A dynamical approach to generalized Weil's Riemann hypothesis

Joint work with Tuyen Truong

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Introduction

Weil's Riemann Hypothesis

X_0: a smooth projective variety defined over a finite field $\mathbf{F}_q$

$X = X_0 \times_{\mathbf{F}_q} \overline{\mathbf{F}}_q$: the base change of X_0 to an algebraic closure $\overline{\mathbf{F}}_q$ of $\mathbf{F}_q$

F: the Frobenius endomorphism of X (with respect to $\mathbf{F}_q$)

Grothendieck–Lefschetz trace formula expresses the number of $\mathbf{F}_{q^n}$-points of X_0 in terms of the traces of F^n on ℓ-adic étale cohomology groups $H^i_{\text{ét}}(X, \mathbf{Q}_\ell)$ which infers that

$$Z(X_0, t) = \prod_{i=0}^{2 \dim X_0} \det(1 - t\,\mathsf{F}^* | H^i_{\text{ét}}(X, \mathbf{Q}_\ell))^{(-1)^{i+1}}.$$

Weil's Riemann Hypothesis

Weil's Riemann hypothesis (Deligne, 1974)

The "characteristic polynomial"

$$P_i(t) := \det\left(1 - t\,\mathsf{F}^*|H^i_{\text{ét}}(X, \mathbf{Q}_\ell)\right)$$

has integer coefficients independent of $\ell\,(\neq p)$ and the eigenvalues of F^* on $H^i_{\text{ét}}(X, \mathbf{Q}_\ell)$ are algebraic integers of absolute value $q^{i/2}$.

Remark

Equivalently, the eigenvalues $\alpha \in \overline{\mathbf{Q}}_\ell$ of the Frobenius action $\mathsf{F}^*|H^i_{\text{ét}}(X, \mathbf{Q}_\ell)$ have the property that $|\iota(\alpha)| \leqslant q^{i/2}$ for every field isomorphism $\iota\colon \overline{\mathbf{Q}}_\ell \simeq \mathbf{C}$.

A motivating observation on complex surfaces

Let $f\colon S \to S$ be an automorphism of a smooth complex projective surface S.

Let $\mathrm{NS}(S)$ be the Néron–Severi group of S. Note that we have a natural inclusion $\mathrm{NS}(S) \hookrightarrow H^2(S, \mathbf{Z})$.

Consider the natural pullback actions of f on $H^2(S, \mathbf{Q})$ and $\mathrm{NS}(S)_{\mathbf{Q}}$.

Observe that

$$\rho(f^*|\,\mathrm{NS}(S)_{\mathbf{Q}}) = \rho(f^*|H^{1,1}(S, \mathbf{C})) = \rho(f^*|H^2(S, \mathbf{Q})). \tag{1.1}$$

Here, given a linear map φ over $\mathbf{R}$ or $\mathbf{C}$, the spectral radius

$$\rho(\varphi) := \max\left\{|\lambda| : \lambda \text{ is an eigenvalue of } \varphi\right\}.$$

A motivating observation on complex surfaces

Reason:

$$\rho(f^*|\,\mathrm{NS}(S)_{\mathbf{Q}}) = \lim_{m\to\infty} \big((f^m)^* H_S \cdot H_S\big)^{1/m},$$

where H_S is an arbitrary ample divisor on S, and

$$\rho(f^*|H^2(S,\mathbf{Q})) = \lim_{m\to\infty} \big((f^m)^*\omega_S \cup \omega_S\big)^{1/m},$$

where ω_S is an arbitrary Kähler form on S (viewed a compact Kähler surface).

Openness of ample/Kähler cone implies the independence of choices of ample divisor/Kähler form.

Choose $\omega_S = c_1(\mathcal{O}_S(H_S))$.

Automorphisms of surfaces in positive characteristic

Theorem (Esnault–Srinivas, 2013)

S_0: *a smooth projective surface over a finite field* $\mathbf{F}_q$

S: *the base change of* S_0 *to an algebraic closure* $\overline{\mathbf{F}}_q$ *of* $\mathbf{F}_q$.

H_S: *a fixed ample divisor on* S

$f \in \mathrm{Aut}(S_0)$: *an automorphism of* S_0

Then

$$\rho(f^*|H^\bullet_{\acute{e}t}(S, \mathbf{Q}_\ell)) = \rho(f^*|\,\mathrm{NS}(S)_\mathbf{Q}).$$

In particular, $\rho(f^*|H^2_{\acute{e}t}(S, \mathbf{Q}_\ell)) = \rho(f^*|\,\mathrm{NS}(S)_\mathbf{Q})$.

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Idea of Proof: Enriques–Kodaira–Bombieri–Mumford classification of surfaces, lifting of automorphisms of K3 surfaces, Tate conjecture/theorem for divisors on Abelian varieties.

A motivating question

What about an arbitrary self-morphism, rational self-map, or more generally, self-corrrespondence f of a smooth projective variety X over a field of arbitrary characteristic?

Question

Or rather, can we compare $f^*|H^{2k}_{\text{ét}}(X)$ and $f^*|\mathsf{N}^k(X)_{\mathbf{Q}}$, in some sense?

$$\mathsf{A}^k(X)_{\mathbf{Q}_\ell} := (\mathsf{Z}^k(X)/\sim_{\text{hom}}) \otimes_{\mathbf{Z}} \mathbf{Q}_\ell \xhookrightarrow{\overline{\text{cl}}_X} H^{2k}_{\text{ét}}(X)$$

$$\downarrow \qquad \overset{?D?}{\dashrightarrow}$$

$$\mathsf{N}^k(X)_{\mathbf{Q}_\ell} := (\mathsf{Z}^k(X)/\equiv) \otimes_{\mathbf{Z}} \mathbf{Q}_\ell.$$

Dynamical degrees in complex dynamics

Definition

Let (X, ω_X) be a compact Kähler manifold of dimension n and f a dominant meromorphic self-map of X. For $0 \leqslant k \leqslant n$, the k**-th dynamical degree** $d_k(f)$ of f is defined by

$$\begin{aligned} d_k(f) &:= \lim_{m\to\infty} \left((f^m)^*\omega_X^k \cup \omega_X^{n-k}\right)^{1/m} \\ &= \lim_{m\to\infty} \left\|(f^m)^*|H^{k,k}(X, \mathbf{C})\right\|^{1/m} \\ &= \lim_{m\to\infty} \left\|(f^m)^*|H^{2k}(X, \mathbf{C})\right\|^{1/m}. \end{aligned}$$

Remark

The existence of limits and the equivalence are due to Dinh and Sibony.

Equivalence of the two definitions of entropy

The following result due to Gromov and Yomdin asserts that the topological entropy of a holomorphic self-map of a compact Kähler manifold is an algebraic invariant.

Theorem (Gromov, 1977 and Yomdin, 1987)

Let f be a holomorphic self-map of a compact Kähler manifold X. Then

$$h_{\mathrm{top}}(f) = h_{\mathrm{alg}}(f) := \max_{0 \leqslant k \leqslant n} \log d_k(f).$$

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Theorem (Dinh–Sibony, 2005)

Let f be a dominant meromorphic self-map of a compact Kähler manifold X. Then

$$h_{\text{top}}(f) \leqslant h_{\text{alg}}(f) := \max_{0 \leqslant k \leqslant n} \log d_k(f).$$

Remark

Let X be a smooth complex projective variety of dimension n and f a dominant rational self-map of X. Let H_X be a fixed ample divisor on X. For $0 \leqslant k \leqslant n$, the k**-th dynamical degree** $d_k(f)$ of f is also equal to

$$\begin{aligned} &\lim_{m\to\infty} \left((f^m)^* H_X^k \cdot H_X^{n-k} \right)^{1/m} \\ &= \lim_{m\to\infty} \left\| (f^m)^* | \mathsf{N}^k(X)_{\mathbf{R}} \right\|^{1/m} \\ &= \lim_{m\to\infty} \left\| (f^m)^* | H^{2k}(X, \mathbf{C}) \right\|^{1/m}. \end{aligned}$$

Choose $\omega_X = c_1(\mathcal{O}_X(H_X))$. Again, openness of various positive cones plays an important role.

Remark

Let X be a smooth complex projective variety of dimension n and f a dominant rational self-map of X. Let H_X be a fixed ample divisor on X. For $0 \leqslant k \leqslant n$, the k-**th dynamical degree** $d_k(f)$ of f is also equal to

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Choose $\omega_X = c_1(\mathcal{O}_X(H_X))$. Again, openness of various positive cones plays an important role.

Complex absolute value of ℓ-adic numbers

Let X be a smooth projective variety over an algebraically closed field $\mathbf{k}$ of arbitrary characteristic.

Let $\mathsf{N}^k(X) := \mathsf{Z}^k(X)/\equiv$ be the group of integral algebraic cycles of codimension k on X modulo numerical equivalence $\equiv$, which is a finitely generated Abelian group.

We endow a norm $\|\cdot\|$ on the finite-dimensional $\mathbf{R}$-vector space $\mathsf{N}^k(X)_{\mathbf{R}} := \mathsf{N}^k(X) \otimes_{\mathbf{Z}} \mathbf{R}$.

Choose a field isomorphism $\iota\colon \overline{\mathbf{Q}}_\ell \simeq \mathbf{C}$ so that we may speak of the complex absolute value of an element of $\overline{\mathbf{Q}}_\ell$: for any $\alpha \in \overline{\mathbf{Q}}_\ell$,

$$|\alpha|_\iota := |\iota(\alpha)|.$$

We endow a norm $\|\cdot\|_\iota$ on the finite-dimensional $\mathbf{Q}_\ell$-vector space $H^\bullet_{\text{ét}}(X)$, the étale cohomology $H^\bullet_{\text{ét}}(X, \mathbf{Q}_\ell)$.

Dynamical degrees in arbitrary characteristic

Definition (Numerical/cohomological dynamical degrees)

Let X be a smooth projective variety of dimension n, H_X an ample divisor on X, and f a dynamical correspondence of X, all defined over an algebraically closed field $\mathbf{k}$ of arbitrary characteristic.

For $0 \leqslant k \leqslant n$ and $0 \leqslant i \leqslant 2n$, two ways to define dynamical degrees:

$$\lambda_k(f) := \lim_{m\to\infty} \left((f^{\Diamond m})^* H_X^k \cdot H_X^{n-k} \right)^{1/m}, \tag{1.2}$$

$$= \lim_{m\to\infty} \left\| (f^{\Diamond m})^* |\mathsf{N}^k(X)_{\mathbf{R}} \right\|^{1/m}, \tag{1.3}$$

$$\chi_i(f)_\iota := \limsup_{m\to\infty} \left\| (f^{\Diamond m})^* | H^i_{\text{ét}}(X, \mathbf{Q}_\ell) \right\|_\iota^{1/m}. \tag{1.4}$$

Remark

For λ_k, the existence of limits and the equivalence are due to Truong.

Dynamical degree comparison (DDC)

Conjecture (Truong, 2016)

For any $0 \leqslant k \leqslant n$, the k-th numerical dynamical degree $\lambda_k(f)$ coincides with the $2k$-th cohomological dynamical degree $\chi_{2k}(f)_\iota$.

Remark

It's true for complex projective varieties (i.e., $\mathbf{k} = \mathbf{C}$).

$$\lim_{m\to\infty} \left((f^{\lozenge m})^* H_X^k \cdot H_X^{n-k}\right)^{1/m} = \lim_{m\to\infty} \left((f^{\lozenge m})^* \omega_X^k \cup \omega_X^{n-k}\right)^{1/m}$$

$$= \lim_{m\to\infty} \left\|(f^{\lozenge m})^* | H^{k,k}(X, \mathbf{C})\right\|^{1/m} = \lim_{m\to\infty} \left\|(f^{\lozenge m})^* | H^{2k}(X, \mathbf{C})\right\|^{1/m}$$

So we are particularly interested in the case $\mathrm{char}(\mathbf{k}) = p > 0$.

Truong's conjecture $\Longrightarrow$ Weil's Riemann hypothesis

F: the Frobenius endomorphism of $X = X_0 \times_{\mathbf{F}_q} \overline{\mathbf{F}}_q$ (with respect to $\mathbf{F}_q$)

$$\chi_i(\mathsf{F})_\iota = \rho(\mathsf{F}^*|H^i_{\text{ét}}(X, \mathbf{Q}_\ell))_\iota$$

WRH $\Leftrightarrow$ All eigenvalues of $\mathsf{F}^*|H^i_{\text{ét}}(X, \mathbf{Q}_\ell)$ have absolute values at most $q^{i/2}$ for any $\iota \colon \overline{\mathbf{Q}}_\ell \simeq \mathbf{C}$, i.e., $\chi_i(\mathsf{F})_\iota \leqslant q^{i/2}$ for any ι.

Using a standard product trick, it suffices to show that

$$\chi_{2k}(\mathsf{F})_\iota \leqslant q^k \text{ for any } \iota.$$

Now, if we assume that DDC holds for F, then

$$\chi_{2k}(\mathsf{F})_\iota = \lambda_k(\mathsf{F}) = q^k.$$

A conjectural evidence towards DDC

Assuming Standard conjectures by Bombieri and Grothendieck

(in particular, of Lefschetz type B + of Hodge type),

then Serre's argument[1] on Kählerian version of Weil's conjecture works verbatim in positive characteristic for all polarized endomorphisms f (i.e., $f^*H_X \sim qH_X$ for an ample H_X).

So, conjecturally, $\chi_i(f) = q^{i/2}$, independent of ι.

On the other hand, it is easy to see that $\lambda_k(f) = q^k$ by definition.

Remark

In fact, need Standard conjecture of Hodge type for $X \times X$. So, even for surfaces, this argument has not been completely worked out. The case of Abelian 4-folds solved recently by Ancona.

[1] in his letter to Weil in 1959, the starting point of motive theory.

Generalized Weil's Riemann hypothesis

Positive characteristic analogous conjecture of Serre's result:

Conjecture (Generalized Weil's Riemann hypothesis)

Let X be a smooth projective variety of dimension n over $\mathbf{k}$.

Let f be a polarized endomorphism of X, i.e., $f^*H_X \sim qH_X$ for an ample divisor H_X and a positive integer $q \in \mathbf{Z}_{>0}$.

Then for any $0 \leqslant i \leqslant 2n$, the eigenvalues of $f^*|H^i_{\acute{e}t}(X, \mathbf{Q}_\ell)$ are q-Weil numbers of weight i, i.e., algebraic numbers α such that $|\sigma(\alpha)| = q^{i/2}$ for every embedding $\sigma\colon \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}$.

The case of Abelian varieties is known due to Weil (see Mumford's book).

The case of Frobenius endomorphism, i.e., WRH, is known due to Deligne.

Other than these, Katz said "I don't think anything is known...".

Correspondences

Let X, Y be smooth projective varieties of dimension n over an algebraically closed field $\mathbf{k}$ of arbitrary characteristic.

Let $f\colon X \vdash Y$ be a correspondence from X to Y, i.e., a rational algebraic cycle of codimension n, or its equivalence class, on $X \times Y$. Namely, $f \in \mathsf{Z}^n(X \times Y)_{\mathbf{Q}}/\sim$ for some adequate equivalence relation $\sim$.

Let $H^\bullet(X)$ be a Weil cohomology theory with a coefficient field $\mathbf{F}$ of characteristic 0. In particular, we have a cup product

$$\cup\colon H^i(X) \times H^{2n-i}(X) \to H^{2n}(X) \xrightarrow{\sim} \mathbf{F},$$

Poincaré duality, Künneth formula, projection formula, cycle class maps $\mathrm{cl}_X\colon \mathsf{Z}^k(X) \to H^{2k}(X)$, Weak/Hard Lefschetz theorem, etc...

Dynamical correspondences

Definition (Dynamical correspondences)

A correspondence $f \in \mathsf{Z}^n(X \times Y)_{\mathbf{Q}}$ is dominant, if for each irreducible component f_i of f, the natural restriction maps $\mathrm{pr}_j\,|_{f_i}$ induced from the projections $\mathrm{pr}_j \colon X \times Y \to X$ or Y are both surjective for $j = 1, 2$.

An effective and dominant correspondence $f \in \mathsf{Z}^n(X \times Y)_{\mathbf{Q}}$ is called a dynamical correspondence from X to Y.

Remark

Sums of graphs of surjective morphisms, or dominant rational maps.

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Remark

Sums of graphs of surjective morphisms, or dominant rational maps.

We have natural pullback and pushforward actions of correspondences on numerical groups and cohomology groups.

We can compose dynamical correspondences just like how we compose dominant rational maps, denoted by $g \diamond f$.

Caveat

Let $f^{\Diamond m}$ denote the m-th dynamical iterate of a dynamical correspondence f.

In general, like the composition of dominant rational maps, $(f^{\Diamond m})^* \neq (f^*)^m$. Consider $\iota\colon [x:y:z] \mapsto [yz:xz:xy]$.

The non-functorial nature of dynamical composition makes the computation of dynamical degrees very hard. For instance, are dynamical degrees algebraic numbers? A counterexample of a dominant rational self-map of $\mathbf{P}^2$ found quite recently (Bell–Diller–Jonsson, 2020).

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Nonetheless, if f is a morphism, (or rather, the graph of a morphism), then $(f^{\Diamond m})^* = (f^*)^m$ and hence dynamical degrees become spectral radii of certain linear operators.

Main Results

Equivalence of entropies

Theorem (Truong, 2016)

Let f be a surjective self-morphism of a smooth projective variety X of dimension n over $\mathbf{k}$. Then

$$\rho(f^*|H^\bullet_{ét}(X, \mathbf{Q}_\ell))_\iota = \rho(f^*|\mathsf{N}^\bullet(X)_\mathbf{R}).$$

In other words,

$$\max_{0 \leqslant i \leqslant 2n} \chi_i(f)_\iota = \max_{0 \leqslant k \leqslant n} \lambda_k(f).$$

As a consequence,

$$\rho(f^*|H^\bullet_{\text{ét}}(X, \mathbf{Q}_\ell))_\iota = \rho(f^*|H^{2\bullet}_{\text{ét}}(X, \mathbf{Q}_\ell))_\iota.$$

This was reproved by Kadattur Vasudevan using dynamical zeta functions.

Abelian variety case

Theorem (H., 2019)

Let f be a surjective self-morphism of an Abelian variety X of dimension n defined over $\mathbf{k}$. Then for any $0 \leqslant k \leqslant n$,

$$\rho(f^*|H^{2k}_{\acute{e}t}(X, \mathbf{Q}_\ell)) = \rho(f^*|\mathsf{N}^k(X)_\mathbf{R}).$$

Namely, DDC holds on Abelian varieties for surjective self-morphisms.

$$\chi_{2k}(f) = \lambda_k(f).$$

Remark

This answers a question of Esnault raised in the AIM workshop "Cohomological methods in Abelian varieties".

Idea of Proof

The endomorphism $\mathbf{Q}$-algebra $\mathrm{End}^0(X)$ is semisimple $\Longrightarrow$ Eigenvalues of $f^*|H^1_{\text{ét}}(X, \mathbf{Q}_\ell)$ consist of n pairs:

$$\pi_1, \dots, \pi_n, \overline{\pi}_1, \dots, \overline{\pi}_n \in \mathbf{C}.$$

We may assume that

$$|\pi_1| \geqslant \cdots \geqslant |\pi_n| > 0.$$

Since $H^{2k}_{\text{ét}}(X, \mathbf{Q}_\ell) = \bigwedge^{2k} H^1_{\text{ét}}(X, \mathbf{Q}_\ell)$ for Abelian varieties,

$$\chi_{2k}(f) = \rho(f^*|H^{2k}_{\text{ét}}(X, \mathbf{Q}_\ell)) = \prod_{i=1}^{k} |\pi_i|^2 . \tag{2.1}$$

We call $\prod_{i=1}^{n}(t - \pi_i) \in \mathbf{C}[t]$ an Albert polynomial $P_f^{\mathrm{A}}(t)$ of f. It satisfies

$$P_f(t) = P_f^{\mathrm{A}}(t) \cdot \overline{P_f^{\mathrm{A}}(t)}.$$

It's a positive analog of the characteristic polynomial of the analytic representation of an endomorphism of a complex Abelian variety.

Idea of Proof

Recall

$$\lambda_k(f) = \lim_{m\to\infty} \Big((f^m)^* H_X^k \cdot H_X^{n-k} \Big)^{1/m}. \qquad (2.2)$$

Let $f^\dagger \colon \phi^{-1} \circ \widehat{f} \circ \phi$ be the Rosati involution of f, where $\phi \colon X \to \widehat{X}$ is a fixed polarization associated to the ample divisor H_X. Then the Albert polynomial $P^{\mathrm{A}}_{f^\dagger \circ f}(t)$ of $f^\dagger \circ f$ is unique satisfying that if we write

$$P^{\mathrm{A}}_{f^\dagger \circ f}(t) = \sum_{k=0}^{n} (-1)^k c_k\, t^{n-k}, \qquad (2.3)$$

then for any k,

$$c_k = \binom{n}{k} \frac{f^* H_X^k \cdot H_X^{n-k}}{H_X^n}. \qquad (2.4)$$

End of the proof

Applying the above to f^m yields that

$$\binom{n}{k}\frac{(f^m)^* H_X^k \cdot H_X^{n-k}}{H_X^n} = e_k(\sigma_1(f^m)^2, \ldots, \sigma_n(f^m)^2), \tag{2.5}$$

where e_k is the k-th elementary symmetric polynomial and the σ_i are singular values. Now,

$$\begin{aligned}
\lambda_k(f) &= \lim_{m\to\infty}\Big(e_k(\sigma_1(f^m)^2, \ldots, \sigma_n(f^m)^2)\Big)^{1/m} && \leftsquigarrow (2.2) + (2.5)\\
&= \max_{1\leqslant i_1 < \ldots < i_k \leqslant n} \lim_{m\to\infty} \sigma_{i_1}(f^m)^{2/m} \cdots \lim_{m\to\infty} \sigma_{i_k}(f^m)^{2/m} \\
&= \max_{1\leqslant i_1 < \ldots < i_k \leqslant n} |\pi_{i_1}|^2 \cdot |\pi_{i_2}|^2 \cdots |\pi_{i_k}|^2 && \leftsquigarrow \text{(linear algebra)}\\
&= |\pi_1|^2 \cdot |\pi_2|^2 \cdots |\pi_k|^2 \\
&= \chi_{2k}(f). && \leftsquigarrow (2.1)
\end{aligned}$$

Homological correspondence γ_r

For any fixed $r \in \mathbf{Q}_{>0}$. There always exists a homological correspondence γ_r of X, i.e.,

$$\gamma_r \in H^{2n}(X \times X) = \bigoplus_{i=0}^{2n} H^i(X) \otimes H^{2n-i}(X) \simeq \bigoplus_{i=0}^{2n} \mathrm{End}_{\mathbf{Q}_\ell}(H^i(X)),$$

so that γ_r^* on $H^i(X)$ is the multiplication-by-r^i map for each i.

If Standard conjecture C holds, then γ_r can be represented by a correspondence $\sum_{i=0}^{2n} r^i \Delta_i$, where $\Delta_i \in \mathsf{Z}^n(X \times X)_{\mathbf{Q}}$ corresponds the i-th Künneth component π_i of the diagonal class $\mathrm{cl}_{X \times X}(\Delta_X)$.

A quantitative version of Standard conjecture C

Conjecture G_r

For any $r \in \mathbf{Q}_{>0}$, the above homological correspondence γ_r of X is algebraic and represented by a rational algebraic n-cycle G_r on $X \times X$, i.e., $\gamma_r = \mathrm{cl}_{X\times X}(G_r)$; moreover, there exists a constant $C > 0$ independent of r, so that for any effective correspondence f of X, we have

$$\|G_r \circ f\| \leqslant C \deg(G_r \circ f),$$

where $\|G_r \circ f\|$ denotes any norm of $G_r \circ f$ as an element in $\mathsf{N}^n(X \times X)_{\mathbf{R}}$ and $\deg(g) := g \cdot (\mathrm{pr}_1^* H_X + \mathrm{pr}_2^* H_X)^n$ for any correspondence $g \in \mathsf{Z}^n(X \times X)_{\mathbf{Q}}$.

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Define $\deg_k(g) := g \cdot \mathrm{pr}_1^* H_X^{n-k} \cdot \mathrm{pr}_2^* H_X^k = g^* H_X^k \cdot H_X^{n-k}$.

Conjecture G_r should be weaker than Standard conjectures

Theorem (H.–Truong, 2021)

Conjecture G_r holds on Abelian varieties.

With Truong, we recently also verify that for polarized endomorphisms:

Standard conjectures $\Rightarrow$ Conjecture G_r $\Rightarrow$ Generalized Weil's Riemann hypothesis

So proving Conjecture G_r could be an alternative way to approach generalized Weil's Riemann hypothesis, DDC, etc...

We suspect that this is true for more general correspondences.

An application to Kummer surfaces

Besides the Abelian varieties case, we can deal with Kummer surfaces.

Theorem (H.–Truong, 2021)

Let $A \dashrightarrow S$ be a dominant rational map from an Abelian surface A to a smooth projective surface S over $\mathbf{k}$. Then the following assertions hold.

(1) *There is a constant $C > 0$ such that for any dynamical correspondence f of S, we have*

$$\left|\mathrm{Tr}(f^*|H^{2k}_{\acute{e}t}(S))\right| \leqslant C \deg_k(f).$$

(2) *Generalized Weil's Riemann hypothesis holds on S.*

Idea of the proof

Consider the following diagram (by resolution of singularities):

$$
\begin{array}{ccccc}
A & \xleftarrow{\ \tau\ } & \widetilde{A} & \xrightarrow{\ \pi\ } & S \\
{\scriptstyle h}\big| & & {\scriptstyle g}\big| & & \big|{\scriptstyle f} \\
A & \xleftarrow{\ \tau\ } & \widetilde{A} & \xrightarrow{\ \pi\ } & S.
\end{array}
$$

Let $g := (\pi \times \pi)^{\star}(f)$ be the dynamical pullback of f under $\pi \times \pi$ and $h := (\tau \times \tau)_{*}(g)$ the proper pushforward of g under $\tau \times \tau$.

Trace formula asserts that

$$d^2 \left| \mathrm{Tr}(f^* | H^i(S)) \right| = d^2 \left| f \cdot \Delta_{4-i} \right| = \left| (\pi \times \pi)_*(g) \cdot \Delta_{4-i} \right| = \left| g \cdot (\pi \times \pi)^* \Delta_{4-i} \right|.$$

Note that g coincides with the dynamical pullback of h under $\tau \times \tau$ but the pullback $(\tau \times \tau)^*(h)$ may contain some exceptional classes, denoted by e.

Idea of the proof

It follows that

$$\begin{aligned} d^2 \left| \mathrm{Tr}(f^*|H^i(S))\right| &= \left|((\tau\times\tau)^*(h) - e)\cdot(\pi\times\pi)^*\Delta_{4-i}\right| \\ &\leqslant \left|(\tau\times\tau)^*(h)\cdot(\pi\times\pi)^*\Delta_{4-i}\right| + \left|e\cdot(\pi\times\pi)^*\Delta_{4-i}\right| \\ &= \left|h\cdot(\tau\times\tau)_*(\pi\times\pi)^*\Delta_{4-i}\right| + \left|e\cdot(\pi\times\pi)^*\Delta_{4-i}\right|. \end{aligned}$$

Using the fact that Conjecture G_r holds on Abelian varieties

$$\begin{aligned} r^i \left|h\cdot(\tau\times\tau)_*(\pi\times\pi)^*\Delta_{4-i}\right| &= \left|(G_r\circ h)\cdot(\tau\times\tau)_*(\pi\times\pi)^*\Delta_{4-i}\right| \\ \lesssim \left\|G_r\circ h\right\| \lesssim \deg(G_r\circ h) &\sim \max_{0\leqslant j\leqslant 2} r^{2j}\deg_j(h) \sim \max_{0\leqslant j\leqslant 2} r^{2j} d^2 \deg_j(f). \end{aligned}$$

When $i = 2$, the second term can be bounded above by

$$\deg_1(e) \leqslant \deg_1((\tau\times\tau)^*(h)) \lesssim \deg_1(h) \sim d^2\deg_1(f).$$

Thank You!

Pullback/pushforward of correspondences

Pullback and pushforward actions of correspondences on cycle class groups and cohomology groups:

$$f^*\colon \mathsf{Z}^k(Y)_{\mathbf{Q}}/\!\sim\,\longrightarrow \mathsf{Z}^k(X)_{\mathbf{Q}}/\!\sim, \quad \beta \mapsto \mathrm{pr}_{1,*}(f\cdot \mathrm{pr}_2^*\,\beta),$$
$$f_*\colon \mathsf{Z}^k(X)_{\mathbf{Q}}/\!\sim\,\longrightarrow \mathsf{Z}^k(Y)_{\mathbf{Q}}/\!\sim, \quad \alpha \mapsto \mathrm{pr}_{2,*}(f\cdot \mathrm{pr}_1^*\,\alpha),$$

where the pr_i denote the natural projections from $X\times Y$ to X and Y, respectively. Similarly, if $\sim$ is an equivalence relation finer than, or equal to, homological equivalence relation $\sim_{\mathrm{hom}}$, we can define natural pullback f^* and pushforward f_* on cohomology groups $H^i(X)$ as follows:

$$f^*\colon H^i(Y)\longrightarrow H^i(X), \quad \beta\mapsto \mathrm{pr}_{1,*}(\mathrm{cl}_{X\times Y}(f)\cup \mathrm{pr}_2^*\,\beta),$$
$$f_*\colon H^i(X)\longrightarrow H^i(Y), \quad \alpha\mapsto \mathrm{pr}_{2,*}(\mathrm{cl}_{X\times Y}(f)\cup \mathrm{pr}_1^*\,\alpha),$$

where $\mathrm{cl}_{X\times Y}\colon \mathsf{Z}^k(X\times Y)_{\mathbf{Q}}\longrightarrow H^{2k}(X\times Y)$ is the cycle class map, which factors through $\mathsf{Z}^k(X\times Y)_{\mathbf{Q}}/\!\sim$ by assumption.

Dynamical composition of dynamical correspondences

Let $f \in \mathsf{Z}^n(X \times Y)_{\mathbf{Q}}$ and $g \in \mathsf{Z}^n(Y \times Z)_{\mathbf{Q}}$ be two irreducible dynamical correspondences.

By generic flatness, there are nonempty Zariski open subsets U_X, U_Y of X and Y such that over which the projections $\mathrm{pr}_1|_f \colon f \to X$ and $\mathrm{pr}_1|_g \colon g \to Y$ are flat, hence have finite fibers. By shrinking U_X, we may assume that the strict image $f(U_X) := \mathrm{pr}_2(f \cap \mathrm{pr}_1^{-1}(U_X))$ of U_X under f is contained in U_Y. It follows that for any point $x \in U_X$, the strict image $f(x)$ of x under f is a finite subset of U_Y, whose strict image under g is still finite.

The dynamical composition $g \lozenge f$ is then defined as the closure of the graph $\{(x, g(f(x))) : x \in U_X\}$ in $X \times Z$. Hence $g \lozenge f \in \mathsf{Z}^n(X \times Z)_{\mathbf{Q}}$.

Geometrically, $g \lozenge f$ is the same as $\mathrm{pr}_{13}((f \times Z) \cap (X \times g))$ with components of dimension greater than n and components whose projections to the factors of $X \times Z$ are not surjective removed.

(Motivic) composition of correspondences

Correspondences can be naturally composed in intersection theory.

More precisely, given two arbitrary correspondences $f\colon X \vdash Y$ and $g\colon Y \vdash Z$, the composite correspondence, denoted by $g \circ f$, is defined by

$$g \circ f := \mathrm{pr}_{13,*}(\mathrm{pr}_{12}^{*} f \cdot \mathrm{pr}_{23}^{*} g) \in \mathsf{Z}^n(X \times Z)_{\mathbf{Q}}/\sim, \qquad (2.6)$$

where the pr_{ij} denote the natural projections from $X \times Y \times Z$ to the appropriate factors, respectively.

(1) $(g \circ f)_* = g_* \circ f_*$ and $(g \circ f)^* = f^* \circ g^*$.

(2) $(f^{\mathsf{T}})_* = f^*$ and $(f^{\mathsf{T}})^* = f_*$.

(3) If f is the graph Γ_π of a flat morphism $\pi\colon X \to Y$, then $f_* = \pi_*$ and $f^* = \pi^*$.

Standard conjecture of Hodge type

For any $0 \leqslant k \leqslant n/2$, denote by

$$\mathsf{A}^k_{\mathrm{prim}}(X) := \mathsf{A}^k(X) \cap P^{2k}(X) = \{\alpha \in \mathsf{A}^k(X) : L^{n-2k+1}(\alpha) = 0\}$$

the set of primitive classes in $H^{2k}(X)$ generated by algebraic cycles of codimension k. Define a symmetric bilinear form on $\mathsf{A}^k_{\mathrm{prim}}(X)$ as follows:

$$\begin{array}{ccc} \mathsf{A}^k_{\mathrm{prim}}(X) \times \mathsf{A}^k_{\mathrm{prim}}(X) & \longrightarrow & \mathbf{Q} \\ (\alpha, \beta) & \mapsto & (-1)^k L^{n-2k}(\alpha) \cup \beta. \end{array} \tag{2.7}$$

Standard conjecture of Hodge type

The above bilinear form is positive definite whenever $k \leqslant n/2$.

Asymptotic behavior of roots of singular values

Schur decomposition + Cauchy interlacing theorem

Lemma 2

Let $\mathbf{A} \in \mathrm{M}_n(\mathbf{C})$, whose eigenvalues are $\pi_1, \ldots, \pi_n \in \mathbf{C}$ so that $|\pi_1| \geqslant \cdots \geqslant |\pi_n|$. For each $m \in \mathbf{N}$, let $\sigma_1(\mathbf{A}^m) \geqslant \cdots \geqslant \sigma_n(\mathbf{A}^m)$ denote the singular values of $\mathbf{A}^m$. Then for any $1 \leqslant i \leqslant n$,

$$\lim_{m \to \infty} \sigma_i(\mathbf{A}^m)^{1/m} = |\pi_i|.$$