K-stability of Smooth Fano SL_2-Threefolds

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Kähler-Einstein Metrics

X compact manifold, $\dim = 2n$. *Kähler* metrics: compatible Riemannian, complex and symplectic structures

Question: If (X, ω) Kähler, $\exists$ *canonical* $\omega' \in [\omega]$?

Definition

(X, ω) *Kähler-Einstein* if $\exists\ \omega' \in [\omega]$, $\lambda \in \mathbb{R}$ with

$$\operatorname{Ric} \omega' = \lambda \omega'.$$

Existence of KE Metrics

New question: do Kähler-Einstein metrics always exist?

$$c_1(X) = \frac{1}{2\pi}[\mathrm{Ric}\,\omega] \in H^2_{dR}(X,\mathbb{R}) \text{ independent of } \omega$$

If X is KE, $c_1(X)$ is **definite**. Split into cases:

$c_1(X) < 0$: KE metrics exist by Aubin ('76), Yau ('78);

$c_1(X) = 0$: KE metrics exist by Yau ('78);

$c_1(X) > 0$: Obstructions exist by Matsushima ('57). $\mathrm{Aut}\,X$ not reductive $\implies X$ not KE. Ex: $\mathrm{Bl}_P\,\mathbb{P}^2$.

K-stability

Yau-Tian-Donaldson conjecture: Translate $c_1(X) > 0$ case into algebraic geometry.

Theorem (Chen, Donaldson, Sun '12)

A smooth complex Fano variety X admits a Kähler-Einstein metric if and only if (X, K_X^{-1}) is K-stable.

K-stability is an entirely **algebro-geometric** condition!

K-stability

Definition

(X, L) complex polarised variety. A *test configuration* for (X, L) is:

- a flat family $\pi\colon \mathcal{X} \to \mathbb{C}$;
- a relatively ample line bundle $\mathcal{L} \to \mathcal{X}$;
- a $\mathbb{C}^\times$-action on $(\mathcal{X}, \mathcal{L})$;

such that everything $\mathbb{C}^\times$-equivariant, and fibres over $t \neq 0$ isomorphic to X.

TC *special* if central fibre X_0 normal.

K-stability

$(\mathcal{X}, \mathcal{L})$ special TC for (X, L). *Donaldson-Futaki invariant* associated to $\mathbb{C}^\times \curvearrowright H^0(X_0, L_0)$.

Definition

A polarised variety (X, L) is *K-semistable* if $\mathrm{DF}(\mathcal{X}, \mathcal{L}) \geq 0$ for every **special** test configuration $(\mathcal{X}, \mathcal{L})$ and *K-stable* if equality holds only for the trivial configuration $\mathcal{X} = X \times \mathbb{A}^1$.

Note: can ignore non-special TCs!

New question: How to check K-stability?

Equivariant K-stability

Infinitely many test configurations $\implies$ K-stability hard to check.

Exploit symmetry: reductive group $G \curvearrowright X$.

Look for **equivariant** test configurations.

Theorem (Datar, Székelyhidi, '16)

Let a reductive algebraic group G act on a smooth complex Fano variety X. Then X is Kähler-Einstein if (X, K_X^{-1}) is K-stable with respect to equivariant special test configurations.

Considerably easier to check!

Complexity

Definition

Connected reductive $G \curvearrowright X$. Borel subgroup $B \subseteq G$. *Complexity*

$$c_G(X) = \min_{x \in X} \operatorname{codim} B \cdot x = \operatorname{trdeg} \mathbb{C}(X)^B.$$

e.g. if $G =$ torus, G-varieties of complexity $0 =$ toric varieties.

For $c_G(X) \leq 1$, **combinatorial description** is possible.

Goal: use combinatorics to show K-stability.

Conditions for Equivariant K-stability

Criteria found in following cases:

	$c_G(X) = 0$	$c_G(X) = 1$
$G =$ torus	Wang/Zhu '04	Ilten/Suess '15
$G \neq$ torus	Delcroix '16	Unsolved: We are here

Ex: Smooth Fano toric X is K-stable iff barycentre of polytope P_X° is 0.

Combinatorial Description in Complexity One

K finitely generated extension of $\mathbb{C}$, $G \curvearrowright K$ connected reductive algebraic group

Luna-Vust theory classifies normal G-varieties X with $\mathbb{C}(X) = K$

Valuations of G- and B-stable divisors:

$\mathcal{V} = \{\nu_D \mid D \subseteq X \text{ G-stable}\}$, *$G$-valuations.*

$\mathcal{D}^B = \{D \subseteq X \mid \text{B stable but not G-stable}\}$, *colours*.

Timashev ('97) applied LV-theory to give **combinatorial** description in complexity one.

Combinatorial Description in Complexity One

Semi-invariants: $f \in K$ s.t. $\forall b \in B$, $b \cdot f = \chi(b) f$, where $\chi\colon B \to \mathbb{C}^\times$.

$K^{(B)} = \{\text{semi-invariants}\}$, $K_\chi^{(B)} = \{\text{semi-invariants of weight } \chi\}$.

Weight lattice: $\Lambda = \{\chi\colon B \to \mathbb{C}^\times \mid K_\chi^{(B)} \neq 0\}$.

Split exact sequence:

$$0 \to (K^B)^\times \to K^{(B)} \to \Lambda \to 0.$$

Splitting map $e\colon \Lambda \to K^{(B)}$ (map χ to some $f \in K_\chi^{(B)}$). *Not* canonical.

Combinatorial Description in Complexity One

Valuations determined by restriction to $K^{(B)}$.

Functional $\ell\colon \Lambda \to \mathbb{Q}$ and restriction to $K^B = \mathbb{C}(\mathbb{P}^1)$

$\nu|_{K^B} = h\nu_p$, $p \in \mathbb{P}^1$, $h \in \mathbb{Q}_{\geq 0}$.

Valuations $\leftrightarrow$ triples (p, ℓ, h): $p \in \mathbb{P}^1$, $\ell \in \Lambda^*$, $h \in \mathbb{Q}_{\geq 0}$.

Central divisors: $\nu|_{K^B} = 0 \implies h = 0$, p arbitrary.

Regular colours $h = 1$.

Subregular colours $h > 1$.

B-Quotient

X *quasihomogeneous* if $K^G = \mathbb{C}$. Open G-orbit, one-parameter family of codim 1 B-orbits. All examples in this case.

$K^B = \mathbb{C}(\mathbb{P}^1)$: rational B-quotient $\pi\colon X \dashrightarrow \mathbb{P}^1$.

Regular colours $D_p = \pi^*(p)$ where $\nu_D|_{K^B} = \nu_p$.

Subregular colours multiplicity > 1 in $\pi^*(p)$, i.e. fibre over p non-reduced.

Central divisors intersect $\pi^*(p)$ for all p.

Hyperfans

Hyperspace

$$\mathcal{H} = \bigcup_{p \in \mathbb{P}^1} \{p\} \times \Lambda^* \times \mathbb{Q}_{\geq 0} / \sim$$

Half-spaces indexed by $\mathbb{P}^1$, boundary hyperplanes glued together.

$\mathcal{V}$, $\mathcal{D}^B \to \mathcal{H}$.

$f \in K_\chi^{(B)}$ functionals: $\langle f, (p, \ell, h) \rangle = h\nu_p(f) + \ell(\chi)$.

X determined by G-subvarieties.

G-subvarieties $Y \subseteq X$ determined by *coloured data*
$\mathcal{V}_Y \cup \mathcal{D}_Y^B = \{\nu_D \in \mathcal{V} \cup \mathcal{D}^B \mid Y \subseteq D\}$.

Coloured data determine *coloured (hyper)cones*:
$\mathcal{C}_Y = \{q \in \mathcal{H} \mid f(\mathcal{V}_Y \cup \mathcal{D}_Y^B) \geq 0 \implies f(q) \geq 0\}$.

G-varieties classified by *coloured hyperfans*.

Smooth Fano SL_2-Threefolds

Cheltsov, Przyjalkowski, Shramov ('19): Classified smooth Fano threefolds with infinite automorphism groups. Simplest examples.

$\dim \mathrm{SL}_2 = 3 \implies \mathrm{SL}_2$-threefolds (generally) complexity one.

Aim:

- Find combinatorial description of smooth Fano SL_2-threefolds with $\mathrm{Aut}\, X$ reductive
- Focus on those without 2- or 3-torus action.
- Use to show K-stability.

Example: $\mathbb{P}^3$ blown up along three disjoint lines

$G = \mathrm{SL}_2 \curvearrowright \mathbb{P}^3 = \mathbb{P}(M_2(\mathbb{C}))$ by left matrix multiplication.

$D =$ singular matrices. Central G-divisor.

B-quotient $\pi\colon \mathbb{P}^3 \dashrightarrow \mathbb{P}^1$, $\left(\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right) \mapsto [z : w]$.

$\Lambda = \mathbb{Z}$

$p = [\alpha : \beta] \in \mathbb{P}^1$, $D_p = \mathcal{Z}(\beta z - \alpha w) = \pi^*(p)$. Regular colours.

$D \cap D_p = Y_p =$ singular matrices with kernel p. G-stable lines.

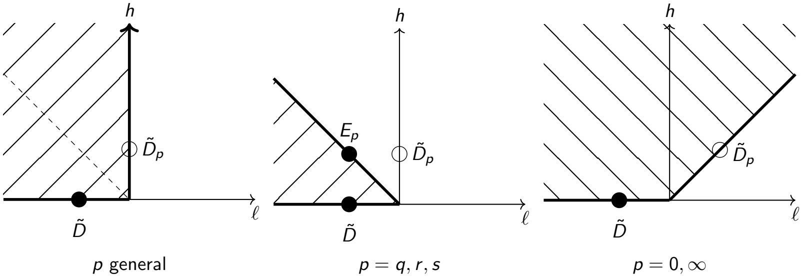
Blow up Y_r, Y_s and Y_q.

Exceptional divisors E_r, E_s and E_q G-stable.

New G-stable curves $\tilde{D} \cap E_i$.

Example

Hyperfan: pictures represent hypercone in 'slice' of $\mathcal{H}$ corresponding to each $p \in \mathbb{P}^1$.



Filled circles = G-divisors
Unfilled circles = colours
Dashed line = boundary of $\mathcal{V}$
Hatched areas = cones of G-stable curves

Volume and β-invariant

$\sigma : Y \to X$ projective birational morphism, Y normal. Prime divisor $F \subseteq Y$ = prime divisor *over* X.

$A_X(F) = \mathrm{ord}_F(K_{Y/X}) + 1$ *log discrepancy* of F over X.

Volume of divisor δ on X is

$$\mathrm{vol}(\delta) = \lim_{k\to\infty} \frac{\dim H^0(X, \mathcal{O}(\delta)^{\otimes k})}{k^n/n!}.$$

β-invariant of Fujita-Li:

$$\beta_X(F) = A_X(F)(-K_X)^n - \int_0^\infty \mathrm{vol}(\sigma^*(-K_X) - xF)\, \mathrm{d}x.$$

β-invariant and K-stability

Theorem (Fujita-Li '16/'15)

A smooth complex Fano variety X is K-stable iff $\beta_X(F) > 0$ for all prime divisors F over X.

Idea: prime divisor F over X

$R =$ section ring of $(X, -K_X)$

F induces filtration $\mathcal{F}$ of R

Rees algebra $\mathcal{A} = \bigoplus_{r \in \mathbb{Z}} \mathcal{F}^r R \cdot z^{-r}$

Embedding $\mathbb{C}[z] \to \mathcal{A}$, so morphism $\pi : \operatorname{Proj} \mathcal{A} \to \mathbb{A}^1$

$X = \operatorname{Proj} R \subseteq \operatorname{Proj} \mathcal{A}$ preimage of $1 \in \mathbb{A}^1$, π is TC.

$\beta_X(F)$ is a positive multiple of Donaldson-Futaki invariant.

Main Result

Theorem (R., Süss)

Let X be a smooth Fano SL_2-threefold. Consider the conditions:

(i) A finite group A acts on $\mathbb{P}^1$ with no fixed points,

(ii) A finite group A interchanges two points in $\mathbb{P}^1$ corresponding to subregular colours of X,

(iii) X has subregular colours lying over three or more distinct points of $\mathbb{P}^1$.

If (i) or (ii) holds and the action of A on $\mathcal{H}$ induced by its action on $\mathbb{P}^1$ fixes the coloured hyperfan of X, or if (iii) holds, then X is K-stable if $\beta_X(F) > 0$ for all **central** *SL_2-stable prime divisors F over X.*

One of these 3 holds in almost every case!

Proof sketch (i)

A-action on $\mathcal{H}$: $a : (p, \ell, h) \mapsto (a \cdot p, \ell, h)$.

Preserves hyperfan of $X \implies (A \times G) \curvearrowright X$.

Non-central prime divisor F lies over point $P_F \in \mathbb{P}^1$.

(i): A-action no fixed points $\implies P_F$ not fixed $\implies F$ not stable under $A \times G$.

Datar-Székelyhidi theorem for $(A \times G)$-stable divisors - all are central.

Proof sketch (ii), (iii)

Cases (ii) and (iii) more difficult.

Idea: two subregular colours at points $\neq P_F$ give non-normality of corresponding TC.

(ii): rule out $P_F =$ interchanged points.

(iii): can always choose two needed points.

Central Divisors

Central divisors mapped to ‘central hyperplane’ Λ^* of $\mathcal{H}$.

Strict convexity, $\operatorname{rk} \Lambda = 1 \implies$ at most one central divisor.

Always exists central divisor over X.

Calculating β

δ divisor on complexity-one variety X.

Formula by Timashev ('00) calculates $\operatorname{vol}\delta$.

Only use root system of G and combinatorial data of δ

Calculating β very easy!

Example: $\mathbb{P}^3$ blown up along 3 lines

X = blow up of $\mathbb{P}^3$ along three lines Y_q, Y_r, Y_s.

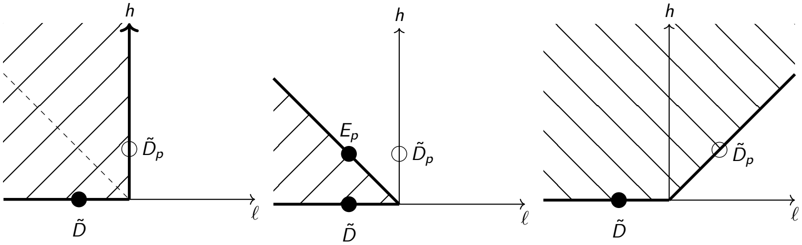
$\mathbb{P}^1$ = sphere

$A = S_3$ = symmetries of triangle qrs.

No fixed points.

Poles are orbit of order 2, all other orbits of order 3 and 6.

Example: $\mathbb{P}^3$ blown up along 3 lines



p general $p = q, r, s$ $p = 0, \infty$

q, r, s = vertices of triangle

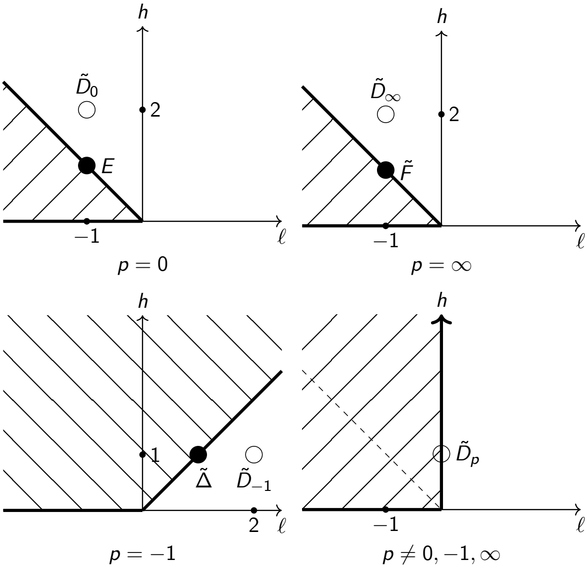
$0, \infty$ = poles of sphere

A-action preserves hyperfan.

X K-stable if $\beta_D(X) > 0$. Formula gives $\beta_D(X) = 11$.

Example: Divisor on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$

X = divisor of tridegree (1,1,1) on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$:



Example: Divisor on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$

$A = \mathbb{Z}_2 \curvearrowright \mathbb{P}^1$: $[\alpha : \beta] \mapsto [\beta : \alpha]$

$1, -1$ fixed.

$0, \infty$ interchanged, subregular colours.

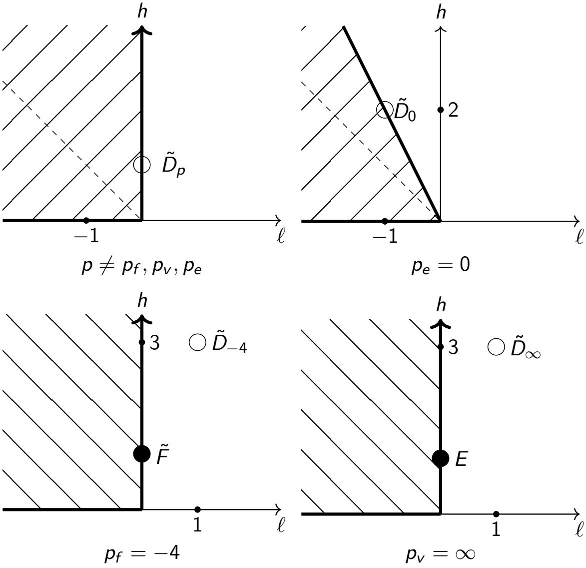
All other points in orbits of order 2.

Hyperfan of X preserved, (ii) implies X K-stable if $\beta > 0$ for central divisor.

Central divisor obtained by blow-ups, $\beta = 28$.

Example: Blow-up of Q along twisted quartic

X = blow up of quadric $Q \subseteq \mathbb{P}^4$ along twisted quartic:



Blow-up of Q along twisted quartic

Three subregular colours over $0, -4, \infty$.

(iii) implies X K-stable if $\beta > 0$ for central divisor.

Central divisor obtained by blow-ups, $\beta = 46$.

Results

Method shows K-stability of (Mori-Mukai numbering):

(1.10)†: V_{22}

(1.15): V_5

(1.16): Quadric hypersurface $Q \subseteq \mathbb{P}^4$

(1.17): $\mathbb{P}^3$

(2.21)†: Blow up of Q along twisted quartic

(2.27): Blow-up of $\mathbb{P}^3$ along twisted cubic

(2.32): Divisor W on $\mathbb{P}^2 \times \mathbb{P}^2$ of bidegree (1,1)

(3.13)†: Blow up of W along curve of bidegree (2,2)

(3.17): Divisor on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$ of tridegree (1,1,1)

(3.25): Blow up of $\mathbb{P}^3$ along two lines

(4.6): Blow up of $\mathbb{P}^3$ along three lines

$\dagger$ = specific examples within a family, green = K-stability not previously known.

Next Steps

-Check applicability to wider classes of complexity one Fanos

-Formula for $-K_X$ would make this easier

-Look for criterion associated with barycentre of a polytope like in $c_G(X) = 0$ case

-Try GL_2-fourfolds