General type results for moduli of hyperkähler varieties

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General type results for moduli of hyperkähler varieties

Joint work in progress with I. Barros, P. Beri & L. Flapan

Goal:
Give new general type results for moduli of hyperkähler varieties and explain challenges for extending.

Main references:
Gritsenko–Hulek–Sankaran (GHS) 2007, 2010, 2011

Plan:

1. Kodaira dimension & hyperkähler varieties
2. Results
3. Sketch proof
4. Comments on proof

- Everything over $\mathbb{C}$

Kodaira dimension

Let X be a smooth connected complete variety, $\omega_X = \Lambda^{\dim X}\Omega_X$. Let $P_m := \dim \mathrm{H}^0(X, \omega_X^{\otimes m})$. The *Kodaira dimension* of X is

$$\kappa(X) = \begin{cases} -\infty & \text{if } P_m = 0 \text{ for all } m > 0; \\ \text{otherwise, the minimal } k \text{ s.t. } \frac{P_m}{m^k} \text{ is bounded} \\ \qquad\qquad (\text{i.e. } P_m \text{ grows like } m^k) \end{cases}$$

1. $\kappa(X)$ is a birational invariant
   - for X singular (non-complete), define $\kappa(X) := \kappa(X')$ for X' a desingularization (completion) of X
2. $\kappa(X) \in \{-\infty, 0, 1, \dots, \dim(X)\}$

X is of *general type* if $\kappa(X) = \dim(X)$.

Examples

- If C is a curve, then $\begin{cases} \kappa(C) = -\infty & \iff g(C) = 0 \\ \kappa(C) = 0 & \iff g(C) = 1 \\ \kappa(C) = 1 & \iff g(C) > 1 \end{cases}$
- $\kappa(\mathbb{P}^n) = -\infty$. More generally, $\kappa(X) = -\infty$ if X is *unirational*, i.e. there is a dominant rational map $\mathbb{P}^n \dashrightarrow X$.
- Severi (1915): the moduli space $\mathcal{M}_g$ of curves of genus g is unirational when $g \leq 10$
- Harris, Mumford, Eisenbud (1980s): $\mathcal{M}_g$ is of general type when $g \geq 24$
- Y.-S. Tai (1982): The moduli $\mathcal{A}_g$ of principally polarized abelian varieties of dim. g is of general type when $g \geq 9$
  - Main ingredient: *Siegel modular forms*

Hyperkähler varieties

A *hyperkähler* (HK) variety is a smooth projective variety X s.t.

1. X is simply connected
2. $\mathrm{H}^0(X, \Omega_X^2)$ is generated by a non-degenerate 2-form.

- $\dim_{\mathbb{C}} X$ is even
- $\omega_X \cong \mathcal{O}_X$
- $\mathrm{H}^2(X, \mathbb{Z})$ is torsion-free & has non-deg. symmetric bilinear form
$$q_X = (\ ,\ )\colon \mathrm{H}^2(X, \mathbb{Z}) \times \mathrm{H}^2(X, \mathbb{Z}) \to \mathbb{Z}$$
i.p. $(\mathrm{H}^2(X, \mathbb{Z}), q_X)$ is a *lattice*.

A *polarization* on X is a primitive ample class $h \in \mathrm{H}^2(X, \mathbb{Z})$.
The *degree* of h is $h^2 = q_X(h, h) \in 2\mathbb{Z}$.

Dimension 2

A HK variety of dim. 2 is called a *K3 surface* (quartic in $\mathbb{P}^3$, double cover of $\mathbb{P}^2$ branched in sextic, ...)

Theorem (Piatetskii-Shapiro, Shafarevich 1971)

For any $d \in \mathbb{Z}_{>0}$, there is a coarse moduli space $\mathcal{F}_{2d}$ of polarized K3 surfaces of degree $2d$.

Remark: $\mathcal{F}_{2d}$ is a 19-dimensional irreducible quasi-projective variety with only finite quotient singularities.

Theorem (GHS 2007)

For $d > 61$ and $d \in \{46, 50, 54, 57, 58, 60\}$, $\mathcal{F}_{2d}$ is of general type.

- Main ingredient: *orthogonal modular forms*

Dimension > 2

Some HK varieties:

1. $X = \mathrm{Hilb}^n(S)$ for a K3 surface $S \rightsquigarrow \dim X = 2n$
2. $X = \mathrm{Kum}^n(A)$ *generalized Kummer*: fibre of summation map $\mathrm{Hilb}^{n+1}(A) \to A$ for abelian surface A $\rightsquigarrow \dim X = 2n$
3. 2 examples by O'Grady:
   - OG10: 10-dim'l example obtained from moduli of sheaves on K3
   - OG6: 6-dim'l ex. obtained from moduli of sheaves on abelian sfc

All known HK varieties are deformation equivalent to one of these.
We say X is of $\mathrm{K3}^{[n]}$ type / OG10 type / ...

Moduli of HK varieties

The *divisibility* $\mathrm{div}(h)$ of h is the number $n \in \mathbb{Z}_{>0}$ such that $\{q_X(h,w) \mid w \in \mathrm{H}^2(X,\mathbb{Z})\} = n\mathbb{Z}$.

- $\deg + \mathrm{div}$ fixes $\mathrm{O}(\mathrm{H}^2(X,\mathbb{Z}))$-orbit of h

Theorem (GHS 2010)

There is a coarse moduli space $\mathrm{M}^{[n]}_{2d,\gamma}$ *(*$\mathrm{M}^{\mathrm{OG10}}_{2d,\gamma}$*) of pairs* (X,h) *where* X *is HK of* $\mathrm{K3}^{[n]}$ *type (*OG10 *type) and* h *is a polarization on* X *with* $h^2 = 2d$ *&* $\mathrm{div}(h) = \gamma$*.*

Remark: If $\mathrm{M}^{[n]}_{2d,\gamma} \neq \emptyset$ ($\mathrm{M}^{\mathrm{OG10}}_{2d,\gamma} \neq \emptyset$), each connected component is an irreducible quasi-projective variety of dim. 20 (21) with finite quotient singularities.

Apostolov, GHS:

- $\mathrm{M}_{2d,\gamma}^{[2]}$ is connected, and non-empty iff $\begin{cases} \gamma = 1, \text{ or} \\ \gamma = 2 \;\&\; d \equiv -1 \ (4) \end{cases}$
- $\mathrm{M}_{2d,\gamma}^{\mathrm{OG10}}$ is nonempty iff $\gamma = 1$, or $\gamma = 3$ & $d \equiv -3$ mod 9.

Theorem (GHS 2010/2011, "split" case)

i) $\mathrm{M}_{2d,1}^{[2]}$ *is of general type when* $d \geq 12$

ii) *Every component of* $\mathrm{M}_{2d,1}^{\mathrm{OG10}}$ *is of general type when* $d \neq 2^n$

Theorem (BBBF, "non-split" case)

i) $\mathrm{M}_{2d,2}^{[2]}$ *is of general type when* $d = 4c - 1$ *with* $c \geq 12$ *or* $c = 10$

ii) *All components of* $\mathrm{M}_{2d,3}^{\mathrm{OG10}}$ *are of general type when* $d = 9k - 3$ *with* $k \geq 4$

Idea of proof, for $M^{[2]}_{2d,\gamma}$

Let $(X, h) \in M^{[2]}_{2d,\gamma}$

1. Let $\Lambda_h :=$ lattice isometric to $h^\perp \subset H^2(X, \mathbb{Z})$.
   Then $M^{[2]}_{2d,\gamma}$ is a dense open of $\mathcal{F}_{\Lambda_h} := \mathcal{D}(\Lambda_h)/\widetilde{O}(\Lambda_h)$, where

$$\mathcal{D}(\Lambda_h) = \{x \in \mathbb{P}(\Lambda_h \otimes \mathbb{C}) \mid x^2 = 0,\ (x.\overline{x}) > 0\}$$
$$\widetilde{O}(\Lambda_h) = \{g \in O(\Lambda_h) \mid g \text{ induces id on } \Lambda_h^\vee/\Lambda_h\}.$$

2. GHS: there is a "nice" compactification $\overline{\mathcal{F}}_{\Lambda_h}$ of $\mathcal{F}_{\Lambda_h}$, i.p. $\overline{\mathcal{F}}_{\Lambda_h}$ has canonical singularities.
   Let Y be a desingularization of $\overline{\mathcal{F}}_{\Lambda_h}$.
3. On $\mathcal{D}(\Lambda_h)$ canonical forms can be obtained from modular forms.

A *modular form* of *weight* $k \in \mathbb{Z}$ and *character* $\chi \colon \widetilde{O}(\Lambda_h) \to \mathbb{C}^\times$ is a holomorphic function $F \colon \mathcal{D}(\Lambda_h)^\bullet \to \mathbb{C}$ s.t.

i) $F(\lambda \cdot Z) = \lambda^{-k} F(Z), \ \lambda \in \mathbb{C}$

ii) $F(g(Z)) = \chi(g) F(Z), \ g \in \widetilde{O}(\Lambda_h)$ ("modularity")

4. Such F gives $\widetilde{O}(\Lambda_h)$-invariant pluricanonical form on $\mathcal{D}(\Lambda_h)$; get pluricanonical form s_F on

$$\mathcal{F}_{\Lambda_h} \setminus (\text{branch locus of } \mathcal{D}(\Lambda_h) \to \mathcal{F}_{\Lambda_h}) \subset \overline{\mathcal{F}}_{\Lambda_h,\mathrm{reg}} \subset Y$$

5. "Low weight cusp form trick": F of weight $a < 20$, $\chi = \det$, vanishing along boundary (*cusp form*) & ramification locus.

   Then for all G of weight $(20 - a)m$, $\chi = 1$, the form $s_{F^m G}$ extends to an element of $\mathrm{H}^0(Y, \omega_Y^{\otimes m})$.

   Fact: the dimension of the space of these G grows like m^{20}.

6. Trick to find cusp form: use embedding $\varphi\colon \Lambda_h \hookrightarrow \Lambda_{2,26}$.
   There is a cusp form Φ_{12} for $O(\Lambda_{2,26})$ of weight 12, $\chi = \det$.

   GHS: the "quasi-pullback" of Φ_{12} along φ is a cusp form of weight < 20 if

   $$0 < \#\{v \in \varphi(\Lambda_h)^\perp \mid v^2 = -2\} < 16 \qquad (1)$$

7. Left to do: find embedding $\varphi\colon \Lambda_h \hookrightarrow \Lambda_{2,26}$ satisfying (1).

What about other HK moduli?

For $\mathrm{M}^{\mathrm{OG10}}_{2d,\gamma}$: similar but for $\gamma = 1$, replace $\widetilde{\mathrm{O}}(\Lambda_h)$ by bigger group G
$\rightsquigarrow$ choose $\Lambda_h \hookrightarrow \Lambda_{2,26}$ s.t. $\Phi_{12}|_{\Lambda_h}$ is modular w.r.t. G

For $\mathrm{M}^{[n]}_{2d,\gamma}$ with $n > 2$:

- When $\gamma = 1, 2$, have to replace $\widetilde{O}(\Lambda_h)$ with bigger group G.
  $\rightarrow$ need modularity w.r.t G;
  $\rightarrow$ there can be "irregular cusps": makes comparing cusp forms and canonical forms harder.
  S. Ma (2018): 1. "refined" low weight cusp form trick;
  2. irregular cusps are rare ($\mathrm{M}^{[2]}_{2d,\gamma}$, $\mathrm{M}^{\mathrm{OG10}}_{2d,\gamma}$)
  [BBBF]: For $\mathrm{M}^{[n]}_{2d,\gamma}$ with $n > 2$, they are still "rare enough"
- Extra variable n

Uniformicity of lower bound

Theorem (BLMNPS 2021)

For any pair (a, b) of coprime integers, there is a unirational 20-dimensional locally complete family of polarized HK varieties of $\mathrm{K3}^{[n+1]}$ *type where* $n = a^2 - ab + b^2$ *and*

$$\text{(degree, divisibility)} = \begin{cases} (6n, 2n) & \text{if } 3 \nmid n \\ (\frac{2n}{3}, \frac{2n}{3}) & \text{if } 3 \mid n \end{cases}$$

[BBBF]: several more such series of unirational families

Corollary: There is no d_0 such that for all n and γ, $\mathrm{M}^{[n]}_{2d,\gamma}$ is of general type for all $d > d_0$.