The Hilbert scheme of points on affine space.

Burt Totaro, Zoom Algebraic Geometry seminar, Oct. 1, 2020.

Based on: M. Hoyois, J. Jelisiejew, D. Nardin, B. Totaro, M. Yakerson:
The Hilbert scheme of infinite affine space and algebraic K-theory.

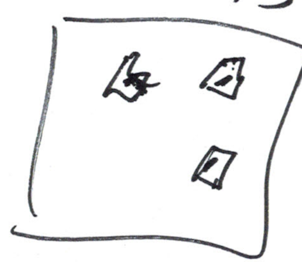
B. Totaro: Torus actions, Morse homology, and the Hilbert scheme of points on affine space.

Both on arXiv.

<u>Def.</u> Let X be a quasi-proj. scheme over a field k. Let $\mathrm{Hilb}_d X :=$ the space of 0-dim. closed subschemes of X of degree d over k. This is a quasi-projective scheme over k.

There is a natural morphism $\mathrm{Hilb}_d X \to S^d X = X^d/S_d$.

<u>Concretely</u>: $\mathrm{Hilb}_d A^n$ is the space of ideals $I \subset k[x_1, \dots, x_n]$ such that $\dim_k(k[x_1, \dots, x_n]/I) = d$.



X

__Example.__ For X a smooth curve,

$$\mathrm{Hilb}_d X \xrightarrow{\cong} S^d X.$$

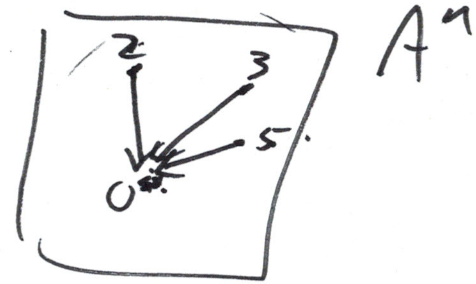
$\underline{x^r = 0}$, some $r \geq 0$.

In particular, for $X = \mathbb{A}^1$,

$$\mathrm{Hilb}_d \mathbb{A}^1 \xrightarrow{\cong} S^d \mathbb{A}^1 \xrightarrow{\cong} \mathbb{A}^d.$$

$$a_1, \ldots, a_d \in \mathbb{A}^1 \longmapsto (x-a_1)\cdots(x-a_n).$$

__Note:__ $S^d \mathbb{A}^n_{\mathbb{C}}$ is always contractible: Use the action of $G_m = \mathbb{C}^*$ on $\mathbb{A}^n$ by scaling, hence on $S^d \mathbb{A}^n$ (and $\mathrm{Hilb}_d \mathbb{A}^n$).



For $\mathrm{Hilb}_d \mathbb{A}^n$, this argument shows at best that

$$\underline{\mathrm{Hilb}_d(\mathbb{A}^n, 0)} \xrightarrow{\simeq} \mathrm{Hilb}_d \mathbb{A}^n.$$

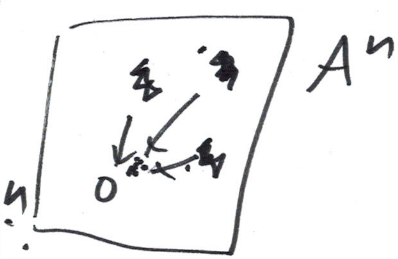
(True, checked in my paper above.)

not just a point.

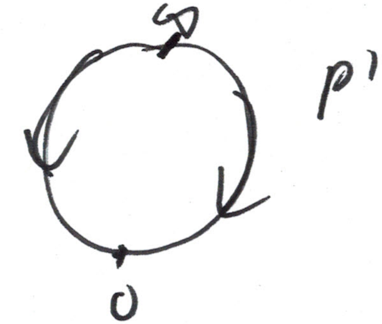
__Note:__ The G_m-action does not extend to a __morphism__

$$\mathbb{A}^1 \times \mathrm{Hilb}_d \mathbb{A}^n \dashrightarrow \mathrm{Hilb}_d \mathbb{A}^n.$$

rational map.



Think of $\mathbb{G}_m$ acting on $\mathbb{P}^1$:



$x \in \mathbb{P}^1 \mapsto \lim_{t \to 0} t(x)$

is **not** continuous.

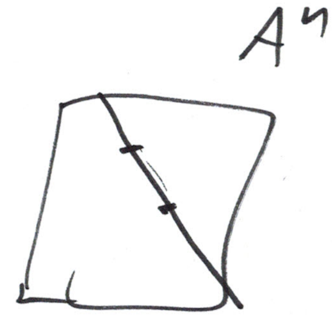
~~A2sa~~ What is $\mathrm{Hilb}_2 \mathbb{A}^n$?

A scheme of degree 2 over k (say alg. closed) is

$\cong \mathrm{Spec}(k \times k)$ or $\cong \mathrm{Spec}(k[x]/(x^2))$.

So

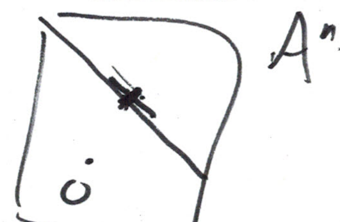
$$\mathrm{Hilb}_2 \mathbb{A}^n = \left\{ \begin{matrix} \text{2 distinct pts} \\ \text{in } \mathbb{A}^n \end{matrix} \right\} \cup \left\{ \begin{matrix} \text{points in } \mathbb{A}^n \\ \text{with a} \\ \text{tangent line} \end{matrix} \right\}.$$



So

$$\mathrm{Hilb}_2(\mathbb{A}^n, 0) \cong \mathbb{P}^{n-1}$$

(not contractible).

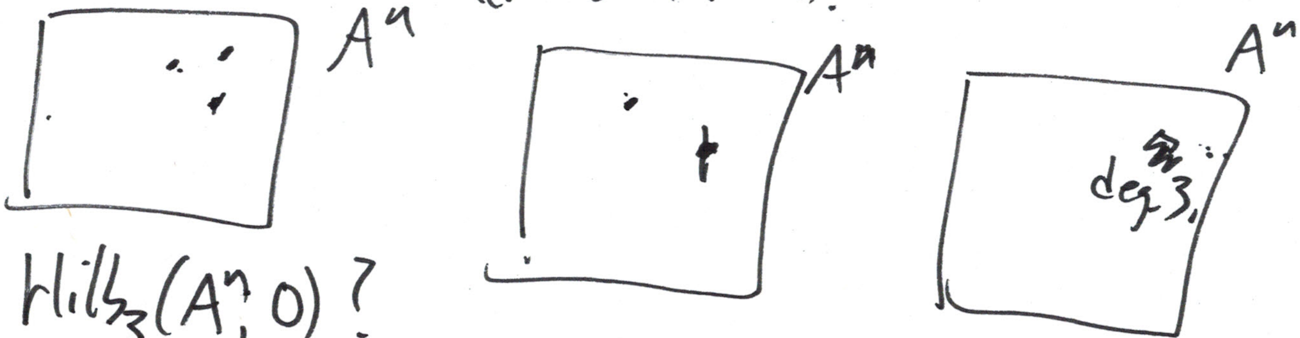


Every degree-2 subscheme of $\mathbb{A}^n$ spans a unique affine line, so:

$$\underset{\cong\, \mathbb{A}^2}{\mathrm{Hilb}_2 \mathbb{A}^1} \to \mathrm{Hilb}_2 \mathbb{A}^n \xrightarrow{\simeq} \left\{ \begin{matrix} \text{affine lines} \\ \text{in } \mathbb{A}^n \end{matrix} \right\} \simeq \mathbb{P}^{n-1}$$

$\boxed{\mathbb{A}^1\text{-homotopy equivalence.}}$

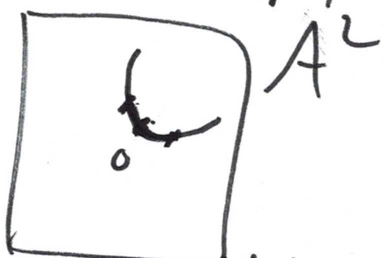
$\mathrm{Hilb}_3 \mathbb{A}^4$. (Cf. notes McKernan at MIT).



What is $\mathrm{Hilb}_3(\mathbb{A}^n, 0)$?

Look at $\mathrm{Hilb}_3(\mathbb{A}^2, 0)$:

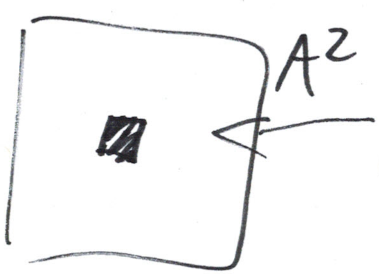
One type of degree-3 subscheme in $\mathbb{A}^2$ supported at 0 is the "curvilinear" type, i.e., contained in a smooth curve.



$\{y=0, x^3=0\} \subset \mathbb{A}^2$.

$\{y = x^2, x^3 = 0\}$

The only other pt. in $\mathrm{Hilb}_3(\mathbb{A}^2, 0)$ is the blob subscheme



$\{x^2=0, xy=0, y^2=0\} \subset \mathbb{A}^2$.

$k[x,y]/(x^2, xy, y^2) = k\cdot\{1, x, y\}$, so this has degree 3.

<u>Conclusion:</u> $\underline{\mathrm{Hilb}_3(\mathbb{A}^2, 0)}$ = the projective cone over a twisted cubic curve $\mathbb{P}^1 \hookrightarrow \mathbb{P}^3$.

$\mathrm{Hilb}_3 \mathbb{A}^2$ is smooth.

Indeed $\underline{\mathrm{Hilb}_3(\mathbb{A}^2, 0) - \{\text{Sing pt.}\}} \xrightarrow{\text{Fibers } \mathbb{A}^1} \mathbb{P}^1$

<u>Bad news.</u> (1). $\mathrm{Hilb}_d \mathbb{A}^n$ is smooth $\iff d \le 3$ or $n \le 2$.
(J. Cheah)

(2). $\underline{\mathrm{Hilb}_d \mathbb{A}^3}$ is reducible if $d \ge 78$ (Iarrobino).

Cartwright–Erman–Velasco–Viray:
$\mathrm{Hilb}_d \mathbb{A}^4$ is reducible $\iff d \ge 8$.

$\mathbb{A}^4$

(3). (Jelisiejew). $\underline{\coprod_{d \ge 0} \mathrm{Hilb}_d \mathbb{A}^{16}}$ satisfies Murphy's law up to retraction

(4). (Hartshorne) $\underline{\mathrm{Hilb}_d \mathbb{P}^n}$ and $\mathrm{Hilb}_d \mathbb{A}^n$ are <u>connected</u>.

<u>Theorem</u> (HJNTY). For each $d \geq 1$,

$$\mathrm{Hilb}_d \mathbb{A}^\infty = \varinjlim_n \mathrm{Hilb}_d \mathbb{A}^n \text{ is}$$

$\mathbb{A}^1$-homotopy equivalent of $BGL(d-1) \simeq Gr_{d-1}(\mathbb{A}^\infty)$.

$$\Rightarrow H^*(\mathrm{Hilb}_d \mathbb{A}^\infty_{\mathbb{C}}; \mathbb{Z}) \cong \underline{\mathbb{Z}[c_1, \dots, c_{d-1}]}, \quad c_i \in H^{2i}.$$

These are the Chern classes of the obvious vector bundle of rank $d-1$:

$$\begin{pmatrix} S \subset \mathbb{A}^n \\ \text{0-dim subscheme} \\ \text{of degree } d \end{pmatrix} \longmapsto \mathcal{O}(S)/k \cdot 1, \quad \text{a } (d-1)\text{-dim v-space.}$$

<u>Theorem</u>. The map

$$\underline{\mathrm{Hilb}_d \mathbb{A}^n_{\mathbb{C}}} \longrightarrow \mathrm{Hilb}_d \mathbb{A}^\infty_{\mathbb{C}}$$

is (over $\mathbb{C}$) $\underline{(2n-2d+2)}$-connected.

Pf. (sketch). Use Poonen's algebraic stack:

$$\mathrm{FFlat}_d = \left[\left\{ \begin{array}{c} \text{space of comm. algebra structures} \\ \text{on } \mathbb{A}^d_k, \text{ compatible} \\ \text{with its given } k\text{-vector} \\ \text{space structure} \end{array} \right\} \Big/ GL(d) \right].$$

↑ affine scheme.

$(d \geq 1)$

There's a morphism

$$\varphi : \underline{\mathrm{Hilb}_d \mathbb{A}^n} \longrightarrow \mathrm{FFlat}_d.$$

$(S \subset \mathbb{A}^n) \longmapsto \mathcal{O}(S)$ as an abstract k-algebra

↕

S as an abstract scheme over k.

(1). Fibers ~~are~~ of φ are

$\mathrm{Emb}_k(S, \mathbb{A}^n)$, a scheme over k,

for a given 0-dim. scheme S of degree d over k.

Here $\underline{\mathrm{Emb}_k(S, \mathbb{A}^n)} \cong \left\{ \begin{array}{l} \text{surjective } k\text{-alg homs.} \\ k[x_1, \ldots, x_n] \twoheadrightarrow \mathcal{O}(S) \end{array} \right\}^{\circ}$

$\underset{\text{open}}{\subset} \left\{ \begin{array}{l} \text{all } k\text{-alg homs.} \\ k[x_1, \ldots, x_n] \to \mathcal{O}(S) \end{array} \right\}^{\circ}$

$\cong \mathcal{O}(S)^{\oplus n} \cong \mathbb{A}_k^{dn}$

If $n \gg d$, then the complement of $\mathrm{Emb}_d(S, \mathbb{A}^n) \subset \mathbb{A}^{dn}$ has high codimension.

So if $n \gg d$, then

$$\mathrm{Hilb}_d \mathbb{A}^n \to \underline{\mathrm{FFlat}_d}$$

is highly connected.

---

(2) Run a $\mathbb{A}^1$ homotopy to show that

$$\mathrm{FFlat}_d \longrightarrow BGL(d-1) = \mathrm{Vect}_{d-1}.$$

$$\underset{\substack{\text{scheme of} \\ \text{degree} \\ d}}{S} \longmapsto \mathcal{O}(S)/k \cdot 1.$$

is $\underline{\mathbb{A}^1\text{-homotopy equivalence}}$. work

We want to degenerate any commutative algebra R of degree d over k, canonically, to the trivial algebra

$$k \oplus V \qquad V \text{ a vector space of dim. } d-1.$$

$$\cong k[x_1, \dots, x_{d-1}]/(x_i x_j = 0 \text{ for all } i, j).$$

Do this with the Rees algebra construction:

~~Given a filtration~~

$$\boxed{0 \subset R_0 \subset R_1 \subset R_2 \subset \cdots \subset R,} \qquad R_i = k\text{-vector spaces,}$$

with $1 \in R_0$, $R_i R_j \subset R_{i+j}$, $R = \bigcup_i R_i$

$\Rightarrow$ get an $\mathbb{A}^1$-family of algebras from R to $gr(R)$.

$$Rees(R) := \bigoplus_{i \geq 0} R_i t^i \subset R[t].$$

$Rees(R)$ is flat over $k[t]$, $Rees/(t) \cong gr(R)$, $Rees/(t-1) \cong R$.

For any R, use the filtration

$$R_0 = k \cdot 1, \quad R_1 = R.$$

$$R \underset{\text{degeneration}}{\rightsquigarrow} \mathrm{gr}(R) = R_0 \oplus R_1/R_0$$
$$= \underset{0}{k} \oplus \underset{1}{(R/k\cdot 1)} \quad \text{("a trivial algebra")}.$$

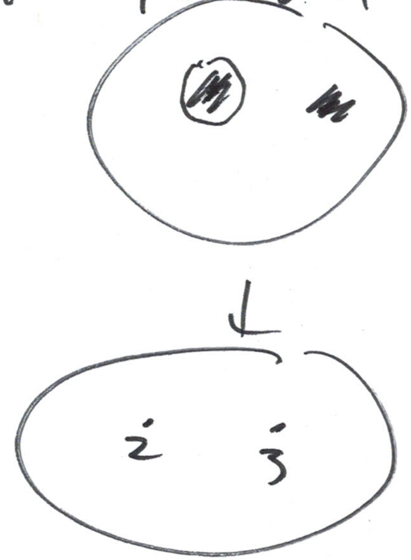
QED:

$$\mathrm{FFlat}_d \xrightarrow{\sim} BGL(d-1).$$

Other things:

Study $\mathrm{Hilb}_d X$ for other smooth varieties X

$$\begin{tikzcd} \coprod \mathrm{Hilb}_{c_i}(\mathbb{A}^n, 0) \arrow[r] & \mathrm{Hilb}_d X \arrow[d] \\ & S^d X \ldots \end{tikzcd}$$



X, $\dim_{\mathbb{C}} X = n$

Y

<u>Dream</u>: Study the "space" of all proper n-dim schemes over a field k.

(cf. Galatius–Madsen–Tillmann–Weiss "Cobordism categories".)