Numerical characterization of torus quotients

Benoît Claudon
(joint work with P. Graf and H. Guenancia)

Université de Rennes 1

May 17 2022

ZAG Seminar

Plan of the talk

Étale quotient of complex tori I

Let T be a complex torus and $G \curvearrowright T$ be a finite group acting on T. We assume first that G acts *freely*.

Étale quotient of complex tori I

Let T be a complex torus and $G \curvearrowright T$ be a finite group acting on T. We assume first that G acts *freely*.

Chern classes

Let $X := T/G$ be the quotient manifold (compact and Kähler). We clearly have:

$$c_i(X) = 0 \in H^{2i}(X, \mathbb{R}), \quad \forall\, i \geq 1.$$

Étale quotient of complex tori II

Let X be a compact Kähler manifold ($n := \dim X$).

Étale quotient of complex tori II

Let X be a compact Kähler manifold ($n := \dim X$).

- if $c_1(X) = 0 \in H^2(X, \mathbb{R})$, then there exists ω Kähler metric such that $\mathrm{Ric}(\omega) = 0$.

Étale quotient of complex tori II

Let X be a compact Kähler manifold ($n := \dim X$).

- if $c_1(X) = 0 \in H^2(X, \mathbb{R})$, then there exists ω Kähler metric such that $\mathrm{Ric}(\omega) = 0$.
- Using the Kähler–Einstein condition for ω:

$$\int_X \left(2nc_2(X,\omega) - (n-1)c_1(X,\omega)^2\right) \wedge \omega^{n-2} \asymp \int_X \|\Theta^\circ(T_X,\omega)\|^2 \geq 0.$$

Étale quotient of complex tori II

Let X be a compact Kähler manifold ($n := \dim X$).

- if $c_1(X) = 0 \in H^2(X, \mathbb{R})$, then there exists ω Kähler metric such that $\mathrm{Ric}(\omega) = 0$.
- Using the Kähler–Einstein condition for ω:

$$\int_X \Big(2nc_2(X,\omega) - (n-1)c_1(X,\omega)^2\Big) \wedge \omega^{n-2} \asymp \int_X \|\Theta^\circ(T_X,\omega)\|^2 \geq 0.$$

Conclusion

If $c_1(X) = 0$ and $c_2(X) \wedge [\omega]^2 = 0$ then $X = T/G$ with T and G as above.

What about the singular case?

If we drop the assumption that G acts freely on T, we are led to consider singular spaces.

Difficulties?

1. Identify a "natural" class of singularities.

What about the singular case?

If we drop the assumption that G acts freely on T, we are led to consider singular spaces.

Difficulties?

1. Identify a "natural" class of singularities. klt!

What about the singular case?

If we drop the assumption that G acts freely on T, we are led to consider singular spaces.

Difficulties?

1. Identify a "natural" class of singularities. klt!
2. Kähler metrics in the singular setting.

What about the singular case?

If we drop the assumption that G acts freely on T, we are led to consider singular spaces.

Difficulties?

1. Identify a “natural” class of singularities. klt!
2. Kähler metrics in the singular setting. Done by Grauert (60’s)!

What about the singular case?

If we drop the assumption that G acts freely on T, we are led to consider singular spaces.

Difficulties?

1. Identify a "natural" class of singularities. klt!
2. Kähler metrics in the singular setting. Done by Grauert (60's)!
3. Define Chern classes/numbers for singular spaces...

What about the singular case?

If we drop the assumption that G acts freely on T, we are led to consider singular spaces.

Difficulties?

1. Identify a "natural" class of singularities. klt!
2. Kähler metrics in the singular setting. Done by Grauert (60's)!
3. Define Chern classes/numbers for singular spaces. . . several definitions giving different theories (McPherson, Baum–Fulton–McPherson, Schwartz. . .)

For torsion–free sheaves I

Definition

Let X be a normal compact complex space, $\mathcal{E}$ be a torsion-free sheaf and $f : \hat{X} \to X$ be a resolution such that $f^\sharp\mathcal{E} := f^*\mathcal{E}/\mathrm{Tor}(f^*\mathcal{E})$ is locally free. We define:

$$c_i(\mathcal{E}) \cdot a := c_i(f^\sharp\mathcal{E}) \cdot f^*(a), \quad \forall\, a \in H^{2n-2i}(X, \mathbb{R}).$$

For torsion–free sheaves I

Definition

Let X be a normal compact complex space, $\mathcal{E}$ be a torsion-free sheaf and $f : \hat{X} \to X$ be a resolution such that $f^\sharp \mathcal{E} := f^*\mathcal{E}/\mathrm{Tor}(f^*\mathcal{E})$ is locally free. We define:

$$c_i(\mathcal{E}) \cdot a := c_i(f^\sharp \mathcal{E}) \cdot f^*(a), \quad \forall\, a \in H^{2n-2i}(X, \mathbb{R}).$$

Same for homogeneous polynomials in the Chern classes; for instance:

$$c_1^2(\mathcal{E}) \cdot a := c_1(f^\sharp \mathcal{E})^2 \cdot f^*(a), \quad \forall\, a \in H^{2n-4}(X, \mathbb{R}).$$

For torsion-free sheaves II

Properties

- Well-defined: does not depend on the choice of f.

For torsion-free sheaves II

Properties

- Well-defined: does not depend on the choice of f.
- Chern classes are defined as linear forms on the cohomology: $c_i(\mathcal{E}) \in H_{2n-2i}(X, \mathbb{R})$ are thus homology classes!

For torsion-free sheaves II

Properties

- Well-defined: does not depend on the choice of f.
- Chern classes are defined as linear forms on the cohomology: $c_i(\mathcal{E}) \in H_{2n-2i}(X, \mathbb{R})$ are thus homology classes!

Warning!

If $c_1(\mathcal{E})$ is defined as a cohomology class (*e.g.* if $\det \mathcal{E}$ is $\mathbb{Q}$-Cartier), we can define $c_1(\mathcal{E})^2 \in H^4(X, \mathbb{R})$ but we have to be very careful. In general:

$$c_1^2(\mathcal{E}) \neq c_1(\mathcal{E})^2 \quad \text{in } H_{2n-4}(X, \mathbb{R})\ldots$$

Compatibility

Example

Let $\mathcal{E} := \mathcal{I}_x$ be the ideal of a point x in a surface X and $f : \hat{X} \to X$ be the blow-up of x with exceptional curve E: $f^*(\mathcal{I}_x) = \mathcal{O}_{\hat{X}}(-E)$ and

$$c_1(\mathcal{E}) = 0 \quad \text{but} \quad c_1^2(\mathcal{E}) = c_1(\mathcal{O}_{\hat{X}}(-E))^2 = -1.$$

Compatibility

Example

Let $\mathcal{E} := \mathcal{I}_x$ be the ideal of a point x in a surface X and $f : \hat{X} \to X$ be the blow-up of x with exceptional curve E: $f^*(\mathcal{I}_x) = \mathcal{O}_{\hat{X}}(-E)$ and

$$c_1(\mathcal{E}) = 0 \quad \text{but} \quad c_1^2(\mathcal{E}) = c_1(\mathcal{O}_{\hat{X}}(-E))^2 = -1.$$

Compatibility 1

If $\mathcal{E}$ is locally free in codimension 2 (*e.g.* $\mathcal{E}$ reflexive), $\det \mathcal{E}$ is $\mathbb{Q}$-Cartier and X is smooth in codimension 2, we then have:

$$c_1^k(\mathcal{E}) = c_1(\det \mathcal{E})^k \quad \text{for } k = 1, 2.$$

For spaces I

Proposition/Definition

Let X be a normal complex space and assume that there exists a resolution $f : Y \to X$ that is *minimal in codimension 2*. Then the quantity

$$c_2(X) \cdot a := c_2(Y) \cdot f^*(a) \quad \text{for } a \in H^{2n-4}(X, \mathbb{R})$$

is independent of the chosen resolution f.

For spaces I

Proposition/Definition

Let X be a normal complex space and assume that there exists a resolution $f : Y \to X$ that is *minimal in codimension 2*. Then the quantity

$$c_2(X) \cdot a := c_2(Y) \cdot f^*(a) \quad \text{for } a \in H^{2n-4}(X, \mathbb{R})$$

is independent of the chosen resolution f.

- Proven by Graf and Kirschner (2020).

For spaces I

Proposition/Definition

Let X be a normal complex space and assume that there exists a resolution $f : Y \to X$ that is *minimal in codimension 2*. Then the quantity

$$c_2(X) \cdot a := c_2(Y) \cdot f^*(a) \quad \text{for } a \in H^{2n-4}(X, \mathbb{R})$$

is independent of the chosen resolution f.

- Proven by Graf and Kirschner (2020).
- It applies when X has klt singularities: in that case, there exists $Z \subset X$ with $\operatorname{codim}_X(Z) \geq 3$ such that

$$X \setminus Z \overset{\text{loc}}{\simeq} \left(\mathbb{C}^2/G\right) \times \mathbb{C}^{n-2} \quad (G < \mathrm{GL}_2(\mathbb{C}) \text{ a finite group}).$$

For spaces II

Compatibility 2

If X is smooth in codimension 2, we have:

$$c_2(X) = c_2(T_X) \quad \text{in } H_{2n-4}(X, \mathbb{R}).$$

For spaces II

Compatibility 2

If X is smooth in codimension 2, we have:

$$c_2(X) = c_2(T_X) \quad \text{in } H_{2n-4}(X, \mathbb{R}).$$

Kummer surface (as always)

Let $X := T/\pm 1$ be a Kummer surface and $f : \hat{X} \to X$ be its minimal resolution (E is the disjoint union of 16 (-2)-curves) with $\hat{X}$ a K3 surface. A direct computation gives $f^* T_X = T_{\hat{X}}(-\log E)$

For spaces II

Compatibility 2

If X is smooth in codimension 2, we have:

$$c_2(X) = c_2(T_X) \quad \text{in } H_{2n-4}(X, \mathbb{R}).$$

Kummer surface (as always)

Let $X := T/\pm 1$ be a Kummer surface and $f : \hat{X} \to X$ be its minimal resolution (E is the disjoint union of 16 (-2)-curves) with $\hat{X}$ a K3 surface. A direct computation gives $f^* T_X = T_{\hat{X}}(-\log E)$

- $c_2(X) = c_2(\hat{X}) = 24.$

For spaces II

Compatibility 2

If X is smooth in codimension 2, we have:

$$c_2(X) = c_2(T_X) \quad \text{in } H_{2n-4}(X, \mathbb{R}).$$

Kummer surface (as always)

Let $X := T/\pm 1$ be a Kummer surface and $f : \hat{X} \to X$ be its minimal resolution (E is the disjoint union of 16 (-2)-curves) with $\hat{X}$ a K3 surface. A direct computation gives $f^* T_X = T_{\hat{X}}(-\log E)$

- $c_2(X) = c_2(\hat{X}) = 24$.
- $c_2(T_X) = c_2(f^* T_X) = c_2(T_{\hat{X}}(-\log E)) = -8 \neq 24$.

For spaces II

Compatibility 2

If X is smooth in codimension 2, we have:

$$c_2(X) = c_2(T_X) \quad \text{in } H_{2n-4}(X, \mathbb{R}).$$

Kummer surface (as always)

Let $X := T/\pm 1$ be a Kummer surface and $f : \hat{X} \to X$ be its minimal resolution (E is the disjoint union of 16 (-2)-curves) with $\hat{X}$ a K3 surface. A direct computation gives $f^* T_X = T_{\hat{X}}(-\log E)$

- $c_2(X) = c_2(\hat{X}) = 24$.
- $c_2(T_X) = c_2(f^* T_X) = c_2(T_{\hat{X}}(-\log E)) = -8 \neq 24$.
- $c_1(T_X) = 0$ but $c_1^2(T_X) = -32$.

Statements (past and present)

Theorem (Greb–Kebekus–Peternell, 2016)

Let X be a projective klt variety and assume that $K_X \equiv 0$ and $c_2(X) \cdot [H]^{n-2} = 0$ for an ample class H. Then there exists an Abelian variety A and a finite group G acting freely in codimension 2 on A such that $X \simeq A/G$.

Statements (past and present)

Theorem (Greb–Kebekus–Peternell, 2016)

Let X be a projective klt variety and assume that $K_X \equiv 0$ and $c_2(X) \cdot [H]^{n-2} = 0$ for an ample class H. Then there exists an Abelian variety A and a finite group G acting freely in codimension 2 on A such that $X \simeq A/G$.

Theorem (C–Graf–Guenancia, 2021)

Let X be a normal compact Kähler klt space and assume that $c_1(X) = 0$ and $c_2(X) \cdot \alpha^{n-2} = 0$ for a Kähler class α. Then there exists a complex torus T and a finite group G acting freely in codimension 2 on T such that $X \simeq T/G$.

The algebraic setting I

- Consider $S := H_1 \cap \cdots \cap H_{n-2}$ (with $H_i \in |mH|$ for $m \gg 1$): this is a *smooth* surface $S \subset X_{\mathrm{reg}}$.

The algebraic setting I

- Consider $S := H_1 \cap \cdots \cap H_{n-2}$ (with $H_i \in |mH|$ for $m \gg 1$): this is a *smooth* surface $S \subset X_{\mathrm{reg}}$.
- $\mathcal{E} := (T_X)_{|S}$ is semi-stable wrt $H_{|S}$, $c_1(\mathcal{E}) = 0$ and $c_2(\mathcal{E}) = 0$. (Simpson, 1992) $\Rightarrow \mathcal{E}$ is flat, given by $\rho : \pi_1(S) \to \mathrm{GL}_n(\mathbb{C})$.

The algebraic setting I

- Consider $S := H_1 \cap \cdots \cap H_{n-2}$ (with $H_i \in |mH|$ for $m \gg 1$): this is a *smooth* surface $S \subset X_{\mathrm{reg}}$.
- $\mathcal{E} := (T_X)_{|S}$ is semi-stable wrt $H_{|S}$, $c_1(\mathcal{E}) = 0$ and $c_2(\mathcal{E}) = 0$. (Simpson, 1992) $\Rightarrow \mathcal{E}$ is flat, given by $\rho : \pi_1(S) \to \mathrm{GL}_n(\mathbb{C})$.
- Lefschetz Theorem (Hamm–Lê, 1985):

$$\pi_1(S) \xrightarrow{\sim} \pi_1(X_{\mathrm{reg}}).$$

The algebraic setting II

- Up to replacing X with a quasi-étale cover[1]:

$$\begin{tikzcd} \pi_1(X_{\mathrm{reg}}) \arrow[dr, "i_*"'] & \pi_1(S) \arrow[l, "\sim"'] \arrow[r, "\rho"] & \mathrm{GL}_n(\mathbb{C}) \\ & \pi_1(X) \arrow[ur, dashed] & \end{tikzcd}$$

[1] A so-called *maximally quasi-étale* cover (GKP, 2016).

The algebraic setting II

- Up to replacing X with a quasi-étale cover[1]:

$$\begin{tikzcd} \pi_1(X_{\mathrm{reg}}) \arrow[dr, "i_*"'] & \pi_1(S) \arrow[l, "\sim"'] \arrow[r, "\rho"] & \mathrm{GL}_n(\mathbb{C}) \\ & \pi_1(X) \arrow[ur, dashed] & \end{tikzcd}$$

- Bertini type arguments $\Rightarrow$ T_X is flat, in particular locally free.

[1] A so-called *maximally quasi-étale* cover (GKP, 2016).

The algebraic setting II

- Up to replacing X with a quasi-étale cover[1]:

$$
\begin{tikzcd}
\pi_1(X_{\mathrm{reg}}) \arrow[dr, "i_*"'] & \pi_1(S) \arrow[l, "\sim"'] \arrow[r, "\rho"] & \mathrm{GL}_n(\mathbb{C}) \\
& \pi_1(X) \arrow[ur, dashed] &
\end{tikzcd}
$$

- Bertini type arguments $\Rightarrow$ T_X is flat, in particular locally free.
- klt$+T_X$ locally free $\Rightarrow$ X is smooth (Zariski–Lipman conjecture for klt spaces, GKKP11).

[1] A so-called *maximally quasi-étale* cover (GKP, 2016).

The analytic setting

When X is merely a compact Kähler klt space. . . we are stuck.

[2] Locally trivial on the total space: $\mathcal{X}$ is locally isomorphic to $X \times \mathbb{D}$.

The analytic setting

When X is merely a compact Kähler klt space... we are stuck.

Naive idea

Find $\pi : \mathcal{X} \to \mathbb{D}$ a locally trivial[2] family over a smooth base $(\mathbb{D}, 0)$ such that $\mathcal{X}_0 := \pi^{-1}(0) \simeq X$ and $\mathcal{X}_t$ is projective for $t \in \mathbb{D}$.

[2] Locally trivial on the total space: $\mathcal{X}$ is locally isomorphic to $X \times \mathbb{D}$.

The analytic setting

When X is merely a compact Kähler klt space. . . we are stuck.

Naive idea

Find $\pi : \mathcal{X} \to \mathbb{D}$ a locally trivial[2] family over a smooth base $(\mathbb{D}, 0)$ such that $\mathcal{X}_0 := \pi^{-1}(0) \simeq X$ and $\mathcal{X}_t$ is projective for $t \in \mathbb{D}$.

Too naive?

- It is not clear that $c_2(\mathcal{X}_t) \cdot \alpha_t = 0$ for some *Kähler* class α_t on $\mathcal{X}_t$.

[2] Locally trivial on the total space: $\mathcal{X}$ is locally isomorphic to $X \times \mathbb{D}$.

The analytic setting

When X is merely a compact Kähler klt space... we are stuck.

Naive idea

Find $\pi : \mathcal{X} \to \mathbb{D}$ a locally trivial[2] family over a smooth base $(\mathbb{D}, 0)$ such that $\mathcal{X}_0 := \pi^{-1}(0) \simeq X$ and $\mathcal{X}_t$ is projective for $t \in \mathbb{D}$.

Too naive?

- It is not clear that $c_2(\mathcal{X}_t) \cdot \alpha_t = 0$ for some *Kähler* class α_t on $\mathcal{X}_t$.
- It would be the case if $c_2(X) \equiv 0$ as a linear form on $H^{2n-4}(X, \mathbb{R})$ (a family π as above is topologically trivial).

[2] Locally trivial on the total space: $\mathcal{X}$ is locally isomorphic to $X \times \mathbb{D}$.

Semi-positivity of c_2 I

Bogomolov–Gieseker inequality, singular case

Let X be a normal compact Kähler space, α be a Kähler class and $\mathcal{E}$ be a rank r reflexive sheaf on X that is stable wrt α. Then:

$$\Delta(\mathcal{E}) \cdot \alpha^{n-2} := \left(2rc_2(\mathcal{E}) - (r-1)c_1^2(\mathcal{E})\right) \cdot \alpha^{n-2} \geq 0.$$

Semi-positivity of c_2 I

Bogomolov–Gieseker inequality, singular case

Let X be a normal compact Kähler space, α be a Kähler class and $\mathcal{E}$ be a rank r reflexive sheaf on X that is stable wrt α. Then:

$$\Delta(\mathcal{E}) \cdot \alpha^{n-2} := \Big(2rc_2(\mathcal{E}) - (r-1)c_1^2(\mathcal{E})\Big) \cdot \alpha^{n-2} \geq 0.$$

Equality case

If $\Delta(\mathcal{E}) \cdot \alpha^{n-2} = 0$ then $\Delta(\mathcal{E}) \cdot \beta^{n-2}$ for any Kähler class β.
In case X has rational singularities, it amounts to saying that $\Delta(\mathcal{E}) \cdot \beta^{n-2}$ for any $\beta \in H^{1,1}(X, \mathbb{R})$.

Semi-positivity of c_2 II

Corollary

If X is a klt compact Kähler space that is smooth in codimension 2 and if $c_1(X) = 0$ then $c_2(X) \cdot \alpha^{n-2} \geq 0$ for any Kähler class α.
In the equality case, $c_2(X) \cdot \alpha^{n-2} = 0$ for any Kähler class.

[3] Holonomy cover (C–Graf–Guenancia–Naumann, 2022)

Semi-positivity of c_2 II

Corollary

If X is a klt compact Kähler space that is smooth in codimension 2 and if $c_1(X) = 0$ then $c_2(X) \cdot \alpha^{n-2} \geq 0$ for any Kähler class α.
In the equality case, $c_2(X) \cdot \alpha^{n-2} = 0$ for any Kähler class.

Proof

Replace X with a quasi-étale cover[3] such that

$$T_X = \bigoplus_{i \in I} \mathcal{E}_i$$

with $\mathcal{E}_i$ stable with respect to α and $\det \mathcal{E}_i = \mathcal{O}_X$ and use $c_2(X) = c_2(T_X)$. □

[3] Holonomy cover (C–Graf–Guenancia–Naumann, 2022)

Decomposition theorem

Bakker–Guenancia–Lehn, 2022

Let X be a compact Kähler space with klt singularities and $c_1(X) = 0$. Up to replacing X with a quasi-étale cover, we have:

$$X \simeq T \times \prod_{i=1}^{k} Y_i \times \prod_{j=1}^{\ell} Z_\ell$$

where

Decomposition theorem

Bakker–Guenancia–Lehn, 2022

Let X be a compact Kähler space with klt singularities and $c_1(X) = 0$. Up to replacing X with a quasi-étale cover, we have:

$$X \simeq T \times \prod_{i=1}^{k} Y_i \times \prod_{j=1}^{\ell} Z_\ell$$

where

- T is a complex torus,

Decomposition theorem

Bakker–Guenancia–Lehn, 2022

Let X be a compact Kähler space with klt singularities and $c_1(X) = 0$. Up to replacing X with a quasi-étale cover, we have:

$$X \simeq T \times \prod_{i=1}^{k} Y_i \times \prod_{j=1}^{\ell} Z_\ell$$

where

- T is a complex torus,
- Y_i are Calabi–Yau spaces: K_{Y_i} is trivial and

$$H^0\left(Y_i, \Omega_{Y_i}^{[p]}\right) \neq 0 \Leftrightarrow p = 0 \text{ or } \dim Y_i.$$

Reduction to CY/IHS case

$$X \simeq T \times \prod_{i=1}^{k} Y_i \times \prod_{j=1}^{\ell} Z_\ell$$

Reduction to CY/IHS case

$$X \simeq T \times \prod_{i=1}^{k} Y_i \times \prod_{j=1}^{\ell} Z_\ell$$

- Z_j are irreducible holomorphic symplectic varieties (IHS): $Z_{j,\mathrm{reg}}$ is endowed with a holomorphic symplectic form whose powers generate the whole algebra of global holomorphic forms.

Reduction to CY/IHS case

$$X \simeq T \times \prod_{i=1}^{k} Y_i \times \prod_{j=1}^{\ell} Z_\ell$$

- Z_j are irreducible holomorphic symplectic varieties (IHS): $Z_{j,\mathrm{reg}}$ is endowed with a holomorphic symplectic form whose powers generate the whole algebra of global holomorphic forms.

$H^1(Y_i, \mathbb{R}) = H^1(Z_j, \mathbb{R}) = 0 \Rightarrow$ any Kähler class α on X can be decomposed

$$\alpha = \alpha_T + \sum_{i=1}^{k} \alpha_i + \sum_{j=1}^{\ell} \beta_j$$

where α_T, α_i andβ_j are Kähler classes on T, Y_i and Z_j.

Reduction to CY/IHS case

We thus have (X smooth in codimension 2):

$$c_2(X) \cdot \alpha^{n-2} = c_2(T_X) \cdot \left(\alpha_T + \sum_{i=1}^{k} \alpha_i + \sum_{j=1}^{\ell} \beta_j \right)^{n-2}$$

$$= \sum_{i=1}^{k} \lambda_i \underbrace{c_2(T_{Y_i}) \cdot \alpha_i^{n_i-2}}_{\geq 0} + \sum_{j=1}^{\ell} \mu_j \underbrace{c_2(T_{Z_j}) \cdot \beta_j^{m_j-2}}_{\geq 0}$$

where λ_i, $\mu_j > 0$, $n_i = \dim Y_i$ and $m_j = \dim Z_j$.

Reduction to CY/IHS case

We thus have (X smooth in codimension 2):

$$c_2(X)\cdot\alpha^{n-2} = c_2(T_X)\cdot\left(\alpha_T + \sum_{i=1}^{k}\alpha_i + \sum_{j=1}^{\ell}\beta_j\right)^{n-2}$$

$$= \sum_{i=1}^{k}\lambda_i \underbrace{c_2(T_{Y_i})\cdot\alpha_i^{n_i-2}}_{\geq 0} + \sum_{j=1}^{\ell}\mu_j \underbrace{c_2(T_{Z_j})\cdot\beta_j^{m_j-2}}_{\geq 0}$$

where $\lambda_i,\ \mu_j > 0$, $n_i = \dim Y_i$ and $m_j = \dim Z_j$.

Conclusion

In our setting:

$$c_2(X)\cdot\alpha^{n-2} = 0 \Rightarrow c_2(T_{Y_i})\cdot\alpha_i^{n_i-2} = c_2(T_{Z_j})\cdot\beta_j^{m_j-2} = 0\ \forall\, i,\ \forall\, j.$$

CY case

Let Y be Calabi–Yau and smooth in codimension 2 with $c_2(Y) \cdot \alpha^{n-2} = 0$ for a Kähler class α.

CY case

Let Y be Calabi–Yau and smooth in codimension 2 with $c_2(Y) \cdot \alpha^{n-2} = 0$ for a Kähler class α.

Equality case of BG inequality: $c_2(Y) \cdot \beta^{n-2} = 0$ for any Kähler class β. In particular, $\beta = [H]$ since Y is projective (it has no 2-forms).

CY case

Let Y be Calabi–Yau and smooth in codimension 2 with $c_2(Y) \cdot \alpha^{n-2} = 0$ for a Kähler class α.

Equality case of BG inequality: $c_2(Y) \cdot \beta^{n-2} = 0$ for any Kähler class β. In particular, $\beta = [H]$ since Y is projective (it has no 2-forms).

GKP $\Rightarrow$ Y is a torus quotient. Contradiction! $\square$

IHS case

Fujiki relations

Let Z be IHS ($\dim Z = 2n$) and smooth in codimension 2. Then $H^2(Z, \mathbb{Q})$ is endowed with a quadratic form q_Z (Beauville–Bogomolov form). There exist constants μ_0 and μ_1 st:

$$a^{2n} = \mu_0 q_Z(a)^n \quad \text{and} \quad c_2(Z) \cdot a^{2n-2} = \mu_1 q_Z(a)^{n-1}$$

for any $a \in H^2(Z, \mathbb{Q})$.

IHS case

Fujiki relations

Let Z be IHS ($\dim Z = 2n$) and smooth in codimension 2. Then $H^2(Z, \mathbb{Q})$ is endowed with a quadratic form q_Z (Beauville–Bogomolov form). There exist constants μ_0 and μ_1 st:

$$a^{2n} = \mu_0 q_Z(a)^n \quad \text{and} \quad c_2(Z) \cdot a^{2n-2} = \mu_1 q_Z(a)^{n-1}$$

for any $a \in H^2(Z, \mathbb{Q})$.

If $c_2(Z) \cdot \alpha^{n-2} = 0$ for a Kähler class α, then c_2 vanishes against the whole $H^2(Z, \mathbb{Q})$.

IHS case

Fujiki relations

Let Z be IHS ($\dim Z = 2n$) and smooth in codimension 2. Then $H^2(Z,\mathbb{Q})$ is endowed with a quadratic form q_Z (Beauville–Bogomolov form). There exist constants μ_0 and μ_1 st:

$$a^{2n} = \mu_0 q_Z(a)^n \quad \text{and} \quad c_2(Z)\cdot a^{2n-2} = \mu_1 q_Z(a)^{n-1}$$

for any $a \in H^2(Z,\mathbb{Q})$.

If $c_2(Z)\cdot\alpha^{n-2} = 0$ for a Kähler class α, then c_2 vanishes against the whole $H^2(Z,\mathbb{Q})$.

Deformation argument

(Bakker–Lehn, 2021) Z admits algebraic approximations $\pi : \mathcal{Z} \to \mathbb{D}$ and we can apply GKP on a projective deformation $\mathcal{Z}_t$.

What next?

It would be great to be able to deal with the *orbifold* second Chern class $c_2^{\mathrm{orb}}(X)$ (Graf–Kirschner in the analytic setting).

What next?

It would be great to be able to deal with the *orbifold* second Chern class $c_2^{\mathrm{orb}}(X)$ (Graf–Kirschner in the analytic setting).

Tentative statement

Let X be a compact Kähler space with klt singularities. If $c_1(X) = 0$ and $c_2^{\mathrm{orb}}(X) \cdot \alpha^{n-2} = 0$ for some Kähler class α, then X is a torus quotient (the group acting freely in codimension 1).

What next?

It would be great to be able to deal with the *orbifold* second Chern class $c_2^{\mathrm{orb}}(X)$ (Graf–Kirschner in the analytic setting).

Tentative statement

Let X be a compact Kähler space with klt singularities. If $c_1(X) = 0$ and $c_2^{\mathrm{orb}}(X) \cdot \alpha^{n-2} = 0$ for some Kähler class α, then X is a torus quotient (the group acting freely in codimension 1).

The algebraic analogue holds true with $\alpha = [H]$ an ample class (Lu–Taji, 2016).

What next?

It would be great to be able to deal with the *orbifold* second Chern class $c_2^{\mathrm{orb}}(X)$ (Graf–Kirschner in the analytic setting).

Tentative statement

Let X be a compact Kähler space with klt singularities. If $c_1(X) = 0$ and $c_2^{\mathrm{orb}}(X) \cdot \alpha^{n-2} = 0$ for some Kähler class α, then X is a torus quotient (the group acting freely in codimension 1).

The algebraic analogue holds true with $\alpha = [H]$ an ample class (Lu–Taji, 2016).

To be able to deal with general torus quotients (no assumptions on the action): consider orbifold structure in codimension 1!