Minimal exponent and Hodge filtrations

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Introduction

Set-up. Let $f \in R = \mathbf{C}[x_1, \ldots, x_n]$ nonzero. We want to study the singularities of the hypersurface $H = (f = 0) \subset X = \mathbf{C}^n$ (more generally: a hypersurface in a smooth complex algebraic variety).

A fundamental invariant of singularities is the *log canonical threshold* $\mathrm{lct}(f)$. In this talk I will discuss a refinement of this invariant: the *minimal exponent*. This becomes interesting when the singularities are ideal from the point of view of the log canonical threshold, that is, when $\mathrm{lct}(f) = 1$.

I will focus mostly on describing the invariant, its properties, the tools used in its study, and some open problems.

The log canonical threshold: the definition

Analytic description: for $f \in \mathbf{C}[x_1, \ldots, x_n]$, the log canonical threshold $\mathrm{lct}(f)$ is

$$\sup\left\{\lambda > 0 \mid \frac{1}{|f(x)|^{2\lambda}} \text{ is locally integrable}\right\}$$

If we only put the integrability condition around some P with $f(P) = 0$, then get the local log canonical threshold $\mathrm{lct}_P(f)$.

Note: the "more f vanishes along its zero-locus", the smaller the log canonical threshold.

Algebraic description: let $\pi\colon Y \to \mathbf{C}^n$ be a log resolution of the pair $\big(\mathbf{C}^n, (f = 0)\big)$. Hence π is proper, birational, Y is smooth, and locally on Y we have coordinates $y_1, \ldots, y_n$ such that

$$f \circ \pi = u(y) y_1^{a_1} \cdots y_n^{a_n} \quad \text{and} \quad \det\big(\mathrm{Jac}(\pi)\big) = v(y) y_1^{k_1} \cdots y_n^{k_n},$$

with $u(y)$ and $v(y)$ invertible.

The log canonical threshold: the definition, cont'd

For such a log resolution, we have

$$\mathrm{lct}(f) = \min \frac{k_i + 1}{a_i}$$

with the minimum over all i and all charts as above.

Note: algebraic description a priori depends on resolution, but gives $\mathrm{lct}(f) \in \mathbf{Q}$.
The equivalence of the two definitions follows from the change of variable formula:

$$\int_U \frac{1}{|f|^{2\lambda}} = \int_{\pi^{-1}(U)} \frac{|\det(\mathrm{Jac}(\pi))|^2}{|f \circ \pi|^{2\lambda}}$$

and the fact that the function $\frac{|z|^{2k_i}}{|z|^{2\lambda a_i}}$ on $\mathbf{C}$ is locally integrable if and only if $\lambda a_i - k_i < 1$.

The log canonical threshold: easy properties, examples

1) If we consider on the log resolution the strict transform of a component Z of $(f=0)$ such that $\mathrm{ord}_Z(f)=q$, then $\mathrm{lct}(f)\leq\frac{1}{q}$. In particular, we always have $\mathrm{lct}(f)\leq 1$.

2) If f defines a smooth hypersurface, then $\mathrm{lct}(f)=1$.

3) Behavior under taking powers: $\mathrm{lct}(f^m)=\frac{\mathrm{lct}(f)}{m}$.

4) If $d=\mathrm{mult}_P(f)$, then $\frac{1}{d}\leq\mathrm{lct}_P(f)\leq\frac{n}{d}$.

5) If $f=x_1^{a_1}+\ldots+x_n^{a_n}$, then

$$\mathrm{lct}(f)=\min\left\{1,\frac{1}{a_1}+\ldots+\frac{1}{a_n}\right\}$$

The log canonical threshold in various settings

This invariant comes up in many settings, for example:
1) Birational geometry: it gives the largest c such that $(\mathbf{C}^n, cH)$ is log canonical. There is a more general version when $\mathbf{C}^n$ is replaced by a variety with mild singularities.

2) Vanishing theorems via multiplier ideals: it is the smallest $\lambda > 0$ such that $\mathcal{J}(f^\lambda) \neq \mathcal{O}_{\mathbf{C}^n}$.

3) Jet schemes: gives the asymptotic rate of growth for the dimension of

$$\{u \in \big(\mathbf{C}[t]/(t^m)\big)^n \mid f(u) = 0 \text{ in } \mathbf{C}[t]/(t^m)\}.$$

4) Positive characteristic version defined via Frobenius (F-pure threshold).

5) Criteria for K-semistability of Fano varieties.

Some history: the Archimedean zeta function of f

If $\varphi \in \mathcal{C}_0^\infty(\mathbf{C}^n)$, one can see that the map

$$s \in \mathbf{C}, \mathrm{Re}(s) > 0 \to \int_{\mathbf{C}^n} |f(x)|^{2s} \varphi(x) dx d\overline{x}$$

is well-defined and holomorphic.
Around 1970, Bernstein-S. Gelfand and Atiyah showed that this extends meromorphically to $\mathbf{C}$ (problem of I. Gelfand). Argument: use resolution of singularities to reduce to monomial case, then integration by parts.

We get a meromorphic distribution denoted Z_f, the *Archimedean zeta function* of f. The argument shows: all poles are of the form $-\frac{k_i+1+m}{a_i}$, with $m \in \mathbf{Z}_{\geq 0}$ (in terms of log resolution). Also: the largest pole is $-\mathrm{lct}(f)$.

The Bernstein-Sato polynomial

Soon afterwards, Bernstein gave another proof, using the following result:
Theorem (Bernstein, 1972). There is a nonzero polynomial $b(s)$ such that

$$b(s)f^s = P(s, x, \partial_x) \bullet f^{s+1} \quad \text{for some} \quad P$$

Here f^s is a formal symbol, on which derivations act by

$$\partial_{x_i} \bullet f^s = \frac{s\frac{\partial f}{\partial x_i}}{f} f^s$$

Note: the set of such $b(s)$ is an ideal in $\mathbf{C}[s]$. Its monic generator is the *Bernstein-Sato* polynomial $b_f(s)$.

Remark. By making $s = -1$ in the definition, we see

$$b_f(-1)\frac{1}{f} = P(-1, x, \partial_x) \bullet 1 \in \mathbf{C}[x_1, \ldots, x_n]$$

If f not invertible, then $b_f(-1) = 0$. Put $\widetilde{b}_f(s) = b_f(s)/(s+1)$.

More history: $b_f(s)$ and the Archimedean zeta function

For simplicity, suppose that $f \in \mathbf{R}[x_1, \ldots, x_n]$ and $\varphi \in \mathcal{C}_0^\infty(\mathbf{R}^n)$.

$$b_f(s) \int_{f(x)>0} f(x)^s \varphi(x) dx = \int_{f(x)>0} \big(P(s, x, \partial_x) \bullet f(x)^{s+1}\big) \varphi(x) dx$$

$$= \int_{f(x)>0} f(x)^{s+1} \psi(x) dx$$

for some $\psi \in \mathcal{C}_0^\infty(\mathbf{R}^n)$. This uses integration by parts and the fact that φ has compact support. Note that this is well-defined and holomorphic for $\mathrm{Re}(s) > -1$.

We repeat: multiply by $b_f(s+1)$ etc. Conclusion: Z_f admits meromorphic continuation and all the poles are of the form $\lambda - j$, for some root λ of $b_f(s)$ and some $j \in \mathbf{Z}_{\geq 0}$.

$b_f(s)$: examples

Example 1. If $f = x_1$, then

$$\partial_{x_1} \bullet x_1^{s+1} = (s+1)x_1^s$$

We get $b_f(s) = s + 1$. One can show: this holds iff H is smooth.

Example 2. If $f = x_1^2 + \ldots + x_n^2$, then

$$(\partial_{x_1}^2 + \ldots + \partial_{x_n}^2) \bullet f^{s+1} = 2(s+1)(2s+n)f^s.$$

In fact, we have $b_f(s) = (s+1)(s+\frac{n}{2})$.

Example 3 (Cayley). If $f = \det(x_{i,j}) \in \mathbf{C}[x_{i,j} \mid 1 \leq i,j \leq n]$, then

$$\det(\partial_{i,j}) \bullet f^{s+1} = (s+1)(s+2)\cdots(s+n)f^s.$$

In fact, $b_f(s) = (s+1)(s+2)\cdots(s+n)$.

$b_f(s)$: basic properties

Theorem (Kashiwara, 1976). For every f, all roots of $b_f(s)$ are in $\mathbf{Q}_{<0}$.

Idea: relate the $\mathcal{D}_X$-module $\mathcal{O}_X[1/f, s]f^s$ and the push-forward of the corresponding $\mathcal{D}$-module on a log resolution.

Theorem (Lichtin, 1989). With our notation for a log resolution of $(\mathbf{C}^n, H)$, every root of $b_f(s)$ is equal to $-\frac{k_i+1+j}{a_i}$ for some i and some $j \in \mathbf{Z}_{\geq 0}$; in particular, all roots are $\leq -\mathrm{lct}(f)$.

Idea: refine Kashiwars's method, bringing in the picture $K_{Y/X}$.

Theorem (Kollár, 1997). For every f, we have $b_f\big(-\mathrm{lct}(f)\big) = 0$.

Idea: use integration by parts as in the application of $b_f(s)$ to $Z_f(s)$.

By the last two theorems: the largest root of $b_f(s)$ is $-\mathrm{lct}(f)$.

The minimal exponent

Definition (Saito). The *minimal exponent* $\widetilde{\alpha}(f)$ of f is the negative of the largest root of $\widetilde{b}_f(s)$. Also local version $\widetilde{\alpha}_P(f)$ for $P \in H$.

Convention: if $\widetilde{b}_f(s) = 1$ (i.e. H is smooth), then $\widetilde{\alpha}(f) = \infty$.

Note: the results of Lichtin and Kollár imply $\mathrm{lct}(f) = \min\{\widetilde{\alpha}(f), 1\}$. Hence $\widetilde{\alpha}(f)$ gives new information precisely when $\mathrm{lct}(f) = 1$.

Theorem (Saito, 1993). The hypersurface H has rational singularities iff $\widetilde{\alpha}(f) > 1$.

Note: if the log resolution is such that $\widetilde{H}$ is smooth, then H has rational singularities iff $k_i + 1 \geq a_i$ for all i, with equality iff $(y_i = 0)$ corresponds to a component of $\widetilde{H}$.

Invariant studied in the 80s for f with isolated singularities by Varchenko, Steenbrink, Loeser (called *Arnold exponent*).

Minimal exponent: examples

The minimal exponent is a refinement of the log canonical threshold that gives an interesting measure of rational singularities. Many properties extend from $\mathrm{lct}(f)$, but there are still many open questions.

Example 1. If $f = x_1^{a_1} + \ldots + x_n^{a_n}$, with $a_i \geq 2$ for all i, then

$$\widetilde{\alpha}(f) = \frac{1}{a_1} + \ldots + \frac{1}{a_n}$$

Example 2. If f is homogeneous, of degree $d \geq 2$, with isolated singularity, then

$$\widetilde{\alpha}(f) = \frac{n}{d}$$

Example 3. If $f = \det(x_{i,j})$, then $\widetilde{\alpha}(f) = 2$.

Properties of the minimal exponent

Theorem (M.-Popa). The minimal exponent satisfies the following properties:

i) If $d = \mathrm{mult}_P(f) \geq 2$, then

$$\widetilde{\alpha}_P(f) \leq \frac{n}{d}$$

with equality if f has an ordinary singularity at P.

ii) The invariant goes down under restriction to a smooth hypersurface: if $g = f(x_1, \ldots, x_{n-1}, 0)$, then $\widetilde{\alpha}_0(g) \leq \widetilde{\alpha}_0(f)$.

iii) We have $\widetilde{\alpha}_P(f+g) \leq \widetilde{\alpha}_P(f) + \widetilde{\alpha}_P(g)$.

iv) Given a family $(f_t)_{t\in T}$ and points P_t such that $f_t(P_t) = 0$, the function $T \ni t \to \widetilde{\alpha}_{P_t}(f_t)$ is lower semicontinuous.

Remark. In the case of isolated singularities, the above statements follow from the work of Varchenko, Steenbrink, and Loeser.

Higher direct images of sheaves of differential forms

Let $\pi\colon Y \to X$ be a log resolution of (X, H), isomorphism over $X \smallsetminus H$, and let $E = \pi^*(H)_{\rm red}$. Suppose H is singular.

Theorem (M.-Popa) We have

$$R^i\pi_*\Omega_Y^{n-i}(\log E) = 0 \quad \text{for all} \quad i > n - 1 - \lceil \widetilde{\alpha}(f) \rceil$$

Remarks. 1) We have $R^i\pi_*\Omega_Y^j(\log E) = 0$ for $i + j > n$ (Saito).

2) In particular, we always get $R^{n-1}\pi_*\Omega_Y^1(\log E) = 0$ (easy).

3) First interesting case: $R^{n-2}\pi_*\Omega_Y^2(\log E) = 0$ if H has rational singularities (M.-Olano-Popa, Kebekus-Schnell).

4) For quasi-homogeneous, isolated singularities, the result is sharp.

Minimal exponent and Hodge ideals

The proofs of the above theorems make use of the connection between minimal exponent and two other objects: *Hodge ideals* and *V-filtration*. From now on: suppose that H is reduced. Consider the inclusion map

$$j\colon U = X \setminus H \hookrightarrow X = \mathbf{C}^n \quad \text{and} \quad \mathcal{O}_X[1/f] = j_*\mathcal{O}_U.$$

$\mathcal{O}_X[1/f]$ is not just a module over the sheaf $\mathcal{D}_X$ of differential operators, but underlies a mixed Hodge module in the sense of Saito's theory. The filtered pieces:

$$F_k\mathcal{O}_X[1/f] = \frac{1}{f^{k+1}} I_k(f) \quad \text{for an ideal} \quad I_k(f) \subseteq \mathcal{O}_X \quad k^{\text{th}}\text{Hodge ideal}$$

Similarly, by considering the push-forward of $\mathbf{Q}_V^H[n]$ from a suitable étale cover V of U, one puts a Hodge filtration on $\mathcal{O}_X[1/f]f^{-\lambda}$, for $\lambda \in \mathbf{Q}_{>0}$ and define $I_k(f^\lambda)$.

Minimal exponent and Hodge ideals, cont'd

This notion extends that of multiplier ideal

$$I_0(f^\lambda) = \mathcal{J}(f^{\lambda-\epsilon}) \quad \text{for} \quad 0 < \epsilon \ll 1$$

and many properties of multiplier ideals extend to Hodge ideals.

Theorem (M.-Popa, Saito). If $\lambda \in (0,1]$ and $k \geq 0$, then

$$I_k(f^\lambda) = \mathcal{O}_X \quad \text{if and only if} \quad k + \lambda \leq \widetilde{\alpha}(f)$$

This in turn follows from a result describing all $I_k(f^\lambda)$ in terms of the V-filtration on $\mathcal{O}_X[1/f, s]f^s$ constructed by Malgrange. For example:

$$F_k\mathcal{O}_X[1/f] = \{P(x,-1)f^{-1} \mid P(x,s)f^s \in F_{k+1}V^1\}$$

Open problem: description via resolutions/valuations

We begin with a result. Suppose that H is reduced and $\pi\colon Y \to \mathbf{C}^n$ is a log resolution of $(\mathbf{C}^n, H)$ such that the strict transform $\widetilde{H}$ is smooth. Write

$$\pi^*(H) = \widetilde{H} + \sum_i a_i E_i \quad \text{and} \quad K_{Y/\mathbf{C}^n} = \sum_i k_i E_i$$

Theorem (M.-Popa, Dirks-M.) We have the following inequality:

$$\widetilde{\alpha}(f) \geq \min_i \frac{k_i + 1}{a_i}$$

Note: if $\widetilde{\alpha}(f) > 1$, then by successively blowing-up the intersection of $\widetilde{H}$ with exceptional divisors, can make RHS in theorem approach 1, hence can't expect equality in general.

Open problem: description via resol/val, cont'd

For the questions that follow may assume H has rational singularities.

Question 1. Given f, is there always a log resolution of $(\mathbf{C}^n, H)$ as above for which we have

$$\widetilde{\alpha}(f) = \min_i \frac{k_i + 1}{a_i}?$$

Question 2. Given an arbitrary log resolution of $(\mathbf{C}^n, H)$, is there always an i such that

$$\widetilde{\alpha}(f) = \frac{k_i + 1}{a_i}?$$

Question 3. Given an arbitrary log resolution of $(\mathbf{C}^n, H)$, what is the data needed to read off $\widetilde{\alpha}(f)$?

Open problem: connection to zeta functions

We begin again with a known result. It is known that for every non-constant f, the Archimedean zeta function Z_f has (at least) simple poles at the negative integers. The other poles: *non-trivial*.

Theorem (Loeser, 1985). If H has isolated singularities, then the largest non-trivial pole of Z_f is at $-\widetilde{\alpha}(f)$.

Question. Does the result remain true for arbitrary singularities?
Remark. Unlike the analytic characterization of $\mathrm{lct}(f)$, the above result seems much harder to use to prove properties of $\widetilde{\alpha}(f)$.

It would be very interesting to prove a non-Archimedean analogue of the above result (for *Igusa* or *motivic* zeta functions).

Igusa zeta function: $Z_{f,p}(s) = \int_{\mathbf{Z}_p^n} |f(x)|_p^s d\mu_p$ for $f \in \mathbf{Z}[x_1, \ldots, x_n]$.

Open problem: connection to zeta functions, cont'd

Denef-Loeser motivic zeta function: if $\mathcal{M}_0 = K_0(\mathrm{Var}/\mathbf{C})[\mathbf{L}^{-1}]$, then

$$Z_f^{\mathrm{mot}} = \int_{\mathbf{C}[\![t]\!]^n} \mathbf{L}^{-s\cdot \mathrm{ord}_t f(-)} \in \mathcal{M}_0[\![\mathbf{L}^{-s}]\!]$$

Equivalent information with $\sum_{m\geq 0}[H_m]\mathbf{L}^{-(m+1)n}T^m \in \mathcal{M}_0[\![T]\!]$, where $H_m = \{u \in (\mathbf{C}[t]/(t^{m+1}))^n \mid f(u) = 0\}$.

Denef-Loeser: Z_f^{mot} is a rational function, with denominators products of $1 - \mathbf{L}^{-a_i s - k_i - 1}$. The $-\frac{k_i+1}{a_i}$ that "have to appear" are the poles of Z_f^{mot}.

Question. Is the largest pole of $(1 - \mathbf{L}^{-s-1}) \cdot Z_f^{\mathrm{mot}}$ equal to $-\widetilde{\alpha}(f)$?

Recall: the Strong Monodromy Conjecture predicts that all poles of Z_f^{mot} are roots of $b_f(s)$.

Open problem: $\widetilde{\alpha}(f)$ as log canonical threshold

The log canonical threshold can be defined also for an ideal $\mathfrak{a} = (f_1, \ldots, f_r)$, either:

- by the same formula, involving a log resolution of $\big(\mathbf{C}^n, V(\mathfrak{a})\big)$, or
- $\mathrm{lct}(\mathfrak{a}) = \sup\left\{\lambda > 0 \mid \frac{1}{\big(|f_1|^2+\ldots+|f_r|^2\big)^\lambda} \text{ is locally integrable}\right\}$.

Question. Given f having a singular point at P, is there a "natural" ideal $\mathfrak{a}$ associated to f such that

$$\widetilde{\alpha}_P(f) = \mathrm{lct}(\mathfrak{a})?$$

Going in the opposite direction is OK:

Theorem (M.). If $\mathfrak{a} = (f_1, \ldots, f_r)$ and $g = f_1y_1 + \ldots + f_ry_r$, then

$$\mathrm{lct}(\mathfrak{a}) = \widetilde{\alpha}(g)$$

Open problem: analogue in positive characteristic

Suppose now that $g \in \mathbf{F}_p[x_1, \ldots, x_n]$, with $g(0) = 0$. In this case, Takagi and Watanabe defined the *F-pure threshold* $\mathrm{fpt}_0(g)$:

$$\mathrm{fpt}_0(g) = \lim_{e \to \infty} \frac{\nu(e)}{p^e}$$

where $\nu(e) = \max\{r \mid g^r \not\in (x_1^{p^e}, \ldots, x_n^{p^e})\} \leq p^e - 1$.

This is an analogue of $\mathrm{lct}_0(f)$ and there are important results and conjectures, when $f \in \mathbf{Z}[x_1, \ldots, x_n]$, regarding the relation between $\mathrm{lct}_0(f)$ and $\mathrm{fpt}_0(f \bmod p)$ for various primes p.

Question. Is there an analogue of $\widetilde{\alpha}(f)$ in this setting? This would be > 1 precisely when $(g = 0)$ has F-rational singularities.