Lattice Polytopes Associated With The Sequence of Grassmannians $Gr(k, k+r)_{k=1}$

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Outline

- Introduction
- Grassmannians
- 2-filling sets
- Symbolic polynomials

WHY DO WE CARE?

Lattice Polytopes

Lattice Polytopes

Some Definitions

A set $X \subset \mathbb{R}^n$ is said to be convex if $\lambda\mathbf{a} + (1-\lambda)\mathbf{b} \in X$ for every pair $\mathbf{a}, \mathbf{b} \in X$ and for every $\lambda \in [0, 1]$.

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A polytope $\mathbf{P}$ is the convex hull of a finite set of points in $\mathbb{R}^n$. That is,

$$\mathbf{P} = \{\sum_{i=1}^{r} \lambda_i \mathbf{x_i} : \ \sum_{i=1}^{r} \lambda_i = 1, \ \lambda_i \in [0,1], \ \mathbf{x_i} \in \mathbb{R}^n\}$$

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The dimension of a polytope **P** is its affine dimension. Given a *d*-dimensional polytope **P**, the **faces** of dimensions 0, 1 and d-1 are called **vertices, edges and facets** of **P** respectively.

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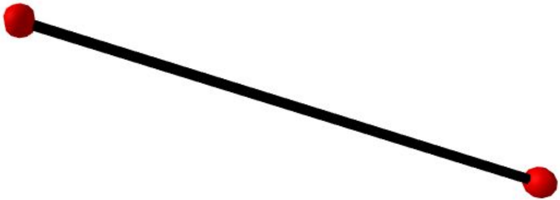
A lattice polytope **P** is the polytope whose vertices have integer coordinates.

d-simplex

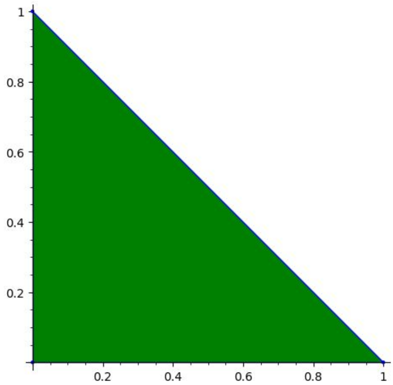
A d-simplex $\triangle_d$ is a d-dimensional polytope which has d+1 vertices. That is, a generalized triangle.

Lattice Polytopes

1-simplex

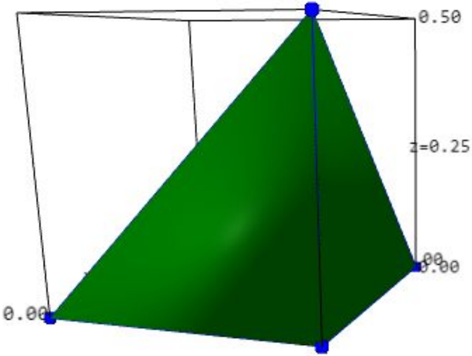


2-simplex



Lattice Polytopes

3-simplex



Combinatorial Problem

Abstractly, the d-simplex $\triangle_d$ can be viewed as the set of all subsets of $[d+1] = \{1, \dots, d+1\}$.
The geometric realization of an abstract d-simplex is the convex hull over all the the standard vectors in $\mathbb{R}^d$ union e_0, where e_0 is the zero vector.

Standard d-simplex

The focus is on the dilations $k\triangle_d$ of the d-standard simplex $\triangle_d$

$$\triangle_d = \{x \in \mathbb{R}^d : x \cdot e_i \geq 0, \ \sum_{i=0}^{d} x \cdot e_i \leq 1\}$$

and

$$k\triangle_d = \{x \in \mathbb{R}^d : x \cdot e_i \geq 0, \ \sum_{i=0}^{d} x \cdot e_i \leq k \ \ k \in \mathbb{Z}_{\geq 1}\}$$

Combinatorial Question

How many integral solutions satisfy the inequality

$$\sum_{i=0}^{d} x \cdot e_i \leq k \text{ where } k \in \mathbb{Z}_{\geq 1}?$$

Combinatorial Question

Ehrhart Theorem

If $\mathbf{P} \subset \mathbb{R}^d$ is a lattice polytope and $k \in \mathbb{Z}_{\geq 1}$ then

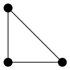
$$\mathrm{ehr}_{\mathbf{P}}(k) = \#(k\mathbf{P} \cap \mathbb{Z}^d)$$

evaluates to a polynomial in k. Equivalently, the Ehrhart series of $\mathbf{P}$ evaluates to a rational function

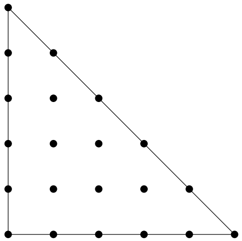
$$\mathrm{Ehr}_{\mathbf{P}}(t) = 1 + \sum \mathrm{ehr}_{\mathbf{P}}(k) t^k = \frac{h^*_{\mathbf{P}}(t)}{(1-t)^{\dim \mathbf{P}+1}}$$

for some polynomial $h^*_{\mathbf{P}}(t)$ of degree at most $\dim\ \mathbf{P}$

k=1, Three integral solutions

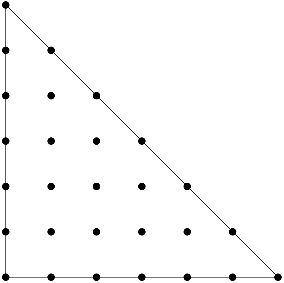


k=5, there are 21 integral solutions

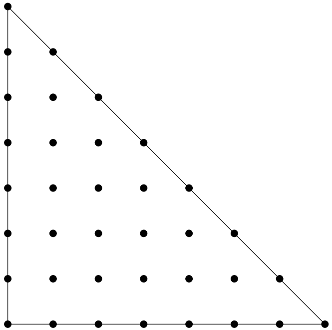


k=6, there are 28 integral solutions

polytopes



k=7, there are 36 integral solutions



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Grassmannians and Flag varieties

Let $V = \mathbb{C}^n$. The set of all k-subspaces of V

$$Gr(k,n) := \{V_k \subseteq V \ : \dim(V_k) = k\}$$

is called Grassmannian variety.

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- $\mathcal{F}\ell_n(\mathbb{C}) \cong GL_n(\mathbb{C})/B$ and $Gr(k,n) \cong GL_n(\mathbb{C})/P_k$

Grassmannians

where P_k is called parabolic subgroup. That is,

$$P_k = \left\{ \begin{bmatrix} \alpha \in GL_k & \beta \\ 0 & \rho \in GL_{n-k} \end{bmatrix} \right\} \tag{1}$$

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- $\pi^{-1}(X_\lambda(V_\bullet)) = X_{w(\lambda)}(V_\bullet)$, where $X_\lambda(V_\bullet)$ is a Schubert variety on the Grassmannian $Gr(k,n)$ indexed by fitted partition λ. The permutation $w(\lambda) \in W(GL_n(\mathbb{C}))/W(P_k) = S_n/S_k \times S_{n-k}$ identified with the partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ is given by

$$w_i = i + \lambda_{k+1-i}, \; 1 \leq i \leq k \;\; and \;\; w_j < w_{j+1}, \;\; k+1 \leq j \leq n \tag{3}$$

This is called Grassmannian permutation associated with λ

Grassmannians

Definition

Let $w \in S_n$. The code $c(w)$ of w is the vector $(c_1(w), \ldots, c_n(w))$ where $c_i(w) = | \{(i,j) : 1 \leq i < j \leq n \text{ and } w(i) > w(j)\} |$. For example, if $w = 316425$ the $c(w) = (2, 0, 3, 1, 0)$. Notice that $c_i(w) \leq n - i$.

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Lemma

Let $\mathbb{Z}^n_{\geq 0}$ be the set of n-tuples of nonnegative integers. There exists an injection of S_n into $\mathbb{Z}^n_{\geq 0}$, sending w to code $c(w)$. Moreover, every code $c(w)$ of w where $w \neq 123 \cdots n$ is the linear combination of the standard vectors.

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Lemma

Let $w(\lambda) = (w_1, \ldots, w_k, \ldots, w_n)$ be a Grassmannian permutation whose descent is at the position k and λ, its associated partition. Then the code $c(w(\lambda))$ is given by $(w_1 - 1, w_2 - 2, \ldots, w_k - k, 0, \ldots, 0)$ and λ is realized as $(w_k - k, \ldots, w_1 - 1)$ where $0 \leq w_i - i \leq k$.

Grassmannians

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- The map (2) induces a monomorphism at the level of cohomology ring

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Borel Presentation

$$\begin{aligned} H^*(Gr(k,n)) &\cong \mathbb{Z}[s_\lambda(x_1,\dots,x_n)]/\langle s_\lambda : \lambda \subsetneq k \times (n-k)\rangle \\ &\cong \mathbb{Z}[x_1,\dots,x_n]^{S_k \times S_{n-k}}/\langle e_1,\dots,e_n\rangle \end{aligned}$$

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Poincaré polynomial

$P(Gr(k,n),t) = \frac{(t^n-1)\cdots(t^{n-k+1}-1)}{(t^k-1)\cdots(t-1)}$

2-filling set $\mathcal{C}^2_{(k,2)}$

The interest is in the combinatorial geometry of the sequence $(Gr(k, 2+k)_{k=1}$ of Grassmannians in type A.

To each term of the sequence we associate a distinguished symbol $\lambda^* = (k, 2)$ and say that the sequence is of the class $\lambda^* = (k, 2)$. The 2-set $\mathcal{C}_{(k,2)}$ identified with λ^* is given by

$$\mathcal{C}_{(k,2)} = \{\square_{1\times d} : \ 1 \leq d \leq k\} \cup \emptyset.$$

That is, $\mathcal{C}_{(k,2)}$ is the set of all $1 \times d$ horizontal dominoes $\square_{1\times d}$ whose number of boxes cannot exceed k. We adjoin the empty set $\emptyset$ to the 2-set.

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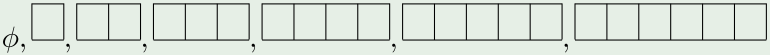
Example

if $\lambda^* = (6, 2)$ then the elements of $\mathcal{C}_{(6,2)}$ are



2-filling set $\mathcal{C}^2_{(k,2)}$

The set $\mathcal{C}_{(k,2)}$ is finite and $\#\mathcal{C}_{(k,2)} = k+1$

Definition

The boxes of the members of $\mathcal{C}_{(k,2)}$ are filled with numbers from the set $[2] := \{1,2\}$ such that the numbers weakly increase from left-to-right. The set of such fillings for $\mathcal{C}_{(k,2)}$ is called 2-filling set denoted by $\mathcal{C}^2_{(k,2)}$ and each filling is called semi-filling domino and denoted by s_f-dominoes.

Notice that by convention the semi-filling of the empty set $\emptyset$ is $\boxed{0}$.

Theorem

The set $\mathcal{C}^2_{(k,2)}$ of s_f-dominoes is finite and its cardinality $L^2(k)$ is $\binom{k+2}{2}$. Moreover, the sequence $(L^2(k))_{k=1}^{\infty}$ of the cardinalities is recorded by the generating function $P(C^2_{(k,2)}, z) = \dfrac{1}{(1-z)^3}$

These are 10 members of $\mathcal{C}^2_{(3,2)}$

$\boxed{0}$, $\boxed{1}$, $\boxed{2}$, $\boxed{1}\boxed{1}$, $\boxed{1}\boxed{2}$, $\boxed{2}\boxed{2}$

$\boxed{1}\boxed{1}\boxed{1}$, $\boxed{1}\boxed{1}\boxed{2}$, $\boxed{1}\boxed{2}\boxed{2}$, $\boxed{2}\boxed{2}\boxed{2}$,

We introduce triangular polynomials $T_k(t) \in \mathbb{Q}[t]$ given by

$$T_k(t) = \sum_{r=0}^{k} (r+1)t^r$$

. They register certain statistics about the elements of $\mathcal{C}^2_{(k,2)}$. Precisely, each term $\mathbf{a}t^{\mathbf{b}}$ of $T_k(t)$ is interpreted as the horizontal rectangle with **b** boxes has **a** ways of fillings.

2-filling set $\mathcal{C}^2_{(k,2)}$

Example

The triangular polynomial $T_5(t)$ of the 2-filling set $\mathcal{C}^2_{(5,2)}$ is given by $T_5(t) = 1 + 2t + 3t^2 + 4t^3 + 5t^4 + 6t^5$

Theorem

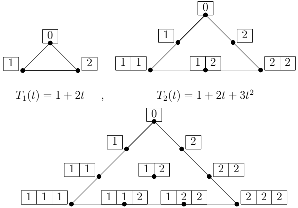
The sequence $\{T_k(t)\}_{k=1}^{\infty}$ *of triangular polynomials in* $\mathbb{Q}[t]$ *as* k *grows is given by recurrence relation* $T_{k+1}(t) = \dfrac{1 - t^{k+2}}{1 - t} + tT_k(t)$ *given that* $T_0(t) = 1$*. Moreover, the series* $\sum_{k=0}^{\infty} T_k(t)z^k = T_0(t) + T_1(t)z + T_2(t)z^2 + T_2(t)z^3 + \cdots$*, is given by the rational function*

$$G(t,z) = \frac{1}{(1-z)(1-tz)^2}\,.$$

.

2-filling set $\mathcal{C}^2_{(k,2)}$

The description of $\mathcal{C}^2_{(k,2)}$ as k grows is the sequence of noded triangles Δ_k each of whose nodes are labeled by s_f dominoes. These triangles are captured by triangular polynomials. $T_1(t)$, $T_2(t)$ and $T_3(t)$ are illustrated below.



$T_1(t) = 1 + 2t$, $T_2(t) = 1 + 2t + 3t^2$

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Newton polygon

Symbolic polynomial $P_k(t_1, t_2)$

Newton polygon

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• The monomial $\prod_{i=1}^{d} t_{r_i}$ is called the weight of s_f- domino

r_1	r_2	$\cdots$	r_d

in $\mathcal{C}^2_{(k,2)}$. The symbolic polynomial $P_k(t_1, t_2)$ is given by

$$P_k(t_1, t_2) = \sum_{\mathcal{C}^2_{(k,2)}} \prod_{k=1}^{d} t_{r_i} \tag{5}$$

Example

The symbolic polynomial associated with $\mathcal{C}^2_{(2,2)}$ is given by

$$P_k(t_1, t_2) = \sum_{\mathcal{C}^2_{(k,2)}} \prod_{i=1} t_{r_i} = 1 + t_1 + t_2 + t_1^2 + t_1 t_2 + t_2^2 \tag{6}$$

Newton Polygon

The Newton polygon $\triangle$ of a polynomial $P_*(t_1, t_2)$ lies in $\mathbb{R}^2$ as the convex hull of the points whose coordinates are exponents of powers that occur in the non-zero coefficient monomials of the polynomial. So, the notion of Newton polygon generalizes that of the degree of its defining polynomial.

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standard 2-simplex

$\bullet \triangle(P_1(t_1, t_2)) = \triangle_2$

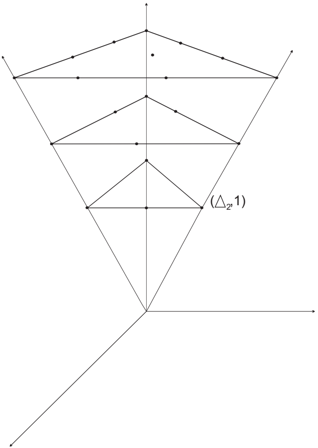
Definition

The cone $\mathcal{C}(\triangle_2)$ over $\triangle_2$ is the set of all non-negative real linear combinations of elements of $(\triangle_3, 1)$. That is,

$$\mathcal{C}(\triangle_2) := \{\sum_{r=0}^{s} \lambda_r d_r : \ s \in \mathbb{N}, \ \lambda \in \mathbb{R}^+ \ d_r \in (\triangle, 1)\} \tag{7}$$

where we set $(\triangle_2, 1) := \{(d, 1) : d \in \triangle_2\} \subset \mathbb{R}^3$

Cone over $\triangle_2$



Polynomial ring

The $\mathbb{C}$-algebra of $\triangle_2$ is the polynomial ring $\mathbb{C}[\triangle_2]$ in three variables t_1, t_2, t^k given by

$$\mathbb{C}[\triangle_2] := \{\sum b_{c_1,c_2,k} t_1^{c_1} t_2^{c_2} t^k : \quad c_1 + c_2 \leq k, \quad b_{c_1,c_2,k} \in \mathbb{C}^*\} \qquad (8)$$

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• This counts the number of monomials in the k-th graded component of $\mathbb{C}[\triangle_2]$.

Hilbert Series

The Hilbert series $H(\mathbb{C}[\triangle_2], z)$ of $\triangle_2$ coincides with the Ehrhart series $\mathrm{Ehr}_{\triangle_2}(z)$

THANK YOU FOR YOUR ATTENTION!