On local-global principles and Galois cohomology

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Zoom Algebraic Geometry
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Goal

Ziel (goal):

Algebraic and

Geometric

aspects of local-global principles

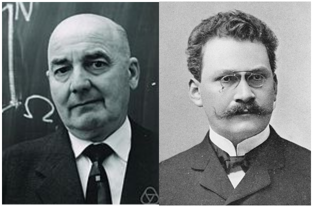
Goal

Zeil (goal):

Algebraic (fields, complexity, which cohomological local-global principles over global and semi-global fields), and
Geometric (zero-cycles for varieties over finite fields and cohomological invariants)
aspects of local-global principles

INTRODUCTION

Hasse-Minkowski: quadratic forms



Theorem

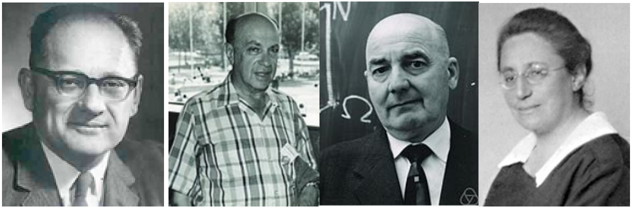
$n \geq 2$, $q(x_1, \ldots x_n)$ a quadratic form with rational coefficients. If the equation

$$q(x_1, \ldots x_n) = 0$$

has a (non-trivial) solution in $\mathbb{Q}_p$ for all p, and in $\mathbb{R}$, then it has a (non-trivial) solution in $\mathbb{Q}$.

(holds over number fields)

Albert-Brauer-Hasse-Noether: central simple algebras



Notations:

- For K a field, $H^i(K, \mu)$ the Galois cohomology group (μ is a $Gal(\bar{K}/K)$-module);
- $i = 2$ then $H^2(K, \mu_n) = Br(K)[n]$ (classifies central simple algebras of order n, up to equivalence).

For a global field:

- K a number field,
- $\Omega_K = \{K_v\}$ are completions at v places of K.
- **Theorem:** $H^2(K, \mu_n) \to \prod_v H^2(K_v, \mu_n)$ **is injective.**

H^1: torsors under linear algebraic groups

K a number field
G a linear algebraic group over K.

Question

Is the map $H^1(K, G) \to \prod_v H^1(K_v, G)$ injective?

$G = O(q)$: quadratic forms
$G = PGL_n$: central simple algebras.

True for G a semisimple simply connected group (Kneser, Harder, Chernousov...), but not in general for a connected linear algebraic group.

For this talk

I higher degree H^i, finite coefficients μ, semi-global fields;

II local-global principles for zero-cycles over global fields of positive characteristic.

Set-up, I

K a field,
$\Omega = \{K \subset K_v\}_v$ a collection of overfields,
μ is a $Gal(\bar{K}/K)$-module;

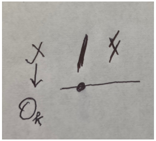
$$Ш^n_\Omega(K,\mu) = \ker\left[H^n(K,\mu) \to \prod_{v\in\Omega} H^n(K_v,\mu)\right]$$

Question

When $Ш^n_\Omega(K,\mu) = 0$*?*

Example: $Ш^2_{\Omega_K}(K,\mu_n) = 0$ if K is a number field and Ω_K corresponds to places of K (Albert-Brauer-Hasse-Noether)

Curves over global and semi-global fields

1. global: K a number field, Ω_K are completions wrt places.
2. semi-global: (Harbater, Hartmann, Krashen; Colliot-Thélène, Parimala, Suresh)
   E is a function field of a curve over a local field k, with a regular proper model $\mathcal{X} \to \mathcal{O}_k$, Ω_E are completions at discrete valuations on E.
   
3. (Kato) $L = K(Y)$, where Y/K is a geometrically integral curve, $\Omega_L = \{K_v(Y_{K_v})\}_{v \in \Omega_K}$;
4. (Harbater-Krashen-P.) $F = E(C)$, where C/E is a geometrically integral curve, $\Omega_F = \{E_v(C_{E_v})\}_{v \in \Omega_E}$;

Ш vs complexity

$* = K$ global, $L = K(Y)$; E semi-global, $F = E(C)$.
$Ш^n_\Omega(*) = \ker\left[H^n(*, \mu) \to \prod_{v\in\Omega} H^n(*_v, \mu)\right]$ for $\mu = \mu_\ell^{\otimes(n-1)}$

	K	L	E	F
$n=1$	$Ш^1_{\Omega_K}(K) = 0$	$Ш^1_{\Omega_L}(L) = 0$	$Ш^1_{\Omega_E}(E) \neq 0$	$Ш^1_{\Omega_F}(F) \neq 0$
$n=2$	$Ш^2_{\Omega_K}(K) = 0$	$Ш^2_{\Omega_L}(L) \neq 0$	$Ш^2_{\Omega_E}(E) = 0$	$Ш^2_{\Omega_F}(F) \neq 0$
$n=3$		$Ш^3_{\Omega_L}(L) = 0$	$Ш^3_{\Omega_E}(E) = 0$	$Ш^3_{\Omega_F}(F) = 0$
$n=4$				$Ш^4_{\Omega_F}(F) = 0$

$Ш^2_{\Omega_L}(L, \mu_\ell) \stackrel{Y(F)\neq 0}{=} Ш^1_{\Omega_K}(K, Jac(Y))[\ell] \neq 0$

If we look at $\Omega =$ all discrete valuations on F, then
$Ш^n_{\Omega_F}(F) = Ш^n_\Omega(F)$ for $n \geq 3$ (uses $Ш_{\Omega_E}(E)$)

$Ш^n_{\Omega_F}(F)$

- $Ш^1_{\Omega_F}(F) \neq 0$ and $Ш^2_{\Omega_F}(F) \neq 0$, $F = E(C)$.
- Take $C = \mathbb{P}^1_E$:

$$0 \to Ш^n_{\Omega_E}(E) \to Ш^n_{\Omega_F}(F) \to \prod_{y \in \mathbb{A}^1_E} Ш^{n-1}_{\Omega_E}(\kappa(y), \mu(-1)) \to 0.$$

Ш vs complexity

$* = K$ global, $L = K(Y)$; E semi-global, $F = E(C)$.
$Ш^n_\Omega(*) = \ker\left[H^n(*, \mu) \to \prod_{v\in\Omega} H^n(*_v, \mu)\right]$ for $\mu = \mu_\ell^{\otimes(n-1)}$

	K	L	E	F
$n = 1$	$Ш^1_{\Omega_K}(K) = 0$	$Ш^1_{\Omega_L}(L) = 0$	$Ш^1_{\Omega_E}(E) \neq 0$	$Ш^1_{\Omega_F}(F) \neq 0$
$n = 2$	$Ш^2_{\Omega_K}(K) = 0$	$Ш^2_{\Omega_L}(L) \neq 0$	$Ш^2_{\Omega_E}(E) = 0$	$Ш^2_{\Omega_F}(F) \neq 0$
$n = 3$		$Ш^3_{\Omega_L}(L) = 0$	$Ш^3_{\Omega_E}(E) = 0$	$Ш^3_{\Omega_F}(F) = 0$
$n = 4$				$Ш^4_{\Omega_F}(F) = 0$

$Ш^n_{\Omega_F}(F)$

- $Ш^1_{\Omega_F}(F) \neq 0$ and $Ш^2_{\Omega_F}(F) \neq 0$, $F = E(C)$.
- Take $C = \mathbb{P}^1_E$:

$$0 \to Ш^n_{\Omega_E}(E) \to Ш^n_{\Omega_F}(F) \to \prod_{y \in \mathbb{A}^1_E} Ш^{n-1}_{\Omega_E}(\kappa(y), \mu(-1)) \to 0.$$

- Use that $Ш^1_{\Omega_E}(E)$ could be nonzero.
  (precisely: when the reduction graph of the special fiber of a regular model $\mathcal{X}$ of E is not a tree.)
- Remark: works if E is a function field of a curve over a complete discretely valued field (no need to assume the residue field is finite).

$Ш^n_{\Omega_F}(F)$

- $Ш^3_{\Omega_F}(F) = 0$ and $Ш^4_{\Omega_F}(F) = 0$.
- $F/E/k$, k local:
  - Take $\mathcal{C} \to \mathcal{O}_k$ a regular proper model of F, such that the special fiber of $\mathcal{C}$ is a simple normal crossings divisor (exists up to de Jong - Gabber alterations of degree prime to ℓ)
  - Use that

$$H^i_{nr}(F/\mathcal{C}) = \cap_{x \in \mathcal{C}^{(1)}} \ker \left[H^i(F) \xrightarrow{\partial^i_x} H^{i-1}(\kappa(x), \mu(-1)) \right] = 0$$

    is zero: Saito and Sato, $i = 3$; Kerz and Saito, $i = 4$.
  - (lemma): $Ш^n_{\Omega_F}(F) \subset H^n_{nr}(F/\mathcal{C})$, $n \geq 3$.

$\square$

Unramified cohomology

X/k is an integral variety

Definition

$$H^i_{nr}(X/k, \mu_n^{\otimes j}) = \cap_v ker \left[H^i(k(X), \mu_n^{\otimes j}) \xrightarrow{\partial_v} H^{i-1}(\kappa(v), \mu_n^{\otimes (j-1)}) \right]$$

where v runs over all discrete valuations of $k(X)$ of rank 1, trivial on k.

$H^i_{nr}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(j)) = \varinjlim H^i_{nr}(X, \mu_{\ell^r}^{\otimes j})$.

Other questions/Remarks

1. other sets of overfields for F:
   - Recall: $\Omega_F = \{E_v(C_{E_v})\}_{v \in \Omega_E}$
   - If we do not include v centered on the closed fiber of $\mathcal{X}$ a model of E then $Ш^i$ could be nonzero for $i = 2, 3, 4$.
2. Next:

$$F \text{ (a function field of an } \textbf{arithmetical} \text{ 3-fold)}$$
$$\downarrow$$
$$\text{a 3-fold } V \text{ over a finite field } \mathbb{F} \text{ (or a surface over } K\text{)}$$
$$\downarrow$$
$$\text{Goal: study } H^3_{nr}(S \times C).$$

Set-up, II: existence of zero-cycles from a local data

K a global field, Ω is the set of places of K, K_v is a completion
X/K a smooth, projective, geometricaly integral variety, $X_v = X_{K_v}$

Question

- *Is* $X(K) \neq \emptyset$*?*
- *Is there a zero-cycle* $z \in CH_0(X)$ *of degree* 1*:* $z = \sum_i n_i P_i$*, with* $\sum_i n_i[\kappa(P_i) : K] = 1$*?*
  I.e. if $I(X) = g.c.d$ *of degrees of closed points of* X*, what is*

$$Ш_I(X) = ker[\mathbb{Z}/I(X) \to \prod_v \mathbb{Z}/I(X_v)].$$

Over number fields

K a number field, Ω is the set of places,
X/K is a **geometrically rational surface**
$\text{Ш}^3_{nr}(X, \mathbb{Q}/\mathbb{Z}(2)) \stackrel{def}{=} ker\left[H^3_{nr}(X, \mathbb{Q}/\mathbb{Z}(2) \to \prod H^3_{nr}(X_v, \mathbb{Q}/\mathbb{Z}(2)\right]$

Theorem (Colliot-Thélène - Kahn)

$\text{Ш}_l(X) = 0$ *if*

1. *K is totally imaginary;*
2. $H^1(K, \mathrm{Pic}\,\bar{X}) = 0$*;*
3. $\text{Ш}^3_{nr}(X, \mathbb{Q}/\mathbb{Z}(2)) = 0$*.*

Conjecture. If K is a global field, X/K is a smooth projective geometrically rational surface, then $\text{Ш}^3_{nr}(X, \mathbb{Q}/\mathbb{Z}(2)) = 0$.

Brauer-Manin obstruction

K a global field, Ω is the set of places of K, K_v is a completion
X/K a smooth, projective, geometricaly integral variety, $X_v = X_{K_v}$

$Br(X) = H^2(X, \mathbb{G}_m)$ the Brauer group of X.
$inv_v : CH_0(X_v) \times Br(X) \to Br(K_v)$
Reciprocity: if $z \in CH_0(X), A \in Br(X)$ one has $\sum_v inv_v(z, A) = 0$.

K a global field, Ω is the set of places of K, K_v is a completion
X/K a smooth, projective, geometricaly integral variety
$inv_v : CH_0(X_v) \times Br(X) \to Br(K_v)$

Conjecture (Colliot-Thélène - Sansuc (81), Kato-Saito (85))

Brauer-Manin obstruction for 0-cycles is the only one: if there is a family $z_{v,v\in\Omega}$ of zero cycles of degree 1 such that

$$\forall A \in Br(X), \sum_v inv_v(z_v, A) = 0,$$

then X has a zero-cycle of degree 1.

Open in general. Progress by Salberger (conic bunldes over $\mathbb{P}^1$ over a number field), Colliot-Thélène, Swinnerton-Dyer, Skorobogatov, Salberger, Frossard, van Hamel, Wittenberg, Yongqi Liang...

Connection with the integral Tate conjecture

$K = \mathbb{F}(C)$, Ω is the set of places, where
$\mathbb{F}$ is finite, $C/\mathbb{F}$ a smooth projective geometrically connected curve
$V \to C$ smooth projective
X/K generic fiber, smooth, $d + 1 = \dim V$
$\ell \neq char(\mathbb{F})$.

Theorem (Saito)

Assume $CH^d(V) \otimes \mathbb{Z}_\ell \to H^{2d}_{ét}(V, \mathbb{Z}_\ell(d))$ is surjective.
If $(z_v)_{v\in\Omega} \in CH_0(X_v)$ of degree 1 with
$\forall A \in Br(X)$, $\sum_v inv_v(z_v, A) = 0$, then X has a zero-cycle of degree prime to ℓ.

No known counterexamples for the integral Tate conjecture (above) for 1-cycles.

Integral Tate conjecture and H^3_{nr}

$V/\mathbb{F}$ smooth projective, $(n, char\,\mathbb{F}) = 1$.

Theorem (Colliot-Thélène - Kahn)

$Coker\big[CH^2(V) \otimes \mathbb{Z}_\ell \to H^4_{ét}(V, \mathbb{Z}_\ell(2))\big]_{tors} \simeq$
$\simeq H^3_{nr}(V, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))/$maximal divisible subgroup.

Case of varieties of dimension 3

$K = \mathbb{F}(X)$, $V \to C$ smooth projective
X/K generic fiber, smooth, $\dim V = 3$.

Theorem (Colliot-Thélène - Kahn)

Assume:

- *Tate conjecture holds for divisors on V;*
- $H^3_{nr}(V, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) = 0$.

If $(z_v)_{v\in\Omega} \in CH_0(X_v)$ of degree 1 with $\forall A \in Br(X)$, $\sum_v inv_v(z_v, A) = 0$, then X has a zero-cycle of degree prime to ℓ.

This motivates:

Question

$V/\mathbb{F}$ smooth projective of dimension 3. Is $H^3_{nr}(V, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) = 0$?

Cases

Question

$V/\mathbb{F}$ smooth projective of dimension 3. Is $H^3_{nr}(V, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) = 0$?

1. Yes, if V is fibered in conics over a surface (Parimala-Suresh).
2. Yes if $V = S \times C$ + some assumptions (e.g. S a "close to rational" surface), (P., Colliot-Thélène-Scavia, Scavia)
3. (Conjecture, Colliot-Thélène - Kahn)
   Yes if V is geometrically uniruled
   (by analogy with a result of Colliot-Thélène - Voisin: $H^3_{nr} = 0$ for $V/\mathbb{C}$ uniruled of dimension 3).
4. Open for $V = E_1 \times E_2 \times E_3$ where E_i is an elliptic curve: e.g. if $f_i \in H^1(E_i, \mathbb{Z}/2)$ is $f_1 \cup f_2 \cup f_3 \in H^3(\mathbb{F}(V))$ non zero?
5. Open if $\dim V = 4$.
6. (P.) Counterexamples if $\dim V = 5$.
7. $H^3_{nr}(V) = 0$ if $dim V = 1$ (trivial) or 2 (Colliot-Thélène - Sansuc - Soulé).

Results for $S \times C$

$(\ell, char(\mathbb{F})) = 1$
$C/\mathbb{F}$ a smooth projective curve
$S/\mathbb{F}$ a smooth projective surface
Assume S is geometrically CH_0-trivial:
$deg : CH_0(S_{\mathbb{K}})_{\mathbb{Q}} \xrightarrow{\sim} \mathbb{Q}$ for any algebraically closed $\mathbb{K} \supset \mathbb{F}$.
Examples: S geometrically rational, supersingular K_3, Enriques

Theorem

$H^3_{nr}(S \times C, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) = 0$ *in the following cases:*

1. *(P.)* $H^1(S, \mathcal{O}_S) = 0$ *and* $NS(\bar{S})$ *has no torsion.*
2. *(Colliot-Thélène - Scavia)* $Hom_{Gal(\bar{\mathbb{F}}/\mathbb{F})}(NS(\bar{S})\{\ell\}, J_C(\bar{\mathbb{F}})\{\ell\}) = 0$ *and Tate conjecture for divisors on surfaces holds.*
3. *(Scavia) if* $Hom_{Gal(\bar{\mathbb{F}}/\mathbb{F})}(NS(\bar{S})\{\ell\}, J_C(\bar{\mathbb{F}})\{\ell\}) = 0$ *and* $Hom_{\mathbb{F}-gr}(Pic^0_{S/\mathbb{F},red}, J_C) = 0.$

Enriques surface x elliptic curve

- (Scavia) $H^3_{nr}(S \times C, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) = 0$ if $Hom_{Gal(\bar{\mathbb{F}}/\mathbb{F})}(NS(\bar{S})\{\ell\}, J_C(\bar{\mathbb{F}})\{\ell\}) = 0$ and $Hom_{\mathbb{F}-gr}(Pic^0_{S/\mathbb{F},red}, J_C) = 0$:
  - example: S Enriques, $C = E$ elliptic curve with $E(\mathbb{F})[2] = O_E$ (i.e. $E : y^2 = f(x)$ with f irreducible of degree 3).

  The integral Tate conjecture for 1-cycles holds in this case:

  $$CH^2(S \times E) \otimes \mathbb{Z}_\ell \to H^4_{\acute{e}t}(S \times E, \mathbb{Z}_\ell(2)) \text{ is surjective.}$$

- (Benoist-Ottem) Integral Hodge conjecture for 1-cycles does not hold for $S/\mathbb{C}$ Enriques surface and E a very general elliptic curve:

  $$CH^2(S \times E) \to Hdg^4(S \times E, \mathbb{Z}) \text{ is not surjective.}$$

On proofs: case $H^1(S, \mathcal{O}_S) = 0$ and $NS(\bar{S})$ has no torsion

1. General facts:
   - For X/k smooth projective, $H^i_{nr}(X) = \cap_v ker(\partial^i_v)$ where it is enough to take v corresponding to $X^{(1)}$.
   - Hence one has a map $\tau : H^i_{ét}(X, \mu) \to H^i_{nr}(X, \mu) \subset H^i(k(X))$.
   - In general $ker\tau$ is mysterious if $i \geq 3$ (if $i = 2$ we use $Br(X) \subset Br(k(X))$ and $H^2_{nr}(X, \mu_\ell) = Br(X)[\ell]$).
     Reminder: $E_1 \times E_2 \times E_3$.
2. Strategy: for $S \times C$, $\mu = \mu_{\ell r}^{\otimes 2}$
   $H^3_{ét}(S \times C, \mu) \curvearrowleft H^3_{ét}(S_{\mathbb{F}(C)}, \mu) \curvearrowleft H^3_{nr}(S \times C/\mathbb{F}, \mu)$.

Bloch-Ogus formalism and Gersten conjecture

X/k smooth projective, geometrically integral, $\mu = \mu_n^{\otimes j}$, $(n, char k) = 1$

1. $\mathcal{C}$:

   $$H^i(k(X),\mu) \to \oplus_{x\in X^{(1)}} H^{i-1}(\kappa(x),\mu(-1)) \to \ldots \oplus_{x\in X^{(i)}} H^0(\kappa(x),\mu(-i)) \to 0$$

   is a resolution of (Zariski) sheaf $\mathcal{H} : U \mapsto H^i_{ét}(U, \mu)$.
2. $H^0(\mathcal{C}) = H^i_{nr}$, $H^i(\mathcal{C}) = CH^i(X)/n$ if $\mu = \mu_m^{\otimes i}$.
3. There is a spectral sequence:

   $$E_2^{pq} = H^p(X, \mathcal{H}^q(\mu)) \Rightarrow H^n_{ét}(X, \mu).$$

4. This gives:

   $$H^3_{ét}(X, \mu_n^{\otimes 2}) \to H^3_{nr}(X, \mu_n^{\otimes 2}) \to CH^2(X)/n \to H^4_{ét}(X, \mu_n^{\otimes 2}).$$

Bloch's method and Gersten conjecture (Quillen)

1. $\mathcal{D} : K_i k(X) \to \oplus_{x \in X^{(1)}} K_{i-1} \kappa(x) \to \ldots \oplus_{x \in X^{(i)}} \mathbb{Z} \to 0$

   is a resolution of a (Zariski) sheaf $\mathcal{K}_i : U \mapsto K_i(H^0(U, \mathcal{O}_X))$
2. Merkurjev-Suslin theorem: $K_2 K/n \xrightarrow{\sim} H^2(K, \mu_n^{\otimes 2})$
3. This gives
   - $Pic(X) \otimes k^* \to H^1(X, \mathcal{K}_2)$
   - $0 \to H^1(X, \mathcal{K}_2)/n \to NH^3_{\acute{e}t}(X, \mu_n^{\otimes 2}) \to CH^2(X)[n] \to 0$

   where $NH^3_{\acute{e}t}(X) = Ker[H^3_{\acute{e}t}(X) \to H^3(K(X))]$

Lifting to $S_{\mathbb{F}(C)}$

Recall:

$$H^3_{ét}(X,\mu_n^{\otimes 2}) \to H^3_{nr}(X,\mu_n^{\otimes 2}) \to CH^2(X)/n \to H^4_{ét}(X,\mu_n^{\otimes 2}).$$

In our setting:
$C/\mathbb{F}$ a smooth projective curve, $K=\mathbb{F}(C)$
$S/\mathbb{F}$ a smooth projective surface, T the torus dual to $Pic(\bar{S})$.

$$H^3_{ét}(S_K,\mu_n^{\otimes 2}) \to H^3_{nr}(S_K/K,\mu_n^{\otimes 2}) \to CH^2(S_K)/n \to H^4_{ét}(S_K,\mu_n^{\otimes 2}).$$

Enough: $A_0(S_K) \subset CH^2(S_K)$ (zero-cycles of degree 0) is trivial.

- We have:

$$\begin{array}{ccccccc} & & & & A_0(S_K) & \longrightarrow & \prod_v A_0(S_{K_v}) \\ & & & & \downarrow \Phi^{\tau}_K & \searrow^{\Psi} & \downarrow \prod \Phi^{\tau}_{K_v} \\ 0 & \longrightarrow & Ш^1(K,T_K) & \longrightarrow & H^1(K,T_K) & \longrightarrow & \prod_v H^1(K_v,T_{K_v}). \end{array} \quad (1)$$

Φ^{τ}_K is injective and Ψ is zero.

$Ш^1(K, T_K)$

- T has a flasque resolution (over $\mathbb{F}$!)

$$0 \to F \to P \to T \to 0$$

  where F is a direct factor of quasi-trivial (since we over $\mathbb{F}$, in general F is flasque), and P is quasi-trivial.
- Hence $Ш^1(K, T_K) \subset Ш^2(K, F_K)$
- Enough: $Ш^2(K, R_{K'/K}\mathbb{G}_m) = 0$ where K'/K is a finite extension,
- i.e. that $Ш^2(K', \mathbb{G}_m) = 0$. This is Albert-Brauer-Hasse-Noether for central simple algebras!

Where we are now

$$H^3_{\acute{e}t}(S \times C, \mu) \curvearrowleft H^3_{\acute{e}t}(S_{\mathbb{F}(C)}, \mu) \curvearrowleft H^3_{nr}(S \times C/\mathbb{F}, \mu)$$

Lifting to $S \times C$

We have

$$H^3_{ét}(S_K, \mu_{\ell^r}^{\otimes 2}) \twoheadrightarrow H^3_{nr}(S_K/K, \mu_{\ell^r}^{\otimes 2}) \supset H^3_{nr}(S \times C/\mathbb{F}, \mu_{\ell^r}^{\otimes 2})$$

$$\begin{array}{ccc} H^3_{ét}(S_K, \mu_{\ell^r}^{\otimes 2}) & \xrightarrow[d^{1,2}]{\simeq} & H^1(K, \mathrm{Pic}\, \bar{S}/\ell^r(1)) \\ \uparrow & & \uparrow \\ H^3_{ét}(S \times C, \mu_{\ell^r}^{\otimes 2}) & \twoheadrightarrow & H^1_{ét}(C, \mathrm{Pic}\, \bar{S}/\ell^r(1)) \end{array}$$

Where we are now

$$H^3_{ét}(S \times C, \mu) \curvearrowleft H^3_{ét}(S_{\mathbb{F}(C)}, \mu) \curvearrowleft H^3_{nr}(S \times C/\mathbb{F}, \mu)$$

Image of $H^3_{ét}(S \times C, \mu)$ is 0 in $H^3(\mathbb{F}(S \times C))$

Uses:

- Enough to consider $\mu = \mu_\ell^{\otimes 2}$ (by Merkurjev-Suslin theorem $H^3_{nr}(S \times C, \mu_\ell^{\otimes 2})$ is the ℓ-torsion of $H^3_{nr}(S \times C, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)))$ and that $\mu_\ell \subset \mathbb{F}$ (restriction-corestriction).
- $Pic(\bar{S}) \otimes H^1_{ét}(\bar{C}, \mathbb{Z}/\ell) \simeq H^3_{ét}(\bar{S} \times \bar{C}, \mathbb{Z}/\ell)$;
- $Pic(\bar{S}) \otimes \bar{\mathbb{F}}(C)^* \to H^1(S_{\bar{\mathbb{F}}(C)}, \mathcal{K}_2)$
  $H^1(X_{\bar{\mathbb{F}}(C)}, \mathcal{K}_2)^G/\ell \xleftarrow{\sim} H^1(X_{\mathbb{F}(C)}, \mathcal{K}_2)/\ell \subset NH^3_{ét}(X, \mu_\ell^{\otimes 2})$

Overview of Scavia's proof: global strategy

- Goal: For $V = S \times C$ one has $H^3_{nr}(V, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) = 0$ if

$$(*)\ Hom_{Gal(\bar{\mathbb{F}}/\mathbb{F})}(NS(\bar{S})\{\ell\}, J_C(\bar{\mathbb{F}})\{\ell\}) = 0 \text{ and } Hom_{\mathbb{F}-gr}(Pic^{\mathbf{0}}_{S/\mathbb{F},red}, J_C) = 0$$

- enough: $CH^2(V) \otimes \mathbb{Z}_\ell \to H^4_{\acute{e}t}(V, \mathbb{Z}_\ell(2))$ is surjective (correspondences, uses that S is geometrically trivial);
- enough: $CH^2(V) \otimes \mathbb{Z}_\ell \to H^4_{\acute{e}t}(\bar{V}, \mathbb{Z}_\ell(2))^G$ is surjective (subtle analysis of Hochschild-Serre, Kunneth, and properties of S)
- (*) $\Rightarrow H^4_{\acute{e}t}(\bar{V}, \mathbb{Z}_\ell(2))^G \simeq H^4_{\acute{e}t}(\bar{S}, \mathbb{Z}_\ell(2))^G \oplus H^2_{\acute{e}t}(\bar{S}, \mathbb{Z}_\ell(1))^G$.
- Finally:

$$\begin{array}{ccc} CH^2(S) \oplus Pic(S) & \longrightarrow & CH^2(V) \\ {\color{red}\downarrow} \quad {\color{blue}\downarrow} & & \downarrow \\ H^4_{\acute{e}t}(\bar{S}, \mathbb{Z}_\ell(2))^G \oplus H^2_{\acute{e}t}(\bar{S}, \mathbb{Z}_\ell(1))^G & \longrightarrow & H^4_{\acute{e}t}(\bar{V}, \mathbb{Z}_\ell(2))^G. \end{array}$$

${\color{red}\downarrow}$ is surjective by Lang-Weil;

${\color{blue}\downarrow}$ is surjective since $b_2 = \rho$.

THANK YOU!!!