On Compactifying Moduli & Degenerations of K-triv. var

Yuji Odaka

ZAG seminar
Nov 26th (2020)

Overview of today's talk

§ Review from K-moduli perspectives

("= Moduli of K-stable var")

§ K3 surface case

(↑ could be regarded as independent (more) explicit story)

§ "Galaxy"

§ Open K-polystability

Brief!

↔ { log CY / open CY
& non-cpt complete CY metric

based on preprints this year (2020)

§ Some background review.

Yau-Tian-Donaldson conjecture

X : proj. variety w/ "mild" sing

L : ample l.b.

$\exists(!)$ Const Scalar Curv. Kähler metric ω $\in 2\pi c_1(L)$ $\Longleftrightarrow$ (X, L) : K-poly stable.

⋮

the case $L \propto K_X$ is Kähler-Einstein $\Leftarrow$ "Building block" "fund. piece"

(i.e. $\exists a \in \mathbb{R}$ $aL \equiv K_X$)

Thm (O'09–11)

MMP use (BCHM & its ext'n)

(i) (X, L): K-semist $\Rightarrow$ X: semi log canonical.

(ii) $L = K_X$ case (ample)

(X, L): K-stable $\Leftrightarrow$ X: slc

($\Leftrightarrow$ K-semist)

$\Leftrightarrow$ $\boxed{\text{KSBA stable var.}}$

(iii) $K_X \equiv 0$ case ("Calabi-Yau")

- (X, L): K-stable $\Leftrightarrow$ K-polystable $\Leftrightarrow$ X: klt
- (X, L): K-semistable $\Leftrightarrow$ X: slc

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($\Leftrightarrow$ K-semist)

$\Leftrightarrow$ KSBA stable var.

$\rightsquigarrow$ interpret KSBA moduli as "K-moduli := moduli of K-(poly) st polvars"

(iii) $K_X \equiv 0$ case ("Calabi-Yau")

- (X, L): K-stable $\Leftrightarrow$ K-polystable $\Leftrightarrow$ X: klt
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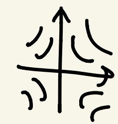
Recap (classical GIT)

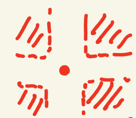


(strict) stable
$\Downarrow$
polystable
$\Downarrow$
semistable

$$SL(V) \curvearrowright (\mathrm{Hilb}, \text{ ample})$$

$$\Rightarrow \quad [\underbrace{\mathrm{Hilb}^{ss}}_{\text{semistable locus}} / SL(V)]$$

$$\downarrow \exists$$



polystable locus (NOT open!)

$$\overline{M} = \mathrm{Hilb}^{ss} /\!/ SL(V) = \mathrm{Hilb}^{ps} / SL(V)$$

cpt moduli

projective sch.

In particular,

$\forall$ semist. pt isotrivially (by $\mathbb{C}^*$-action) limit to $\exists!$ polyst pt.

Recent K-moduli of $\mathbb{Q}$-Fano. $L = -K_X$. (cf. ZAG many talks)

- "K-moduli works étale locally the same way as classical GIT" ("KE moduli stack" of O-Spotti-Sun'12, "good moduli" in Alper's sense.)

- should be proper/compact (Donaldson-Sun'12, recent progress: Blum-HL-Liu-Xu'20)

explainable by the Metric space limit/behaviour (Gromov-Hausdorff)

2 suggestions.

① Replace $\mathbb{Q}$-Gor family functor (normal base S)

$\mathbb{Q}$-Fano

by Anti-can. POLarized $\mathbb{Q}$-Fano family functor.

$$\begin{array}{c} (\mathcal{X}, \boxed{\mathcal{L}}) \\ \downarrow \pi \\ S \end{array}$$

ample $\mathbb{Q}$-l.b

s.t. $\forall \mathcal{X}_s$: $\mathbb{Q}$-Fano

K-semist

$\mathcal{L}|_{\mathcal{X}_s} = -mK_{\mathcal{X}_s}$

(modulo $\pi^* \mathrm{Pic}(S)$)

$\Rightarrow$

- [Lem (O-Spotti-Sun 2.4) cf.
  Automatically $\mathbb{Q}$-Gor deformation
  (when S is normal/sm)]

- allow general S.

2 suggestions.

① Replace $\mathbb{Q}$-Gor family functor (normal base)

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by Anti-can. POLarized $\mathbb{Q}$-Fano family functor.

$\mathcal{M}^{\mathrm{pol}}_{K,\mathrm{Fano},V}$

$$(\mathcal{X},\mathcal{L}) \quad \text{ample } \mathbb{Q}\text{-l.b}$$
$$\downarrow \pi$$
$$S$$

s.t. $\forall \mathcal{X}_s$: $\mathbb{Q}$-Fano K-semist

$\mathcal{L}|_{\mathcal{X}_s} = -mK_{\mathcal{X}_s}$

volume $\overset{(?)}{=}$ V (>0)

(modulo $\pi^*\mathrm{Pic}(S)$)

Aim: to put scheme str on the K-mod & deformation theory

(cf. Kaloghiros-Petracci '20, Ito-Sano.)

Lee-Nakayama '16

[Refined K-moduli conj. : $\mathcal{M}^{\mathrm{pol}}_{K,\mathrm{Fano},V}$ is rep by w/ coarse proj. Schemes (NOT necess. reduced)

KE moduli stack / Good moduli sp]

② [NOT necessarily Fano]

Conjecture

$$\begin{tikzcd} & & & S_2 \\ & & & (\mathcal{X}', \mathcal{L}') \arrow[dll, dashed, "\text{vert. bir}"'] \arrow[ddl, bend left, "\pi'"] \\ (\mathcal{X}, \mathcal{L})^{\text{rel. ample}} \arrow[d, "\pi"', "\text{flat proj.}"] & \arrow[l] \arrow[dr, phantom, "\square"] & \text{fiber prod} \arrow[d] & \\ C & \arrow[l, "\text{fin}"'] & C' & \end{tikzcd}$$

Fix.

"CM minimization"

π' minimizes $\dfrac{\deg(\mathrm{CM\ l.b})}{\deg(C'/C)}$

$\iff {}^{\forall}\pi'$-fiber is K-semistable.

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\end{tikzcd}$$

$(\mathcal{X},\mathcal{L})$ rel. ample.

Fix.

S_2

"CM minimization"

π' minimizes $\dfrac{\deg(\text{CM l.b})}{\deg(C'/C)}$

$\Leftrightarrow$ $\forall$ π'-fiber is K-semistable.

Progress

- "KE" case Proven when K: ample (Wang-Xu, O 2013), K: triv (O'13)

- General:

② [NOT necessarily Fano]

Conjecture

S_2

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Progress

- "KE" case

  Proven when K: ample (Wang-Xu, O 2013), K: triv (O'13)

  Fano case
  - $\Leftarrow$: known to Blum, Li, Wang, Xu (unpublished)
  - $\Rightarrow$: NOT yet ($\leftarrow$ properness conj)

- General: partial progress by K. Ohno arXiv: 1811.12229.
  to appear in Math. Proc. Cambridge.

What about K-moduli of CY?

Recap again:

Thm (O'09–11)

$K_X \equiv 0$ case ("Calabi-Yau")

- (X, L): K-stable $\Downarrow$ $\Leftrightarrow$ K-polystable $\Leftrightarrow$ X: klt
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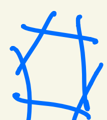
ex
- elliptic curve
- K3 w/ ADE sing

ex
- I_N  degeneration (N-gon of $\mathbb{P}^1$'s)
- type II / III Kulikov degen of K3

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ex
• elliptic curve
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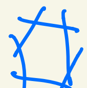
PROBLEM.

• K-semist one does NOT degen to K-polyst ones

• I_N-degen for $\forall N$

ex
• I_N  degeneration (N-gon of $\mathbb{P}^1$'s)
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PROBLEM

- K-$\boxed{\text{Semi}}$st one does NOT degen to K-$\boxed{\text{poly}}$st ones

$\Rightarrow$ then how about (strictly) K-Stable locus ?

($\Leftrightarrow$ Moduli of polarized klt CY)

Fix

$M^\circ :=$ connected moduli of $\mathbb{Q}$-pol smooth K-triv var. (X, L)

($\curvearrowright$ quasi-proj. by Viehweg 90s)

Still problem

PROBLEM

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$\Rightarrow$ then how about (strictly) K-Stable locus ?

($\Leftrightarrow$ Moduli of polarized klt CY)

Fix

$M^\circ :=$ connected moduli of $\mathbb{Q}$-pol smooth K-triv var. (X, L), $L \in \frac{1}{N}\mathrm{Pic}(X)$

(quasi-proj. by Viehweg 90s)

Still problem

For all klt CY degen (X, L), $\exists$? upper bound of the index N ? ($L \in \frac{1}{N}\mathrm{Pic}(X)$)

Still problem (Boundedness)

For all klt CY degen (X, L), $\exists$? upper bound of the index N ?
$L \in \frac{1}{N}\mathrm{Pic}(X)$

$$\Longleftrightarrow \quad \begin{array}{ccccccccc} M^{\circ} & & & & & & & & \\ \cap & \cap & & & \subset & & & & \\ M^{(1)} & \subset & M^{(2)} & \cdots & & \subset & M^{(N)} & \cdots & \text{stabilizes?} \end{array}$$

q-proj.
(Viehweg)

Still problem

For all klt CY degen (X,L), $\exists$? upper bound of the index N ?
$\in \frac{1}{N}\mathrm{Pic}(X)$

$\Leftrightarrow$

$$\begin{array}{ccccccccc} & M^{o} & & & & & & & \\ & \cap & \cap & & \subset & & & & \\ & M^{(1)} \subset & M^{(2)} & \cdots & \subset & M^{(N)} & \cdots & \text{stabilizes?} \end{array}$$

q-proj. (Viehweg)

use { wt2 Hodge str, Verbitsky Torelli, Namikawa simul resol. $\Downarrow$

Thm $\boxed{\text{YES}}$ for AV, K3, HyperKähler.
N=1 classical
[O-Oshima '18, §8.3]

& CY metrics are conti (Gromov-Hausdorff top)

Anyhow K-polystable pol CYs do NOT form Cpt Moduli.

BUT sometimes

$\exists$ "weak K-moduli compactification" $M \subset \overline{M}$

$\underset{\text{def}}{\iff}$ $\partial \overline{M} = \overline{M} \setminus M$ parametrizes K-(only!) semistable pol CYs

i.e. slc pol CYs

& $\overline{M}$ is often dominated by toroidal (AMRT) / semi-toric (Looijenga) cptif

(when M is conn. "Shimura var")

[Renowned example $A_g \subset \overline{A_g}^{\mathrm{Vor}}$

(Namikawa, Hulek-Kahn-Weintraub, Nakamura, Alexeev)]

One recent approach for weak K-moduli : log KSBA

i.e.

for each $\underset{\substack{\cap \\ M}}{(X, L)}$ $K_X \equiv 0$, we put $\underset{\substack{\cap \\ |mD|}}{D}$ ample div

$\Rightarrow$ consider can. degen of $((X, D), L)$ & corresponding $M \subset \overline{M}^{\log \mathrm{KSBA}}$

One recent approach

for weak K-moduli : log KSBA

i.e.

for each (X, L), we put D
$K_X \equiv 0$ ... ample div

$\underset{M}{\cap}$ ... $(\underset{|mD|}{\cap})$

EXTRA

additional str.

$\Rightarrow$ consider **can. degen** of $((X, D), L)$

& corresponding $M \subset \overline{M}^{\log KSBA}$

"PROBLEM" : depends on D !

e.g. Shah '81 $\neq$ Alexeev-Engel-Thompson '18

(both cptify $M = \mathcal{F}_2$)

How we can get "canonical" cptif ?

(of the moduli M of pol CY)

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(of the moduli M of pol CY)

Recall the Diff. Geom idea :

[**Thm** (Donaldson-Sun)

$$\mathcal{X}^* = \bigcup X_t \quad \text{KE Fano family}$$
$$\downarrow$$
$$\Delta^*_t$$

$$\Rightarrow \operatorname{GH\,lim}_{t\to 0} X_t \text{ is KE } \mathbb{Q}\text{-Fano} \ ]$$

※ i.e. use "**canonical**" **metric!**

How we can get "canonical" cptif ?

(of the moduli M of pol CY)

Recall the Diff. Geom idea :

Thm (Donaldson-Sun) 2012.

$$\begin{array}{ll} \mathcal{X}^* = \bigcup X_t & \text{KE Fano} \\ \downarrow & \text{family} \\ \Delta^*_t & \end{array}$$

$\Rightarrow$ $\mathrm{GH}\lim_{t\to 0} X_t$ is KE $\mathbb{Q}$-Fano

※ NO AG proof yet ⋯

$\rightsquigarrow$ try to immitate !

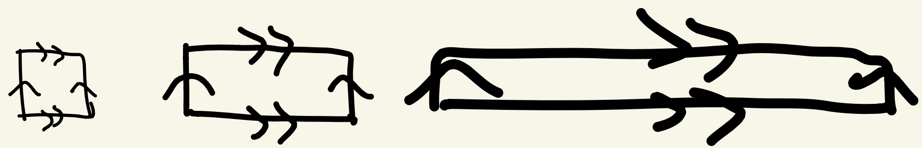
Elliptic curve case:

$$X_t = \mathbb{C}^*/t^{\mathbb{Z}} \quad (\text{for } |t|<1) \qquad \text{“Tate curve”}$$

$$= \mathbb{C}/2\pi i\mathbb{Z} + \log t\,\mathbb{Z}$$

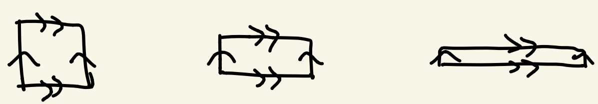
for $t \to 0$

① fix inj.rad (rescale)



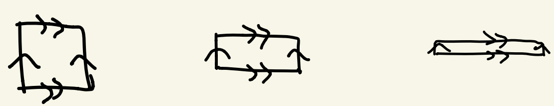
$\dashrightarrow$?

② fix vol



$\dashrightarrow$?

③ fix diam (rescale)



$\dashrightarrow$?

Elliptic curve case:

$$X_t = \mathbb{C}^*/t^{\mathbb{Z}} \quad (\text{for } |t| < 1) \qquad \text{"Tate curve"}$$

$$= \mathbb{C}/2\pi i\mathbb{Z} + \log t\,\mathbb{Z}$$

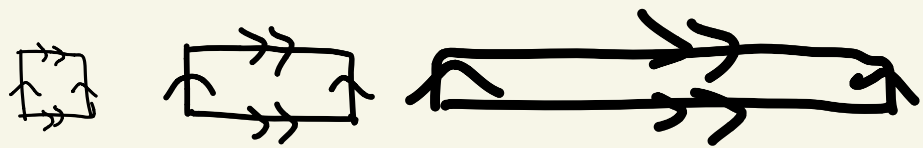
cylinder

$\mathbb{P}^1 \setminus \{0,\infty\} = \mathbb{C}^*$

$\| $

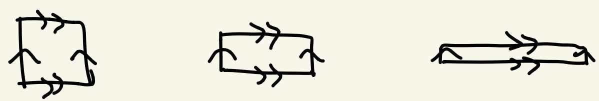
for $t \to 0$

① fix inj. rad (rescale)



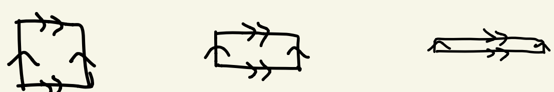
$\dashrightarrow \mathbb{C}/\mathbb{Z}$

② fix vol



$\dashrightarrow$?

③ fix diam (rescale)



$\dashrightarrow$?

Elliptic curve case:

$$X_t = \mathbb{C}^*/t^{\mathbb{Z}} \quad (\text{for } |t|<1) \qquad \text{"Tate curve"}$$

$$= \mathbb{C}/2\pi i\mathbb{Z} + \log t\ \mathbb{Z}$$

for $t \to 0$

① fix inj.rad (rescale)

② fix vol

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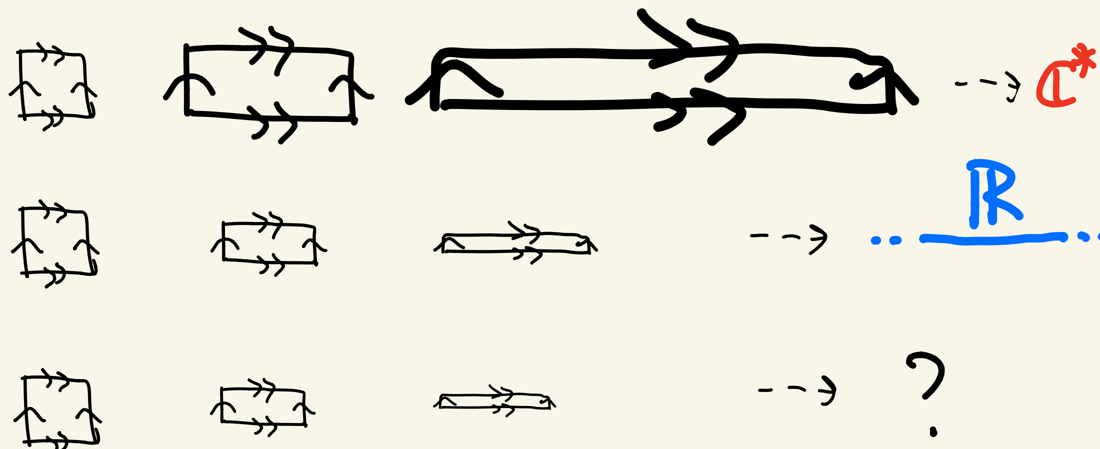


Elliptic curve case:

$$X_t = \mathbb{C}^* / t^{\mathbb{Z}} \quad (\text{for } |t| < 1)$$ "Tate curve"

$$= \mathbb{C} / 2\pi i \mathbb{Z} + \log t \ \mathbb{Z}$$

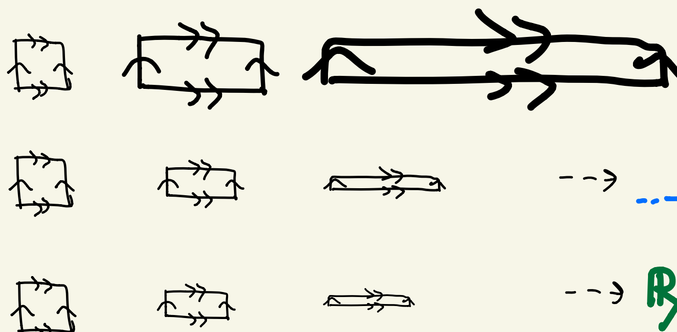
Néron model $\Downarrow$

for $t \to 0$

① fix inj.rad (rescale) $\dashrightarrow \mathbb{C}^*$

② fix vol $\dashrightarrow \mathbb{R}$

③ fix diam (rescale) $\dashrightarrow \mathbb{R}/\mathbb{Z} = S^1$



trop. ell curve $\Uparrow$

Konts-Soib "conj." $\left\{ \begin{array}{l} = \text{dual int. cpx (of } I_N\text{-degen)} \\ = \text{ess. skeleton} \end{array} \right.$

Elliptic curve case:

$$X_t = \mathbb{C}^*/t^{\mathbb{Z}} \quad (\text{for } |t|<1) \qquad \text{"Tate curve"}$$

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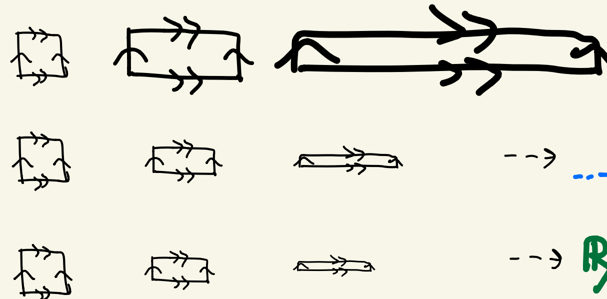


① fix inj.rad (rescale) $\dashrightarrow \mathbb{C}^*$ (Néron model $\Downarrow$)

② fix vol $\dashrightarrow \mathbb{R}$

③ fix diam (rescale) $\dashrightarrow \mathbb{R}/\mathbb{Z} = S^1$ (trop. ell curve $\Uparrow$)

From here in the talk, we use ③-perspective (& ① a bit) for the Moduli cptif.

Konts-Soib "conj."
- = dual int. cpx (of I_N-degen)
- = ess. skeleton

Slogan: KE metrics know a lot about degen!

§ K3 surface case.

Recall

- $\mathcal{F}_{2d} = \left\{ (X,L) \;\middle|\; \begin{array}{l} X: \text{possibly ADE K3} \\ L: \text{ample \& primitive.} \\ \quad \text{w/ } (L^2) = 2d \end{array} \right\}$

Zar open dense.

$\left(\supset \mathcal{F}_{2d}^{\circ} = \left\{ \text{"} \;\middle|\; X: \text{smooth K3} \right\} = \mathcal{F}_{2d} \setminus \text{finite Heegner divs.} \right)$

$\xrightarrow{\sim} \Gamma_{2d} = \tilde{O}^{+}(\Lambda_{2d}) \Big\backslash \boxed{\left\{ \mathbb{C}\sigma \;\middle|\; \begin{array}{l} \sigma \in \Lambda_{2d} \otimes \mathbb{C} \setminus \{0\} \\ \sigma^2 = 0, \ (\sigma, \bar{\sigma}) > 0 \end{array} \right\}^{\circ}} \ \mathcal{D}_{\Lambda_{2d}}$

discrete arith. group

loc. Herm. Sym. space of IV/orthog. typ

(Pyatetski-Shapiro; Shafarevich "Torelli" type thm)

- I. Satake constructed finite compactifications
around 1957–60
to each loc. sym. sp (associated to "types" of (highest wt rep) irr Rep. of G)
$\Gamma\backslash(\mathcal{D}=G/K)$

- probably most famous example is "SBB"

when G/K is $\boxed{\text{hermitian (i.e. } \exists\, \mathbb{C}\text{-str)}}$ & Baily-Borel
1964–66

---- the cptif is projective (var)

We use "Non-variety" Satake compactifications!

$$\partial \overline{\mathcal{F}_{2d}}^{\mathrm{Sat,\ adjoint\ rep}} = \coprod_{\ell :\ \text{isotrop line} \subset \Lambda_{2d}\otimes\mathbb{Q}\ (\mathrm{mod}\ \Gamma_{2d})} \Gamma_{2d} \cap \mathrm{stab}(\ell) \Big\backslash \left\{ v \in \ell^{\perp}/\ell \otimes \mathbb{R} \ \middle|\ v^2 > 0 \right\}$$

(NON variety)

"type III"

$\mathcal{F}_{2d}(\ell)$: (real) 18-dim ball quot

$$\amalg \coprod_{p :\ \text{isotrop plane} \subset \Lambda_{2d}\otimes\mathbb{Q}\ (\mathrm{mod}\ \Gamma_{2d})} \Gamma_{2d} \cap \mathrm{stab}(p) \backslash \text{1-pt}$$

"type II"

$\boxed{\text{1-pt}}$ $\mathcal{F}_{2d}(p)$ 0

$\Updownarrow$ compare

SBB cptf

$$\partial \overline{\mathcal{F}_{2d}}^{\mathrm{SBB}} = \coprod_{p :\ \text{isotrop plane} \subset \Lambda_{2d}\otimes\mathbb{Q}\ (\mathrm{mod}\ \Gamma_{2d})} \Gamma_{2d} \cap \mathrm{stab}(p) \backslash \mathbb{H}$$

(variety)

$\boxed{\text{Modular curve}}$ 2

type II degen.

$$\amalg \coprod_{\ell :\ \text{isotrop } \ell \subset \Lambda_{2d}\otimes\mathbb{Q}\ (\mathrm{mod}\ \Gamma_{2d})} \Gamma_{2d} \cap \mathrm{stab}(\ell) \backslash \text{1-pt}$$

$\boxed{\text{1-pt}}$ 0

type III degen.

Contents of [0018, arXiv:1810.07685 (175pp)] :
§4

- construct

Geometric realization map $\Phi_{alg} : \overline{\mathcal{F}_{2d}}^{\text{Sat, adj}} \to \left\{ \text{cpt metric sp } (\text{+ additional str}) \right\}$

$\cup$

38

$\mathcal{F}_{2d} \longrightarrow$

$\Downarrow$ (rotated $\in$)

$(X, L) \longmapsto$ KE met on X^4 / diameter

(hyper Kähler)

"polyst"?

$\mathbb{R}$-Monge-Ampère eqn (analogue of RFK)

$g_{ij} = \dfrac{\partial^2 \phi}{\partial x_i \partial x_j}$ $\boxed{\det g_{ij} = \text{cst}}$

metric lim ("collapse")

"typeIII" $\mathcal{F}_{2d}(\ell) \to \left\{ \text{certain (explicit) metrized } S^2 \ (\cong \mathbb{CP}^1) \right\}$

18

"typeII" $\mathcal{F}_{2d}(p) \to [0,1]$

0

metric lim (collapse)

Important: this gives "canonical cptifs" (thou we want "justification" i.e. continuity etc)

Geometric realization map $\Phi_{alg} : \overline{\mathcal{F}_{2d}}^{\,\mathrm{Sat,adj}} \to \left\{ \text{cpt metric sp} \atop \text{(+ additional str)} \right\}$

$\cup$

(38)

$\mathcal{F}_{2d} \longrightarrow$

Ψ

$(X, L) \longmapsto$ KE met on X^4 / diameter

(hyper Kähler.)

metric lim ("collapse")

"type III" $\mathcal{F}_{2d}(\ell) \to \{$ certain (explicit) metrized S^2 $(\cong \mathbb{CP}^1)$ $\}$

(18)

"type II" $\mathcal{F}_{2d}(p) \to$ [0,1]

(0)

metric lim (collapse)

Cor Gross-Wilson / Kontsevich-Soibelman Conj. for AV/K3.

[Conj whole Φ_{alg} is $\boxed{\text{conti (w.r.t. GH top)}}$ Thm (0018) true for direction (also true for AV)]

$\boxed{\text{sketch proof}}$ of K3 $\rightsquigarrow$ S^2 continuity (collapsing)

If $(X_i, L_i) \in \mathcal{F}_{2d}$ ($i=1,2,\dots$) approaches to $\mathcal{F}_{2d}(\ell)$

e.g. type III degeneration

(e.g. $[x_0x_1x_2x_3 + t F_4 = 0]$ )

$\Rightarrow$ for $i \gg 0$

$\exists$ (canonical) special Lagrangian fibration $X_i \twoheadrightarrow S^2$ (as expected in Mirror symmetry)

$\left(\Leftrightarrow \text{"hK rot"} \; X_i \cong X_i^\vee \twoheadrightarrow \mathbb{CP}^1\right)$ (different $\mathbb{C}$-str) $\boxed{\text{ellip. K3}}$ π_i

Moreover we can specify $\boxed{\text{Explicit nbhd}}$ of $\mathcal{F}_{2d}(\ell)$

[0018, §4.4 4.14, 4.18]

sketch proof of K3 $\rightsquigarrow$ S^2

continuity (collapsing)

If $(X_i, L_i) \in \mathcal{F}_{2d}$ approaches to $\mathcal{F}_{2d}(\ell)$
$i = 1, 2, \dots$

e.g. type III degeneration

(e.g. $[x_0x_1x_2x_3 + t F_4 = 0]$ )

$\Rightarrow$ for $i \gg 0$

$\exists$ (canonical) special Lagrangian fibration

$$X_i \twoheadrightarrow S^2$$

(as expected in Mirror symmetry)

($\Leftrightarrow$ "hK rot" (different $\mathbb{C}$-str) $\check{X_i} \twoheadrightarrow \mathbb{CP}^1$)

$\boxed{\text{ellip. K3}}$ π_i

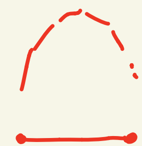
$\Rightarrow$ if $i \to \infty$, ellip. fiber of this shrinks!

(geom. analysis after Yau, Gross-Wilson / Gross-Tosatti-Zhang...)

Recent developments : "type II" case (cf. arXiv : 2010.00416 (O) & [Osh, in preparation])

(w/ Oshima)

- associate $\boxed{\text{(explicit) PL density fun } \overset{\text{convex}}{V : [0,1] \to \mathbb{R}_{\geq 0}}}$ to type II degen family / seq.
- partially prove it captures <u>limit measure</u> of the KE seq



basic shape is classified by Dynkin diags.

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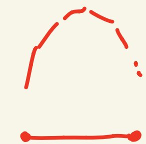

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Abst. Existence Thm (Honda-Sun-Zhang '19)

For $\{X_i : K3\}_{i=1,2,\ldots} \xrightarrow[\lim]{mGH} X_\infty$ = $[0,1]$ (roughly, $\exists \varphi_i : X_i \to X_\infty$ map almost keeping {measure, metric})

$\exists$ PL fun V s.t. limit metric $= \sqrt{V}\, dx$

$\exists$ nat aff str ∇_{HSZ} ($\Leftrightarrow dx$) limit measure $= V\, dx$.

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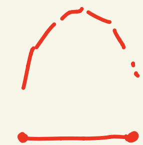

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$\exists$ nat aff str ∇_{HSZ} ($\Leftrightarrow dx$) limit measure $= V\, dx$.

"mirror"?

(Berkovich type aff. str. $\neq$)

What we did:

- consider the real 17-dim ball quot

$$\underbrace{O(\Lambda_{seg}) \backslash \left\{ v \in \Lambda_{seg} \otimes \mathbb{R} \;\middle|\; v^2 > 0 \right\} / \mathbb{R}_{>0}}$$

(isot. plane $P \subset \Lambda_{K3.}$ $(3,19)$

$\Lambda_{seg} := P^{\perp}/P$

$= \mathrm{II}_{1,17}$)

$(=: \mathcal{M}_{K3}(a)^{\tau}$ in op. cit$)$

and $\underbrace{O(\Lambda_{seg}) \backslash \left\{ v \in \Lambda_{seg} \otimes \mathbb{R} \;\middle|\; v^2 > 0 \right\} / \mathbb{R}_{>0}}_{\text{real 17-dim ball quot}} \xrightarrow{\tilde{\Phi}} \{V\}$ def by $\overbrace{\text{Alexeev-Brunyate-Engel '20}}$

& Osh

(independent)

What we did:

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$(=: \mathcal{M}_{K3}(a)^{\tau}$ in op. cit$)$

log KSBA

for Moduli cptif

of Ellip. K3s.

def by $\overbrace{\text{Alexeev-Brunyate-Engel '20}}$

& Osh

(independent)

for this

(lim. measure) purpose.

What we did:

isot. plane $P \subset \Lambda_{K3.}$ $(3,19)$

$\Lambda_{seg} := P^{\perp}/P = \mathrm{II}_{1,17}$

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$(=: \mathcal{M}_{K3}(d)^{\tau}$ in op. cit$)$

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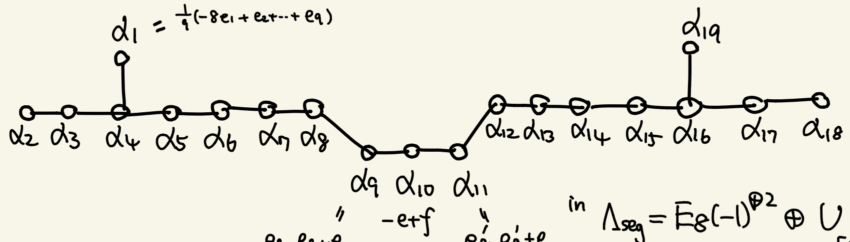
real 17-dim ball quot

- for any type II degen/seq (of K3) $(\mathcal{X}^*, \mathcal{L}^*)$ $\{X_i\}$ polarized

$\rightsquigarrow$ we take limit X_∞ in $\square$ & $V(\mathcal{X}^*, \mathcal{L}^*) := \tilde{\Phi}(X_\infty)$

Some picture of Φ

... controlled by the roots



in $\Lambda_{seg} = E_8(-1)^{\oplus 2} \oplus U$

$\subset \langle e_1, \ldots e_9 \rangle$, $\langle ', \ldots ' \rangle$, $\langle e, f \rangle$

&

$3\beta := \alpha_1 - 2\alpha_2 - \alpha_3.$

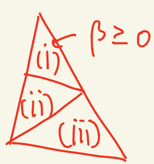
real (17)-dim ball quot

- we identify $O(\Lambda_{seg}) \backslash \{ v \in \Lambda_{seg} \otimes \mathbb{R} \mid v^2 > 0 \} / \mathbb{R}_{>0}$ $=$ Left-Right invol $\backslash \left\{ \lambda \in \Lambda_{seg, \mathbb{R}} \;\middle|\; \begin{array}{l} (\alpha_i, \lambda) \geq 0 \;\; (\forall i) \\ \lambda^2 > 0 \end{array} \right\}$

  (by Vinberg)

- divide into 3×3 chambers.

  (L) (R)



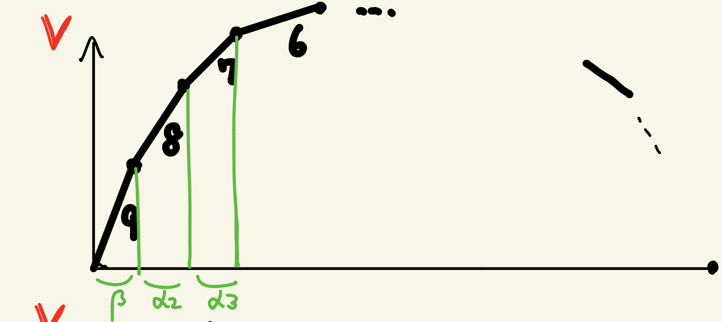
$$\begin{Bmatrix} (i)_L \\ (ii)_L \\ (iii)_L \end{Bmatrix} \times \begin{Bmatrix} (i)_R \\ (ii)_R \\ (iii)_R \end{Bmatrix}$$

(L) decides left end of V

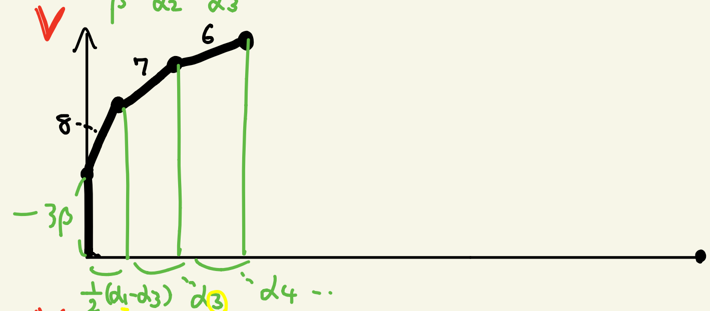
(R) " right "

part (i)$_L$

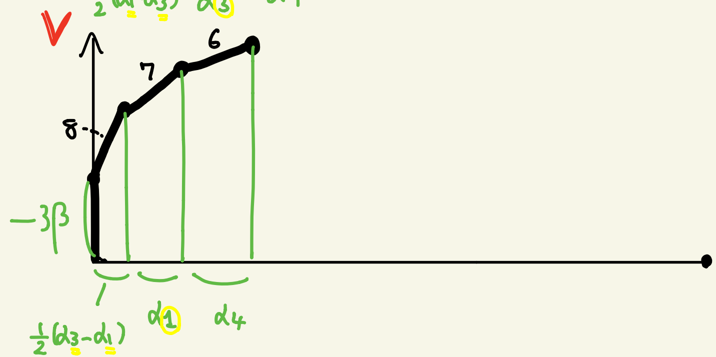
$(\beta, \lambda) \geq 0$



(ii)$_L$

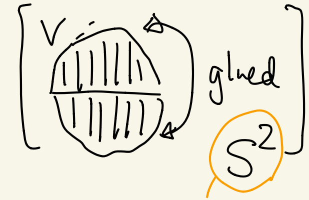


(iii)$_L$



Rmk

[ABE] used



"dumpling"

for Ellip K3 degen.

the same for Right side.

the way of taking limit X_∞ in $\square$

--- Consider in

general Kähler setting:

$$\mathcal{F}_{2d} \longrightarrow \mathcal{M}_{K3} := \left\{ \text{all } \left(\begin{array}{l}\text{KE-metrized} \\ \text{possibly ADE}\end{array}\right) \text{ Kähler K3} \right\} \Big/ \begin{array}{l}\text{certain} \\ \text{change of } \mathbb{C}\text{-str} \\ \text{(hK rot)}\end{array}$$

38 57

$$\bigcup_{\text{all K3}} \text{Kähler cone} \Big/ \sim$$

//

$$\underset{\text{(Kobayashi-Todorov)}}{\cong} \quad SO^{\circ}(3,19) \big/ SO(3) \times SO(19)$$

the way of taking limit in $\square$
X_∞

--- Consider in general Kähler setting:

$$\bigcup_{\text{all K3}} \text{Kähler cone} \,/\sim$$

$$\parallel$$

$$\mathcal{F}_{2d} \;(38) \longrightarrow \mathcal{M}_{K3} \;(57) := \left\{ \begin{array}{l} \text{all } \left(\begin{array}{l}\text{KE-metrized}\\ \text{possibly ADE}\end{array}\right) \\ \text{Kähler K3} \end{array} \right\} \Big/ \begin{array}{l}\text{certain}\\ \text{change of } \mathbb{C}\text{-str}\\ (\text{hK rot})\end{array}$$

($\square$: $\mathbb{R}$-dim)

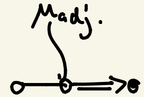
$$\mathcal{M}_{K3} \underset{\text{(Kobayashi-Todorov)}}{\cong} SO^\circ(3,19)\big/ SO(3)\times SO(19)$$

$$\cap \qquad\qquad\qquad \cap$$

$$\overline{\mathcal{M}_{K3}}^{\text{Satake, adj. rep}} \twoheadleftarrow \overline{\mathcal{M}_{K3}}^{\text{Satake}, \tau}$$

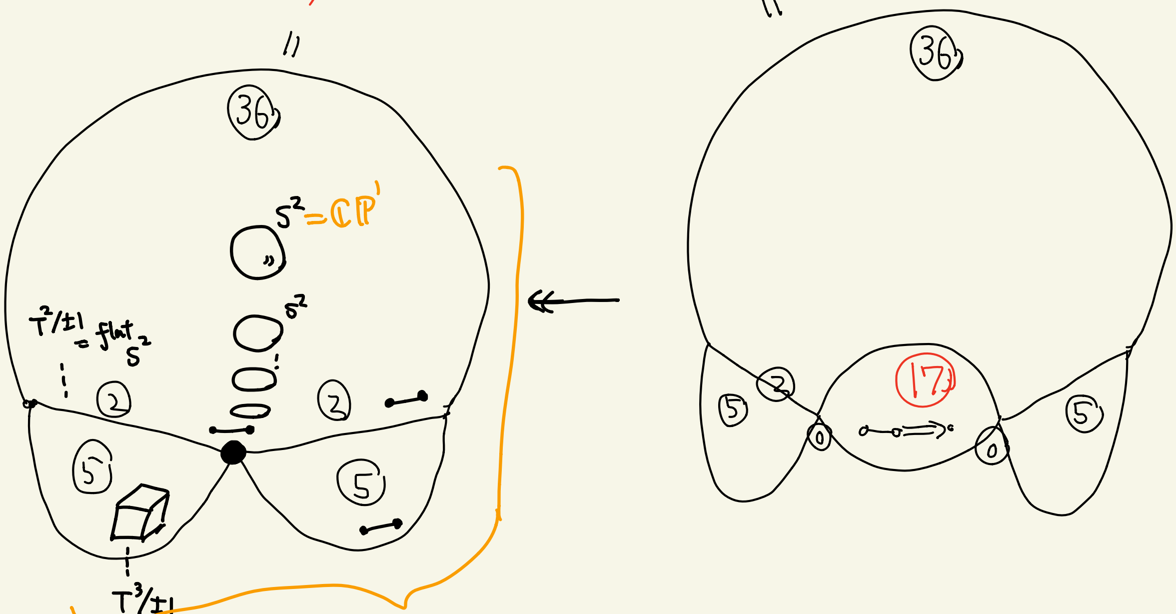
(2 compactifications)

picture



$$\partial \overline{\mathcal{M}_{K3}}^{\text{Satake, adj. rep}} \twoheadleftarrow \partial \overline{\mathcal{M}_{K3}}^{\text{Satake}, \tau}$$

(b'dry) (b'dry)

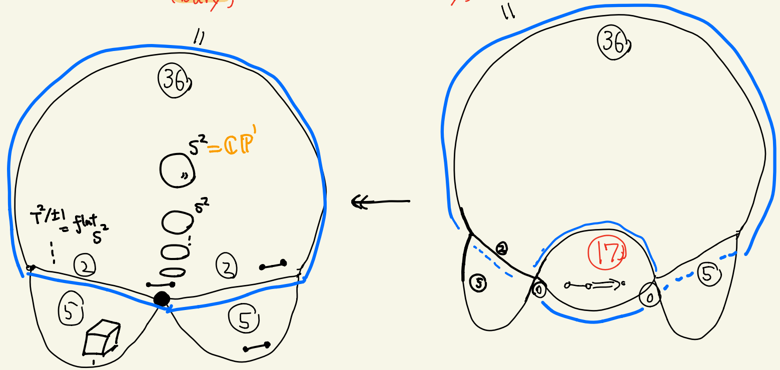


[OO'18 §6] gives this geometric realization.
(& proved continuity at 36)

Extract Blue parts

$\partial \overline{\mathcal{M}_{K3}}^{\text{Satake, adj. rep}} \twoheadleftarrow \partial \overline{\mathcal{M}_{K3}}^{\text{Satake}, \tau}$

(b'dry) (b'dry)



identify

(Weierstrass)

Ellip. K3's Moduli

$\mathcal{M}_W$

(GIT $\Updownarrow$ 0018 §7)

18-dim normal q-pr. var.

(loc. HSD)

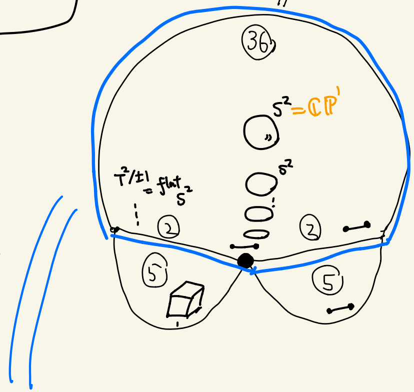
& its Satake-Baily-Borel compactification!

Extract Blue parts

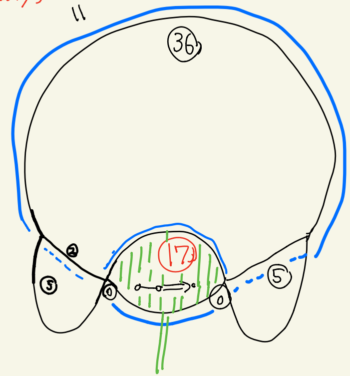
$\partial \overline{\mathcal{M}_{K3}}^{\text{Satake, adj. rep}}$ (bdry) $\twoheadleftarrow$ $\partial \overline{\mathcal{M}_{K3}}^{\text{Satake}, \tau}$ (bdry)

$\parallel$ $\qquad\qquad\qquad\qquad$ $\parallel$

identify



$\twoheadleftarrow$



$\mathcal{M}_{K3}(d)^{\tau}$ $\leftarrow$ Nothing but our concerned parameter sp of V

(Weierstrass)

Ellip. K3's Moduli

M_W

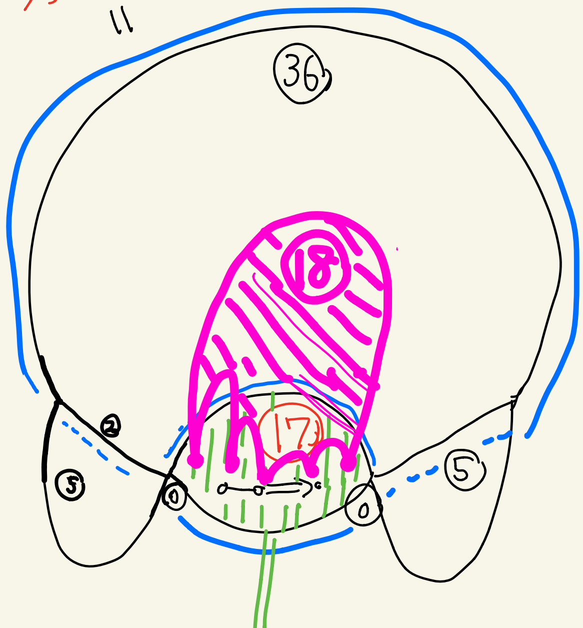
18-dim normal g-pr. var.

(loc. HSD)

$\partial \overline{\mathcal{M}_{K3}}^{\text{Satake}, \tau}$

(bdry)

||



$\mathcal{M}_{K3}(d)^{\tau}$

Limit locus of

$\mathcal{F}_{2d}$ (Alg. K3s)

||

for each d.

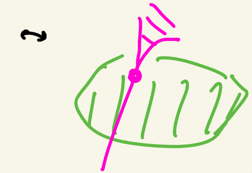
$\#(\text{Its} \cap \text{w/ } 17) < \infty$

"but become dense when $\bigcup_d$"

$\boxed{\text{Associated lattice}}$ ("weaker info than V")

Λ_{per} for type II degen $(\mathcal{X}^*, \mathcal{L}^*) \subset (\mathcal{X}, \mathcal{L})$

$$\begin{array}{ccc} (\mathcal{X}^*, \mathcal{L}^*) & \subset & (\mathcal{X}, \mathcal{L}) \\ \downarrow & & \downarrow \\ \Delta^* & \subset & \Delta \end{array}$$

$\rightsquigarrow$ 

$:= \left(v_{2d}^{\perp} \subset P^{\perp}/P \simeq \Lambda_{\mathrm{seg}} \right)$ $[v_{2d} \in P^{\perp}/P]$

$\cup$

DAA ⋯ AD

or

DA ⋯ AE

or

EA ⋯ AE

Speculated Geom. meaning (7.1 of op cit [O' $\overset{\text{arXiv}}{201000416}$])

$\boxed{V_0 \,/\, V_1} = \mathcal{X}_0$, then term. obj ('s dim) of MMP w/ scaling $\mathcal{L}|_{V_i}$ on V_i may decide D or E ?

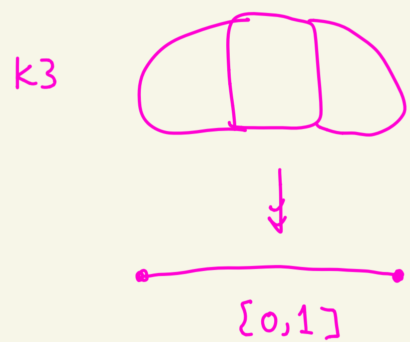
(OK for $d=1$: Friedman 80s)

Ex [EAE-type]

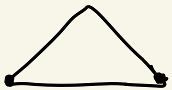
If $\mathcal{X}_0 = \boxed{V_0} \cup \cdots \cup \boxed{V_1}$

w/ V_i : (Anti-can) Del Pezzo surf

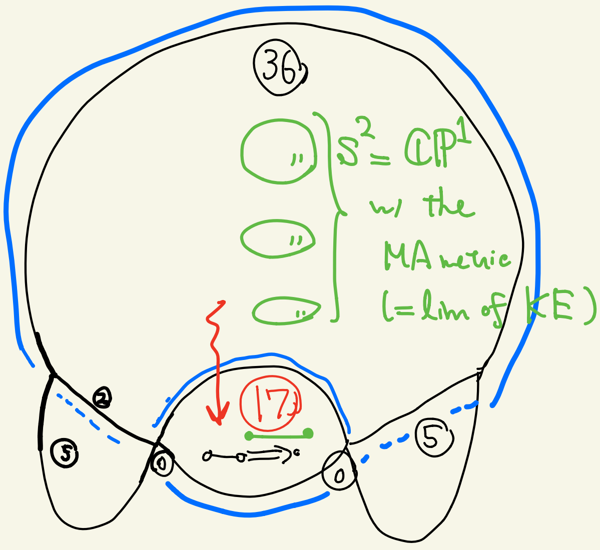
$\Rightarrow$ $\mathcal{X}_t$: (probably) close to glued HK space by Hein-Sun-Viaclovsky-Zhang '18



V $=$



Main Thm (of Part II of op. cit) also [Osh]



$\tilde{\Phi}$ i.e. the V-fns indeed describes (at least) the $\boxed{\text{limit measure}}$ (on •—•) of (S^2, $\overset{\mathbb{R}}{\text{MA}}$ met) seq.

[Conj: of (K3, KE) as well] $\in \mathcal{M}_{K3}$.

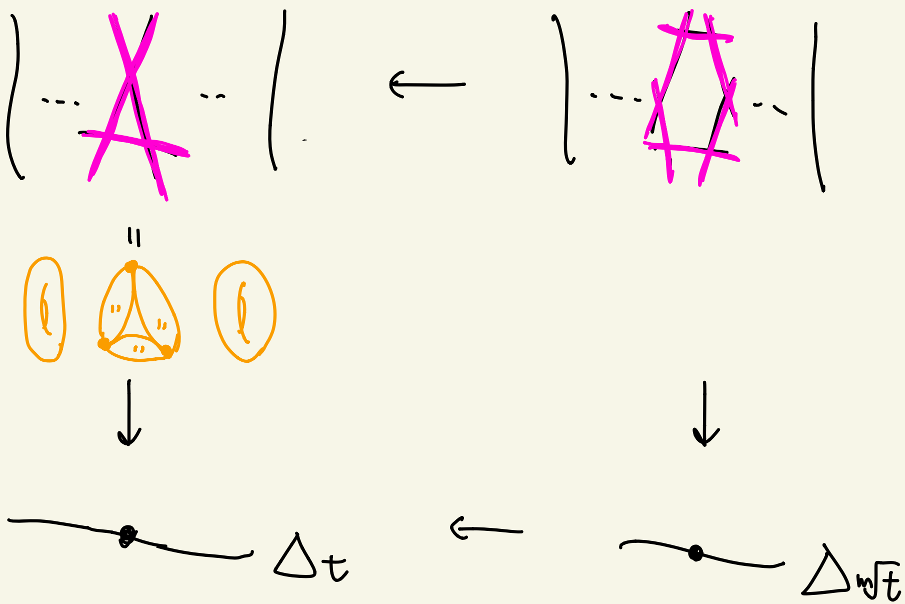
proof ingredients

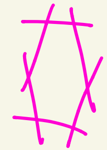
- Asym. behaviour of $\overset{\mathbb{R}}{\text{MA}}$ metrics (cf. [OO18, §7], [Osh])
- 17-dim (open) b'dry strata = dual int. cpx of $\partial \overline{M_W}^{ABE,\nu}$ (toroidal)
- [ABE] & [O, Part I] stable reduction.

§ Brief intro to "Galaxy" (arXiv : 2011.12748 (today))

For $\boxed{I_d}$ - degen (in Kodaira's sense) of Ellip. curve

I_{md} - degen.



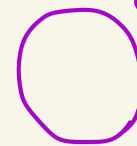
$\Longrightarrow$ Consider $\varprojlim_{m} \Big(\mathrm{Ind}$  $\Big) =: X_\infty$

Simplest Galaxy!

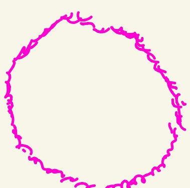
$\mathbb{R}/\mathbb{Z}$

$\parallel$

S^1



X_∞



$\twoheadrightarrow$ Essential skeleton

[<u>Prop</u> $\exists$ conti map]

$$X_\infty \supset \coprod_{a\in \mathbb{Q}/\mathbb{Z}} \left(\mathbb{P}^1\setminus\{0,\infty\}\right) \sqcup \left(\coprod_{\substack{a:\ \text{irr} \\ \text{mod.}\ \mathbb{Z}}} \bullet\right)$$

scale up!

$$X_\infty \supset \coprod_{a\in\mathbb{Q}/\mathbb{Z}} \boxed{\left(\mathbb{P}^1\setminus\{0,\infty\}\right)} \sqcup \left(\coprod_{a:\ \text{irr mod}.\mathbb{Z}} \bullet\right)$$

$S^1 = \mathbb{R}/\mathbb{Z}$

scale up!

scale up!

shape of the totality.

we extend this to

$\forall$ meromorphic degen of K-triv vars. $\begin{matrix}\mathfrak{X}^* \\ \downarrow \\ \Delta^*\end{matrix}$

($\forall$ dim).

$\boxed{\text{Thm}}$ (O. arXiv: 2011.12748)

$$\underset{\text{Huber, adic}}{\mathfrak{X}_{\eta,\mathbb{C}((t^{\mathbb{Q}}))}} \xrightarrow[\exists \text{ conti}]{} \!\!\!\!\rightarrow \overset{\forall \text{ galaxy}}{X_\infty} \xrightarrow[\exists f_{tr} \text{ (conti.)}]{} \!\!\!\!\rightarrow \underset{\text{ess. skeleton}}{S(\mathfrak{X}^*)} \subset \mathfrak{X}_\eta^{\text{Berk, an.}}$$

• $f_{tr}^{-1}(\forall \text{ rat. pt}) \overset{\text{dense}}{\supset} \boxed{\text{open CY}}$ (unique up to log crep. bir)

S^1 for ell curve

S^2 for K3 or

§ Brief intro to Open K-polystability (Details: arXiv: 2009.13876)

.. Non-cpt version of K-polystability

More precisely, motivated by many

Complete Ricci-flat Kähler metrics (e.g. $\dim_{\mathbb{C}}=2$, ALG, ALH grav. instantons)
(KE)

w/ vol. growth dim $\leq \dim_{\mathbb{C}} X^\circ$

on (X°, L°)

$\cap$

$(\underbrace{(X, D)}_{\log CY}, L)$

Thm seems true for many known examples ..... ? $\Updownarrow$?

We introduced Open K-polystabilities of (X°, L°)

Thank you
for listening.

Please keep
safe & healthy

until "the next spring"
春