients which do not posess such expansions).

First we use the identities

$$\binom{a}{j} = \frac{a}{j}\binom{a-1}{j-1} = \frac{a}{a-j}\binom{a-1}{j} \tag{2}$$

to replace $\dbinom{2n+1}{n}$ by $\dfrac{2n+1}{n+1}\dbinom{2n}{n}$ and $\dbinom{\frac{1}{2}}{n+1}$ by $\dfrac{1}{2(n+1)}\dbinom{-\frac{1}{2}}{n}$.

This reduces (1) to

$$\binom{2n}{n} = (-4)^n\binom{-\frac{1}{2}}{n} \tag{3}$$

which we shall establish instead of (1). (Note that (2) can be proved using the identities $\dfrac{d}{dx}(1+x)^a = a(1+x)^{a-1}$ and $(1+x)(1+x)^{a-1} = (1+x)^a$, avoiding expansion of any binomial coefficients).

To prove (3), it is sufficient to show that

$$\sum_0^\infty \binom{2n}{n} z^n = (1-4z)^{-1/2} \tag{4}$$

The left hand side of the equation above is the constant term in the Laurent expansion in x of

$$S = \sum_0^\infty (x^{\frac{1}{2}}+x^{-\frac{1}{2}})^{2n} z^n = \sum (x+1/x+2)^n z^n = \frac{-x/z}{x^2+(2-1/z)x+1}$$

$$= \frac{-x/z}{\alpha-1/\alpha}\left\{\frac{1}{x-\alpha} - \frac{1}{x-1/\alpha}\right\}$$

where $$2\alpha = 1/z - 2 + \left((1/z - 2)^2 - 4\right)^{\frac{1}{2}}$$
$$= z^{-1} - 2 + z^{-1}(1 - 4z)^{\frac{1}{2}} \ .$$

Here convergence is assured by taking $1/2 < x < 2$ and $0 < z < 1/9$, then $\alpha > 4$ and so x/α and $1/(\alpha x)$ are both between 0 and 1. Then $\alpha - 1/\alpha = z^{-1}(1 - 4z)^{\frac{1}{2}}$ and so

$$S = x(1 - 4z)^{-\frac{1}{2}} \left\{ (1/\alpha) \sum_0^\infty \left(\frac{x}{\alpha}\right)^n + (1/x) \sum_0^\infty \left(\frac{1}{x\alpha}\right)^n \right\}$$

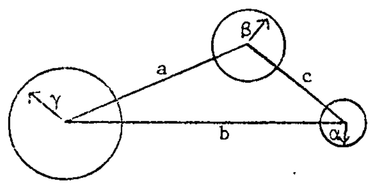
which has constant term $(1 - 4z)^{-\frac{1}{2}}$.

QUERY POSITIVE DEFINITE (JCMN 17)

Given n disjoint spheres in three dimensions, let m_{rr} be the reciproc of the radius of number r, and let m_{rs} = the reciprocal of the centre distance between spheres r and s. Is the matrix positive definite?

For $n = 3$ the answer is YES.

Let the radii (α, β and γ) and centre distances (a, b and c) be as shown. Since the result is easy for two spheres it will be sufficient to show that $D = \det M > 0$.



$$D = \begin{vmatrix} 1/\alpha & 1/c & 1/b \\ 1/c & 1/\beta & 1/a \\ 1/b & 1/a & 1/\gamma \end{vmatrix} = \left(1/(\beta\gamma) - a^{-2}\right)/\alpha + \left(1/(ab) - 1/(\gamma c)\right)/c + \left((1/ac) - (1/b\beta)\right)/b$$

$$D > \frac{1}{\alpha\beta\gamma} - \frac{1}{\alpha a^2} - \frac{1}{c^2\gamma} - \frac{1}{b^2\beta}$$

Since $a > \beta + \gamma$ it follows that $a^{-2} < 1/(4\beta\gamma)$ and so $-1/(\alpha a^2) > -1/(4\alpha\beta\gamma)$ and there is a similar inequality for each of the last two terms.

$$D > \frac{1}{\alpha\beta\gamma} - \frac{3}{4\alpha\beta\gamma} > 0.$$

H. Kestelman.

INTEGRAL CALCULUS

Show that, if m is a positive integer,

$$\int_0^\pi \int_0^\pi \frac{1 - \cos mx \cos my}{2 - \cos x - \cos y}\, dx\, dy = 2\pi\left(1 + 1/3 + \ldots + 1/(2m-1)\right)$$

The origin of this integral is in an electrical network. If a one-ohm resistor connects each adjacent pair of lattice points in the plane, and unit current is taken out from the origin, then the potential difference between the lattice point (m, n) and the origin is:

$$\frac{1}{2\pi^2} \int_0^\pi \int_0^\pi \frac{1 - \cos mx \cos ny}{2 - \cos x - \cos y}\, dx\, dy.$$

AN EXPANSION PROBLEM (JCMN 16)

The question, from *E.O. Tuck*, was whether to any positive integer m corresponds k such that all the coefficients in the power series for $(1 - kx)^{1/m}$ are integers.

Let m be fixed. Write for convenience

$$k = m^{1+\eta} \ . \qquad (1)$$

Let the prime-factorisation of m be

$$m = \prod_{i=1}^{s} p_i^{\alpha_i} \ . \qquad (2)$$

The general term in the expansion of

$$(1 - kx)^{\frac{1}{m}}$$

is

$$\frac{\frac{1}{m}\left(\frac{1}{m} - 1\right) \ldots \left(\frac{1}{m} - r + 1\right)}{r!} (-k)^r x^r \qquad (r \geq 1)$$

Using (1), the modulus of the coefficient of x^r simplifies to

$$\frac{(m-1)\ (2m-1)\ \ldots\left\{(r-1)\ m-1\right\}}{r!}\ m^{\eta r}$$

Let p be a prime. The highest power of it occurring in the denominator $r!$ is

$$t = [r/p] + [r/p^2] + \ldots + [r/p^\lambda] \qquad (3)$$

$$\lambda = \max\{n \mid n \in Z^+ \text{ and } p^n < r\}$$

There are two cases to be considered:

(a) p is coprime to m

(b) p is a factor of m

Start with (a)

Let $1 \le \tau \le \lambda$

Then the congruence

$xm \equiv 1 \pmod{p^\tau}$ has a solution since $(m,p^\tau) = 1$.

Moreover the number of integers x, satisfying the congruence, such that $1 \le x \le r-1$, is either $[r/p^\tau]$ or $[r/p^\tau] + 1$, hence

$\prod_{i=1}^{r-1} (im-1)$ is divisible by <u>at least</u> the t^{th} power of p.

(t defined in (3).)

(It is possible that the product has some factor divisible by p^n where $n > \lambda$).

For settling case (b), making use of (2), write

$$m^{\eta r} = \prod_{i=1}^{s} p_i^{\alpha_i \eta r}.$$

From (3), $t \le r/p_i + r/p_i + \dots + r/p_i^{\lambda_i}$

$< r/(p_i - 1)$.

$r!$ divides $m^{\eta r}$ if for all $i = 1, \dots s$

$r/(p_i - 1) \le \alpha_i \eta r$.

This will be achieved if

$$\eta \ge \max_i \frac{1}{(p_i - 1)\alpha_i} \tag{4}$$

This is the answer for a fixed m.

It is true for all m that $p_i \ge 2$ and $\alpha_i \ge 1$ hence

$$\max \frac{1}{(p_i - 1)\alpha_i} \le 1$$

i.e. if $k = m^2$ the coefficients of $(1-kx)^{\frac{1}{m}}$ are all integers for all m.

Marta Sved

JCMN18.

AN EXPANSION PROBLEM (JCMN 16)

Given m, for what k does $(1-kx)^{1/m}$ have integral coefficients? *Marta Sved* gave a sufficient condition for this, which showed that one could take $k = m^2$. *Alf van der Poorten* answered a slightly more general question by showing that $(1-kx)^{\ell/m}$, (where $\gcd(\ell,m) = 1$), has integral coefficients if and only if $m\Pi$ divides k, where Π is the product of the prime factors of m. Here is the proof:

All that is needed is two remarks. The first is that if, $\nu_p(n!)$ is the exact power of p (p prime) that divides $n!$, then $\nu_p(n!) < n$. The second is that if we eliminate common factors in the quotient

$$\frac{\ell \cdot (m-\ell) \cdot (2m-\ell) \cdot \dots \cdot ((n-1)m-\ell)}{n!}$$

then the only primes that divide the denominator are primes dividing m. Together these remarks immediately imply that

$$k^n \binom{\ell/m}{n}$$

is an integer iff $m\Pi$ divides k. We will show a little more in each case.

For p prime denote by $\nu = \nu_p(x)$ that integer such that $p^\nu | x$ but $p^{\nu+1} \nmid x$ (suppose x to be an integer). Then

$$\nu_p(n!) = (n - \sigma_p(n)) / (p-1),$$

where $\sigma_p(n) = \sum a_i$, the sum of the digits in the p-ary expansion of n. Indeed it is easy to see that

$$\nu_p(n!) = \sum_{i \ge 1} \left[\frac{n}{p^i}\right] \quad ([a] \text{ is the integral part of } a).$$

Write $n = a_0 + a_1 p + \dots + a_k p^k$, $0 \le a_i < p$ (this is the p-ary expansion mentioned above). Then we have

$$\nu_p(n!) = (n - a_0)/p + (n - (a_0 + a_1 p))/p^2 + \ldots.$$

and noting $1/p + 1/p^2 + \ldots. = 1/(p-1)$, this yields the assertion.

Now let q be prime and $(m,q) = 1$. Then $q^\nu | n!$ implies $q^\nu | \ell \cdot (m-\ell) \cdot (2m-\ell) \cdot \ldots. \cdot (n-1)m-\ell$. A proof is harder to write out than to think out. But notice that the congruence $xm-\ell \equiv 0 \pmod{q^c}$ has a solution x_0, $0 \le x_0 < q^c$, hence $q^c < n$ implies q^c divides some term of the product $\ell \cdot (m-\ell) \cdot \ldots \cdot (n-1)m-\ell$. Were we now to note that certainly $(x_0 + rq^{c'})m - \ell \equiv 0 \pmod{q^{c'}}$ for all $c' \le c$ we would notice that to each distinct integer $1, \ldots, n$ divisible by some power of q there corresponds some distinct integer $rm - \ell$ $0 \le r < n$ divisible by at least that power of q and the assertion would follow immediately.

It should be no surprise that we can find a k: For a general power series $\sum a_n z^n$ with rational coefficients which represents an algebraic function, a theorem of *Eisenstein* (Werke p. 765 - 770) states that there is an integer k such that all the $k^n a_n$ are integers.

Van der Poorten also suggests the following as an exercise. It is related to *Apéry's* proof of the irrationality of $\zeta(3)$ (see *v.d.P's* forthcoming article in Maths. Intelligencer Vol. 2, part 1).

ANOTHER EXPANSION PROBLEM

Show that $(1 - 6x + x^2)^{-\frac{1}{2}} = \sum b_n x^n$ has integer coefficients b_n, and that furthermore $b_n = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}$, and they satisfy

$$nb_n + (n-1)b_{n-2} = (6n-3)b_{n-1}.$$

MORE IDENTITIES FOR BINOMIAL COEFFICIENTS

$$\binom{\frac{1}{2}}{n+1} = \frac{(-1)^n}{2^{2n+1}(2n+1)}\binom{2n+1}{n} \tag{0}$$

This identity, slightly rearranged, is given on the first page of JCMN no. 17, Nov. 1978, with a request for an elegant proof to replace the "slogging" method. Perhaps the following goes some way towards meeting that request.

1. Proof of (0) The foundation of most of the following work is Legendre's duplication formula for the gamma function,

$$\Gamma(\tfrac{1}{2})\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma(z+\tfrac{1}{2}). \tag{1}$$

We also use for (0) the obvious identity

$$\begin{aligned} z(z-1)(z-2)\cdots(z-n) &= (-1)^{n+1}(-z)(1-z)(2-z)\cdots(n-z) \\ &= (-1)^n z \frac{\Gamma(n+1-z)}{\Gamma(1-z)}. \end{aligned} \tag{2}$$

For (0) we only use (1) in the case where $2z$ is an odd positive integer; this case can be proved by algebra almost as elementary as the above proof of (2).

To prove (0),

$$\begin{aligned} \binom{\frac{1}{2}}{n+1} &= \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)\cdots(\frac{1}{2}-n)}{(n+1)!} \\ &= \frac{(-1)^n\frac{1}{2}}{(n+1)!}\frac{\Gamma(n+\frac{1}{2})}{\Gamma(\frac{1}{2})} && \text{by (2) with } z=\tfrac{1}{2}, \quad (3) \\ &= \frac{(-1)^n\frac{1}{2}}{(n+1)!}\frac{\Gamma(n+\frac{1}{2})}{\Gamma(\frac{1}{2})}\frac{\Gamma(n+1)}{n!} \\ &= \frac{(-1)^n\frac{1}{2}}{(n+1)!\,n!}\frac{\Gamma(2n+1)}{2^{2n}} && \text{by (1) with } z = n+\tfrac{1}{2}, \\ &= \frac{(-1)^n}{2^{2n+1}}\frac{1}{2n+1}\frac{(2n+1)!}{(n+1)!\,n!} && \text{as required.} \end{aligned}$$

2. Two related identities A different use of the duplication formula (1) leads to another such identity. The denominator in (3) is

$$\begin{aligned} (n+1)!\,\Gamma(\tfrac{1}{2}) &= (n+1)\Gamma(n+1)\Gamma(\tfrac{1}{2}) \\ &= (n+1)\,2^n\,\Gamma(\tfrac{1}{2}n+\tfrac{1}{2})\Gamma(\tfrac{1}{2}n+1) && \text{by (1) with } z = \tfrac{1}{2}n+\tfrac{1}{2}. \end{aligned}$$

This with (3) gives

$$\binom{\frac12}{n+1} = \frac{(-1)^n}{(n+1)2^{n+1}}\,\frac{\Gamma(n+\frac12)}{\Gamma(\frac12 n+\frac12)\Gamma(\frac12 n+1)}\,. \tag{4}$$

We thus have the identity (or identities?)

$$\binom{\frac12}{n+1} = \frac{1}{2^{n+1}(n+1)}\binom{n-\frac12}{\frac12 n} \qquad \text{if } n \text{ is even}, \tag{5}$$

$$\binom{\frac12}{n+1} = \frac{-1}{2^{n+1}(n+1)}\binom{n-\frac12}{\frac12 n-\frac12} \qquad \text{if } n \text{ is odd}. \tag{6}$$

The subdivision of (4) into (5) and (6) is unnecessary if we do not object to the lower element in a binomial symbol having a non-integral value. The combined identity can then be written as (4) with the quotient of gamma functions replaced by either of the binomial coefficients on the right sides of (5) and (6).

On the other hand, replacement of n by $2n$ in (5) and by $2n+1$ in (6) gives two identities bearing a remarkable resemblance to (0). These are, for all positive integers n,

$$\binom{\frac12}{2n+1} = \frac{1}{2^{2n+1}(2n+1)}\binom{2n-\frac12}{n}, \tag{7}$$

$$\binom{\frac12}{2n+2} = \frac{-1}{2^{2n+2}(2n+2)}\binom{2n+\frac12}{n}. \tag{8}$$

3. <u>Further outlook</u> One wonders how many more such identities there are! The following complement to (0), (7) and (8),

$$\binom{\frac12}{n} = \frac{(-1)^{n-1}}{2^{2n}(2n-1)}\binom{2n}{n}, \tag{9}$$

is easily derived from (0), as follows.

$$\binom{2n}{n} = \frac{n+1}{2n+1}\binom{2n+1}{n} = \frac{n+1}{2n+1}\,\frac{2^{2n+1}(2n+1)}{(-1)^n}\binom{\frac12}{n+1}$$
$$= \frac{2^{2n+1}(n+1)}{(-1)^n}\,\frac{\frac12-n}{n+1}\binom{\frac12}{n} = \frac{2^{2n}(2n-1)}{(-1)^{n-1}}\binom{\frac12}{n}.$$

Similar operations performed on (0), (7), (8), (9) yield more identities; but the only such one having the same order of simplicity, and distinct from them, seems to be

$$\binom{\frac12}{n+1} = \frac{(-1)^n}{2^{2n}(n+1)}\binom{2n-1}{n}. \tag{10}$$

This may be obtained most directly from (0), as follows.

$$\binom{2n-1}{n} = \frac{(n+1)n}{(2n+1)(2n)}\binom{2n+1}{n}$$
$$= \frac{n+1}{2(2n+1)}\,\frac{2^{2n+1}(2n+1)}{(-1)^n}\binom{\frac12}{n+1} = \frac{2^{2n}(n+1)}{(-1)^n}\binom{\frac12}{n+1},$$

giving (10). A natural deduction from (10) is

$$\binom{\frac12}{n} = \frac{n+1}{\frac12-n}\binom{\frac12}{n+1} = \frac{(-1)^n}{2^{2n}(\frac12-n)}\binom{2n-1}{n} = \frac{(-1)^{n-1}}{2^{2n-1}(2n-1)}\binom{2n-1}{n},$$

but this is not distinct from (0); in fact it follows from (0) by replacing n by $n-1$ and using the symmetry of binomial coefficients.

4. <u>Summary</u> We have obtained identities (0), (7), (8), (9), (10), involving (not respectively)

$$\binom{2n-1}{n},\quad \binom{2n-\frac12}{n},\quad \binom{2n}{n},\quad \binom{2n+\frac12}{n},\quad \binom{2n+1}{n}.$$

Some other, less simple, identities alluded to but not explicitly stated in the previous paragraph are as follows.

$$\binom{\frac12}{2n+1} = \frac{2n-\frac12}{2^{2n}(2n-1)(2n+1)}\binom{2n-\frac32}{n},$$

$$\binom{\frac12}{n+1} = \frac{(-1)^n(2n-1)}{2^{2n-2}(2n-2)(2n+2)}\binom{2n-2}{n},$$

$$\binom{\frac12}{2n+1} = \frac{(2n-\frac12)(2n-\frac32)}{2^{2n-1}(2n-3)(2n-1)(2n+1)}\binom{2n-\frac52}{n};$$

and similarly with - replaced by + in the upper elements of the binomial symbols on the right. So it seems unlikely that more simple identities will be found by pressing in these directions.

5. <u>Connection with Legendre polynomials</u> The anonymous author of (0), JCMN no. 17 page 1, obtained it by equating two evaluations of

$$\int_0^1 P_{2n+1}(\mu)\,d\mu\,;$$

he writes that "some calculations in fluid mechanics ... indicate that" this integral "ought to be equal to" the left side of (0). One way of removing the possible shadow of doubt which seems to lie over this statement is to integrate term by term the two equations

$$\sum_{n=0}^{\infty} r^n P_n(\mu) = \frac{1}{\sqrt{1-2r\mu+r^2}}, \qquad \sum_{n=0}^{\infty} (-r)^n P_n(\mu) = \frac{1}{\sqrt{1+2r\mu+r^2}},$$

subtract, re-expand and equate coefficients.

Another way uses only the recurrence relations

$$(m+1)P_{m+1}(\mu) + mP_{m-1}(\mu) = (2m+1)\mu P_m(\mu), \tag{11}$$

$$\frac{d}{d\mu}\{P_{m+1}(\mu) - P_{m-1}(\mu)\} = (2m+1)P_m(\mu), \tag{12}$$

which hold for positive integers m. It goes as follows.

$$\begin{aligned}\int_0^1 P_m(\mu)\,d\mu &= \frac{1}{2m+1}\left[P_{m+1}(\mu) - P_{m-1}(\mu)\right]_0^1 \quad \text{by (12)},\\ &= \frac{1}{2m+1}\left(P_{m-1}(0) - P_{m+1}(0)\right)\\ &= \frac{P_{m-1}(0)}{2m+1}\left\{1 + \frac{m}{m+1}\right\} \quad \text{by (11) with } \mu = 0,\\ &= \frac{P_{m-1}(0)}{m+1}.\end{aligned}$$

By (11) with $\mu = 0$ again (or else using the generating equation),

$$P_{2n}(0) = (-1)^n \frac{2n-1}{2n}\,\frac{2n-3}{2n-2}\cdots\frac{1}{2} = \binom{-\frac{1}{2}}{n}, \tag{13}$$

and so
$$\int_0^1 P_{2n+1}(\mu)\,d\mu = \frac{P_{2n}(0)}{2n+2} = \frac{\frac{1}{2}}{n+1}\binom{-\frac{1}{2}}{n} = \binom{\frac{1}{2}}{n+1}. \tag{14}$$

This evaluates the integral. However this evaluation is not used in obtaining the identities (0), (7), (8), (9), (10), which are the main results of this note.

6. Postscript The last step in (14) simplifies (10), giving

$$\binom{-\frac{1}{2}}{n} = \frac{(-1)^n}{2^{2n-1}}\binom{2n-1}{n}. \tag{15}$$

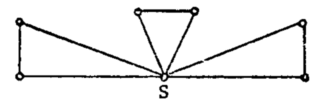
This is the simplest of all our results. But a direct verification shows that it is little more than a disguised form of the duplication formula (1) with $z = n$.

E.R. Love.

FRIENDSHIP AND LOVE

The difference is that friendship is mutual, if A is a friend of B then B is a friend of A. Algebraists call such a relation "symmetric". Love does not have this property, for we may find A loving B but B not loving A.

The Friendship Theorem comes from asking "If every two people have just one common friend, then what?". The answer (JCMN 16) is that there is somebody (whom we might call S) who is a friend of everybody else, and all the others have just two friends, S and one other. In fact the pattern of friendships has to be as shown in Figure 1, which gives the case of seven people, any other odd number is possible.



Now can we look for a Love Theorem like the Friendship Theorem? The condition of every pair having a common friend, when we try to adapt it to a relation on a directed graph, gives two alternatives. Hypothesis G is "Given any two people there is just one other that loves them both." and hypothesis H is "Given any two people A and B, there is just one other, C, such that A loves C and C loves B." We therefore ask what directed graphs satisfy one of the hypotheses G and H.

Trivially all the friendship graphs, such as that of Fig. 1, will satisfy both G and H if each edge is understood as a pair of directed edges, one going in each direction. But are there any other possiblities? Two are known. They are sketched below using the convention that an edge curves to the left when you follow it in the positive direction. The graph of Fig. 2 was given by *G. Szekeres* in JCMN 16 and that of Fig. 3 was explained to me by *J.M. Hammersley*.

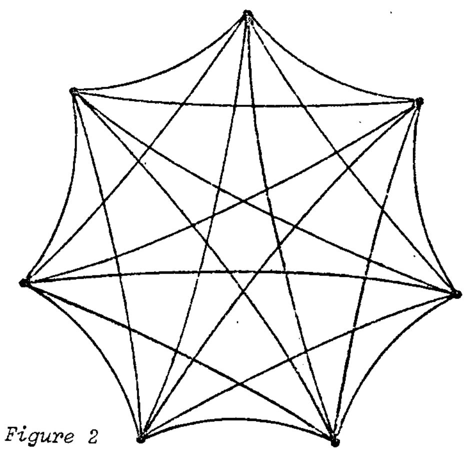


Figure 2

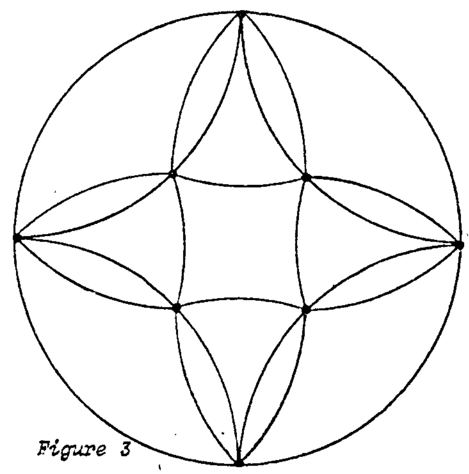


Figure 3

The conditions G and H may also be expressed in matrix notation. Let M be the incidence matrix of a directed graph, it is a square matrix with components all either zero or one, and with zeros on the principal diagonal, and $M_{ij} = 1$ if there is an edge from node i to node j. Hypothesis G is that $M^T M$ has components equal to one everywhere off the principal diagonal, and H is that M^2 has components one everywhere off the principal diagonal.

Can you think of any other examples?

SOME MATRIX POLYNOMIAL QUESTIONS

H. Kestelman

(a) Let J be the matrix with ones on the first superdiagonal, that is with (r, s) element $= 1$ when $r + 1 = s$ and zero otherwise.

Show that every matrix commuting with J is a polynomial in J.

(b) With complex square matrices use an asterisk to denote Hermitean conjugate. Let A be normal (i.e. $AA^* = A^*A$); show that A^* is a polynomial in A.

(c) Is it true in general that a matrix commuting with A is a polynomial in A?

EXPONENTIAL PROGRAMMING

The following non-linear programming problem arose in connection with a number-theoretic question:

$$\text{Minimise} \quad y^{\lambda}$$

$$\text{subject to} \quad x^{1-\lambda} y^{\lambda} \geq 1$$

$$(1-x)^{1-\lambda}(y-1)^{\lambda} \geq 1$$

$$0 \leq \lambda \leq 1, \quad 0 \leq x \leq 1, \quad y \geq 1.$$

C.J. Smyth.

THE TIMES FRIDAY NOVEMBER 17 1978

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From Our Correspondent
Accra, Nov 16

Lieutenant-General F W K Akuffo, Ghana's Head of State, announced today that his Government has ordered the immediate release of all people jailed for their involvement in plots to overthrow the government headed by his predecessor, General Kutu Acheampong.

The 24 men comprise, both civilians and soldiers, most of whom were serving life terms after General Acheampong had commuted their death sentences.

General Akuffo said their release was a gesture of "mercy and goodwill" to "promote national reconciliation".

General Akuffo made his announcement while receiving the report of the Constitutional Commission which has been writing a draft version of the constitution. Under this the country will have a "transitional interim national government" in October next year for not less than four years.

The commission was originally created by General Acheampong to decide details of his controversial "union government" idea and started work in May with a membership of 25.

When General Acheampong was overthrown in July, however, there were calls for its dissolution, but General Akuffo's Government detained it and enlarged the membership to 55.

The Ghana Bar Association has been one of the commission's most vociferous critics

by Dalai Lama

From Our Own Correspondent
Delhi, Nov 16

The release by the Chinese of 24 Tibetan prisoners, including former high officials, which was announced yesterday, has been welcomed by the Dalai Lama in his Dharamsala retreat in the Indian Himalayas.

Peking, Nov 16.—Chagyari Namgyal Gyagso, a former leader of Tibet's rebellion against Chinese rule 19 years ago, who was one of those released on November 4, was quoted today by the New China news agency as saying that the uprising failed because communism offered an alternative to serfdom.

"We lost because we had inflicted so many hardships on the Tibetan people", he said. Reuter.

Prime number record broken

Hayward, California, Nov 15.—Two 18-year-old American students have discovered with the help of a computer at California State University the biggest known prime number, the number two to the 21,701st power.

Laura Nickel and Curt Noll received congratulations from Dr Bryant Tuckerman, an American who discovered the previous record-holder among prime numbers, two to the 19,937th power.—Agence France-Presse.

ending his speech with an appeal for unity, an MP interrupted to ask why he also "failed to make the small gesture" of ordering an inquiry against his son. "You are possessed of an evil spirit," Mr Desai retorted. "What you need is an exorcist." But Mr Desai then told the MP he was prepared to discuss anything he wished to raise but not before a party forum.

Mr Desai was evidently conscious that the criticisms of Janata's poor performance which were expressed behind closed doors at the Ujjain meeting in Madhya Pradesh all end up in the Prime Minister's office.

His tone has changed rapidly from his initial reaction to Mrs Gandhi's Chikmagalur by-election victory: that it would not make any difference to the Government's working. All the Janata breast-beating at Ujjain during the past two days has been caused by the warning sounded from southern India.

Mr Chandra Shekhar, the party's president, who has already said he does not wish to continue in the post or join the Cabinet in the impending reshuffle, attacked the Government on both economic and social fronts.

Child labour ban

Hongkong, Nov 16.—Hongkong plans to ban children under the age of 14 from working in any sector of its economy by next year, Mr Neil Henderson, Labour Commissioner, said.

doing so only because of their dependence upon regular work from the Provisional IRA. This was a most contemptuous response. Indeed one wonders where Mr Mason's twisted attitude would place the Lord Chief Justice, since he ordered the releases. Perhaps the Secretary of State would observe that Sir Robert also depends on regular work from terrorists?

What Mr Mason has betrayed is an attitude in which the law has no status independent of those who use or pronounce upon it, that it is there to be manipulated according to one's partisan creed and that anyone who disagrees with him must be part of a terrorist plot.

The time has come for Mr Callaghan to consider whether Mr Mason is to be taken as the best available ambassador for the British Parliamentary tradition.
Yours faithfully,
JOHN MEEHAN, Northern Ireland Convenor,
Legal Action Group,
Flat 4, 6 College Park,
Belfast, Northern Ireland.

Prime number

From Mr D. A. Simpson

Sir, On Friday afternoon having read your news item headlined "Prime number record broken", I walked into one of my mathematical classes of 11-year-old boys, and wrote the article on the blackboard.

When I got to "2 to the 21,701st power", I stopped. There was a slight pause.

First boy: "What a big number, Sir." Second boy: "But, Sir, that number can't possibly be prime; 2 goes into it."

Could you have meant 2 to the 21,701st power minus 1?
Yours sincerely,
DONALD SIMPSON,
Winchester House School,
Brackley,
Northamptonshire.
November 20.

Yes, we meant 2 to the power of 21,701 minus 1. Our report on November 17 was wrong, and Mr Simpson's second boy was right.

nal Procedure places obligation upon the co

The Old Vic

From the Secretary-Gen Arts Council

Sir, I am reluctant to discussion in your colu it might appear (wrong Arts Council's sole crit matter is the quality of Company.

However, I must c Hugh Manning's curiou tion (letter, November late Lord Chandos's when the National Com to the South Bank t Theatre would be free its own course". This c be taken to mean th Council would subsidi ing. The council's given to performing which create product. T own buildings, but we dize venues alone to e who run them to be buying in artistic produ

BOUND VOLUME

Issues 1 to 17 inclusive of the James Cook Mathematical Notes are now being reprinted in a single volume. This will be available at a price of $5.00 (including postage). Customers in Australia are asked to send cheques payable to James Cook University. Those overseas are invited to send any kind of currency of roughly equivalent value (for example $5.50 (U.S.A.) or £2.60 (United Kingdom) etc.)

Your editor would like to hear from you anything connected with mathematics or with James Cook.

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James Cook University of
North Queensland,
Townsville, 4811,
Australia.