Smooth projective rational varieties with non-finitely generated discrete automorphism group

— joint work with Professors Tien-Cuong Dinh & Xun Yu

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§1 Introduction

$/\mathbb{C}$ X : smooth proj. var.

Question

Is $\mathrm{Aut}\,X/\mathrm{Aut}^\circ X$ finitely generated (FG) as groups?

Is $\mathrm{Aut}\,X$ finitely generated if $\mathrm{Aut}^\circ X = \{id\}$ (discrete)?

Main Theorem (Dinh - O — - Yu)

(1) $\forall n \geq 3$ $\exists V$ smooth projective rational variety s.t. $\dim V = n$ & $\mathrm{Aut}\,V$ is discrete but not finitely generated (NFG)

(2) X smooth proj. var. $\dim X = n \geq 3$. If $\mathrm{Aut}\,X/\mathrm{Aut}^\circ X$ is NFG, then $\kappa(X) \leq n-2$.

(3) Conversely $\forall n \geq 3$ $\forall \kappa \in \{-\infty, 0, \cdots, n-2\}$ $\exists$ smooth proj. var X s.t. $\dim X = n$, $\kappa(X) = \kappa$ & $\mathrm{Aut}\,X$ is discrete & NFG.

Background for Question & Main Thm

1) $\underline{\dim X = 1}$ $\Rightarrow$ Question is affirmative

- $g(X) = 0 \Rightarrow X = \mathbb{P}^1$ $\mathrm{Aut}\,X = \mathrm{Aut}^0 X = PGL(2, \mathbb{C})$
- $g(X) = 1 \Rightarrow X \cong$ elliptic curve

  $\mathrm{Aut}\,X / \mathrm{Aut}^0 X \cong \mu_n$

  $n = 2, 4, 6$
- $g(X) \geq 2 \Rightarrow |\mathrm{Aut}\,X| < \infty$

2) $\underline{\dim X = 2}$ highly non-trivial (probably the most subtle case as discussed later)

<u>Thm A</u> (Classical, related to <u>Degtyarev-Itenberg-Kharlamov</u>)

$\dim X = 2$ & $\mathrm{Aut}\,X / \mathrm{Aut}^0 X$ NFG

$\Rightarrow$ X is either rational or <u>non-min</u> with $\kappa(X) = 0$.

<u>Our Thm</u> generalization for $\dim X \geq 3$ with partial converse of Thm A.

<u>Thm A</u> "positive direction" for Question (& exceptions are not expected).

3) Negative direction

(fairly recent)

- **Lesieutre (2018)**

$\exists X$ smooth projective var

$\dim X = 6$ & $\kappa(X) = -\infty$

s.t. $\mathrm{Aut}\, X$ discrete & NFG.

$\Leftarrow$ start from $\exists S$ smooth rat proj. surf. of Coble type — ①

$(|-K_S| = \phi$ & $|-2K_S| \neq \phi)$

(also quite crucial for our Main Thm (1). Explain later).

- **Dinh-O– (2019)**

$\forall n \geq 2$ $\exists X$ smooth proj var

$\dim X = n$ & $\kappa(X) = n-2$ s.t.

$\mathrm{Aut}\, X$ discrete & NFG.

$\Leftarrow$ Start from $\exists S \underset{\text{birat}}{\sim}$ K3 of Coble type — ②

(birat to $\exists$ double cover of Coble surf)

(also important for our Main Thm (3). Explain later.)

Some remarks

Open (Most major in this Topics)

$\exists$ smooth projective rational surface S

s.t. $\mathrm{Aut}\,S / \mathrm{Aut}^\circ S$ NFG?

Rem 1 $/ k = \bar{k}$ char $k = p > 0$

Main Thm (1)

true also $\forall k = \bar{k} \supset \mathbb{F}_p(t)$ & $p \geq 3$

Main Thm (3)

true also $\forall k = \bar{k} \supset \mathbb{F}_p(t)$ & $p \geq 3$

Main Thm (2) ?? / char. $p > 0$

(Use finiteness of pluricanonical representation.)

Rem 2 We frequently use:

Proposition 1

G: group & $H < G$ s.t. $[G : H] < \infty$.

Then

(1) G: FG $\Leftrightarrow$ H: FG.

(2) $\forall K < G$ $[K : K \cap H] < \infty$.

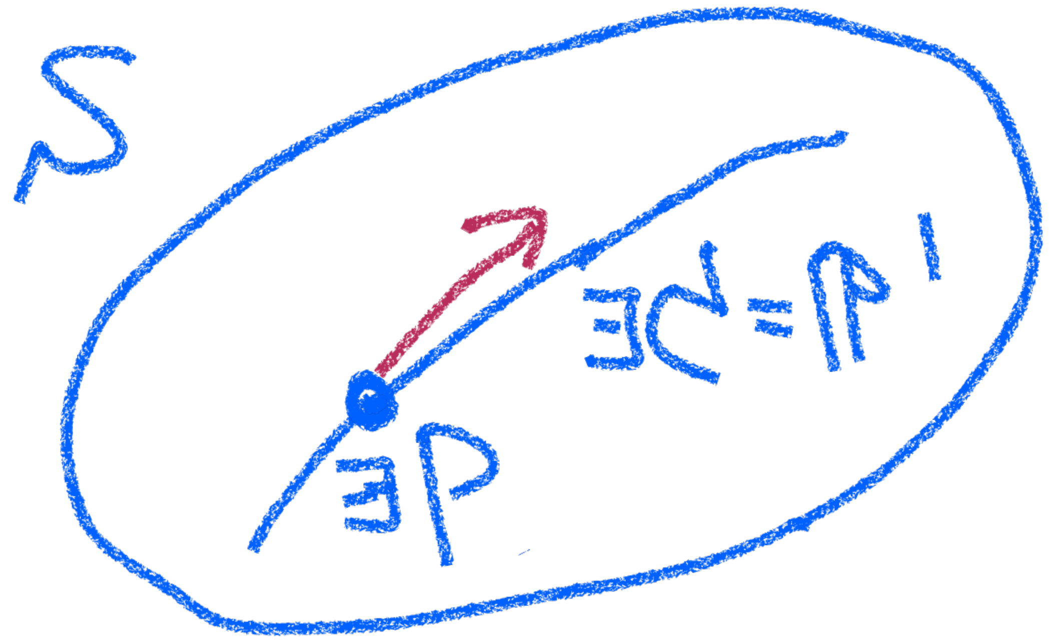
Notation $X \supset Y_1, Y_2$

$$\mathrm{Aut}(X, Y_1, Y_2) = \left\{ f \in \mathrm{Aut}(X) \,\middle|\, \underset{i=1,2}{f(Y_i) = Y_i} \right\}$$

Some strategy for G FNG

(after Lesieutre & Dinh-O_)

Surfaces with 2 assumptions: $\boxed{\text{Ass A}}$ & $\boxed{\text{Ass B}}$



$$\mathrm{Aut}(S,P) \overset{\uparrow\,\boxed{\text{Ass A}}}{=} \mathrm{Aut}(S,C,P) \xrightarrow{r_C} \mathrm{Aut}(\underset{\mathbb{P}^1}{C}, \underset{\infty}{P})$$

$$\mathrm{Aut}(C,P) \supset \{z \mapsto z+c \mid c \in \mathbb{C}\} \cong (\mathbb{C},+) \supseteq r_C(G)$$

$$\mathrm{Aut}(S,C,P) \supset G := \{f \mid (df)|_{T_{C,P}} = \mathrm{id}\} \twoheadrightarrow r_C(G)$$

$$\boxed{\text{Ass B}} \rightarrow \quad r_C(G) \supset \exists \text{ NFG gp}$$

$$\left(\text{eg } \left\langle \tfrac{1}{2^n} \,\middle|\, n \geq 0 \right\rangle\right)$$

Then G : NFG

(☹ $\exists$ NFG $\subset r_C(G)$
$\Rightarrow r_C(G)$ NFG $\Rightarrow G$ NFG
$r_C(G) \subset (\mathbb{C},+)$ abelian; $G \twoheadrightarrow r_C(G)$)

If moreover

$\mathrm{Aut}\, S \supset G$ finite index

$\Rightarrow \mathrm{Aut}\, S$: NFG

Prop 1

Are there such S?

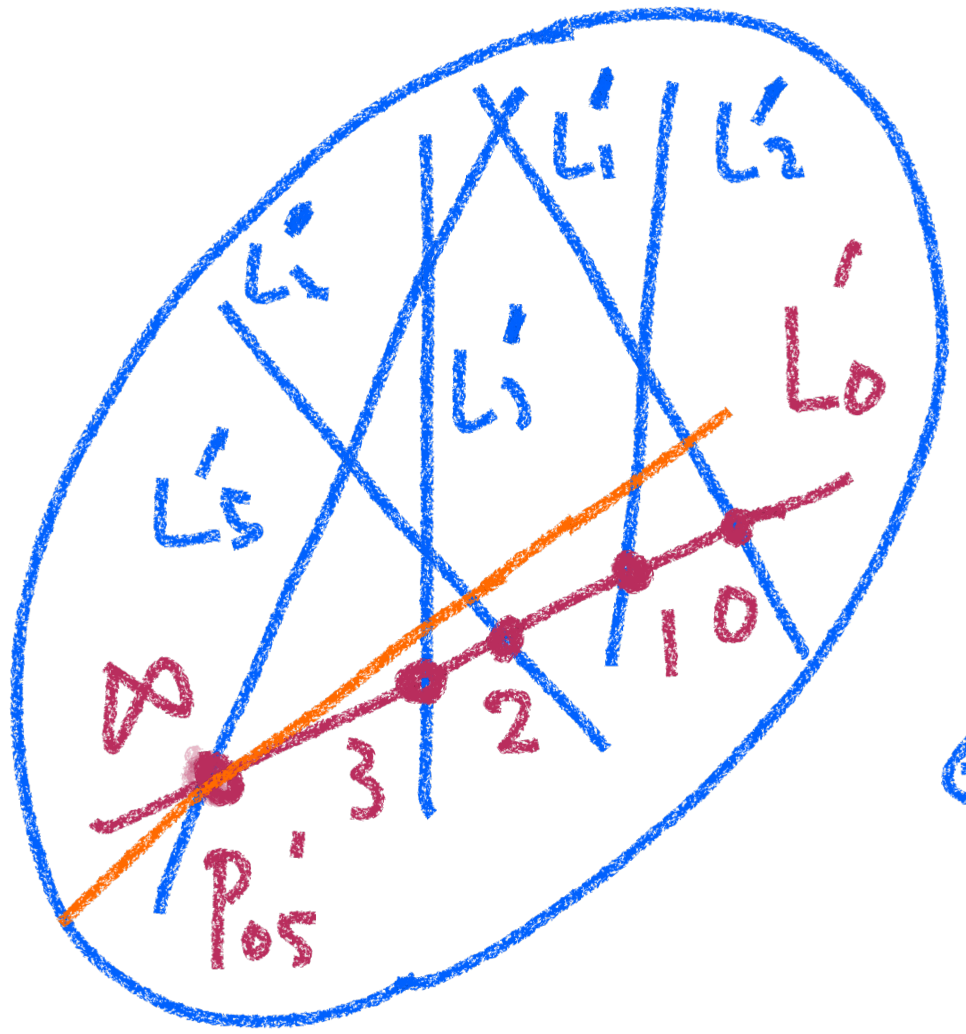
Thm B (Lesieutre)

$L_0', \dots, L_5' \subset \mathbb{P}^2$ suitable 6 lines

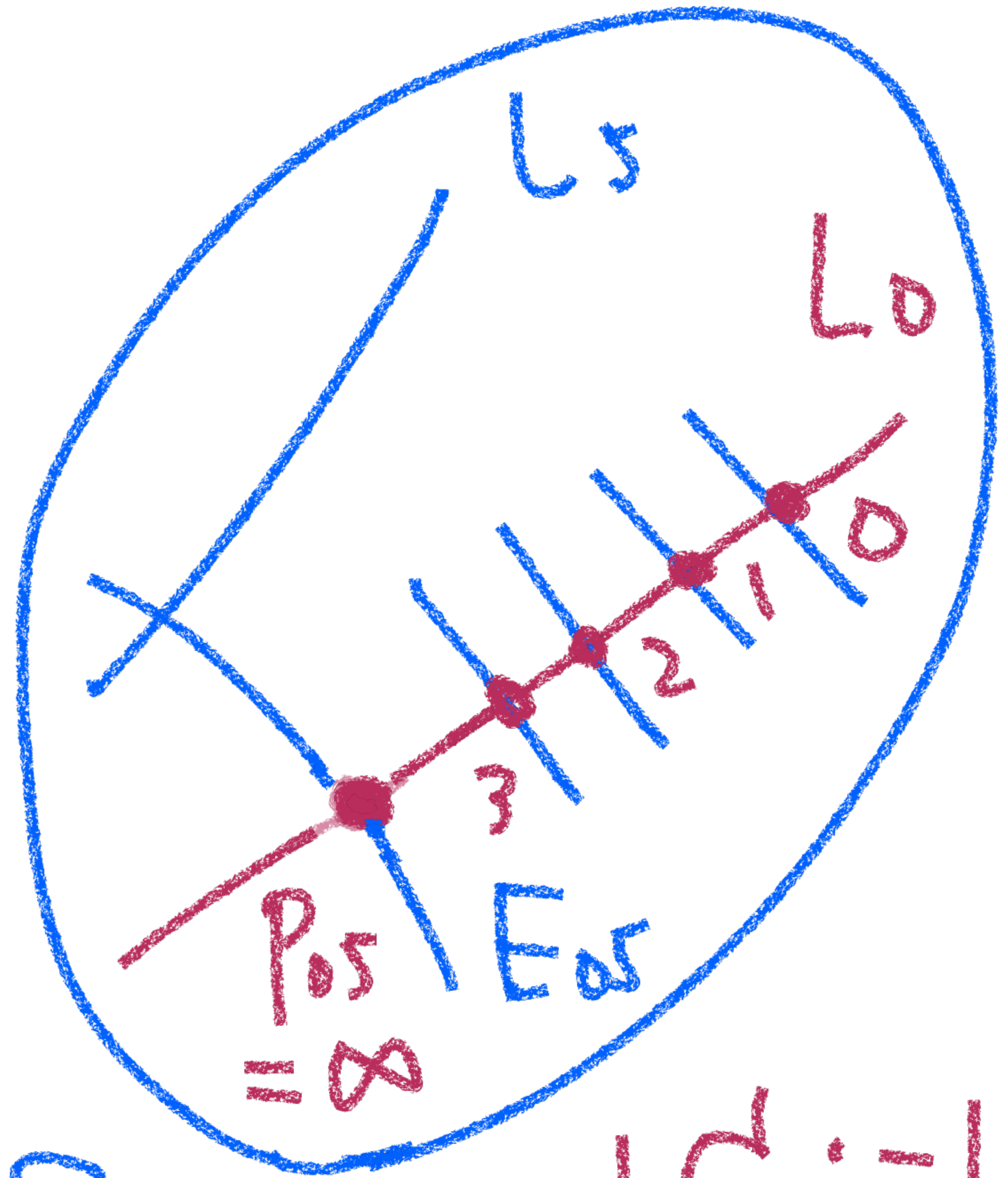
$P_{ij} = L_i' \cap L_j'$ $(i \neq j)$

S: blow up 15 pts P_{ij}

(Lesieutre's surface)



$\mathbb{P}^2$



S

$\begin{cases} C := L_0 \\ P := P_{05} \end{cases}$

Then:

1) $\forall m > 0 \quad |-2mK_S| = \{ \sum_{i=0}^{5} m L_i \}$

2) $\mathrm{Aut}\, S \hookrightarrow GL(NS(S)) \quad g \mapsto g_*$

($\Rightarrow$ discrete)

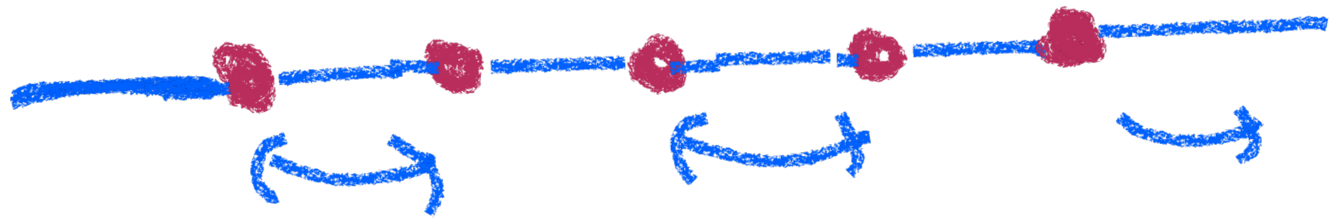
3) (S, C, P) satisfies Ass A & Ass B i.e. with Ass B $\downarrow$

$\mathrm{Aut}(S) \supset \mathrm{Aut}(S,P) = \mathrm{Aut}(S,C,P) \supset G$

(but ∞-index)

Rem Ass A: by (1)

Ass B: use 15 birat inv. on $\mathbb{P}^2$ s.t.



$L_0 = L_0'$

which extend 15 inv. $\in \mathrm{Aut}\, S$

Thm C (Dinh - O ___)

$\exists S \underset{\text{birat}}{\sim} {}^{\exists} Km(E \times F) = S_{min}$

$\exists p \in C = \mathbb{P}^1 \subset S$ with Ass A & B:

$\mathrm{Aut}\, S \supset \underbrace{\mathrm{Aut}(S,P) = \mathrm{Aut}(S,C,P) \supset G}_{\text{finite index}}$

$\bigcap$ (∞-index)

$\mathrm{Bir}\, S_{min} = \mathrm{Aut}\, S_{min}$

Hence Aut S is discrete & NFG

(Dinh-O ___ surface)

(S, G) Lesieutre's surface or Dinh-O ___ surface

Main Strategy for Thm (1) & (3)

Realize G as a finite index subgroup of Aut $^{\exists}$X via product & blow-up.

i.e. to take product $S \times V$ and blow-up so that

1) Keep G as autom.
2) Kill almost all "Aut S \ G" (when S: Lesieutre's surface)
3) Create $\nexists$ new autom

(3) hard in gen

$$\begin{pmatrix} \text{del Pezzo surf} \\ \text{of deg} = 1, \text{ general} \end{pmatrix} \xleftarrow[\text{blow-up}]{\exists \text{pt}} \begin{pmatrix} \text{rational} \\ \text{elliptic} \\ \text{surface} \end{pmatrix}$$

$|\mathrm{Aut}| < \infty$ $\qquad\qquad$ $\mathrm{Aut} \supseteq \mathbb{Z}^8$

On the other hand:

$\boxed{\text{Proposition 2}}$

> X: smooth proj. var $/\mathbb{C}$
> $\Rightarrow \exists\, Q_1, \cdots, Q_N \in X$ s.t.
> $\mathrm{Aut}(X, Q_1, \cdots, Q_N) = \{id\}$.

$\because$) If $H^0(X, T_X) \neq 0$, then choose $Q_1 \in X$ general.

Then

$$\{v \in H^0(X, T_X) \mid v(Q_1) = 0\} \subsetneqq H^0(X, T_X)$$

Then by induction

$\exists\, Q_1, \cdots, Q_{N-1}$ ($\exists\, N-1 \leqq \dim H^0(X, T_X)$)

s.t.

$$\{v \in H^0(X, T_X) \mid v(Q_i) = 0\ \forall i\} = \{0\}.$$

Then

$\mathrm{Aut}(X, Q_1, \cdots, Q_{N-1})$ discrete hence countable.

Finally choose

$$Q_N \in X \setminus \bigcup_{g \in \mathrm{Aut}(X, Q_1, \cdots, Q_{N-1}) \setminus \{id\}} X^g \neq \phi \quad (/\mathbb{C})$$

//

<u>Q</u> Is Prop 2 true for any $k = \bar{k}$? eg $k = \bar{\mathbb{Q}}$?

Proof of Thm (1)

S: Lesieutre's surface

$\mathrm{Aut}(S, C, P) = \mathrm{Aut}(S, P)$

$\cup$

$G = \{f \mid (df)|T_{C,P} = \mathrm{id}\}$ NFG

V: smooth Fano m-fold ($m \geq 1$)

$Q_1, \dots, Q_N \in V$ s.t.

$\mathrm{Aut}(V, Q_1, \dots, Q_N) = \{\mathrm{id}\}$.

(by Prop. 2).

Note $\begin{cases} |-2mC_S| = \{\sum_{i=0}^{5} m L_i\} \text{ one element} \\ |-2mK_V| = \text{very ample} \\ (m \gg 0) \end{cases}$

$$\begin{tikzcd}
W_0 := S \times V \ni & R_i := (P, Q_i)\ (i \geq 2) & R_1 := (P, Q_1) & \\
W_1 \arrow[u, "\text{b.up at } R_1, \dots, R_N"'] \supset & E_i \cong \mathbb{P}^{m+1}, \arrow[u] & E_1 \cong \mathbb{P}^{m+1} = \mathbb{P}(T_{W_0, R_1}) \ni R_\infty := [(v, w)] \arrow[u] & \\
W_2 \arrow[u, "\text{b.up at } R_\infty"'] \supset & E_i \cong \mathbb{P}^{m+1}\ (i \geq 2), \arrow[u, "\cong"'] & E_\infty \cong \mathbb{P}^{m+1}, \arrow[u] & E_1' \not\cong \mathbb{P}^{m+1}
\end{tikzcd}$$

$v \in T_{C,P} \setminus \{0\}$, $w \in T_{V,R_1} \setminus \{0\}$

Thm [DOY] $\mathrm{Aut}\, W_2$ is discrete & NFG.

Note $V = \mathbb{P}^m \Rightarrow W_2$ rat & $\dim W_2 = m+2$.

$\Rightarrow$ Thm (1).

Proof

1°) $\left[\operatorname{Aut} \overset{=S\times V}{\widetilde{W_0}} = \operatorname{Aut} S \times \operatorname{Aut} V\right]$

∵) $\supseteq$ clear. $\subseteq$:

$$|-2mK_{\widetilde{W_0}}| = \{m\textstyle\sum L_i\} + p_2^*|-2mK_V|$$

$$p_2 = \Phi_{|-2mK_{\widetilde{W_0}}|} : \widetilde{W_0} \longrightarrow V \subset |-2mK_{\widetilde{W_0}}|^*$$

$$\circlearrowright \operatorname{Aut} \widetilde{W_0} \implies \circlearrowright \operatorname{Aut} \widetilde{W_0}$$

Hence $f \in \operatorname{Aut} \widetilde{W_0}$ is of the form:

$$f(\underset{\substack{\Uparrow \\ S}}{s}, \underset{\substack{\Uparrow \\ V}}{t}) = (\underbrace{f_t}_{\in \operatorname{Aut} S \text{ (may dep on } t \in V)}(s), f_V(t))$$ and therefore

$$T : V \longrightarrow \operatorname{Aut} S : t \longmapsto f_t.$$

But $\operatorname{Aut} S$ is discrete.

∴ $T = \text{const}$, i.e. $f = f_S \times f_V \in \operatorname{Aut} S \times \operatorname{Aut} V$.

2°) $\widetilde{W_1}$ ← 1st blowup of $\widetilde{W_0}$

$$\left[\begin{array}{l}\operatorname{Aut} \widetilde{W_1} = \operatorname{Aut}(S, P) \times \operatorname{Aut}(V, \{Q_1, \dots, Q_N\}) \\ \qquad\qquad \overset{\shortparallel}{\supseteq} \operatorname{Aut}(S, C, P) \times \{id\}\end{array}\right]$$

finite index

∵) Consider

$$\begin{tikzcd} \mathbb{P}^{m+1} \subset \widetilde{W_1} \arrow[r, "\pi_1"] \arrow[rr, bend left, "\widehat{p}_1"] \arrow[rr, bend right, "\widehat{p}_2"'] & \widetilde{W_0} \arrow[r] \arrow[r] & S \ncong \mathbb{P}^2 \text{ (2-dim)},\ V \text{ (m-dim)} \end{tikzcd}$$

$\Rightarrow \widehat{p}_1(\mathbb{P}^{m+1}) = pt$ & $\widehat{p}_2(\mathbb{P}^{m+1}) = pt$

$\Rightarrow \mathbb{P}^{m+1} \subset \operatorname{Exc}(\pi_1) = \bigcup_{i=1}^{N} E_i \underset{\therefore}{=} \left(\begin{array}{l}\text{union of} \\ \forall \mathbb{P}^{m+1} \text{ in } \widetilde{W_0}\end{array}\right)$

Hence

$$\begin{aligned} \operatorname{Aut} \widetilde{W_1} &= \operatorname{Aut}(\widetilde{W_1}, \{E_1, \dots, E_N\}) \\ &= \operatorname{Aut}(\widetilde{W_0}, \{R_1, \dots, R_N\}) \\ &\underset{1°)}{=} \operatorname{Aut}(S, P) \times \underbrace{\operatorname{Aut}(V, \{Q_1, \dots, Q_N\})}_{\substack{\text{at most order } N! \\ \text{by } \operatorname{Aut}(V, Q_1, \dots, Q_N) = \{id\}}} \\ &\qquad \Vert \\ &\underset{\text{finite index}}{\supseteq} \operatorname{Aut}(S, C, P) \times \{id_V\}. \quad // \end{aligned}$$

3°) [$\mathrm{Aut}\, W_2 \supset G \times \{id\}$ fin. index.]

∵) Again consider

$$\begin{tikzcd} \mathbb{P}^{m+1} \subset W_2 \arrow[r, "\pi_2"] \arrow[rrr, bend left] \arrow[rrr, bend right] & W_1 \arrow[r, "\pi_1"] & W_0 \arrow[r] \arrow[r] & S \\ & & & V \end{tikzcd}$$

$$\Rightarrow \mathbb{P}^{m+1} \subset \mathrm{Exc}\ \pi_1 \circ \pi_2 \quad \not\ni \mathbb{P}^{m+1}$$

$$\qquad = \{\underbrace{E_\infty, E_2, \cdots, E_N}_{\therefore\ \forall\, \mathbb{P}^{m+1} \text{ in } W_2}, E_i'\}$$

Hence

$$\begin{aligned} &\mathrm{Aut}\, W_2 \\ &= \mathrm{Aut}(W_2, \{E_\infty, E_2, \cdots, E_N\}) \\ &\underset{\text{fin. index}}{\supseteq} \mathrm{Aut}(W_2, E_\infty, \{E_2, \cdots, E_N\}) \\ &= \mathrm{Aut}(W_2, E_\infty) \\ &= \mathrm{Aut}(W_1, [(v,w)]) \end{aligned}$$

$$\underset{\substack{\uparrow \\ \text{finite index} \\ \text{by } 2°) \,\&\, \text{Prop 1 (2)}}}{\supseteq} \{(f, id_V) \in \mathrm{Aut}(S, C, P) \times \{id\} \mid \underline{d(f \times id_V)(v,w) \,/\!/\, (v,w)}\}$$

$$\qquad\qquad \Leftrightarrow df(v) = v$$

$$= G \times \{id_V\}.$$

Hence $\mathrm{Aut}\, W_2 \underset{\text{finite index}}{\supset} G$: NFG

Hence $\mathrm{Aut}\, W_2$ is discrete & NFG. //

Proof of Thm (3)

S = Dinh-O surface $\overset{\text{birat}}{\sim}$ K3

$\underbrace{\mathrm{Aut}\, S > \mathrm{Aut}(S, C, P) > G}_{\text{fin. index}}$ NFG

A : abelian var of dim ℓ (≥ 0)

Ψ

$Q_1, \cdots, Q_N$

$\mathrm{Aut}(A, Q_1, \cdots, Q_N) = \{id_A\}$ (Prop 2)

$V \subset \mathbb{P}^{m+1}$ smooth hypersurface of general type ($m \geq 0$)

$\widetilde{S \times A} \xrightarrow{\pi} S \times A$ blow up at

$R_i = (P, Q_i)$ $i = 1, \cdots, N$

& $X = \widetilde{S \times A} \times V \xrightarrow{p} V$

$\dim X = 2 + \ell + m$ & $\kappa(X) = m$

$\mathrm{Aut}(\widetilde{S \times A})$

$= \mathrm{Aut}(S \times A, \{P\} \times \{Q_1, \cdots, Q_N\})$

$\mathbb{P}^{\ell+1}$ or $|K_{\widetilde{S \times A}}|$

$= \mathrm{Aut}(S, P) \times \underbrace{\mathrm{Aut}(A, \{Q_1, \cdots, Q_N\})}_{|\ | \leq N!}$

Albanese map $S \times A \to A$

$\geq G \times \{id\}$

finite index

$\mathrm{Aut}\, X$

$= \mathrm{Aut}(\widetilde{S \times A}) \times \mathrm{Aut}\, V$

$\Phi_{|mK_X|}$

$\geq G \times \{id\} \times \overbrace{\mathrm{Aut}\, V}^{\text{finite}}$

finite index

Hence $\mathrm{Aut}(X) \underset{\text{finite index}}{>} G$: discrete & NFG. //

Proof of Thm(2)

$\kappa(V) = \dim V \Rightarrow |\mathrm{Aut}\, V| < \infty$

by finiteness of pluricanonical representation. ——— ⊛

$\kappa(V) = \dim V - 1$ $\left(\overset{?}{\Rightarrow} \mathrm{Aut}\, V / \mathrm{Aut}^0 V \text{ FG}\right)$

$$\begin{array}{ccc} \Phi_{|mK_V|} : V & \dashrightarrow & B \subset |mK_V|^* \\ \circlearrowleft & & \circlearrowleft \\ \mathrm{Aut}\, V & & \mathrm{Aut}\, V \\ & & \text{finite image} \\ & & \text{by } ⊛ \end{array}$$

$\Rightarrow \mathrm{Aut}^0 V \subset G \underset{\text{fin. index}}{\subset} \mathrm{Aut}\, V$ s.t.

$G \curvearrowright V_{\bar{\eta}} / \overline{\mathbb{C}(B)}$ ell. curve $/ \overline{\mathbb{C}(B)}$

faithfully

$(G = \ker(\mathrm{Aut}\, V \to \mathrm{Aut}\, B))$

∴ G: solvable as so is Aut(elliptic curve)

∴ $G \xrightarrow{r} GL(N^1(V)) = GL(\rho, \mathbb{Z})$

Image = solvable $\subset GL(\rho, \mathbb{Z})$

$\Rightarrow$ fin. gen. Malcev's Thm

$$∴ \quad 1 \to \underbrace{\frac{\ker r}{\mathrm{Aut}^0 V}}_{\text{finite}} \to \underbrace{\frac{G}{\mathrm{Aut}^0 V}}_{∴\ \text{fin. gen}} \to \underbrace{\mathrm{Im}\, r}_{\text{fin. gen.}} \to 1$$

Fujiki-Lieberman's Thm

$$∴ \quad \frac{\mathrm{Aut}\, V}{\mathrm{Aut}^0 V} \underset{\text{finite index}}{\supset} \frac{G}{\mathrm{Aut}^0 V} \text{ fin. gen } //$$

Thank you very much for your attention & please stay safe !

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(Main)