ZAG seminar

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On the top weight rational cohomology of $\mathcal{A}_g$

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Idea: understand the geometry of moduli spaces via tropical modular interpretation of the original space + compactification.

$\hookrightarrow$ • endow comb. data associated to a comp. $\mathcal{M} \subset \overline{\mathcal{M}}$ with tropical modular interpretation $\mathcal{M}^{trop}$

• study $\mathcal{M}^{trop}$ (e.g. $H_\bullet(\mathcal{M}^{trop}, \mathbb{Q})$)

• translate to $\mathcal{M}$ (e.g. $H^\bullet(\mathcal{M}, \mathbb{Q})$)

Today

Def Let $\mathcal{A}_g$ be moduli space of abelian varieties of dim g.

$\mathcal{A}_g$ is a smooth and separated Deligne-Mumford stack of dim $d := \binom{g+1}{2}$, endowed with quasiprojective moduli space A_g.

• $\mathcal{A}_g$ is not proper, but admits several compactifications

- we will consider toroidal compactifications $\overline{\mathcal{A}}_g^{\Sigma}$, Σ comb. data
- $\forall\, \Sigma$, can construct $\overline{\mathcal{A}}_g^{trop,\Sigma}$ (joint with Brannetti, Viviani)

$\longrightarrow$ we will compute $H_\bullet(\overline{\mathcal{A}}_g^{trop,\Sigma}, \mathbb{Q})$, $g \leq 7$.

→ We will compute $H_\bullet(\overline{\mathcal{A}}_g^{trop,\Sigma}, \mathbb{Q})$, $g \leq 7$.

and translate the results to $H^\bullet(\mathcal{A}_g, \mathbb{Q})$.

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<u>Question</u> What is known about $H^\bullet(\mathcal{A}_g, \mathbb{Q})$?

- <u>$g=2$</u> [Igusa'62] $H^k(\mathcal{A}_2, \mathbb{Q}) = \begin{cases} \mathbb{Q}, & k=0,2 \\ 0, & \text{else} \end{cases}$

- <u>$g=3$</u> [Hain'02] $H^k(\mathcal{A}_3, \mathbb{Q}) = \begin{cases} \mathbb{Q}, & k=0,2,4 \\ \mathbb{Q}^2, & k=6 \\ 0, & \text{else} \end{cases}$

- <u>$g=4$</u>, $H^\bullet(\mathcal{A}_4, \mathbb{Q})$ is not fully understood but a lot is known (eg. Hulek-Tommasi)

<u>Question</u> [Grushevsky'09] Are there odd degree classes in $H^\bullet(\mathcal{A}_g, \mathbb{Q})$?

↳ answer should be yes (there are lots of known odd coh. classes in $H^\bullet(\mathcal{M}_g, \mathbb{Q})$

↳ Tommasi'05: $H^5(\mathcal{M}_4, \mathbb{Q}) \cong \mathbb{Q}$)

↳ Our techiques will allow us to compute

$\underline{Gr^W_{2g} H^\bullet(\mathcal{A}_g, \mathbb{Q})}$ ⇝ the top weight cohomology of $\mathcal{A}_g$, for $g \leq 7$.

<u>Recall</u>: by Deligne's theory of weights

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$H^\bullet(\mathcal{A}_g, \mathbb{Q})$ admits a weight filtration

$$\cdots \supset W_j H^k(\mathcal{A}_g, \mathbb{Q}) \supset W_{j-1} H^k(\mathcal{A}_g, \mathbb{Q}) \supset \cdots$$

with graded pieces

$$\mathrm{Gr}^W_j H^k(\mathcal{A}_g, \mathbb{Q}) := W_j H^k(\mathcal{A}_g, \mathbb{Q}) / W_{j-1} H^k(\mathcal{A}_g, \mathbb{Q})$$

with weights $k \leq j \leq 2k$

Def The top weight rational cohomology of $\mathcal{A}_g$ is

$$\mathrm{Gr}^W_{\underbrace{2d}_{\substack{\shortparallel \\ g(g+1)}}} H^\bullet(\mathcal{A}_g, \mathbb{Q})$$

<u>Theorem</u> [BBCMMW]

i) $\mathrm{Gr}^W_6 H^\bullet(\mathcal{A}_2, \mathbb{Q}) = 0$

ii) $\mathrm{Gr}^W_{12} H^k(\mathcal{A}_3, \mathbb{Q}) \cong \begin{cases} \mathbb{Q} & \text{if } k = 6 \\ 0 & \text{else} \end{cases}$ [Hain]

iii) $\mathrm{Gr}^W_{20} H^\bullet(\mathcal{A}_4, \mathbb{Q}) = 0$ [Hulek-Tommasi]

iv) $\mathrm{Gr}^W_{30} H^k(\mathcal{A}_5, \mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } k = 15, 20 \\ 0 & \text{else} \end{cases}$

v) $\mathrm{Gr}^W \ H^k(\mathcal{A}_6, \mathbb{Q}) = \{ \mathbb{Q} \text{ if } k = 30$

v) $Gr^W_{42} H^k(\mathcal{A}_6, \mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } k = 30 \\ 0 & \text{else} \end{cases}$

v) $Gr^W_{56} H^k(\mathcal{A}_7, \mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } k = 28, 33, 37, 42 \\ 0 & \text{else} \end{cases}$

Rmk • Answer to Grushevsky's question is yes.

• Speculate on non-vanishing results in general

e.g. $Gr^W_{2d} H^d(\mathcal{A}_g, \mathbb{Q}) \neq 0$, if g odd ?

• Relevant info for $H^\bullet(\mathcal{A}_g^{sat}, \mathbb{Q})$

↳ proof uses existence of good compactifications of $\mathcal{A}_g$

§ Toroidal compactifications of $\mathcal{A}_g$ and $\mathcal{A}_g^{trop}$

Let $\Omega_g = \{$ positive definite quadratic forms in $\mathbb{R}^g \} \subset \mathbb{R}^{\binom{g+1}{2}}$

$\cap$

$\Omega_g^{rat} = \left\{ \begin{array}{l} \text{pos. semi-definite} \\ \text{quad. forms with} \\ \text{rat'l kernel} \end{array} \right\}$

$GL_g(\mathbb{Z}) \curvearrowright \Omega_g^{rat}$

$(A, Q) \longmapsto AQA^T$

Def An admissible decomposition of Ω_g^{rat} is a family

$\{\sigma\}_{\sigma \in \Sigma}$

of Rat. polyhedral cones s.t.

1) $\Omega_g^{Rat} = \bigcup_{\sigma \in \Sigma} \sigma$

2) Σ is closed under taking faces and intersections

3) Σ is stable under action of $GL_g(\mathbb{Z})$

4) $\# \Sigma / GL_g(\mathbb{Z}) < \infty$

__Facts__ Given Σ, one can construct $\bar{\mathcal{A}}_g^{\Sigma}$ and $\mathcal{A}_g^{\Sigma, trop}$ s.t.

- $\mathcal{A}_g \subset \bar{\mathcal{A}}_g^{\Sigma}$ is a toroidal comp.
- $\bar{\mathcal{A}}_g^{\Sigma} \setminus \mathcal{A}_g$ is stratified according to poset structure on Σ
- $\exists\, \Sigma : \bar{\mathcal{A}}_g^{\Sigma} \setminus \mathcal{A}_g$ is simply normal crossings
- $\mathcal{A}_g^{\Sigma, trop} := \varinjlim \{\sigma\}_{\sigma \in \Sigma}$, arrows given by inclusions of faces comp. w/ action of $GL_g(\mathbb{Z})$

  $\hookrightarrow$ generalized cone complex [BMV]

Rmks: $\bar{\mathcal{A}}_g^{\Sigma}$ has different regularity properties according to Σ (not always simplest Σ will define better behaved $\bar{\mathcal{A}}_g^{\Sigma}$).

- $\forall\, \Sigma,\ \bar{\mathcal{A}}_g^{\Sigma} \longrightarrow \underbrace{\bar{\mathcal{A}}_g^{Sat}}_{\text{Satake}} = \mathcal{A}_g \sqcup \mathcal{A}_g \sqcup \dots \sqcup \mathcal{A}_1 \sqcup \mathcal{A}_0$

... $H^0(\bar{\mathcal{A}}_g^{Sat}, \mathbb{Q})$

Satake

(our results have relevant also in the understanding of $H^\bullet(\bar{\mathcal{A}}_g^{Sat}, \mathbb{Q})$)

Thm [Comparison theorem] Let Σ be as above. Then

$$\boxed{Gr^W_{2d} H^{2d-i}(\mathcal{A}_g, \mathbb{Q}) \cong \tilde{H}_{i-1}(L\mathcal{A}_g^{trop,\Sigma}, \mathbb{Q})}$$

proof (Sketch)

1) Can assume that all cones in Σ are smooth and that $\bar{\mathcal{A}}_g^{\Sigma} \setminus \mathcal{A}_g$ is simple normal crossings.

- Consider a smooth refinement of Σ, Σ' [FC]
- $L\mathcal{A}_g^{trop,\Sigma} \simeq L\mathcal{A}_g^{trop,\Sigma'}$ (top. of the link gets unvaried under taking subdivisions)

2) $\mathcal{A}_g$ is smooth DM

$\mathcal{A}_g \subset \bar{\mathcal{A}}_g^{\Sigma}$ simple normal crossings

$\Delta(\mathcal{A}_g \subset \bar{\mathcal{A}}_g^{\Sigma}) \cong L\mathcal{A}_g^{trop,\Sigma}$ [FC] + const. of $\mathcal{A}_g^{trop,\Sigma}$

3) Deligne's mixed Hodge theory + [CGP] gen'l to smooth sep. DM stacks

$$Gr^W_{2d} H^{2d-i}(\mathcal{A}_g, \mathbb{Q}) \cong \tilde{H}_{i-1}(\Delta(\mathcal{A}_g \subset \bar{\mathcal{A}}_g^{\Sigma}), \mathbb{Q})$$

PD $\wr\wr$ $\qquad\qquad$ $\wr\wr$

$$\tilde{H}_{i-1}(L\mathcal{A}_g^{trop,\Sigma}, \mathbb{Q}).$$

P.D. $\cong$ $\qquad\qquad$ $\tilde{H}_{i-1}(LA_g^{trop,\Sigma}, \mathbb{Q})$.

$Gr_0^W H_c^i(\mathcal{A}_g, \mathbb{Q})^*$ $\blacksquare$

$\cong$

$$Gr_0^W \tilde{H}^{i-1}(\partial \bar{\mathcal{A}}_g^{\Sigma}, \mathbb{Q})^* \cong \tilde{H}^{i-1}(\Delta(\mathcal{A}_g \subset \bar{\mathcal{A}}_g^{\Sigma}), \mathbb{Q}))^*$$

<u>Rmk</u> theorem implies that we can compute $Gr_{2d}^W H^\bullet(\mathcal{A}_g, \mathbb{Q})$ by considering any Σ, even thought $\bar{\mathcal{A}}_g^{\Sigma}$ doesn't have ideal reg. properties

§ <u>Perfect cone decomposition</u> Σ_g^P

Let $Q \in \Omega_g$,

- $M(Q) := \{\bar{x} \in \mathbb{Z}^g \setminus \{0\} : \bar{x} \text{ minimizes } Q\}$
- $\sigma(Q) := \mathbb{R}_{\geq 0}\langle \bar{x}\bar{x}^T, \bar{x} \in M(Q)\}$
- $\Sigma_g^P = \{\sigma(Q) : Q \in \Omega_g\}$

<u>Thm</u> [Voronoi 1908] Σ_g^P is a perfect decomp. of Ω_g^{perf}.

<u>Example</u> $g=2$

Let $Q = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}$, $Q(x,y) = x^2 + xy + y^2$

$M(Q) = \{\pm(1,0), \pm(0,1), \pm(1,-1)\}$

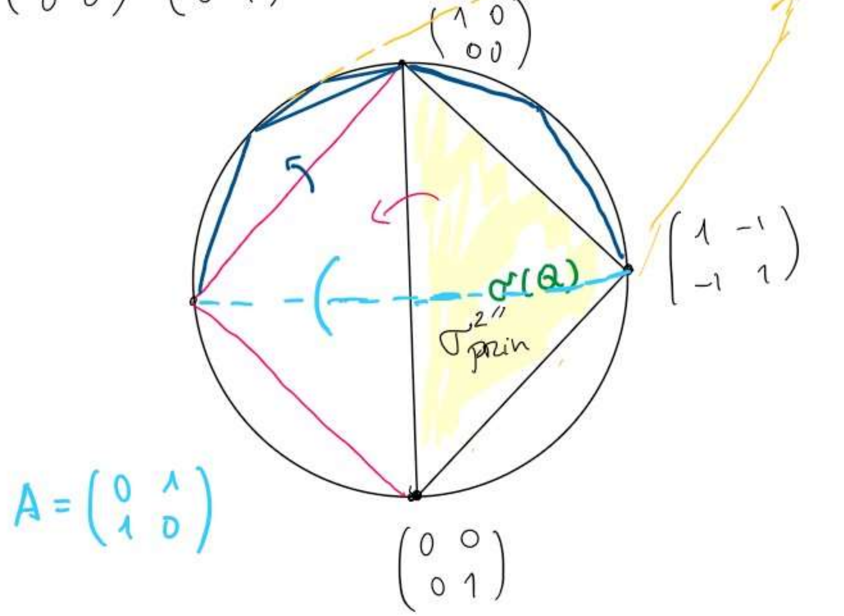
and $\quad$ $\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 \end{pmatrix}$ {

and

$$\sigma(Q) = \mathbb{R}_{\geq 0}\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \right\}$$

$$= \sigma^2_{\text{prin}}$$

slice of Ω_2^{rt}



Def A cone $\sigma \in \Sigma_g^P$ is <u>alternating</u> if given $A \in GL_g(\mathbb{Z})$ s.t. $A \sigma A^T = \sigma$, the action of A is orientation preserving.

Eg. σ^2_{prin} is not alternating!

Def [Perfect complex $P^g_\bullet$]

- $P^g_k = \mathbb{Q}\left\langle \begin{matrix} GL_g(\mathbb{Z})\text{-orbits of alternating} \\ \text{cones in } \Sigma_g^P \text{ of dim } k+1 \end{matrix} \right\rangle$
- $\partial_k([\sigma]) = \sum_{\substack{\tau \subseteq \sigma \\ \text{facet}}} (-1)^{\sqcup} [\tau]$

Prop $\forall i \geq 0$, $H_{i-1}(P^g_\bullet) \cong \tilde{H}_{i-1}(LA_g^{\text{trop},\Sigma}, \mathbb{Q}) \cong Gr^W_{2d} H^{2d-i}(A_g, \mathbb{Q})$

→ We get our results by computing the homology of $P^g_\bullet$

$\underline{g=2}$ $\quad P^2_\bullet = \quad 0 \to P^2_2 \to P^2_1 \to P^2_0 \to P^2_{-1} \to 0$

$$0 \to 0 \to \mathbb{Q} \xrightarrow{\cong} \mathbb{Q} \to 0$$

$$\therefore \; H_k(P^2_\bullet) = 0, \; \forall k$$

$$\Rightarrow \; Gr^W_6 H^\bullet(\mathcal{A}_2, \mathbb{Q}) = 0$$

$\underline{g=3}$

$$0 \to P^3_5 \to P^3_4 \to P^3_3 \to P^3_2 \to P^3_1 \to P^3_0 \to P^3_{-1} \to 0$$

$$0 \to \mathbb{Q}\sigma(K_4) \to 0 \to 0 \to 0 \to 0 \to \mathbb{Q} \xrightarrow{\cong} \mathbb{Q} \to 0$$

$$Gr^W_{12} H^{12-i}(\mathcal{A}_3, \mathbb{Q}) \cong \begin{cases} \mathbb{Q} & \text{if } i=0 \\ 0 & \text{else} \end{cases}$$

$g = 4, 5, 6, 7$ use computations of Elbaz-Vincent, Gangl, Soulé on the cohomology of the modular groups $SL_N(\mathbb{Z})$ and $GL_N(\mathbb{Z})$, $N \leq 7$

In these computations the authors use a complex ("Voronoi complex") closely related to ours

* Our computations give that $H_{\dots}(P^g_\bullet) \neq 0$, $g=3,5,6,7$.

* Our computations give that $H_{2g-1}(P^g_\bullet) \neq 0$, $g = 3, 5, 6, 7$.

$\Rightarrow Gr^W_{(g+1)g} \, H^{g(g-1)}(\mathcal{A}_g, \mathbb{Q}) \neq 0$, $g = 3, 5, 6, 7$

↳ Is this true in general?

(it is true that $H_{2g-1}(G^{(g)}_\bullet) \neq 0$ for $g = 3$ and $g \geq 5$

* Applications to study of $\boxed{\mathcal{A}_g^{SAT}}$

Consider stable cohomology ring $H^\bullet(\mathcal{A}_\infty^{SAT}, \mathbb{Q})$

$$\underbrace{H^\bullet(\mathcal{A}_\infty^{SAT}, \mathbb{Q}) = \langle \lambda_i, \ i \text{ odd}, \ \underbrace{y_{4j+2}}, \ j = 1, 2, 3 \rangle}$$

↗ Charney-Lee ↳ class of degree $4j+2$ and weight 0 (Chen - Looijenga)

Ass. to $\mathcal{A}_g^{SAT} = \mathcal{A}_g \sqcup \mathcal{A}_{g-1} \sqcup \cdots \sqcup \mathcal{A}_1 \sqcup \mathcal{A}$

have S.S. $E^1_{p,q} = H_c^{p+q}(\mathcal{A}_p, \mathbb{Q}) \Rightarrow H^{p+q}(\mathcal{A}_g^{SAT}, \mathbb{Q})$, $p = 0, \ldots g$

y classes imply existence of ∞'ly many weight 0 classes

in $Gr_0^W H_c^j(\mathcal{A}_g, \mathbb{Q})$

and by P.D.

$$Gr_0^W H_c^j(\mathcal{A}_g; \mathbb{Q}) \times Gr^W_{(g+1)g} \, H^{(g+1)g-j}(\mathcal{A}_g; \mathbb{Q}) \to \mathbb{Q}$$

∞'ly many classes in topweight.

<u>Questions</u>: 1* $Gr_0^W H_c^{2g}(\mathcal{A}_g, \mathbb{Q}) \neq 0$, $g = 3$ and $g \geq 5$?

[illegible] produce stable cohomology classes in $Gr_0^W H^\bullet(\mathcal{A}_\infty^{SAT}, \mathbb{Q})$?

2* [illegible]

[illegible]

2 * Do these produce stable cohomology classes in $\mathrm{Gr}^W_0 H^\bullet(\mathcal{A}^{SAT}_\infty, \mathbb{Q})$?

3 * $\mathrm{Gr}^W_0 H^k_0(\mathcal{A}_g, \mathbb{Q}) = 0$, $k < 2g$?

<u>Our results</u> $\Rightarrow$ 1, 3 true for $g \leq 7$

also 2 is true for $g = 3, 5$, i.e.

$\mathrm{Gr}^W_0 H^6_c(\mathcal{A}_3, \mathbb{Q})$ produces y_6 in $\mathrm{Gr}^W_0 H^6(\mathcal{A}^{SAT}_\infty, \mathbb{Q})$

$\mathrm{Gr}^W_0 H^{10}_c(\mathcal{A}_5, \mathbb{Q})$ produces y_{10} in $\mathrm{Gr}^W_0 H^{10}(\mathcal{A}^{SAT}_\infty, \mathbb{Q})$.